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Robust Rudder Roll Damping Control

———Ph. D. Thesis

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Abstract

The results of a systematic research to solve a specific ship motion control problem, simultaneous roll damping and course keeping using the rudder are presented in this thesis. The fundamental knowledge a priori is that rudder roll damping is highly sensitive to the model uncertainty, therefore $\mathcal{H}_\infty$ theory is used to deal with the problem.

The necessary mathematical tools and the $\mathcal{H}_\infty$ theory as the basis of controller design are presented in Chapter 2 and 3. The $\mu$ synthesis and the D-K iteration are introduced in Chapter 3. The ship dynamics and modeling technology are discussed in Chapter 4, two kinds of ship model have been obtained: linear ship model used for designing the controller and nonlinear model used for simulation. The ship model uncertainty is discussed in this chapter and so is a wave model because the ship’s roll motion is caused by waves.

Using an unstructured model of uncertainty, three controllers with different kind of control schemes are designed by the mixed sensitivity method in Chapter 5. Sea-way simulation results show that each of these controllers have good robust stability and performance. The roll damping reduction is above 35% for all of these controllers. Roll reduction of near 70% has been obtained by the cascade controller.

Using structured model of uncertainty, a $\mu$ controller is designed in Chapter 6. The good robust performance has been recognized in the simulation results. It is shown that the $\mu$ controller has the best robust characteristics with respect to model uncertainty and the roll reduction is near 50% with an envelope of model perturbations.

**Keywords:** roll damping, rudder roll damping, ship model, ship control, robust control
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Chapter 1

Introduction

The main reasons for using roll stabilizing systems on merchant ships are to prevent cargo from damage and to increase the working effectiveness of the crew. From a safety point of view it is well known that large roll motions cause people to make more mistakes during operation due to sea sickness and tiredness. For naval ships certain operations such as landing a helicopter or the effectiveness of the crew during combat are of major importance. Therefore, roll reduction is an important area of research.

If motion in a certain degree of freedom is an undesirable feature of the behavior of a ship in rough weather, it is natural to consider ways of reducing it. Methods of motion reduction are often known by the general name of motion ‘stabilization’, although it should be realized that this is usually an incorrect use of the word. In [Lloyd, 1989], it is pointed out that the oscillatory motions of all practical conventional ship designs are already ‘stable’ in that they can generally be expected to return to an equilibrium datum level after some small disturbance. This is ensured by the stiffness terms in the equations of motion. The term ‘stabilization’ implies an increase in the stiffness coefficients, but almost every practical motion stabilization device derives most of its effect by increasing the motion damping. They should therefore more correctly be called motion damping.

Of all ship motions (heave, pitch, roll, etc.) rolling motion has been respected and successfully damped at present. This is so because normal vessels have inherently low damping properties with respect to roll. Therefore in the region of resonance, where severe rolling is expected, consideration for the safety of the ship is important. On the other hand, only moderate force is necessary to induce an adequate damping moment. In comparison to rolling motion, the conditions of resonance for heave and pitch are less well defined, and the forces necessary for damping are rather large.

In the investigation of motion damping it is the forced motion that is of real importance. With reference to the motion equations it can be seen that there are three different ways to reduce forced motions [Bhattacharyya, 1978]:

1. By increasing the damping coefficient. This is called damping stabilization. Since physically “damping” means a dissipation of energy, this kind of motion stabilization is effective in the case of free oscillations especially in the region of resonance, where damping has its maximum effectiveness. The bilge keel is such a damping device for rolling motion.

2. By reducing the natural frequency of the ship (i.e., by increasing the natural period), so that the tuning factor becomes much greater than unity. This type of motion damping, known as tuning stabilization, is effective only for forced oscillations. Since, however, a seaway is composed of waves of all frequencies, it is not always proper to change the natural frequency of a vessel because the modified natural frequency may come across a different encounter frequency of the same magnitude, and resonance will still take place.

3. By reducing directly the exciting force or moment. This type is known as equilibrium damping and is, in principle, applicable to all kinds of motions; in such a case, a stabilizing moment is applied that is opposite in phase to the exciting moment, and this will result in a reduction in the exciting moment. However, the usefulness of this kind of damping depends
entirely on the effectiveness of the control device. Although equilibrium damping is used mainly in the case of forced motions, it is equally effective in the case of free oscillations.

As mentioned above, rolling motion has most successfully been controlled from the point of view of stabilization, since the forces and moments involved in rolling are comparatively small.

The stabilizers and other means used to control rolling motion have been of many kinds: the following are some examples.

a. Bilge keels.
b. Roll damping fins
c. Anti-rolling tanks
d. Rudder roll damping
e. Gyroscopic stabilizers.
f. Movement of weight.
g. Jet flaps.

Because of economical considerations, some of the devices mentioned above are not used at present, whereas others, although they may not be most effective technically, continue in use. We will shortly introduce the first four and make a comparison for them.

1.1 Bilge keels

Bilge keels are the simplest form of roll damping device. They were first demonstrated in about 1870. They are fins in planes approximately perpendicular to the hull near the turn of the bilge. The longitudinal extent varies from about 25 to 50 percent of the length of the ship as shown in Figure 1.1.

![Figure 1.1 Bilge keel notation](image)

Bilge keels are very effective roll damping devices which work well at all speeds. They have the significant advantage that they have no moving parts and require no maintenance beyond that normally given to the hull surface. Their only disadvantage is that they increase the resistance of the ship, but this effect can be minimized by carefully aligning the keels with the flow streamlines around the bilges. This is usually done using some kind of flow visualization technique on a model during the design stage. Correct alignment can only be achieved at one speed (the cruising speed is usually chosen) but the resistance penalty at other speeds is usually small.
1.2 Roll Damping Fins

Roll damping fins are a highly attractive device for roll damping. These are usually mounted on rotatable stocks at the turn of the bilge near the middle of the ship as shown in Fig 1.2. The angle of incidence of the fins is continuously adjusted by a control system, which is sensitive to the rolling motion of the ship. The fins develop lift forces that exert roll moments about the center of gravity of the ship. These roll moments are arranged to oppose the moment applied by the waves and the roll motion is reduced.

At speeds above 10–15 knots active fins are probably the most effective method of damping a ship. Roll motion reductions of at least 50% in roll damping system are usually possible in moderate waves with a well designed system. However, the fins become progressively less effective as the speed is reduced and they are not usually specified for ships which habitually operate at low speed. It should also be understood that fins have a limited capacity and their ability to reduce roll motion decreases in very severe sea states. They are relatively sophisticated and expensive pieces of equipment and require considerable maintenance. Nevertheless, their ability to work well over a wide range of conditions has earned them almost universal acceptance and they are now fitted to many ships.

Retractable fins are often specified for merchant ships. In such case, the fins can be withdrawn into the hull when the ship is operating in calm weather to eliminate their small resistance penalty. This feature is also used to eliminate the risk of grounding when the ship is operating in shallow water or coming alongside. Retractable fins are usually of high aspect ratio and are hydrodynamically very efficient, giving a relatively large lift for a given fin area.

In war ships it is usual to fit non-retractable fins as these have a greater immunity to damage from shock and explosion. It is then necessary to confine the fins to the enclosing rectangle defined by the ship’s maximum beam and draught (see Figure 1.2). This places an effective limit on the area and aspect ratio which can be adopted and these fins are usually rather less efficient than their retractable counterparts. A disadvantage is that the roll damping fins cause drag and underwater noise that cannot be neglected by war ships.

1.3 Anti-rolling Tanks

The fluid in a partially filled tank in a ship will swash backwards and forwards across the tank as the ship rolls. The shifting weight of the fluid will exert a roll moment on the ship and, by suitable design, this can be arranged to damp the roll motion. Figure 1.3 shows some of the types
of anti-rolling tanks which are currently in use. The simplest is the flume of free surface tank that consists of a rectangular tank running athwartships. Sometimes a limited control is exerted over the motion of the fluid by installing a restriction or baffle in the center of the tank.

U-tube tanks have also been fitted in a number of ships. In this case the free surface is confined to the two arms of the U-tube which are connected by a horizontal duct. The tops of the vertical arms may be open to the atmosphere or they may be connected by a horizontal air duct. In this case a throttle valve may be included to exert some control over the motions of the fluid. Some designs incorporate a throttle valve or a pump in the bottom duct.

![Simple flume tank](image1)
![Flume tank with baffle](image2)
![Simple U-tube tank](image3)
![U-tube with air duct and Throttle valve](image4)
![U-tube with throttle valve](image5)
![Active U-tube with pump](image6)

Figure 1.3 Types of Anti-roll tanks

Anti-rolling Tanks work well at low speeds but they are not usually as effective as a well designed active fin system at high speed. For this reason they are often specified for ships like survey vessels or weather ships that must spend the majority of their time hove to.

Tanks have the advantage that they have no moving parts (except perhaps for a pump or controlled throttle valve) and require little maintenance. They also avoid the small resistance penalty associated with fins and bilge keels. They take up a considerable volume of the ship’s hull but it may by possible to use the fresh water supply or some of the fuel oil as the working fluid so this loss of volume may not be serious. The optimum tank position high in the ship often makes access along the ship difficult.

A major disadvantage is that the free surface always reduces the metacentric height so that roll stability will be reduced. As a consequence all tanks amplify roll motions at low encounter frequencies. In certain circumstances this amplification may become a serious problem and it may be necessary to immobilize the tank by draining it or filling it completely. This will invariably take
1. Introduction

a considerable time and anti-rolling tanks are therefore not suitable for the ships which are required to change course frequently (e.g. warships).

1.4 Rudder Roll Damping (RRD)

Roll of a ship is caused by external disturbances, e.g. wave, wind and current, these contribute to the roll by exerting varying forces and moments on the hull. But roll is also caused by movement of the rudder. An alteration of course makes the ship heel and when the ship rights itself, it turns back towards its equilibrium position in a damped oscillation.

Table 1.1 Overall comparison of ship roll damping systems [Sellars and Martin, 1992]

<table>
<thead>
<tr>
<th>Roll Damping Type</th>
<th>General Application</th>
<th>% Roll Reduction</th>
<th>Price ($x1000)</th>
<th>Installation</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fins (small fixed)</td>
<td>mega yachts; naval auxiliaries</td>
<td>90</td>
<td>100-200</td>
<td>hull attachment; supply and install power and control cables</td>
<td>speed loss; largest size about 2m², underwater noise</td>
</tr>
<tr>
<td>Fins (retractable)</td>
<td>passenger; cruise; ferries; naval combatants</td>
<td>90</td>
<td>400-1500</td>
<td>hull attachment; supply and install power and control cables</td>
<td>size range from 2m² to about 15m²</td>
</tr>
<tr>
<td>Fins (large fixed)</td>
<td>naval combatants</td>
<td>90</td>
<td>300-1300</td>
<td>hull attachment; supply and install power and control cables</td>
<td>speed loss; underwater noise</td>
</tr>
<tr>
<td>Tanks (free surface)</td>
<td>work vessels; small passenger and cargo ships; ferries;</td>
<td>75</td>
<td>30-50</td>
<td>install steelwork supply and install power and control cables</td>
<td>includes liquid level monitor</td>
</tr>
<tr>
<td>Tanks (U-tube)</td>
<td>work vessels</td>
<td>75</td>
<td>200-300</td>
<td>install steelwork and piping and valves; install instrument cables</td>
<td>includes heel control system capability</td>
</tr>
<tr>
<td>RRD</td>
<td>small, high speed vessels</td>
<td>50-75</td>
<td>50-250</td>
<td>install power and control cable</td>
<td>new development; more robust steering gear may be required</td>
</tr>
<tr>
<td>Bilge Keels</td>
<td>universal</td>
<td>25-50</td>
<td></td>
<td>hull attachment</td>
<td>speed loss</td>
</tr>
</tbody>
</table>

Roll damping by means of rudders are relatively inexpensive compared to roll damping fins, has approximately the same effectiveness, and causes no drag or underwater noise if the system is
turned off. However, RRD requires relatively fast rudder to be effective, typically $\delta_{\text{max}} = 5 - 20$ (deg/s). Another disadvantage is that the RRD will not be effective if the ship’s speed is low.

Roll damping by means of rudders has been analyzed by numerous authors since 1972 [Cowley and Lambert, 1972]. Rudder Roll Damping system design for naval vessels and the reports of sea experiments can be found in Baitis, Woolaver and Beck (1983, 1989), Källström, Wessel and Sjolander (1988), Amerongen, Klugt and Pieffers, (1987). A lot of papers had been published since then to introduce the results of different control and modeling methods for RRD, for instance, LQ control [Blanke, Haals and Andreasen, 1989]; LQG control [Katebi et al., 1989]; Neural Network Control [Tiaon and Zhou, 1992]; Fuzzy control [Zacharias and Pfister, 1995]; Multivariate Auto Regressive Model [Oda et al., 1992; Oda, Ohtsu and Hotta, 1995]. Investigations have shown that the roll reduction by rudder could be 50%~70% for a specific vessel. Early in 1990, commercial controllers for RRD presented in the Europe market [Källström and Schultz, 1990]. However, the advance of RRD technique is slow and most of the research is concentrated to navy vehicles. The main reason is that large power and rudder rate of the steering machine is needed for RRD, and it is easy to satisfy such requirements by navy vehicles because of their powerful maneuverability. Contrarily, it will need to update the steering machine for commercial ships for RRD. The second reason is that the effect of RRD highly depends on the dynamics of the ship. Experiments have been carried out in which the same RRD controller performed satisfactorily on one ship but unsatisfactorily on the other, although both of them are sister ships and have almost the same hull geometry. The small difference lies in the form of bilge keels, rudder shape and loading conditions [Blanke and Christensen, 1993]. This means that RRD is highly sensitive to the model uncertainty.

Practically, the model uncertainty is not only produced by the variation of the ship structure, but also by the change of ship’s speed, the ship’s load and fouling of the ship hull. It will reduce the roll reduction, even makes RRD fail. This clearly indicates that it is necessary to design a RRD controller with respect to an appropriate robust performance to deal with model uncertainties. The $\mathcal{H}_{\infty}$ theory is a popular and available robust control method in controller design. A $\mathcal{H}_{\infty}$ controller designed with Multi Objective Sensitivity method was done by Stoustrup et al., (1995), but no simulation results was presented. Another research was done by Christiansen, (1995), the simulation result is good. Since they only consider to deal with the output disturbances in the design by multi objective sensitivity, model uncertainty was not considered neglected, so the results are very conservative.

This thesis considers model uncertainty. Both unstructured and structured model uncertainty is discussed. Three types of $\mathcal{H}_{\infty}$ controller are designed for unstructured uncertainty. The first one is a cascade controller consisting of two controllers, heading controller and roll damping controller [Yang and Blanke, 1997; Yang, Jia and Bi, 1998]. The second one is a parallel of two controllers [Yang and Jia, 1998], with their outputs connected together to control the rudder in superposition of the command signals. The third is a multi-input single-output MISO controller [Yang, 1997; Yang, Jia and Bi, 1998], which course keeping and roll damping combines. All of the three types of controllers are designed by the mixed sensitivity approach.

The main point of this thesis is structured model uncertainty control for RRD. For structured uncertainty, $\mu$ analysis and $\mu$ synthesis are better methods currently in use for controller design. In the last part of the thesis, a multi-input single-output controller is designed by $\mu$ synthesis [Yang, 1997; Yang and Blanke, 1998; Yang and Jia]. Seaway simulations have been done and the results show that all the controllers have good robust stability and robust performance, among them the best one is the controller designed by $\mu$ synthesis.
Chapter 2
Spaces and Norms

In order to fully comprehend and appreciate modern robust control theory, some mathematical prerequisites from functional analysis are necessary. In many fine textbooks on robust control, this prior knowledge is either assumed or discussed only very briefly. However, it is believed that an introduction to the relevant spaces and norms is appropriate for the later discussion in the thesis. The necessary mathematical tools are introduced in this chapter, although some of them are conceptually quite straightforward and others are computationally involved.

All the proofs of theorems stated in this thesis are not given explicitly. They can be found in [Morari and Zafiriou, 1989; Skogestad and Postlethwaite, 1996; Tøffner-Clausen, 1995; Zhou, Doyle and Glover, 1995] and other references.

2.1 Eigenvalues and Eigenvectors

Let $A$ be a square $n \times n$ matrix. The eigenvalues $\lambda_i, i = 1, \ldots, n,$ are the $n$ solutions to the $n$'th order characteristic equation

$$\det(A - \lambda I) = 0 \quad \text{(2.1)}$$

The (right) eigenvector $t_i$ corresponding to the eigenvalue $\lambda_i$ is the nontrivial solution ($t_i \neq 0$) to

$$(A - \lambda_i I)t_i = 0 \iff At_i = \lambda_i t_i \quad \text{(2.2)}$$

The corresponding left eigenvectors $q_i$ satisfy

$$q_i^*(A - \lambda_i I) = 0 \iff q_i^* A = \lambda_i q_i^* \quad \text{(2.3)}$$

When we just say eigenvector we mean the right eigenvector.

Remark. The left eigenvectors of $A$ are the (right) eigenvectors of $A^*$. $A^*$ denotes the complex conjugate transpose of $A$, $A^* = AT$. The eigenvalues are sometimes called characteristic gains. The set of eigenvalues of $A$ is called the spectrum of $A$, the largest of the absolute values of the eigenvalues of $A$ is the spectral radius of $A$, $\rho(A) = \max_i |\lambda_i(A)|$.

Note that if $t$ is an eigenvector then so is $\alpha t$ for any constant $\alpha$. Therefore, the eigenvectors are usually normalized to have unit length, i.e. $t_i^* t_i = 1$. An important result for eigenvectors is that eigenvectors corresponding to distinct eigenvalues are always linearly independent. For repeated eigenvalues, this may not always be the case, that is, not all $n \times n$ matrices have $n$ linearly independent eigenvectors (these are the so-called “defective” matrices).

The eigenvectors may be collected as columns in the matrix $T$ and the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ as diagonal elements in the matrix $\Lambda$:

$$T = [t_1, t_2, \ldots, t_n]; \quad \Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \quad \text{(2.4)}$$

We may then write (2.2) in the following form
\[ AT = TA \quad (2.5) \]

Let us now consider using \( T \) for “diagonalization” of the matrix \( A \). That is possible when the eigenvectors are linearly independent and \( T^{-1} \) exists. This always happens if the eigenvalues are distinct, or when the eigenvectors corresponding to a multiple eigenvalue are linearly independent. From (2.5) it follows that the eigenvector matrix diagonalizes \( A \) in the following manner

\[ A = T^{-1}AT \quad (2.6) \]

Let \( \lambda_i \) denote the eigenvalues of \( A \), the following are the properties of eigenvalues:

1. The sum of the eigenvalues of \( A \) is equal to the trace of \( A \) (sum of the diagonal elements):
   \[ \text{Trace}(A) = \sum \lambda_i. \]

2. The product of the eigenvalues of \( A \) is equal to the determinant of \( A \): \[ \det A = \prod \lambda_i. \]

3. The eigenvalues of an upper or lower triangular matrix are equal to the diagonal elements of the matrix.

4. For a real matrix the eigenvalues are either real, or occur in complex conjugate pairs.

5. \( A \) and \( A^T \) have the same eigenvalues (but different eigenvectors in general).

6. The inverse \( A^{-1} \) exists if and only if all eigenvalues of \( A \) are non-zero. The eigenvalues of \( A^{-1} \) are then \( 1/\lambda_1, \ldots, 1/\lambda_n \).

7. The matrix \( A + cI \) has eigenvalues \( \lambda_i + c \).

8. The matrix \( cA^k \) where \( k \) is an integer has eigenvalues \( c\lambda_i^k \).

9. Consider the \( l \times m \) matrix \( A \) and the \( m \times l \) matrix \( B \). Then the \( l \times l \) matrix \( AB \) and the \( m \times m \) matrix \( BA \) have the same non-zero eigenvalues. To be more specific assume \( l > m \). Then the matrix \( AB \) has the same \( m \) eigenvalues as \( BA \) plus \( l - m \) eigenvalues which are identically equal to zero.

10. Eigenvalues are invariant under similarity transformations, that is, \( A \) and \( DAD^{-1} \) have the same eigenvalues.

11. The same eigenvectors matrix diagonalizes the matrix \( A \) and the matrix \( (I+A)^{-1} \).

12. **Gershgorin’s theorem.** The eigenvalues of the \( n \times n \) matrix \( A \) lie in the union of \( n \) circles in the complex plan, each with center \( a_{ii} \) and radius \( r_i = \sum_{j \neq i} |a_{ij}| \) (sum of off-diagonal elements in row \( i \)). They also lie in the union of \( n \) circles, each with center \( a_{ii} \) and radius \( r_i = \sum_{j \neq i} |a_{ji}| \) (sum of off-diagonal elements in column \( i \)).

13. A matrix is positive definite if and only if all its eigenvalues are real and positive.

### 2.2 Vector Norms and Matrix Norms

It is useful to have a single number, to give an overall measure of the size of a vector, a matrix, a signal or a system. For this purpose, functions called norms are used. The most commonly used norm is the Euclidean vector norm, \[ \| e \|_2 = \sqrt{e_1^2 + e_2^2 + \cdots + e_n^2}. \] This is simply the distance between two points \( y \) and \( x \), where \( e_i = y_i - x_i \) is the difference in their \( i \)th coordinates.

A norm of \( x \), which may be a vector, a matrix, a signal or a system, is a real number denoted by \( \| x \| \), that satisfies the following properties:

1. \( \| x \| \geq 0 \) (positivity);
2. \( \| x \| = 0 \) if and only if \( x = 0 \) (positive definiteness);
3. \( \| \alpha x \| = |\alpha| \| x \| \), for any scalar \( \alpha \) (homogeneity);
4. \( \| x + y \| \leq \| x \| + \| y \| \) (triangle inequality)

More precisely, \( x \) and \( y \) are elements in a vector space \( V \) over the field \( C \) of complex numbers, and the properties above must be satisfied \( \forall x, y \in V \) and \( \alpha \in C \). Given a linear space \( H \) there may
be many possible norms on $H$. Given a linear space $H$ and a norm $\| \cdot \|$ on $H$, the pair $(H, \| \cdot \|)$ is called a normed space.

**Remark** The same notation $\| \cdot \|$ denote entirely different norms for different elements. For example, consider the infinity-norm, $\| x \|_\infty$. If $x$ is a constant vector, then $\| x \|_\infty$ is the largest element in the vector (we often use $\| x \|_{\text{max}}$ for this). If $x(t)$ is a scalar time signal, then $\| x(t) \|_\infty$ is the peak value of $|x(t)|$ as a function of time. If $X$ is a constant matrix then $\| X \|_\infty$ may denote the largest matrix element (we use $\| X \|_{\text{max}}$ for this) and in this thesis we use $\| X \|_\infty$ to denote the largest matrix row-sum. Finally, if $X(s)$ is a stable proper system (transfer function), then $\| X \|_\infty$ is the $H_\infty$ norm which is the peak value of the maximum singular value of $X$, $\| X(s) \|_\infty = \max_{\omega} \sigma(X(j\omega))$.

### 2.2.1 Vector Norms

Let $V$ be a vector space over the field $K$, where $K$ is either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers. Then $x \in V$ means that $x = (x_1, x_2, \ldots, x_n)$ with $x_i \in K$, $\forall i$. In this thesis, we will use $\mathbb{C}^n$ to represent a $n$ dimension vector subspace of $V$. Clearly, $\mathbb{C}^n$ is the space of complex $n$-vectors. For $x \in \mathbb{C}^n$ the $p$-norms are defined by:

$$\| x \|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \quad (2.7)$$

where $p \geq 1$ must be taken to satisfy the triangle inequality (property 4 of a norm).

In control theory, the $1$-, $2$- and $\infty$-norm are most important since they have obvious physical interpretations:

**Vector 1-norm (sum norm).** This simple norm just gives the sum of absolute values of all elements of the vector

$$\| x \|_1 = \sum_{i=1}^{n} |x_i| \quad (2.8)$$

**Vector 2-norm (Euclidean norm).** This is the most common vector norm, and corresponds to the shortest distance between two points

$$\| x \|_2 = \sqrt{\sum_{i=1}^{n} |x_i|^2} = \sqrt{x^* x} \quad (2.9)$$

**Vector $\infty$-norm (max norm).** This is the largest element magnitude in the vector.

$$\| x \|_\infty = \| x \|_{\text{max}} = \max_{i} |x_i| \quad (2.10)$$

Since the various vector norms only differ by constant factors, all norms on $\mathbb{C}^n$ are equivalent norms which means that if $\| \cdot \|_\alpha$ and $\| \cdot \|_\beta$ are norms on $\mathbb{C}^n$, then there exists a pair $c_1, c_2 > 0$ so that

$$c_1 \| x \|_\alpha \leq \| x \|_\beta \leq c_2 \| x \|_\alpha \quad \forall x \in \mathbb{C}^n: \quad (2.11)$$

In particular, $\forall x \in \mathbb{C}^n$:

$$\| x \|_2 \leq \| x \|_1 \leq \sqrt{n} \| x \|_2 \quad (2.12)$$

$$\| x \|_\infty \leq \| x \|_2 \leq \sqrt{n} \| x \|_\infty \quad (2.13)$$

$$\| x \|_\infty \leq \| x \|_{\text{max}} \leq n \| x \|_\infty \quad (2.14)$$

In Figure 2.1 the differences between the vector norms are illustrated by plotting the contours for $\| x \|_p=1$ for the case with $n = 2$. 
2.2.2 Matrix Norms

Now let us consider the space $H \in \mathbb{C}^{m \times n}$, namely the space of $m \times n$ complex matrices. $\mathbb{C}^{m \times n}$ is also a linear space. A norm on a matrix $\| A \|$ is a matrix norm, if in addition to the four norm properties in (1) ~ (4), it also satisfies the multiplicative property

(5). $\| AB \| \leq \| A \| \cdot \| B \|$ (multiplicative property)

Property (5) is very important when combining systems, and forms the basis for the small gain theorem. Note that there exist norms on matrices, which are not matrix norms in case they satisfy the first four properties of norm but do not satisfy the fifth. Such norms are called generalized matrix norms.

Consider the following equation which is illustrated in Figure 2.2

$$y = Gu$$

(2.15)

We may think of $u$ as the input vector and $y$ as the output vector and the “gain” of the matrix $A$ as defined by the ratio $\| y \| / \| u \|$. The maximum gain for all possible input directions is of particular interest. This is given by the matrix $p$-norm. Matrix $p$-norms on $\mathbb{C}^{m \times n}$ are defined in terms of the $p$-norms for vectors on $\mathbb{C}^n$:

$$\| A \|_p = \sup_{x \in \mathbb{C}^n, \| x \|_p = 1} \| Ax \|_p, \quad \forall A \in \mathbb{C}^{m \times n}$$

(2.16)

Notice that the matrix $p$-norms are induced norms. They are induced by the corresponding $p$-norms on vectors. One can think of $\| A \|_p$ as the maximum gain of the matrix $A$ measured by the norm ratio of vectors before and after multiplication by $A$. In general matrix $p$-norms are difficult to compute. However, for $p = 1, 2, \text{ or } \infty$, there exist simple algorithms to compute $\| A \|_p$ exactly. If $A = [a_{ij}] \in \mathbb{C}^{m \times n}$ we have

$$\| A \|_1 = \max_j \sum_{i=1}^m |a_{ij}| \quad \text{maximum column sum}$$

Figure 2.1 Contours for the vector $p$-norm, $\| x \|_p = 1$ for $p = 1, 2, \infty$

Figure 2.2 A simple input-output relation
\[ \|A\|_2 = \sqrt{\rho(A^*A)} = \sigma_1(A) \] spectral norm or maximum singular value

\[ \|A\|_\infty = \max_{i,j} |a_{ij}| \] maximum row sum

where the spectral radius \( \rho(A) = \max \lambda_i(A) \) is the largest eigenvalue of the matrix \( A \).

It is easy to prove that all induced norms \( \|A\|_p \) are matrix norms and thus satisfy the multiplicative property represented in Figure 2.3.

\[
\begin{array}{ccc}
    & u & \rightarrow & B & \rightarrow & v & \rightarrow & A & \rightarrow & y \\
\end{array}
\]

Figure 2.3 Representation of multiplicative property

The fourth matrix norm that is important in modern control theory is the \( F \)-norm. It is given simply as the root sum of squares of the magnitude of all the matrix elements:

\[
\|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2}, \quad \forall A \in \mathbb{C}^{m \times n} \tag{2.17}
\]

Notice that the \( F \)-norm is not an induced norm.

The \( F \)- and \( \rho \)-norms on \( \mathbb{C}^{m \times n} \) are also equivalent norms. Thus there are upper and lower bounds on the ratio between any two different norms applied to the same matrix. If one norm for a given matrix tends towards zero or infinity, so do all other norms. Let \( A = [a_{ij}] \in \mathbb{C}^{m \times n} \). Then

\[
\begin{align*}
\sigma_1(A) = \|A\|_2 & \leq \|A\|_F \leq \sqrt{\min(m,n)} \|A\|_2 \\
\|A\|_{\max} & = \max_{i,j} |a_{ij}| \leq \|A\|_2 \leq \sqrt{mn} \max_{i,j} |a_{ij}| \\
\|A\|_2 & \leq \|A\|_1 \leq \sqrt{m} \|A\|_\infty \tag{2.18} \\
\|A\|_2 & \leq \sqrt{n} \|A\|_\infty \tag{2.19} \\
\frac{1}{\sqrt{n}} \|A\|_\infty & \leq \|A\|_2 \leq \sqrt{m} \|A\|_\infty \tag{2.20} \\
\frac{1}{\sqrt{m}} \|A\|_1 & \leq \|A\|_2 \leq \sqrt{n} \|A\|_1 \tag{2.21} 
\end{align*}
\]

Note that from (2.19) the maximum singular value is closely related to the largest element of the matrix. Therefore \( \|A\|_{\max} \) can be used as a simple and readily available estimate of \( \|A\|_2 \). An important property of the matrix 2-norm and \( F \)-norm is that they are invariant under multiplication by unitary or orthogonal matrices. Assume that \( Q^*Q = I \) and \( Z^*Z = I \) for \( Q \in \mathbb{C}^{m \times m} \) and \( Z \in \mathbb{C}^{n \times n} \). Then

\[
\begin{align*}
\|QAZ\|_F & = \|A\|_F \tag{2.23} \\
\|QAZ\|_2 & = \|A\|_2 \tag{2.24}
\end{align*}
\]

This property is crucial to many proofs in robust control theory.

### 2.2.3 Signal Norms

We will consider the temporal norm of a time-varying (or frequency-varying) signal \( e(t) \). In contrast with spatial norms (vector and matrix norm), we find that the choice of temporal norm
makes a big difference. As an example, consider Figure 2.4 which shows two signals, $e_1(t)$ and $e_2(t)$. For $e_1(t)$ the infinity-norm (peak) is one, $\| e_1(t) \|_\infty = 1$ whereas since the signal does not “die out” the 2-norm is infinite, $\| e_1(t) \|_2 = \infty$. For $e_2(t)$ the opposite is true.

The following temporal norms of signals are commonly used:

1. **1-norm in time** (integral absolute error):
   \[
   \| e(t) \|_1 = \int_{-\infty}^{\infty} \sum_i e_i(\tau) \, d\tau
   \]  
   (2.25)

2. **2-norm in time** (quadratic norm, integral square error):
   \[
   \| e(t) \|_2 = \sqrt{\int_{-\infty}^{\infty} \sum_i |e_i(\tau)|^2 \, d\tau}
   \]  
   (2.26)

3. **$\infty$-norm in time** (peak value in time)
   \[
   \| e(t) \|_\infty = \max \left( \max_{\tau} |e_i(\tau)| \right)
   \]  
   (2.27)

The meaning of 1-norm and $\infty$-norm are shown in Figure 2.5. To be mathematically correct we should have used $\sup_\tau$ rather than $\max_\tau$ in (2.27), since the maximum value may not actually be achieved (e.g. if it occurs at $t = \infty$).

2.3 Singular Values

In modern control theory singular values have been used to extend the classical frequency response Bode plot to multivariable systems. Consider the input-output relation shown in figure 2.2, we have the following expression:
where \( G \) is a transfer function matrix. We use the matrix norm to represent the maximum system gain. In particular, if we use 2-norms, then
\[
\|G\|_2 = \sup_{u \neq 0} \frac{\|Gu\|_2}{\|u\|_2} = \overline{\sigma}(G)
\]
(2.29)

where \( \overline{\sigma}^2(G) \) is the maximum eigenvalue of \( G^*G \). Now, if \( G \) has \( m \) rows and \( n \) columns, and \( m \geq n \), then the positive square roots of the eigenvalues of \( G^*G \) are called the singular values of \( G \). (If \( m \leq n \), then the square roots of the eigenvalues of \( GG^* \) are the singular values of \( G \)).

If instead of \( G \), we have \( G(s) \), and set \( s = j \omega \) \((0 \leq \omega < \infty)\), then the singular values of \( G(j\omega) \) are function of \( \omega \) we shall denote them by \( \sigma_i(\omega) \) when we wish to emphasize their dependence on frequency, or by \( \sigma_i(G) \) when we wish to distinguish the principal gains of \( G \) from those of some other system.

How do we then evaluate the frequency response of \( G(j\omega) \)? An obvious way would be to pick one of the induced matrix norms introduced previously. All of the 1-, 2- and \( \infty \)-norm have potential engineering applications. However, the control theory for using them in design or analysis is only well-developed for the 2-norm. Thus let us evaluate the frequency response of \( G(s) \) at the frequency \( \omega \) by
\[
\|G(j\omega)\|_2 = \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{\|G(j\omega)u(j\omega)\|_2}{\|u(j\omega)\|_2} = \overline{\sigma}(G(j\omega))
\]
(2.30)

Letting \( 0 \leq \omega < \infty \) we may compute the matrix 2-norm for every \( \omega \) to obtain a upper bound for the “gain” of the transfer matrix \( G(s) \). However, we would like to have a lower bound on \( G(j\omega) \) as well. This lower bound can be obtained with the minimum singular value given by
\[
\underline{\sigma}(G(j\omega)) = \inf_{u \in \mathbb{C}^n \setminus \{0\}} \frac{\|G(j\omega)u(j\omega)\|_2}{\|u(j\omega)\|_2}
\]
(2.31)

Thus if we measure the “gain” of the system \( G(s) \) as 2-norm ratio of the input and output, then the maximum and minimum singular values of \( G(s) \) will constitute upper and lower bound on this gain. In fact, we may assess the system “gain” even in more detail using the singular value decomposition. Let us, first, introduce the following important relation between the singular values and the eigenvalues of a complex matrix.

The singular values of a complex matrix \( \forall G \in \mathbb{C}^{m \times n} \), denoted by \( \sigma_i(G) \), are the \( k \) nonnegative square roots of the eigenvalues of \( G^*G \), where \( k = \min\{n, m\} \). Thus
\[
\sigma_i(G) = \sqrt{\lambda_i(G^*G)} \quad i = 1, 2, \ldots, k
\]
(2.32)

It is usually ordering the singular values in the sequence of \( \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k \) and denote \( \sigma_1, \sigma_k \) by \( \overline{\sigma} \) and \( \underline{\sigma} \) respectively, then the largest and smallest singular value are
\[
\overline{\sigma}(G) = \sigma_1(G) = \sup_{u \in \mathbb{C}^n \setminus \{0\}} \frac{\|Gu\|_2}{\|u\|_2} = \|G\|_2
\]
(2.33)
\[
\underline{\sigma}(G) = \sigma_k(G) = \inf_{u \in \mathbb{C}^n \setminus \{0\}} \frac{\|Gu\|_2}{\|u\|_2} = \|G^{-1}\|_2^{-1} \quad \text{if } G^{-1} \text{ exists}
\]
(2.34)

The ratio between the maximum and minimum singular value is called the condition number \( \kappa \).
2. Space and Norms

Let us then introduce the **singular value decomposition**. Let \( G \in \mathbb{C}^{m \times n} \) be a complex matrix. Then there exist two unitary matrices \( V \in \mathbb{C}^{m \times m}, U \in \mathbb{C}^{n \times n} \) and a diagonal matrix \( \Sigma \in \mathbb{R}^{m \times n} \) such that

\[
G = V \Sigma U^* \tag{2.36}
\]

where

\[
\Sigma_k = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_k).
\]

\( v_1 \rightarrow v_m \) are the \( m \) columns of \( V \).

\( u_1^* \rightarrow u_n^* \) are the \( n \) rows of \( U^* \).

This is known as the singular-value decomposition (SVD) of the matrix \( G \).

An interpretation for the SVD of a real matrix \( G \) is as follows. Any real matrix \( G \), looked at geometrically, maps a unit radius hyper-sphere into a hyper-ellipsoid. The right unitary matrix \( U^* \) and the left unitary matrix \( V \) make major axis rotations in the unit radius hyper-sphere and in the hyper-ellipsoid respectively. The singular values give the lengths of the principal axes of the ellipsoid when mapping from hyper-sphere to hyper-ellipsoid. The following example shows the SVD of a real matrix [Morari and Zafiriou, 1989].

**Example 2.2** Let \( G \) be given by

\[
G = \begin{bmatrix}
0.8712 & -1.3195 \\
1.5783 & -0.0947
\end{bmatrix}
\]

The SVD of \( G \) is \( G = V \Sigma U^* \) where

\[
V = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}, \quad \Sigma = \begin{bmatrix}
2 & 0 \\
0 & 1
\end{bmatrix}, \quad U = \frac{1}{\sqrt{3}} \begin{bmatrix}
\sqrt{3} & 1 \\
-1 & \sqrt{3}
\end{bmatrix}
\]

It is interpreted geometrically in Figure 2.6

In the following some of the important properties of singular values are stated:
\[ \sigma(G) \leq |\lambda_i(G)| \leq \sigma(G) \]  \hspace{1cm} (2.39)

\[ \sigma(G) = \frac{1}{\sigma(G^{-1})} \]  \hspace{1cm} if \( G^{-1} \) exists \hspace{1cm} (2.40)

\[ \sigma(G) = \frac{1}{\sigma(G^{-1})} \]  \hspace{1cm} if \( G^{-1} \) exists \hspace{1cm} (2.41)

\[ \sigma(\alpha G) = |\alpha| \sigma(G) \]  \hspace{1cm} (2.42)

\[ \sigma(G + H) \leq \sigma(G) + \sigma(H) \]  \hspace{1cm} (2.43)

\[ \sigma(GH) \leq \sigma(G)\sigma(H) \]  \hspace{1cm} (2.44)

\[ \max \{\sigma(G), \sigma(H)\} \leq \sigma([G H]) \leq \sqrt{2} \max \{\sigma(G), \sigma(H)\} \]  \hspace{1cm} (2.45)

\[ \sum_{i=1}^{n} \sigma_i^2 = \text{trace}(G^*G) \]  \hspace{1cm} (2.46)

where \( \lambda_i(G) \) is the \( i \)’th eigenvalue of \( G \), \( \alpha \) is a constant (complex) scalar.

Consider the input-output matrix equation:
\[ y(j\omega) = G(j\omega)u(j\omega) \]  \hspace{1cm} (2.47)

where the input \( u \) is a column of \( U \) in equation (2.36). Using equation (2.38), we can formulate it as
\[ y(j\omega) = \sum_{i=1}^{k} v_i \sigma_i u_i^*(j\omega) \]  \hspace{1cm} (2.48)

Since \( U \) is unitary, \( u_i^* u_j \) will be orthogonal to each other so that \( u_i^* u_j = 0 \), for \( i \neq j \) and \( u_i^* u_i = 1 \). Now assume that input \( u(j\omega) = \alpha u_i \). The input-output equation then becomes:
\[
y(j\omega) = \sum_{i=1}^{k} v_i^* \sigma_i u_j^* u_j \alpha
\]

This illustrates that the gain of the system is precisely \( \sigma_i \) if the input signal is in the direction of \( u_j \). The set \( \{u_1, u_2, \ldots, u_n\} \) is called the set of input principal direction of \( G \). In particular, the greatest possible gain \( \sigma = \sigma_1 \) occurs if the input signal is in the direction of \( u_1 \), and the smallest possible gain \( \sigma = \sigma_n \) occurs if it is in the direction of \( u_n \). Note that the principal directions are orthogonal to each other since \( U^* U = I \).

If the input vector is in the direction \( u_j \), then (2.50) illustrates that the output vector is in the direction of \( y_j \). The set \( \{y_1, y_2, \ldots, y_n\} \) is called the set of output principal direction. Again, these are orthogonal to each other. The set of singular value \( \{\sigma_1, \sigma_2, \ldots, \sigma_n\} \) are also called the principal gains or directional gains of the system matrix \( G \).

Thus when \( G(s) \) is a transfer function matrix we can plot the singular values \( \sigma(G(j\omega)) \), for \( i = 1, 2, \ldots, k \) as functions of frequency \( \omega \). These curves are the multivariable generalization of a system amplitude-ratio Bode plot. For multivariable systems, the amplification of the output vector with respect to the input in the form of sinusoid \( u e^{j\alpha t} \) depends on the direction of the complex vector \( u \) as illustrated above. The amplification is at least \( || G(s)|| \) and at most \( \sigma(G(j\omega)) \). The condition number \( \kappa(G(j\omega)) \), plotted versus frequency \( \omega \) outlines the system gain sensitivity to the direction of the input vector. If \( \kappa(G(j\omega)) \gg 1 \) the gain of the transfer function matrix will vary considerably with the input direction and \( G(s) \) is said to be ill-conditioned. Conversely, if \( \kappa(G(j\omega)) \approx 1, \forall \omega \geq 0 \), the gain of the transfer matrix will be insensitive to the input direction and the system is said to be well-conditioned. A well-conditioned multivariable system behaves much like a single-variable system and controller design for well-conditioned systems is fairly straightforward. For ill-conditioned systems, however, much more care has to be taken in both design and analysis.

### 2.4 Function Space

Let \( \mathcal{H} \) is a linear space over the field \( \mathbb{K} \). An inner product on \( \mathcal{H} \) is a complex valued function denoted by \( < \cdot, \cdot > \) from \( \mathcal{H} \times \mathcal{H} \) to \( \mathbb{K} \), which has the following properties:

\[
< f, g + h > = < f, g > + < f, h >
\]

\[
< f, \alpha h > = \alpha < f, h >
\]

\[
< f, h > = < h, f >
\]

\[
< f, f > \geq 0
\]

\[
< f, f > = 0 \quad \text{if and only if } f = 0
\]

where \( f, g, h \in \mathcal{H} \) and \( \alpha \in \mathbb{K} \). A inner product \( < \cdot, \cdot > \) induces a norm, namely, \( || f || = < f, f >^{1/2} \).

Given the linear space \( \mathcal{H} \) and an inner product \( < \cdot, \cdot > \) on \( \mathcal{H} \), the pair \( (\mathcal{H}, < \cdot, \cdot >) \) denotes an inner product space.

Let \( (\mathcal{H}, \| \cdot \|) \) be a normed space. Having defined a norm \( \| \cdot \| \) we can assess convergence in \( \mathcal{H} \). A sequence \( \{ f_n \}, n = 1, 2, \ldots \) in \( \mathcal{H} \) converges to \( f \in \mathcal{H} \), and \( f \) is the limit of the sequence, if the sequence of real positive numbers \( \| f - f_n \| \) converges to zero. If such \( f \) exists, then the sequence is convergent.

A sequence \( \{ f_n \}, n = 1, 2, \ldots \) in \( \mathcal{H} \) is called a Cauchy sequence if

\[
(\forall \varepsilon > 0) \exists \text{ integer } N \ni k > n \Rightarrow \| f_k - f_n \| < \varepsilon
\]

Intuitively, the elements in a Cauchy sequence eventually cluster around each other. They are trying to “converge”. Clearly every convergent sequence is a Cauchy sequence. If every Cauchy
sequence in $H$ is convergent (that is, if every sequence which is trying to converge actually does converge), then $H$ is complete. A complete normed space is called a Banach space. An inner space $(H, \langle \cdot, \cdot \rangle)$ is said to be completed if it is complete with respect to the norm induced by the inner product $(H, \langle \cdot, \cdot \rangle)$ which is called Hilbert space. Obviously, a Hilbert space is also a Banach space. For example, $\mathbb{C}^n$ with the usual inner product is a finite dimensional Hilbert space.

2.4.1 Time Domain Spaces

Consider vector-valued functions of a continuous time variable, we define norms for such functions which are fully analogous to the corresponding vector norms. Consider a function $f(t)$ defined on an interval $-\infty < t < \infty$ and taking values in $\mathbb{C}^n$. Restrict $f(t)$ to be square-Lebesgue integrable:

$$\int_{-\infty}^{\infty} \| f(t) \|_2^2 dt < \infty$$

where the norm $\| f(t) \|_2$ is vector 2-norm. The set of all such functions is a Banach space under the norm

$$\| f \|_2 = \sqrt{\int_{-\infty}^{\infty} \| f(t) \|_2^2 dt} = \sqrt{\int_{-\infty}^{\infty} f^*(t)f(t)dt}$$

This space is called the Lebesgue space $L^2(\mathbb{R}, \mathbb{R}^n)$. Note that it may be confused with other 2-norms. For a given value $t_0$ of $t$, the vector 2-norm is

$$\| f(t_0) \|_2 = \sqrt{f^*(t_0)f(t_0)}$$

and the operator 2-norm is

$$\| f \|_2 = \sqrt{\int_{-\infty}^{\infty} f^*(t)f(t)dt}$$

The subspace of $L^2(\mathbb{R}, \mathbb{R}^n)$ for which $f(t) = 0, \forall t < 0$ (causal time functions) is called a Hardy space $H^2(\mathbb{R}, \mathbb{R}^n)$ under the operator norm.

Similar to the operator 2-norm we may construct operator $\infty$-norm.

$\mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^n)$ is the Banach space of bounded real vector-valued functions of time:

$$\sup_{t \in \mathbb{R}} \max_i \left| f_i(t) \right| < \infty$$

with norm

$$\| f \|_\infty = \sup_{t \in \mathbb{R}} \max_i \left| f_i(t) \right|$$

$\mathcal{H}_\infty(\mathbb{R}, \mathbb{R}^n)$ is the subspace of $\mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^n)$ for which $f(t) = 0, \forall t < 0$.

A large class of causal time domain signals have finite values for the above two norms and thus are members of $H^2(\mathbb{R}, \mathbb{R}^n)$ and $H_\infty(\mathbb{R}, \mathbb{R}^n)$.

2.4.2 Frequency Domain Spaces

Consider the vector valued function $f(j\omega)$ which is defined for all frequencies $-\infty < \omega < \infty$ (that is, on the imaginary axis), taking values in $\mathbb{C}^n$ and is square-Lebesgue integrable on the imaginary axis. The space of all such functions is a Hilbert space under the inner product

$$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(j\omega)g(j\omega)d\omega$$
It is denoted by the Lebesgue space $L_2(j\mathbb{R}, \mathbb{C}^n)$. The corresponding induced norm is
\[
\| f \|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(j\omega) f(j\omega) d\omega}
\] (2.64)

Next, $H_2(\mathbb{C}, \mathbb{C}^n)$ is the space of all functions $f(s)$ which are analytic in $\mathbb{R} s > 0$, (i.e. no right half plane poles) taking values in $\mathbb{C}^n$, and satisfy the uniform square-integrability condition
\[
\| f \|_2 = \sup_{\xi > 0} \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(\xi + j\omega) f(\xi + j\omega) d\omega < \infty
\] (2.65)

This makes $H_2$ a Banach space. Functions in $H_2$ are not defined a priori on the imaginary axis, but we can get there in the limit.

**Theorem 2.1.** If $f(s) \in H_2(\mathbb{C}, \mathbb{C}^n)$ then the limit
\[
\tilde{f}(j\omega) = \lim_{\xi \to 0} f(\xi + j\omega)
\] (2.66)
exists and $\tilde{f}(j\omega)$ belongs to $L_2(j\mathbb{R}, \mathbb{C}^n)$. Moreover, the mapping $f(s) \to f(j\omega)$ from $H_2(\mathbb{C}, \mathbb{C}^n)$ to $L_2(j\mathbb{R}, \mathbb{C}^n)$ is linear, injective and norm-preserving.

It is customary to identify $f(s)$ in $H_2(\mathbb{C}, \mathbb{C}^n)$ and its boundary function $\tilde{f}(j\omega)$ in $L_2(j\mathbb{R}, \mathbb{C}^n)$. Therefore, we consider $H_2(\mathbb{C}, \mathbb{C}^n)$ as a closed subspace of the Hilbert space $L_2(j\mathbb{R}, \mathbb{C}^n)$ under the inner product (2.63) and induced norm (2.64).

The $L_2$ spaces defined above in the frequency domain can be related to the $L_2$ spaces defined in the time domain. The following definition and lemma are necessary for showing this relation.

**Definition 2.1.** Let $H_1$ and $H_2$ be normed space. An operator $U$ from $H_1$ into $H_2$ is called an isometry, if $\mathcal{D}(U) = H_1$ (the domain of $U$ is $H_1$) and $\| Uf \|_2 = \| f \|_1$ (the norm of $f$ on $H_1$ equals the norm of $Uf$ on $H_2$) for all $f \in H_1$. An isometry $U$ from $H_1$ into $H_2$ is called an isomorphism of $H_1$ onto $H_2$ if $\mathcal{D}(U) = H_2$ (the range of $U$ is $H_2$). Thus an isomorphism is a one-to-one mapping from one normed space to another which preserves norms.

**Lemma 2.1** The Fourier transform is a Hilbert space isomorphism from the time domain Lebesgue space $L_2(\mathbb{R}, \mathbb{R}^n)$ to the frequency domain Lebesgue space $L_2(j\mathbb{R}, \mathbb{C}^n)$ and from the time domain Hardy space $H_2(\mathbb{R}, \mathbb{R}^n)$ to the frequency domain Hardy space $H_2(\mathbb{C}, \mathbb{C}^n)$.

It can be shown that this Fourier (or bilateral Laplace) transform yields an isometric isomorphism between the $L_2(\mathbb{R}, \mathbb{R}^n)$ spaces in the time domain and the $L_2(j\mathbb{R}, \mathbb{C}^n)$ spaces in the frequency domain and between the $H_2(\mathbb{R}, \mathbb{R}^n)$ spaces in the time domain and the $H_2(\mathbb{C}, \mathbb{C}^n)$ spaces in the frequency domain. Furthermore, if $f(t) \in H_2(\mathbb{R}, \mathbb{R}^n)$, then its Laplace transfer $F(s) \in H_2(\mathbb{C}, \mathbb{C}^n)$ and
\[
\sqrt{\int_0^{\infty} f(t)^* f(t) dt} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(j\omega) F(j\omega) d\omega}
\] (2.67)

Since they are related by a norm preserving isomorphism. This equation also is recognized as a result of Parceval’s theorem.

Finally we present 4 frequency domain spaces of matrix-valued functions.
\[ L_2(\mathbb{R}, \mathbb{C}^{m \times n}) \] is a Hilbert space of matrix-valued (or scalar-valued) functions on the imaginary axis \( j\mathbb{R} \) and consists of all complex matrix functions \( F(j\omega) \) such that the integral below is bounded, i.e.
\[
\int_{-\infty}^{\infty} \text{Trace}\{F^*(j\omega)F(j\omega)\}d\omega < \infty
\]

The inner product for this Hilbert space is defined as
\[
\langle F, G \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}\{F^*(j\omega)G(j\omega)\}d\omega
\]
for \( F, G \in L_2(\mathbb{R}, \mathbb{C}^{m \times n}) \), and the inner product induced norm is given by
\[
\| F \|_2 = \sqrt{\langle F, F \rangle}
\]

\[ H_\infty(\mathbb{C}^{m \times n}) \] is a closed subspace of \( L_2(\mathbb{R}, \mathbb{C}^{m \times n}) \) with matrix functions \( F(s) \) analytic in \( \mathbb{R}(s) > 0 \) (open right-half plane). The corresponding norm is defined as
\[
\| F \|_\infty = \sup_{\xi > 0} \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}\{F^*(j\xi)F(j\xi)\}d\xi \right)
\]

Notice that frequency domain functions in \( L_2(\mathbb{R}, \mathbb{C}^{m \times n}) \) and \( H_\infty(\mathbb{C}^{m \times n}) \) are defined only on the imaginary axis, their domain is \( j\mathbb{R} \). It is thus not meaningful to talk about their poles and zeros of their stability. They are simply frequency responses, not transfer functions. Conversely, the domain of functions in \( H_2(\mathbb{C}^{m \times n}) \) and \( H_\infty(\mathbb{C}^{m \times n}) \) is the entire complex plane \( \mathbb{C} \). They are transfer functions, not just frequency responses.

\[ H_2(\mathbb{C}, \mathbb{C}^{m \times n}) \] and \( H_\infty(\mathbb{C}, \mathbb{C}^{m \times n}) \) are stable spaces since they do not allow poles in the right-half plane. Transfer functions in \( H_2(\mathbb{C}, \mathbb{C}^{m \times n}) \), however, must roll off in all frequencies to satisfy
In contrast, transfer functions in $\mathcal{H}_\infty(C, C^{m \times n})$ may maintain non-zero gain as $\omega \to \infty$. In terms of state-space realizations $(A, B, C, D)$, the $D$ matrix must be zero for a transfer function in $\mathcal{H}_2(C, C^{m \times n})$, i.e. the system must be strictly proper.

### 2.5 System Norms

Consider the system in Figure 2.7, where $G(s)$ is a stable transfer function matrix and $g(t)$ is the corresponding impulse response matrix. To evaluate the performance we ask the question: given information about the allowed input signals $w(t)$, how large can the outputs $z(t)$ become? To answer this, we must evaluate the relevant system norm.

![Figure 2.7 A simple system](image)

We will here evaluate the output signal in terms of the usual $2$-norm

$$\|z(t)\|_2 = \sqrt{\sum_i \int_{-\infty}^{\infty} |z_i(\tau)|^2 \, d\tau}$$  \hspace{1cm} (2.77)

and consider two kind of the inputs:

- $w(t)$ is a series of unit impulses.
- $w(t)$ is any signal satisfying $\|w(t)\|_2 = 1$.

The relevant system norms in the two cases are the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms respectively.

#### 2.5.1 System $\mathcal{H}_2$ Norms

First, we have the following definition

**Definition 2.2** A system $G(s)$ is strictly proper if $G(s) \to 0$ as $s \to \infty$. If $G(s) \to \text{constant} \neq 0$ as $s \to \infty$, we say $G(s)$ is semi-proper. A system $G(s)$ that is strictly proper or semi-proper is proper.

Consider a strictly proper system $G(s)$, i.e. $D=0$ in a state-space realization, we use the Euclidean norm and integrate over frequency for the $\mathcal{H}_2$ norm

$$\|G(s)\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}(G^*(j\omega)G(j\omega)) \, d\omega}$$  \hspace{1cm} (2.78)

We see that $G(s)$ must be strictly proper, otherwise the $\mathcal{H}_2$ norm is infinite. The $\mathcal{H}_2$ norm can be given another interpretation. By Lemma 2.1, equation (2.78) is equal to the $\mathcal{H}_2$ norm of the impulse response

$$\|g(t)\|_2 = \sqrt{\int_{0}^{\infty} \text{Trace}(g^T(\tau)g(\tau)) \, d\tau}$$  \hspace{1cm} (2.79)

Note that $G(s)$ and $g(t)$ are dynamic systems while $G(j\omega)$ and $g(\tau)$ are constant matrices (for given value of $\omega$ or $\tau$).
Because the $\text{Trace}(g^T(\tau)g(\tau)) = \sum_{i,j} |g_{i,j}(\tau)|^2$, we can change the order of integration and summation in (2.79) to get
\[
\|g(t)\|_2 = \sqrt{\sum_{i,j} \int_0^\infty |g_{i,j}(\tau)|^2 d\tau}
\] (2.80)
where $g_{i,j}(\tau)$ is the $ij$'th element of the impulse response matrix, $g(\tau)$. From this we see that the $\mathcal{H}_2$ norm can be interpreted as the 2-norm output resulting from applying unit impulses $\delta_j(t)$ to each input, one after another.

In summary, we have the following deterministic performance interpretation of the $\mathcal{H}_2$ norm
\[
\|G(s)\|_2 = \max_{w(t)\equiv\text{unit impulses}} \|z(t)\|_2
\] (2.81)

The $\mathcal{H}_2$ norm can also be given a stochastic interpretation since it measures the expected root mean square (RMS) value of the output in response to white noise excitation. All quadratic-norm control scheme like LQG optimal control and Kalman optimal filtering minimizes the $\mathcal{H}_2$ norm of a closed loop transfer function matrix. In classical LQG theory, cost functions are given in the time domain. However, it is now well known that they have frequency domain interpretations. Minimizing the $\mathcal{H}_2$ norm of a system (transfer function) matrix thus controls the output for a specific input signal, namely vector white noise. In contrast, minimizing the $\mathcal{H}_\infty$ norm of a system (transfer function) matrix controls the output for a specified set of bounded input signals.

### 2.5.2 System $\mathcal{H}_\infty$ Norm

Consider a proper linear stable system $G(s)$ (i.e. $D \neq 0$ is allowed). For the $\mathcal{H}_\infty$ norm we use the singular value (induced 2-norm) spatially (for the matrix) and pick out the peak value as a function of frequency
\[
\|G(s)\|_\infty = \sup_{\omega} \sigma(G(j\omega))
\] (2.82)

In terms of performance we see from (2.82) that the $\mathcal{H}_\infty$ norm is the peak of the transfer function “magnitude”, and by introducing weights, the $\mathcal{H}_\infty$ norm can be interpreted as the magnitude of some closed-loop transfer function relative to a specified upper bound. This leads to specifying performance in terms of weighted sensitivity, mixed sensitivity, and so on.

However, the $\mathcal{H}_\infty$ norm also has several time domain performance interpretations. First, it is the worst-case steady-state gain for sinusoidal inputs at any frequency. Furthermore, the $\mathcal{H}_\infty$ norm is equal to the induced (worst-case) 2-norm in the time domain:
\[
\|G(s)\|_\infty = \sup_{w(t)\neq0} \frac{\|z(t)\|_2}{\|w(t)\|_2} = \sup_{\|w(t)\|_2=1} \|z(t)\|_2
\] (2.83)

The $\mathcal{H}_\infty$ norm is also equal to the induced power norm, and also has an interpretation as an induced norm in terms of the expected values of stochastic signals. All these various interpretations make the $\mathcal{H}_\infty$ norm useful in engineering applications.

The $\mathcal{H}_\infty$ norm is usually computed numerically from a state-space realization of $G(s)$, as the smallest value of $\gamma$ such that the Hamiltonian matrix $H$ has no eigenvalues on the imaginary axis, where
\[
H = \begin{bmatrix}
A + BR^{-1}D^*C & BR^{-1}B^* \\
-C^*(I + DR^{-1}D^*)B & -(A + BR^{-1}D^*C)^*
\end{bmatrix}
\] (2.84)
and \( R = \gamma I - D^*D \). [Zhou, Doyle and Glover, 1995]. This is an iterative procedure, where one may start with a large value of \( \gamma \) and reduce it until imaginary eigenvalues for \( H \) appear.

### 2.6 Summary

There are many spaces mentioned in this section. The **linear space** is the essential space in all of them. Given a linear space \( H \) and a norm \( \| \cdot \| \) on \( H \), the pair \((H, \| \cdot \|)\) is called a **normed space**. Given the linear space \( H \) and an inner product \( \langle \cdot, \cdot \rangle \) on \( H \), the pair \((H, \langle \cdot, \cdot \rangle)\) is called an **inner product space**. A complete normed space is called a **Banach space**. A completed inner space is called **Hilbert space**. Restrict \( f(t) \) to be square-Lebesgue integrable, the subspace of a Banach space under the norm \( \| f(t) \|_2 \) is called the **Lebesgue space** \( \mathcal{L}_2(\mathbb{R}, \mathbb{R}^n) \). The subspace of \( \mathcal{L}_2(\mathbb{R}, \mathbb{R}^n) \) for which \( f(t) = 0, \; \forall \; t < 0 \) (causal time functions) is called a **Hardy space** \( \mathcal{H}_2(\mathbb{R}, \mathbb{R}^n) \). \( \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^n) \) is the Banach space of bounded real vector-valued functions of time. \( \mathcal{H}_\infty(\mathbb{R}, \mathbb{R}^n) \) is the subspace of \( \mathcal{L}_\infty(\mathbb{R}, \mathbb{R}^n) \) for which \( f(t) = 0, \; \forall \; t < 0 \).

There were also many norms mentioned in this section:

- **Vector norm** is a number which measures the size of a vector
  \[ \| x \|_p = \left( \sum_{i=1}^{n} |x_i|^p \right)^{1/p} \]

- **Matrix norm** is a number which measures the size of a matrix
  \[ \| A \|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} \]

- **Signal norm** is a number which measures a temporal norm, it is the size of a signal
  \[ \| e(t) \|_p = \left( \int_{-\infty}^{\infty} \left( \sum_{i} |e_i(\tau)|^p \right)^{1/p} d\tau \right) \]

- A system norm is a number which measures the size of a system. For unit impulses, the \( \mathcal{H}_2 \) norm is used. It is
  \[ \| G(s) \|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Trace}(G^*(j\omega)G(j\omega))d\omega} \]

  For any other inputs, the \( \mathcal{H}_\infty \) norms is used
  \[ \| G(s) \|_\infty = \sup_{\omega} \sigma(G(j\omega)) \]
Chapter 3

Robust Stability and Robust Performance

Most control design is based on the use of a design model. The relationship between models and the reality they represent is subtle and complex. A mathematical model provides a map from inputs to responses. The quality of a model depends on how closely its responses match those of the true plant. Since no single fixed model can respond exactly like the true plant, we need, at the very least, a set of maps. However, the modeling problem is much deeper- the universe of mathematical models from which a model set is chosen is distinct from the universe of physical systems. Therefore, a model set that includes the true physical plant can never be constructed. It is necessary for the engineer to make a leap of faith regarding the applicability of a particular design based on a mathematical model. To be practical, a design technique must help make this leap small by accounting for the inevitable inadequacy of models. A good model should be simple enough to facilitate design, yet complex enough to give the engineer confidence that designs based on the model will work on the true plant.

In particular, the difference between the model and the plant is called model uncertainty, which deteriorates the behavior of control systems. To deal with problems correlate to the model uncertainty, instead of the nominal model $P$, we may study the behavior of a class of models, $P_p=P+E$, where the “uncertainty” or “perturbation” $E$ is bounded, but otherwise unknown. This is what robust control mainly to cover. In this case, weighting functions $w(s)$ are used to express $E$ in terms of normalized perturbations $\Delta$, $E=w\Delta$ where the magnitude (norm) of $\Delta$ is less than 1.

3.1 Survey of Robust Control with Uncertainty

Practically, two types of uncertainty are commonly involved in robust control: unstructured uncertainty and structured uncertainty.

**Structured uncertainty** represents parametric variations in the plant dynamics, for examples:
1. Uncertainties in certain entries of state-space matrices ($A$, $B$, $C$), e.g. the uncertain variations in a system’s stability and control derivatives.
2. Uncertainties in specific poles and/or zeros of the plant transfer function.
3. Uncertainties in specific loop gains/phases.

**Unstructured uncertainty** usually represents frequency-dependent elements such as actuator saturation and unmodelled structural models in the high frequency range or plant disturbances in the low frequency range. Their relation to the nominal plant can be either additive or multiplicative. Considering that the multiplicative uncertainty can either be modeled as output or input forms, with feed forward or inverse forms, there are six possible forms of unstructured uncertainties. (see Figure 3.1).

<table>
<thead>
<tr>
<th>Type of Uncertainty</th>
<th>Model Expression</th>
<th>Perturbation Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>Additive uncertainty</td>
<td>$P_p = P + E_A$; $E_A = w_A \Delta_A$</td>
<td>(3.1)</td>
</tr>
<tr>
<td>Input multiplicative uncertainty</td>
<td>$P_p = P(I + E_I)$; $E_I = w_I \Delta_I$</td>
<td>(3.2)</td>
</tr>
<tr>
<td>Output multiplicative uncertainty</td>
<td>$P_p = (I + E_O)P$; $E_O = w_O \Delta_O$</td>
<td>(3.3)</td>
</tr>
<tr>
<td>Inverse additive uncertainty</td>
<td>$P_p = P(I - E_A)^{-1}$; $E_A = w_A \Delta_A$</td>
<td>(3.4)</td>
</tr>
<tr>
<td>Inverse input multiplicative</td>
<td>$P_p = P(I - E_I)^{-1}$; $E_I = w_I \Delta_I$</td>
<td>(3.5)</td>
</tr>
</tbody>
</table>
Inverse output multiplicative uncertainty

\[ P_p = (I - E_{iO})^{-1}P; \quad E_{iO} = w_{iO}\Delta_{iO} \quad (3.6) \]

Figure 3.1 Six types of model uncertainties

where \( P_p \) is the perturbed plant. The negative sign in front of the \( E \)'s does not really matter here since we assume that \( \Delta \) can have any sign. \( \Delta \) denotes the normalized perturbation and \( E \) the “actual” perturbation. In here \( w \) is a scalar weights, so \( E = w\Delta = \Delta w \). In some case, we use matrix weights, \( E = W_2\Delta W_1 \) where \( W_1 \) and \( W_2 \) are given transfer function matrices. In practice, we may have several perturbations those are unstructured, e.g. \( \Delta \) at the input and \( \Delta \) at the output. These may be combined into a larger perturbation, \( \Delta = \text{diag}\{\Delta_i, \Delta_o\} \). For a matrix unstructured uncertainty, \( \Delta \) should be a full matrix so that if \( \Delta = \text{diag}\{\Delta_i, \Delta_o\} \), it is a block-diagonal matrix, not a strict unstructured uncertainty.

Figure 3.2 General feedback control system

Consider the general multivariable feedback control system illustrated in Figure 3.2. In this system, output disturbances \( d \) and reference signals \( r \) have apart from the sign the same effect on the control error \( e \), and measurement noises \( n \) of the disturbed plant outputs \( y \) have the same effect with \( d \). If the model uncertainty is included explicitly, we can express it with a general control configuration shown in Figure 3.3. This is convenient for synthesize of a controller.

When the attention is concentrated to the design of the controller and \( \Delta \) is omitted (or assume \( \Delta = 0 \)), we use a block diagram for the standard output control problem, shown in Figure 3.4. Assuming that generalized plant \( G \) is partitioned as
3. Robust Stability and Robust Performance

![Diagram of control configuration](image)

Figure 3.3 General control configuration (for controller synthesis)

\[ G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}. \]

The closed loop transfer function from \( w \) to \( z \), \( z = Nw \), is denoted by

\[ N = F_1(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21} \]

(3.7)

\( F_1(G, K) \) is called a Linear Fractional Transformation (LFT) of \( G \) and \( K \). The subscript \( l \) stands for “lower” and indicates that \( K \) is below \( G \) (see Figure 3.4). We will have upper LFT with the subscript \( u \) indicating that \( \Delta \) is over \( N \) in the following.

![Diagram of closed loop system](image)

Figure 3.4 General closed loop system for controller synthesis

![Diagram of N-\( \Delta \) structure](image)

Figure 3.5 \( N-\Delta \) structure for robust performance analysis

When the controller has been designed, we want to analyze the uncertain system, then the \( N-\Delta \) structure is shown in Figure 3.5 is used, where the controller is included in \( N \). This block diagram represents a standard perturbation problem.

Similar to the standard control problem, \( N \) is partitioned as

\[ N = \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix} \]

The closed loop transfer function from \( w \) to \( z \), \( z = Fw \), is denoted by

\[ F = F_2(N, \Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12} \]

(3.8)

For a perturbed system, we need to check whether it is stable and if it has acceptable performance for all perturbed plants that are included in the model uncertainty set.

There are different definitions for stability and performance of a system with and without model uncertainty. The following definitions are used and the related discussions will be given in the subsequent sections respectively.

**Nominal Stability (NS):** The system is stable without model uncertainty.

**Nominal Performance (NP):** The system satisfies the performance specifications without model uncertainty.

**Robust Stability (RS):** The system is stable for all perturbed plants about the nominal model up to the worst-case model uncertainty.

**Robust Performance (RP):** The system satisfies the performance specifications for all perturbed plants about the nominal model up to the worst-case model uncertainty.
3. Robust Stability and Robust Performance

3.2 Nominal Stability

There are many ways in which nominal stability may be defined. In this section the definition of the internal stability is given first so that NS definition could be given. Then a Multivariable Nyquist criterion will be introduced.

**Definition 3.1:** A system is **internally stable** if the origin state $x = 0$ is globally asymptotically stable, i.e. the states $x$ go to zero from all initial states when input $u = 0$.

This means that bounded external input signals result in bounded output signals in an internally stable system.

Now we regroup the external input signals in Figure 3.2 into the feedback loop as $w_1$ and $w_2$, and regroup the input signals of the plant and controller as $e_1$ and $e_2$. Then the feedback loop with the plant and the controller can be simply represented as in Figure 3.6.

![Figure 3.6 Internal Stability Analysis Diagram](image)

where $w_1$ is input disturbances of the plant and $w_2$ is output disturbances of the plant. For a given nominal system shown in Figure 3.6, if the feedback controller $K$ internally stabilizes the nominal model $P$, the closed loop feed back system is said to be **NS**.

The stability of a multivariable feedback control system is determined by the extended Nyquist stability criterion [Morari and Zafiriou, 1989].

**Theorem 3.1 (Extended Nyquist Stability Criterion):** If the open loop transfer function matrix $P(s)K(s)$, has $p$ poles in the right-half s-plane, then the closed loop system is stable if and only if the map of $\text{det}(I+P(s)K(s))$, as $s$ traverses along the Nyquist $\mathcal{D}$ contour, encircles the origin $p$ times anticlockwise, assuming no right-half s-plane zero-pole cancellations have occurred when the product $P(s)K(s)$ is formed.

Remember that the Nyquist $\mathcal{D}$ contour goes up the imaginary axis from the origin to infinity, then along a semicircular arc in the right-half plan until it meets the negative imaginary axis and finally up to the origin. If any poles of $P(s)K(s)$ are encountered on the imaginary axis the contour is indented so as to exclude these poles.

The extended Nyquist stability criterion will be used in assessing not only nominal stability but also robust stability of an uncertain closed loop system.

3.3 Nominal Performance

For the nominal performance, we can simply state that: for a given nominal system shown in Figure 3.6, if the performance objectives are satisfied for the nominal plant $P$, the closed loop feed back system is said **NP**.
To assess the performance of a multivariable system, it is necessary to consider the sensitivity function and the complementary sensitivity function of the system. From Figure 3.2, if the input disturbance \( d(s) \) is ignored, it is easy to see that
\[
y(s) = S(s)d(s) + T(s)r(s) - T(s)n(s) \tag{3.9}
\]
\[
u(s) = K(s)S(s)[r(s) - n(s) - d(s)] \tag{3.10}
\]
where, \( S(s) \) is the sensitivity function and \( T(s) \) is the complementary sensitivity function
\[
S(s) = (I + P(s)K(s))^{-1} \tag{3.11}
\]
\[
T(s) = P(s)K(s)(I + P(s)K(s))^{-1} \tag{3.12}
\]
\( S(s) \) is the closed loop transfer matrix from \( d \) to \( y \) or from \( r \) to \( e \); \( T(s) \) is the closed loop transfer matrix from \( r \) to \( y \), see Figure 3.2.

From (3.9) and (3.10), we know that

- To attenuate the effect of disturbance \( d(s) \) the sensitivity \( S(s) \) should be small.
- To track the reference input \( r(s) \), the sensitivity \( S(s) \) should be small. The complementary sensitivity function \( T(s) \) should be equal to 1.
- To reject measurement noise \( n(s) \), the complementary sensitivity function \( T(s) \) should be small.
- To reduce the control energy, \( K(s)S(s) \) should be small.

The “sizes” of frequency responses \( S(j\omega) \) and \( T(j\omega) \) can be measured by means of the largest singular value for multivariable systems.

From the requirement of good disturbance attenuation, the following is needed
\[
\overline{\sigma}(S(j\omega)) = \left\| S(j\omega) \right\|_\infty < 1 \tag{3.13}
\]
From the requirement of good measurement noise rejection, the following is needed
\[
\overline{\sigma}(T(j\omega)) = \left\| T(j\omega) \right\|_\infty < 1 \tag{3.14}
\]
However, it is impossible that both \( S \) and \( T \) are small in the same frequency range because \( S + T = I \). A trade off is necessary between disturbance attenuation and measurement noise rejection or between reference tracking and measurement noise rejection. Fortunately, it can be done by considering that the spectra of disturbance \( d(s) \) are usually concentrated at low frequencies, while the spectra of measurement noise \( n(s) \) are concentrated at higher frequencies. Therefore, we can loop shape the system, \( P(s)K(s) \) such that \( \overline{\sigma}(S(j\omega)) \) is small at low frequencies and \( \overline{\sigma}(T(j\omega)) \) is small at high frequencies.

A typical performance specification for robust control is then given as
\[
\sup \overline{\sigma}(S_w(j\omega)) = \left\| S_w(j\omega) \right\|_\infty \leq 1 \tag{3.15}
\]
where \( S_w(s) \) is the weighted sensitivity function (see Figure 3.7)
\[
S_w(s) = W_{p2}(s)S(s)W_{p1}(s) \tag{3.16}
\]

Figure 3.7: Nominal performance problem
The weight function $W_p(s)$ is used to trade off the relative importance of the individual error in $e(s)$ and to weight the frequency range of interest. The input weight function $W_p(s)$ is used to perform any necessary scaling. The nominal performance is then defined as follows:

**Definition 3.2 (Nominal Performance)** When the weighted function $W_p(s)$ and $W_{p2}(s)$ are given, to design a stabilizing controller $K(s)$ makes the weighted sensitivity function satisfy

$$\left\| S_p(j\omega) \right\|_\infty < 1$$

(3.17)

If such a $K(s)$ exists, we say that the closed loop system has **nominal performance**.

The block diagram is convenient to analyse the nominal performance problem. The block diagram of Figure 3.7 is shown in Figure 3.4. Note that $G(s)$ is the generalized plant for the system weighted by $W_{p2}(s)$ and $W_{p1}(s)$. The transfer function from $w(s)$ to $z(s)$ is given as

$$z(s) = F_l(G(s), K(s))w(s)$$

$$= (G_{11}(s) + G_{12}(s)K(s)(I - G_{22}(s)K(s))^{-1}G_{21}(s))w(s)$$

$$= -W_{p2}(s)S(s)W_{p1}(s)w(s)$$

$$= -S_p(s)w(s)$$

(3.18)

The third equation is obvious from Figure 3.7.

Now, the nominal performance problem is to find a controller such that

$$\left\| F_l(G(s), K(s)) \right\|_\infty < 1$$

(3.19)

### 3.4 Robust Stability

For a system shown in Figure 3.2, if a feedback controller $K$ internally stabilizes all perturbed models $P_p$ with a perturbation $\Delta$, the closed loop feedback system is said **RS**.

Consider the uncertain $N$-$\Delta$ system in Figure 3.5. Suppose that the system is **NS** (when $\Delta=0$ the system is stable), that is, the whole $N$ is stable. We also suppose that $\Delta$ is stable. From equation (3.8) we know that if we want the $N$-$\Delta$ system is **RS**, $(I - N_{11}\Delta)^{-1}$ must be stable.

The block diagram representing $(I - N_{11}\Delta)^{-1}$ is called **M**-$\Delta$ structure and is shown in Figure 3.8, where $M=N_{11}$. The following theorem will find whether the **M**-$\Delta$ structure is stable.

**Theorem 3.2 (Robust Stability)** Assume that $M$ is stable and that the perturbation $\Delta$ is stable, then the perturbed closed-loop system is stable if and only if the map of $\det(I - M\Delta(j\omega))$, as $s$ traverses along the Nyquist $\mathcal{D}$ contour, does not encircle the origin. Then the closed-loop system in Figure 3.8 is stable for all perturbations $\Delta$ with $\sigma(\Delta(j\omega)) \leq 1$ if and only if one of the following four equivalent conditions is satisfied:

$$\det(I - M\Delta(j\omega)) \neq 0 \quad \forall \omega, \forall \Delta \left| \sigma(\Delta) \leq 1 \right. \quad (3.20)$$

$$\Leftrightarrow \rho(M\Delta(j\omega)) < 1 \quad \forall \omega, \forall \Delta \left| \sigma(\Delta(j\omega)) \leq 1 \right. \quad (3.21)$$

![Figure 3.8 M-Δ structure for robust stability analysis](image-url)
\[ \Leftrightarrow \sigma(M(j\omega)) < 1 \quad \forall \omega \] (3.22)
\[ \Leftrightarrow \|M\|_\infty < 1 \] (3.23)

where the notation \( | \) means “such that”.

If we break the connection of \( M-\Delta \) structure at the point of \( y_\Delta \) in Figure 3.8, the \( M-\Delta \) forms a stable open-loop transfer function \( L(s)=M(s)\Delta(s) \). Theorem 3.2 with (2.19) is also the celebrated Small Gain Theorem.

**Theorem 3.3 (Small Gain Theorem)** Assume that \( L(s) \) is stable. Then the closed loop system shown in Figure 3.8 is stable if the spectral radius \( \rho(L(j\omega)) < 1 \), \( \forall \omega \).

Theorem 3.2 states that if \( \| M(s) \|_\infty < 1 \), there is no perturbation \( \Delta(s) \) \( (\sigma(\Delta(j\omega)) \leq 1) \) which makes \( \det(I-M(s)\Delta(s)) \) encircle the origin as \( s \) traverses the Nyquist \( \mathcal{D} \) contour. Notice that we assumed that the absence of encirclements is necessary and sufficient to maintain stability. Any one of these assumptions is standard in robust control. The \( \infty \)-norm constraint (3.23) in theorem 3.2 is not conservative since we have bounded the uncertainty in terms of the spectral norm (maximum singular value). Thus if \( \| M(s) \|_\infty \geq 1 \), there exists a perturbation \( \Delta(s) \) for which \( \sigma(\Delta'(j\omega)) \leq 1 \) that will destabilize the closed loop system. If the uncertainty is tightly represented by \( \Delta(s) \), then the singular value bound on \( M(j\omega) \) is thus a tight robustness bound.

Theorem 3.3 states that for an open-loop stable system, a sufficient condition for closed loop stability is to keep the loop “gain” measured by \( (\rho(j\omega)) \) less than unity. Fortunately this is only a sufficient condition for stability. Otherwise the usual performance requirement of high controller gain for low frequencies could not be achieved. Then we should use this theorem to find a controller \( K(s) \) to make the \( N-\Delta \) system robust stable.

Assume a system with a multiplicative perturbation \( \Delta \) (see Figure 3.9). Two diagonal weight matrices \( W_{u_1}(s) \) and \( W_{u_2}(s) \) are introduced such that
\[ \Delta(s) = W_{u_2}(s)\Delta(s)W_{u_1}(s) \] (3.24)
where \( \Delta(s) \) satisfies
\[ \sigma(\Delta(j\omega)) \leq 1 \quad \Leftrightarrow \|\Delta\|_\infty \leq 1 \] (3.25)

Similar to \( W_{p_1}(s) \) and \( W_{p_2}(s) \), \( W_{u_1}(s) \) is used to perform the necessary scaling and \( W_{u_2}(s) \) is used as a frequency weighting for the interesting frequency range. By choosing these two weight matrices, \( \|M\|_\infty \leq 1 \) can be satisfied (see the following). Figure 3.9 also can be represented in the \( M-\Delta \) structure. \( M(s) \) is the transfer matrix seen from \( u_\Delta \) to \( y_\Delta \) in Figure 3.9.
We know that $y = -W_u(P(s)K(s)y$ and $y = (I + P(s)K(s))^{-1}W_u$, therefore
\[
M(s) = -W_u(s)P(s)K(s)(I + P(s)K(s))^{-1}W_u(s)
\]
\[
= -W_u(s)T(s)W_u(s)
\]  \hspace{1cm} (3.26)
From Theorem 3.2, it is easy to see that the closed-loop $N$-$\Delta$ system is RS if and only if
\[
\|M\|_\infty < 1
\]  \hspace{1cm} (3.27)
\[
\iff \|W_u(s)T(s)W_u(s)\|_\infty < 1
\]  \hspace{1cm} (3.28)
If a general block diagram in Figure 3.3 is used for this problem, the robust stability controller design could be simplified. From Figure 3.9, a modified general block diagram is shown in Figure 3.10.

![Figure 3.10 General closed system with uncertainty](image)

Using this structure, the transfer function from $u_\Delta$ to $y_\Delta$ is given by LFT form:
\[
F_i(G(s), K(s)) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}
\]
\[
= -W_uPK(I + PK)^{-1}W_u
\]
\[
= -W_uT(s)W_u
\]  \hspace{1cm} (3.29)
Note that in (3.29), $G(s)$ is the generalized plant of the perturbed system weighted by $W_u(s)$ and $W_u$ in Figure 3.9.
The robust stability problem is to find a controller $K(s)$ such that
\[
\|F_i(G(s), K(s))\|_\infty < 1
\]  \hspace{1cm} (3.30)
Comparing Equation (3.30) with Equation (3.19), it is found that they are the same in form. The difference is that for NP, $F_i(G,K)$ is the weighted sensitivity function, and the weighting matrices $W_p$ are included in $G$; however, for RS, $F_i(G,K)$ is the weighted complementary sensitivity function, and the weighting matrices $W_u$ are included in $G$.

3.5 Robust Performance

Consider a system shown in Figure 3.2 while $P_p$ is used instead of $P$ (see Figure 3.1). If the performance objectives are satisfied for all perturbed plant $P_p$ with a perturbation $\Delta$, where $\Delta(s)$ satisfies (3.25), the closed loop feed back system is said RP.

By definition, the perturbed weighted sensitivity $\tilde{S}_w$ instead of $S_w$ in (3.17) should be used in the investigation of robust performance problem, here $\tilde{S}_w = W_{p2}\tilde{S}_w W_{p1}$, we obtain the following conclusion:

The system is RP if for the perturbed plant about the nominal model up to the worst-case model uncertainty, the following criterion is satisfied
\[
\|\tilde{S}_w(j\omega)\|_\infty < 1
\]  \hspace{1cm} (3.31)
Form the general block diagrams in Figure 3.3 and Figure 3.5, the transfer function from \( w \) to \( z \) is

\[
F(s) = F_y(N(s), \Delta(s)) = N_{22}(s) + N_{21}(s)\Delta(s)(I - N_{12}(s)\Delta(s))^{-1}N_{12}(s)
\]

where

\[
N(s) = F_y(G(s), K(s))
\]

Consider the multiplicative perturbation shown in Figure 3.9, and the input and output weight functions shown in Figure 3.7, the robust performance problem is illustrated in Figure 3.11.

From that, we get

\[
N_{11} = -W_{u1}PK(1 + PK)^{-1}W_{u2} = M(s)
\]

(3.34)

\[
N_{12} = -W_{u1}PK(1 + PK)^{-1}W_{p1}
\]

(3.35)

\[
N_{21} = -W_{p2}(1 + PK)^{-1}W_{u2}
\]

(3.36)

\[
N_{22} = -W_{p2}(1 + PK)^{-1}W_{p1} = S_w(s)
\]

(3.37)

Insert (3.34)~(2.38) into (3.32), then the closed loop transfer function \( F \) with perturbation should be

\[
F = W_{p2}(1 + PK + W_{p2}\Delta W_{u1}PK)^{-1}W_{p1}
\]

\[= W_{p2}(1 + \tilde{P}K)^{-1}W_{p1}
\]

\[= W_{p2}\tilde{S}W_{p1}
\]

\[= S_w
\]

(3.38)

where \( \tilde{P} = (1 + W_{u1}\Delta W_{u1})P \) is the generalized plant with an output multiplicative weighted uncertainty, and \( \tilde{S} = (1 + \tilde{P}K)^{-1} \) is the perturbed sensitivity function.

Therefore, the robust performance problem is to find a controller to satisfy

\[
\| F_u(N(s), \Delta(s)) \|_{\infty} \leq 1 \quad \forall \Delta(s) \quad \text{and} \quad \| \Delta \|_{\infty} \leq 1
\]

(3.39)

Comparing Equations (3.39) and (3.23), it is found that they are similar because of (3.26). If a norm bounded matrix perturbation \( \Delta_p(s) \) with \( \| \Delta_p(s) \|_\infty \leq 1 \) is introduced, Theorem 3.2 can be used to describe robust performance problem for an interconnection \( F_u(N(s), \Delta(s)) \). \( \Delta_p(s) \) is the structure that the perturbations come from [Zhou et al., 1995]. Then lump \( \Delta_p(s) \) and \( \Delta(s) \) into one block \( \hat{\Delta} \)

\[
\hat{\Delta}(s) = \text{diag}(\Delta(s), \Delta_p(s))
\]

(3.40)

(see Figure 3.12), a similar theorem can be stated for robust performance:
Theorem 3.4 (Robust Performance). Assume that the interconnection \( N = F_u(G(s), K(s)) \) is stable and that the perturbation \( \hat{\Delta}(s) \) is of such a form that the perturbed closed loop system in Figure 3.12 is stable if and only if the map of \( \det(I - N(s)\hat{\Delta}(s)) \), as \( s \) traverses along the \( \mathcal{D} \) contour, does not encircle the origin. Then the system \( F_u(N(s), \Delta(s)) \) will satisfy the robust performance criterion if and only if \( N(s) \) is stable for all perturbations \( \hat{\Delta}(s) \) with \( \bar{\sigma}(\Delta(j\omega)) \leq 1 \).
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\( \det(I - N(j\omega)\hat{\Delta}(j\omega)) \neq 0 \quad \forall \omega, \forall \hat{\omega} \in \sigma(\hat{\Delta}(j\omega)) \leq 1 \)

\( \Leftrightarrow \rho(N(j\omega)\hat{\Delta}(j\omega)) < 1 \quad \forall \omega, \forall \hat{\omega} \in \sigma(\hat{\Delta}(j\omega)) \leq 1 \)

\( \Leftrightarrow \|N(s)\|_{\infty} < 1 \) \hspace{1cm} (3.41)

### 3.6 General \( \mathcal{H}_\infty \) Control Problem

There are many ways in which feedback design problems can be analysed as \( \mathcal{H}_\infty \) optimization problems. A standard problem shown in Figure 3.4 is described by

\[
\begin{bmatrix}
    z \\
    y
\end{bmatrix} = G(s)
\begin{bmatrix}
    w \\
    u
\end{bmatrix} = 
\begin{bmatrix}
    G_{11}(s) & G_{12}(s) \\
    G_{21}(s) & G_{22}(s)
\end{bmatrix}
\begin{bmatrix}
    w \\
    u
\end{bmatrix}
\]

\( u = K(s)y \) \hspace{1cm} (3.42)

As shown before, the closed-loop transfer function from \( w \) to \( z \) is given by LFT

\[
z = F_l(G,K)w
\]

\( = (G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21})w \) \hspace{1cm} (3.44)

Let \( G(s) \) be given by its state-space matrices \( A, B, C \) and \( D \) and introduce the notation:

\[
G = \begin{bmatrix}
    A & B_1 & B_2 \\
    C_1 & D_{11} & D_{12} \\
    C_2 & D_{21} & D_{22}
\end{bmatrix}
\]

\( \mathcal{H}_\infty \) control involves the minimization of \( \mathcal{H}_\infty \) norm of \( F_l(G,K) \). The following requirements must be satisfied by \( \mathcal{H}_\infty \) problem to a plant \( G \).

1. (R1) \( (A, B_2, C_3) \) is stabilizable and detectable.
2. (R2) \( D_{12} \) and \( D_{23} \) have full rank.
3. (R3) \( \begin{bmatrix}
    A - j\omega I \\
    C
\end{bmatrix}
\) has full column rank for all \( \omega \).
4. (R4) \( \begin{bmatrix}
    A - j\omega I \\
    C
\end{bmatrix}
\) has full row rank for all \( \omega \).
5. (R5) \( D_{11} = 0 \) and \( D_{22} = 0 \).
6. (R6) \( D_{12} = \begin{bmatrix}
    0 \\
    I
\end{bmatrix} \) and \( D_{21} = \begin{bmatrix}
    0 & I
\end{bmatrix} \)
7. (R7) \( D_{12}^T C_1 = 0 \) and \( B_2 D_{21}^T = 0 \)
8. (R8) \( (A, B_1) \) is stabilizable and \( (A, C_1) \) is detectable.

Assumptions (R5)–(R7) are not necessary but they significantly simplify the algorithm formulas. If (R7) holds then (R3) and (R4) may be replaced by (R8).
From Figure 3.4 and theorem 3.4, it will be known that the standard $\mathcal{H}_\infty$ optimal control problem is to find all stabilizing controllers $K$, which minimize

$$\|F_j(G,K)\|_\infty = \max_\omega \sigma(F_j(G(j\omega),K(j\omega)))$$

(3.46)

In general, the $\mathcal{H}_\infty$ algorithm is to find a sub-optimal controller. That is, for a specified $\gamma$ a stabilizing controller is found for which $\|FL(G,K)\|_\infty < \gamma$. If an optimal controller is required then the algorithm can be used iteratively, reducing $\gamma$ until the minimum is reached within a given tolerance. In practice, it is usually not necessary to obtain an optimal controller for $\mathcal{H}_\infty$ problem, and it is often computationally simpler to design a sub-optimal one. Let $\gamma_{\text{min}}$ be the minimum value of $\|F_j(G,K)\|_\infty$ over all stabilizing controllers $K$, then the $\mathcal{H}_\infty$ sub-optimal control problem is:

Given a $\gamma > \gamma_{\text{min}}$, find all stabilizing controllers $K$ such that

$$\|F_j(G,K)\|_\infty < \gamma$$

(3.47)

This can be solved efficiently using MATLAB robust control toolbox. An optimal solution is approached by reducing $\gamma$ iteratively. The algorithm is summarized below with all the simplifying assumptions.

For the general control configuration of Figure 3.4 described by equation (3.42)–(3.45), with assumption (R1)–(R8), there exists a stabilizing controller $K(s)$ such that $\|FL(G,K)\|_\infty < \gamma$ if and only if

(i) $X_\infty \geq 0$ is a solution to the algebraic Riccati equation

$$A^T X_\infty + X_\infty A + C_1^T C_1 + X_\infty (\gamma^{-2}B_1^T B_1 - B_2 B_2^T) X_\infty = 0$$

(3.48)

such that $\text{Re} \lambda_i [A + (\gamma^{-2}B_1^T B_1 - B_2 B_2^T) X_\infty] < 0$, $\forall i$;

(ii) $Y_\infty \geq 0$ is a solution to the algebraic Riccati equation

$$AY_\infty + Y_\infty A^T + B_1^T B_1 + Y_\infty (\gamma^{-2}C_1^T C_1 - C_2^T C_2) Y_\infty = 0$$

(3.49)

such that $\text{Re} \lambda_i [A + Y_\infty (\gamma^{-2}C_1^T C_1 - C_2^T C_2)] < 0$, $\forall i$;

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$

(3.50)

All such controllers are then given by $K = F_j(J,Q)$ where

$$J(s) = \begin{bmatrix}
A_\infty & -Z_\infty L_\infty & L_\infty B_2 \\
F_\infty & 0 & I \\
-C_2 & I & 0
\end{bmatrix}$$

(3.51)

$$F_\infty = -B_2^T X_\infty$$

(3.52)

$$L_\infty = -Y_\infty C_2^T$$

(3.53)

$$Z_\infty = (I - \gamma^{-2}Y_\infty X_\infty)^{-1}$$

(3.54)

$$A_\infty = A + \gamma^{-2}B_1^T B_1 X_\infty + B_2 F_\infty + Z_\infty L_\infty C_2$$

(3.55)

and $Q(s)$ is any stable proper transfer function matrix with $\|Q\|_\infty < \gamma$, so that

$$K(s) = J_{11}(s) + J_{12}(s)Q(s)(I - J_{22}(s)Q(s))^{-1} J_{21}(s)$$

(3.56)

The $\infty$-norm of the closed loop system $F_j(G(s),J(s),Q(s))$ satisfies:

$$\|F_j(G(s),J(s),Q(s))\|_\infty < \gamma$$

(3.57)

The controller obtained for $Q(s) = 0$ is known as the central $\mathcal{H}_\infty$ controller.
3. Robust Stability and Robust Performance

Given all the assumptions (R1)−(R8) the above expression is the simplest form of the general \( \mathcal{H}_\infty \) algorithm. For the more general situation, some of the assumptions are relaxed. For example, if (R7) is not available, the Riccati equations (i) and (ii) are more complex so that the \( K \) is more complex too. If \( D_{22} \neq 0 \) in \( G \), first apply the above mentioned algorithm to the following generalized plant

\[
\hat{G} = G - \begin{bmatrix} 0 & 0 \\ 0 & D_{22} \end{bmatrix}
\]  

and obtain a controller \( \hat{K} \) for \( G \). Then form the controller

\[
K = \hat{K}(I + D_{22}\hat{K})^{-1}
\]  

If \( \hat{K} \) stabilizes \( \hat{G} \), then \( K \) will stabilize \( G \). This conclusion can be obtained directly from the structure plot of \( \hat{G} \) and \( \hat{K} \) (see Figure 3.13), and that

\[
F_j(G, K) = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}
= \hat{G}_{11} + \hat{G}_{12}\hat{K}(I + D_{22}\hat{K})^{-1}[I - (\hat{G}_{22} + D_{22}\hat{K})\hat{K}(I + D_{22}\hat{K})]^{-1}\hat{G}_{21}
= \hat{G}_{11} + \hat{G}_{12}\hat{K}[I + D_{22}\hat{K}] - (\hat{G}_{22} + D_{22}\hat{K})\hat{K}^{-1}\hat{G}_{21}
= \hat{G}_{11} + \hat{G}_{12}\hat{K} - (\hat{G}_{22} + D_{22}\hat{K})\hat{G}_{21} = F_j(\hat{G}, \hat{K})
\]  

The necessary condition of this transform is only \( \det(I + D_{22}\hat{K}(\infty)) \neq 0 \).

Figure 3.13 Controller for a plant which is not strictly proper.

If a controller that achieves \( \gamma_{\text{min}} \) to within a specified tolerance is desired, then we can perform a bisection on \( \gamma \) until its value is sufficiently accurate. The above result provides a test for each value of \( \gamma \) to determine whether it is less than \( \gamma_{\text{min}} \) or greater than \( \gamma_{\text{min}} \). It is called \( \gamma \)-iteration.

3.7 Mixed Sensitivity Approach (\( \gamma \)-iteration)

The mixed sensitivity approach is a direct and effective way of achieving multivariable loop shaping. Mixed sensitivity is the name given to transfer function shaping problems in which the sensitivity function \( S = (I + PK)^{-1} \) is shaped along with one or more other closed-loop transfer functions such as \( K(s)S(s) \) and/or complementary sensitivity function \( T \), e.g.
3. Robust Stability and Robust Performance

\[
\begin{bmatrix}
W_1 S \\
W_2 KS
\end{bmatrix}
\]

or

\[
\begin{bmatrix}
W_1 S \\
W_3 T
\end{bmatrix}
\]

(3.62)

where the \( P \) is a controlled plant and \( K \) is the controller.

The **Mixed sensitivity approach** means to find a stabilizing controller \( K \) which minimizes above criterion (3.61) or (3.62).

The General feedback control system is redrawn in Figure 3.14. \( S \) is the transfer function from disturbance \( d \) to output \( y \). It will be perfect for disturbance rejection if the maximum singular value \( \sigma(S) \) is made small over the frequency of disturbance. Considering that the disturbance is typically a low frequency signal, a low pass filter \( W_1(s) \) with a bandwidth equal to that of disturbance can be selected to shape \( S \).

\( KS \) is a transfer function between \( d \) and the control signal \( u \). It is important to include \( KS \) as a mechanism for limiting the size and bandwidth of the controller output, and hence the control energy used. The weight \( W_2(s) \) is usually designed as a high pass filter with a crossover frequency approximately equal to that of the desired closed-loop bandwidth.

The ability to shape \( T \) is desirable for tracking problem and noise attenuation. It is also important for robust stability with respect to output multiplicative perturbations. The complementary sensitivity weight matrix \( W_3(s) \) can be chosen according to the requirement of frequency characteristic of the performance.

\[
\begin{bmatrix}
G \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
W_1 \\
W_2 \\
W_3
\end{bmatrix}
\]

\[
\begin{bmatrix}
z_1 \\
z_2 \\
z_3
\end{bmatrix}
\]

\[
y
\]

Figure 3.15 \( S/KS/T \) weighted mixed sensitivity problem
The system structure plot with these three weight functions is shown in Figure 3.15. The shaping of the closed loop transfer functions of this system is more difficult than the shaping of two functions. For RRD control, we only consider a system with two weight functions that is given in Figure 3.16. So that for the structure of mixed sensitivity, the elements of the corresponding generalized plant $P$ are

$$
G_{11} = \begin{bmatrix} W_1 \\ 0 \end{bmatrix} \quad G_{12} = \begin{bmatrix} -W_1 P \\ W_2 P \end{bmatrix} \\
G_{21} = I \quad G_{22} = -P
$$

(3.63)

![Figure 3.16 S/T weighted mixed sensitivity problem](image)

A difficult matter is to select these weight functions $W_1$, $W_3$ in the design of $\mathcal{H}_\infty$ controller. Several suggestions have been given in [Skogestad and Postlethwaite, 1996]. One of them is:

Weight $W_i$ must be stable. If not, the requirement (R1) is not satisfied. Therefore when we need an integral action in weight, for example, a $\frac{1}{s+\varepsilon}$ is used to approximate $\frac{1}{s}$, where $\varepsilon << 1$.

By the mixed sensitivity approach, robust stability and nominal performance is actually achieved, if a controller $K$ which makes (3.62) small than 1 is found. From (3.29) and (3.18) it is easily seen that in this case, the weight input functions $W_{p1}$ and $W_{u2}$ are $I$, which means that the same weight for the inputs in the whole frequency range are adopted and there is no scaling to inputs. $W_1$ and $W_3$ in Figure 3.6 correspond to $W_{p2}$ and $W_{u1}$ in Figure 3.10. Therefore the meaning of $W_1$ and $W_3$ are clear now. We can say that if a stabilizing $K$ is found to make

$$\begin{bmatrix} W_1 S \\ W_3 T \end{bmatrix}_{\infty} < \gamma ,$$

then by the mixed sensitivity approach the robust performance can be obtained as well.

### 3.8 Structured Singular Value

The controller designed by the above mentioned methods have the disadvantage of conservation because the uncertainty is unstructured. If the uncertainty is structured, a better control objective can be desired. This purpose can be reached by the structured singular value $\mu$ synthesis. The
structured singular value $\mu$ is a function, which provides a generalization of the singular value $\sigma$ and the spectral radius $\rho$.

First, consider a standard feedback interconnection of a stable $M(s)$ and a normalized perturbation $\Delta(s)$ shown in Figure 3.17. One important question we may ask is how large $\Delta$ (in the sense of $\|\Delta\|_\infty$) can be without destabilizing the feedback system.

![Figure 3.17 A standard feedback interconnection](image)

Since the closed-loop poles are given by $\det(I - M\Delta) = 0$, the feedback system becomes unstable if $\det(I - M(s)\Delta(s)) = 0$ for some $s \in \overline{C}_+$. ($\overline{C}_+$ means closed right-half complex plane).

Now let $\alpha > 0$ be a sufficiently small number such that the closed-loop system is stable for all stable $\|\Delta\|_\infty < \alpha$. Next increase $\alpha$ until $\alpha_{\text{max}}$ so that the closed-loop system becomes unstable (see Figure 3.18). So $\alpha_{\text{max}}$ is the **robust stability margin**.

![Figure 3.18 Robust stability margin $\alpha_{\text{max}}$](image)

By small gain theorem,

$$\frac{1}{\alpha_{\text{max}}} = \|M\|_\infty = \sup_{s \in \overline{C}_+} \sigma(M(s)) = \sup_{\omega} \sigma(M(j\omega))$$

(3.64)

if $\Delta$ is unstructured. Note that for any fixed $s \in \overline{C}_+$, $\sigma(M(s))$ can be written as

$$\sigma(M(s)) = \frac{1}{\min\{\sigma(\Delta) \mid \det(I - M(s)\Delta) = 0, \Delta \text{ is unstructured}\}}$$

(3.65)

In other words, the reciprocal of the largest singular value of $M$ is a measure of the smallest unstructured $\Delta$ that causes instability of the feedback system.

To quantify the smallest destabilizing structured complex $\Delta$, the concept of singular values needs to be generalized. In view of the characterization of the largest singular value of a matrix $M(s)$ given by (3.65), we shall define

$$\mu(M(s)) = \frac{1}{\min\{\sigma(\Delta) \mid \det(I - M(s)\Delta) = 0, \Delta \text{ is structured}\}}$$

(3.66)

as the largest structured singular value of $M(s)$ with respect to the structured complex $\Delta$. Then it is obvious that the robust stability margin of the feedback system with structured complex uncertainty $\Delta$ is

$$\frac{1}{\alpha_{\text{max}}} = \|M\|_\infty = \sup_{s \in \overline{C}_+} \mu(M(s)) = \sup_{\omega} \mu(M(j\omega))$$

(3.67)
The formal definition of structured singular value $\mu$ is

$$
\begin{align*}
\mu_\lambda(M) &= \frac{1}{\min \{ \sigma(\Delta) | \Delta \in \Lambda, \det(I - M\Delta) = 0 \}} \\
\mu_\lambda(M) &= 0 \quad \text{if no } \Delta \in \Lambda \text{ let } \det(I - M\Delta) = 0
\end{align*}
$$

(3.68)

where

$$
\Lambda = \{ \text{diag}[\delta_1, I_{r_1}, \ldots, \delta_S, I_{r_S}, \Delta_1, \ldots, \Delta_F] | \delta_i \in C, \Delta_j \in C^{m_j \times m_j} \}
$$

(3.69)

There are two types of perturbation (uncertainty) blocks in the above expression: repeated scalar blocks and full blocks. The $i$'th repeated scalar block is $r_i \times r_i$, while the $j$'th full block is $m_j \times m_j$. Two non-negative integers $S$ and $F$ represent the numbers of repeated scalar blocks and full blocks respectively. All the dimensions should satisfy

$$
\sum_{i=1}^{S} r_i + \sum_{j=1}^{F} m_j = n
$$

so that $\Lambda \subset C^{n \times n}$.

The $1/\mu_\lambda(M)$ is the “size” of the smallest perturbation measured by its maximum singular value, which makes $I - M\Delta$ singular. If $M(s)$ is a transfer function, we can interpret $1/\mu_\lambda(M(j\omega))$ as the “size” of the smallest perturbation $\Delta(j\omega)$ which shift the characteristics of the transfer matrix $M(s)$ to the Nyquist point $(-1+j0)$ at $s$-plane.

Clearly $\mu_\lambda(M)$ depends not only on $M$ but also on the structure of $\Lambda$. The following example demonstrates that $\mu$ also depends on whether the perturbation is real or complex.

**Example 1. $\mu$ of a scalar** [Skogestad and Postlethwaite, 1996]. If $M$ is a scalar then in most cases $\mu(M) = |M|$. This follows from (3.68) by selecting $|\Delta| = (1/|M|)$ such that $I - M\Delta = 0$. However, this requires that we can select the phase of $\Delta$ such that $M\Delta$ is real, which is impossible when $\Delta$ is real and $M$ has an imaginary component, so in this case $\mu(M) = 0$. In summary, we have

$$
\begin{align*}
\Delta \text{ complex:} & \quad \mu(M) = |M| \\
\Delta \text{ real:} & \quad \mu(M) = \begin{cases} |M| & \text{for real } M \\ 0 & \text{otherwise} \end{cases}
\end{align*}
$$

From above discussions, we know that when all the blocks in $\Delta$ are complex, $\mu$ may be computed relatively easy. So for most cases, we only consider complex $\Delta$ in this thesis.

The robust stability and robust performance with $\mu$ need to be redefined since the structured uncertainty replaces the unstructured uncertainty in the plant model.

**Theorem 3.5 (Robust Stability with $\mu$).** Assume that the interconnection $M(s)$ and the perturbation $\Delta(s)$ are stable, then the closed loop system in Figure 3.8 is stable for all perturbations $\Delta(s) \in \Lambda(s)$, $\sigma(\Delta(j\omega)) > 1$ if and only if

$$
\mu_\lambda(M(j\omega)) < 1 \quad \forall \omega \in \mathbb{R}
$$

(3.70)

More generally, let $\beta > 0$, the closed loop system in Figure 3.8 is stable for all perturbations $\Delta(s) \in \Lambda(s)$, $\sigma(\Delta(j\omega)) < \frac{1}{\beta}$ if and only if

$$
\mu_\lambda(M(j\omega)) < \beta \quad \forall \omega \in \mathbb{R}
$$

(3.71)

If $\mu > 1$, what does it mean? For example, a value of $\mu = 1.1$ for robust stability means that all the uncertainty blocks $\mu$ must be decreased in magnitude by factor 1.1 in order to guarantee
stability. Otherwise that will make $I - M\Delta$ singular. A larger value of $\mu$ is “bad” as it means that a smaller perturbation makes $I - M\Delta$ singular, whereas a smaller value of $\mu$ is “good”.

![Diagram](image)

**Figure 3.19 N-Δ structure for RP with $\mu$**

**Theorem 3.6 (Robust Performance with $\mu$).** Rearrange the uncertain system into the $N$-$\Delta$ structure shown in Figure 3.19. The transfer function from $w$ to $z$ is denoted by $F_u(N,\Delta)$. Assume nominal stability such that $N$ is internally stable. Then the system will have robust performance if and only if

$$\|F_u(N,\Delta)\|_\omega < 1, \quad \forall \|\Delta\|_\omega \leq 1 \quad (3.72)$$

$$\Leftrightarrow \mu_3(N(j\omega)) < 1, \quad \forall \omega \quad (3.73)$$

Or, let $\beta > 0$, the system is of robust performance if and only if

$$\|F_u(N,\Delta)\|_\omega < \beta, \quad \forall \|\Delta\|_\omega \leq \frac{1}{\beta} \quad (3.74)$$

$$\Leftrightarrow \mu_3(N(j\omega)) < \beta, \quad \forall \omega \quad (3.75)$$

The $\mu$ condition for Robust Performance involves the enlarged perturbation $\tilde{\Delta} = \text{diag}\{\Delta, \Delta_p | \Delta \in \Lambda, \Delta_p \in C^{k\times k}\}$. Here $\Delta$, which itself may be a block-diagonal matrix, represents the true uncertainty, whereas $\Delta_p$ is a full complex matrix stemming from the $H_\infty$ norm performance specification. For example, for the nominal system (with $\Delta = 0$) we get it from properties of $\mu$ (see next section: two extreme sets) that $\Sigma(N_{22}) = \mu_{\Delta_p}(N_{22})$, because $\Delta_p$ is a full matrix.

Since $\tilde{\Delta} = \text{diag}\{\Delta, \Delta_p\}$, it is clear from the properties of $\mu$ [Morari and Zafiriou, 1989] that,

$$\mu_3(N) \geq \max\{\mu_3(N_{11}), \mu_{\Delta_p}(N_{22})\} \quad (3.76)$$

where as just noted $\mu_{\Delta_p}(N_{22}) = \Sigma(N_{22})$. Equation (3.76) implies that $\text{RS} (\mu_{\Delta}(N_{11}) < 1)$ and $\text{NP} (\Sigma(N_{22}) < 1)$ are automatically satisfied when $\text{RP} (\mu_3(N(j\omega)) < 1)$ is satisfied ($N_{22}$ is a transfer function from $w$ to $z$ (see Figure 3.5)). However, note that $\text{NS}$ (stability of $N$) is not guaranteed by (3.76) and must be tested separately.

### 3.9 $\mu$ Synthesis (D-K iteration)

As we have seen that, the structured singular value $\mu$ is a powerful tool for investigating $\text{RS}$ and $\text{RP}$ of perturbed systems with structured and/or unstructured uncertainty. However the direct computation of $\mu$ from definition is impossible at present. A number of methods have been developed to solve this problem indirectly. One of them is to approximate it using upper bound
and lower bound of $\mu$. Let us first consider an alternative expression for $\mu(M)$ using the notation $BA = \{ \Delta \in \Lambda \mid \sigma(\Delta) \leq 1 \}$

$$\mu(M) = \max_{\Delta \in BA} \rho(M\Delta) \quad (3.77)$$

We can relate $\mu(M)$ to familiar linear algebraic quantities when $\Lambda$ is one of the two extreme sets outlined below.

- If $\Lambda = \{ \delta \mid \delta \in C \} (S = 1, F = 0, r_1 = n)$, then $\mu(M) = \rho(M)$, the spectral radius of $M$.
- If $\Lambda = C^{\text{max}} (S = 0, F = 1, m_1 = n)$, then $\mu(M) = \sigma(M)$, the maximum singular value.

Obviously, for a general $\Lambda$ as in (2.71), we must have

$$\{ \delta \mid \delta \in C \} \subset \Lambda \subset C^{\text{max}} \quad (3.78)$$

Hence directly from the definition of $\mu$ and the two special cases above, we conclude that

$$\rho(M) \leq \mu(M) \leq \sigma(M) \quad (3.79)$$

These bounds alone are not sufficient because the gap between $\rho$ and $\sigma$ can be arbitrarily large. Thus the bounds given in (3.79) must be tightened, otherwise neither $\rho$ nor $\sigma$ provide useful bounds. However, the tightened bounds can be obtained by considering transformations on $M$ that do not affect $\mu(M)$, but do affect $\rho$ and $\sigma$. To do this, the following two subsets of $C^{\text{max}}$ are defined

$$\Theta = \{ Q \in \Lambda \mid QQ^T = I_n \} \quad (3.80)$$

$$\Gamma = \left\{ \left[ \begin{array}{c} \text{diag}(D_1, \ldots, D_S, d_1 I_{m_1}, \ldots, d_F I_{m_F}) \\ D_i \in C^{n \times n}, D_i = D_i^*, d_j > 0, d_j \in R, d_j > 0 \end{array} \right] \right\} \quad (3.81)$$

**Theorem 3.7 (Upper and Lower Bound of $\mu$).** For all $Q \in \Theta$ and $D \in \Gamma$

$$\mu(MQ) = \mu(M) = \mu(DMD^{-1}) \quad (3.82)$$

So the tightened bounds of $\mu$ are

$$\max_{Q \in \Theta} \rho(QM) \leq \mu(M) \leq \inf_{D \in \Gamma} \sigma(DMD^{-1}) \quad (3.83)$$

Unfortunately, the quantity $\rho(QM)$ can have multiple local maxima, which are not global. Thus local search cannot guarantee to obtain $\mu$, but may yield only a lower bound. For upper bound, this optimization is convex in $D$ (i.e. there is only one minimum, the global minimum), then $\inf_{D \in \Gamma} \sigma(DMD^{-1})$ can, in principle, be found. However, the upper bound is not always equal to $\mu$ for some block structures. There exist matrices for which, $\mu$ is less than the infimum [Zhou, Doyle and Glover, 1996]. Nevertheless, even in the case where the upper bound does not equal to $\mu$, the upper bound is still reasonably tight.

For reliable use of the $\mu$ theory, it is essential to have upper and lower bounds. The most important use of the upper bound is taken as a computational scheme when combined with the lower bound. Another important feature of the upper bound is that it can be combined with $\mathcal{H}_\infty$ controller synthesis methods to yield an ad-hoc $\mu$-synthesis method. Note that the upper bound when applied to transfer functions is simply a scaled $\mathcal{H}_\infty$ norm. This is exploited in the $D-K$ iteration procedure to perform $\mu$-synthesis.

The idea of $D-K$ iteration is to find the controller that minimizes the peak value over frequency of this upper bound. This is,

$$\min_k \left( \inf_{D \in \Gamma} \left\| D N(K) D^{-1} \right\|_{\infty} \right) \quad (3.84)$$
\[ \Leftrightarrow \min_K \inf_{D \in \mathcal{G}} \|DF_l(G,K)D^{-1}\|_\infty \tag{3.85} \]

by alternating between minimizing \(\|DF_l(G,K)D^{-1}\|_\infty\) with respect to either \(K\) or \(D\) (while holding the other fixed). To start the iterations, a stable and minimum phase matrix must be selected for \(D\) and \(D\Delta = \Delta D\) also be satisfied. For a fixed scaling transfer matrix \(D\), \(\min_K \|DF_l(G,K)D^{-1}\|_\infty\) is a standard \(\mathcal{H}_\infty\) optimization problem. For a given stabilizing controller \(K\), \(\inf_{D \in \mathcal{G}} \|DF_l(G,K)D^{-1}\|_\infty\) is a standard convex optimization problem and it can be solved pointwise in frequency domain

\[ \sup_{\omega} \inf_{D \in \mathcal{G}} \|DF_l(G,j\omega,K(j\omega))D^{-1}\|_\infty \tag{3.86} \]

The iteration may continue until satisfactory performance is achieved: \(\|DF_l(G,K)D^{-1}\|_\infty < 1\), or until the \(\mathcal{H}_\infty\) norm no longer decreases. The \(D-K\) iteration, which combines \(\mathcal{H}_\infty\) synthesis and \(\mu\) analysis, often yields good results.

In the above discussions only complex perturbation sets and the complex structured singular values are involved to assess stability and performance degradation under these types of perturbations. In specific instances, it may be more natural to model some of the uncertainties with real perturbations. For example, this type of uncertainty takes place on the real coefficients of a linear differential equation. It is possible to simply treat these perturbations as complex and proceed with the complex-\(\mu\) analysis, the results is expected to be conservative.

![Figure 3.20](image)

Suppose that a coefficient \(c\), in a particular system, is constant, but unknown, and the value of \(c\) is modeled to lie in an interval, for instance, \(c \in [0.8 \ 1.6]\). This can be modeled effectively with a real perturbation,

\[ c \in \{1.2 + (0.4)\delta \mid \delta \in \mathbb{R}, |\delta| \leq 1\} \]

Clearly, this set captures the uncertainty in the coefficient \(c\), what is the correct interpretation of the uncertainty set model if \(\delta\) is taken as complex

\[ c \in \{1.2 + (0.4)\delta \mid \delta \in \mathbb{C}, |\delta| \leq 1\} \]

In a linear, time-invariant system, robustness to this constant, complex uncertain parameter \(c\) is mathematically equivalent to robustness to all stable, linear, time-invariant transfer functions, \(\hat{c}(s)\), whose Nyquist plots lie in the disk shown in Figure 3.20.

Consequently, using complex parameters, the uncertain model for \(c\) represents a stable linear system whose characteristics are similar to an uncertain real gain, but deviate in a manner quantified by the disc-shaped constraint on its frequency response. In general, using disk instead of intervals leads to more conservative robustness properties.
Hence researchers have developed algorithms for robustness tests with both real and complex perturbation blocks [Balas et al., 1993; Young, 1994; TØffner-Clausen, 1995]. The theory for a mixed real/complex upper bound is more complicated to describe than the upper bound theory for complex $\mu$. Here we only introduce the results stated by [Young, 1994].

Suppose we have a system matrix $M, M \in \mathbb{C}^{n \times n}$ and a mixed perturbation $\Lambda$, defined by

$$\Lambda = \left\{ \left[ \begin{array}{c} \delta_r^1 I_r, \delta_r^m I_r, \delta_c^i I_c, \delta_c^m I_c, \ldots, \delta_r^i I_r, \delta_r^m I_r, \delta_c^i I_c, \delta_c^m I_c, \Delta_r, \ldots, \Delta_c \end{array} \right] \right\}$$

with $m = m_r + m_c$. Notice that for compatibility with $M$, we must have $\sum_{i=1}^{m} r_i = n$. We define the following set of block diagonal scaling matrices

$$\hat{D} = \{ \text{diag}(D_1, \ldots, D_{m_r+m_c}, d_1 I_{n_r+n_c}, \ldots, d_m I_r) \}$$

$$\hat{G} = \{ \text{diag}(G_1, \ldots, G_{m_r+m_c}, O_{n_r+n_c}, \ldots, O_r) \}$$

**Theorem 3.8 (Upper Bound of Mixed $\mu$)** If there exists a $\beta > 0$, a $D \in \hat{D}$ and a $G \in \hat{G}$ with appropriate block-diagonal structures such that

$$\sigma\left( I + G^2 \right)^{-\frac{1}{4}} \left( \frac{1}{\beta} DMD^{-1} - jG(I + G^2)^{-\frac{1}{4}} \right) \leq 1$$

then

$$\mu(M) \leq \beta$$

This bound is a derivative of an earlier bound in [Fan et al., 1991]. The smallest $\beta > 0$ for which $D$ and $G$ matrices exist and satisfy this constraint is the mixed $\mu$ upper bound for $M$. There is a corresponding $D,G$-$K$ iteration procedure for the synthesis. In practice using MATLAB Toolbox, the upper bound of mixed $\mu$ is obtained by defining $\beta^*$ as:

$$\beta^* = \inf_{\beta \in \mathbb{R}, G \in \hat{G}, D \in \hat{D}} \{ \beta \left| \sigma(M_{DG}) \leq 1 \right. \}$$

with $M_{DG}$ given by

$$M_{DG} = (I + G^2)^{-\frac{1}{4}} \left( \frac{DMD^{-1}}{\beta} - jG \right)(I + G^2)^{-\frac{1}{4}}$$

then

$$\mu(M) \leq \beta^*$$

Another theorem is given for a lower bound of mixed $\mu$ [Fan et al., 1991; TØffner-Clausen, 1995]

**Theorem 3.9 (Lower bound of mixed $\mu$)** Let $\rho_\Lambda(M)$ denote the real spectral radius of $M$:

$$\rho_\Lambda(M) = \max \{ | \lambda_\Lambda (M) | \left| \lambda_\Lambda (M) \right. \text{is a real eigenvalue of } M \}$$

The lower bound of $\mu$ is

$$\rho_\Lambda(M) \leq \mu(M)$$

If $M$ has no real eigenvalues then $\rho_\Lambda(M) = 0$. 

Theorem 3.8 (Upper Bound of Mixed $\mu$)
Therefore, we get the upper bound and lower bound on mixed real/complex $\mu$:

$$\rho_\mu(M) \leq \mu(M) \leq \beta^*$$  \hspace{1cm} (3.94)

For more details please see [Fan et al., 1991; Young, 1994; Tøffner-Clausen, 1995].

In practice of using $\mu$-synthesis, the following suggestions have been given by [Skogestad and Postlethwaite, 1996]:

1. Because of the effort involved in deriving detailed uncertainty descriptions, and the subsequent complexity in synthesizing controllers, the rule is to “start simple” with a crude uncertainty description and then to see whether the performance specifications can be met. Only if they can’t, should one consider more detailed uncertainty descriptions such as parametric uncertainty (with real perturbations).

2. The use of $\mu$ implies a worst-case analysis, so one should be careful about including too many sources of uncertainty, noise and disturbances – otherwise it becomes very unlikely for the worst case to occur, and the resulting analysis and design may be unnecessarily conservative.

3. There is always uncertainty with respect to the inputs and outputs, so it is generally ‘safe’ to include diagonal input and output uncertainty. The relative (multiplicative) form is very convenient in this case.

4. $\mu$ is most commonly used for analysis. If $\mu$ is used for synthesis, then we recommend that one keeps the uncertainty fixed and adjust the parameters in the performance weight until $\mu$ is close to 1.
Chapter 4
Ship Dynamics Modeling

Modeling of marine vehicles involves the study of statics and dynamics. Statics concerns the equilibrium of bodies at rest or moving with constant velocity, whereas dynamics concerns bodies having accelerative motions. Modeling of ship dynamics is much more complex than the statics, and the dynamics modeling may describe the characteristics of ship motion to a great extent.

4.1 Motion Equations of Ships

In this section, the motion of ships will be discussed first, then modeling of ship hydrodynamics will be given.

The motion of a ship has 6 degree of freedom since 6 independent coordinates are necessary to determine the position and orientation of a rigid body. The first three coordinates and their time derivatives correspond to the position and translational motion along the X, Y and Z axes, while the last 3 coordinates and their time derivatives are used to describe the orientation and rotational motion of the body fixed reference frame $A_{xyz}$ with respect to the earth-fixed reference frame $O_{x'y'z'}$. For marine vehicles, the 6 different motion components are conveniently defined as: surge, sway, heave, roll, pitch and yaw (Figure 4.1).

Figure 4.1 Position, force and states for a ship, with reference to inertial and ship body fixed coordinate systems.
The motion of a ship at sea is described relative to an inertial reference frame, it is usually assumed that the accelerations of a point on the surface of the Earth hardly affects the motion description of low speed marine vehicles. As a result, an earth-fixed reference frame $O_{e\,x'y'z'}$ can be considered as inertial. This suggests that the position and orientation of the vehicle should be described relative to the inertial reference frame while the linear and angular velocities of the vehicle should be expressed in the body-fixed coordinate system.

The origin $A$ of the body fixed frame is usually chosen to coincide with the Center of Gravity (CG) when CG is in the principal plane of symmetry. For marine vehicles it is desirable to derive the equations of motion for an arbitrary origin $A$ in a local body fixed coordinate system. The reason is that the CG on a ship is not a certain point. It will be moving as the load of the ship is changed. Since the hydrodynamic and kinematic forces and moments are given in the body-fixed reference frame, we will formulate Newton’s laws in this frame.

When deriving the equations of motion it will be assumed that: (i) the vehicle is rigid and (ii) the earth-fixed reference frame is inertial. The first assumption eliminates the consideration of forces acting between individual elements of mass while the second eliminates forces due to the Earth’s motion relative to a star-fixed reference system.

The motion of a marine vehicle at sea can be considered as the superposition of the rigid body translational motion and rigid body rotating motion [Jia, 1987]. Hence the velocity of the CG of a ship is:

$$ U_G = U_A + \dot{\rho}_G $$

(4.1)

$$ U_A = ui + vj + wk $$

(4.2)

$$ \rho_G = x_Gi + y_Gj + z_Gk $$

(4.3)

where $U_A$ is the speed of origin $A$, $\rho_G$ is a vector from origin to CG, and $i, j$ and $k$ are three orthogonal unit vectors of the body-fixed reference frame.

Due to the fact that the direction can be changed only other than amplitude for $\rho_G$ on a certain point, the following equation holds:

$$ \dot{\rho}_G = \Omega \times \rho_G $$

(4.4)

where $\Omega$ is the angular velocity of the rigid body about the origin

$$ \Omega = pi + qj + rk $$

(4.5)

The rigid body momentum equation based on Newton’s Second Law is:

$$ F = m\dot{U}_G = m(\dot{U}_A + \Omega \times \rho_G + \Omega \times \dot{\rho}_G) $$

(4.6)

where $m$ is the total mass of the ship. It is known from (4.4) that

$$ \dot{i} = \Omega \times i; \quad \dot{j} = \Omega \times j; \quad \dot{k} = \Omega \times k; $$

(4.7)

Therefore, we obtain:

$$ \dot{U}_A = ui + vj + wk + ui + vj + wk $$

$$ = ui + vj + wk + \Omega \times (ui + vj + wk) $$

$$ = ui + vj + wk + \Omega \times U_A $$

(4.8)

Similarly, the following result holds from above equations

$$ \dot{\Omega} = \ddot{p}i + \ddot{q}j + \ddot{r}k $$

(4.9)

Substituting (4.8) (4.9) into (4.6) yields

$$ F = m[ui + vj + wk + \Omega \times U_A + (\dot{q}z_G - \dot{r}y_G)i + (\dot{r}x_G - \dot{p}z_G)j + $$

$$ + (\dot{p}y_G - \dot{q}x_G)k + \Omega \times (\Omega \times \rho_G)] $$

(4.10)

Expanding (4.10) and then projecting it on three orthogonal coordinates of body-fixed $A_{xyz}$, the force expressions on $x, y$ and $z$ axes could be obtained respectively
\[ F_x = m(u + qw - rv - x_G (q^2 + r^2) + y_G (pq - r) + z_G (rp + \dot{q})) \]  
\[ F_y = m(v + ru - pw - y_G (r^2 + p^2) + z_G (qr - \dot{p}) + x_G (pq + \dot{r})) \]  
\[ F_z = m(w + pv - qu - z_G (p^2 + q^2) + x_G (rp - \dot{q}) + y_G (qr + \dot{p})) \]  

The three equations (4.11) represent the translational motion. Now, the other three equations representing rotation of the ship need to be derived.

According to the angular momentum theorem of a rigid body rotating around the center of gravity, we have

\[ M_G = \frac{dh_G}{dt} \]  

\[ M_G \] is the moment relating to the body’s center of gravity, \( h_G \) is an angular momentum about the center of gravity. Based on the definition of angular momentum

\[ h_G = \sum \Delta_m r \times (\Omega \times r) \]  

where \( \Delta_m \) is a mass element of the rigid body; \( r \) is a vector from the mass element to the center of gravity and \( r = \alpha \mathbf{a} + \beta \mathbf{j} + \gamma \mathbf{k} \); \( \Omega \) is the angular velocity of the rigid body.

Expanding the expression (4.13) yields:

\[ h_G = (I_{aa} p - I_{ab} q - I_{ar} r) \mathbf{i} + (-I_{ba} p + I_{bb} q - I_{br} r) \mathbf{j} + \\
+ (-I_{ca} p - I_{cb} q + I_{cr} r) \mathbf{k} \]  

If we use \( I_G \) to express the inertia tensor about the body’s center of gravity

\[ I_G = \begin{bmatrix} I_{aa} & -I_{ab} & -I_{ar} \\ -I_{ba} & I_{bb} & -I_{br} \\ -I_{ca} & -I_{cb} & I_{cc} \end{bmatrix} \]  

where \( I_{aa} = \sum \Delta_m (\beta^2 + \gamma^2) \), \( I_{bb} = \sum \Delta_m (\alpha^2 + \gamma^2) \), \( I_{cc} = \sum \Delta_m (\alpha^2 + \beta^2) \) are the inertia moments of ship (rigid body) around the body attached coordinate system \( A_{xyz} \); \( I_{ab} = \sum \Delta_m \alpha \beta = I_{ba} \), \( I_{ar} = \sum \Delta_m \alpha \gamma = I_{ra} \), \( I_{br} = \sum \Delta_m \beta \gamma = I_{rb} \) are the products of inertia about the same coordinate system.

In this way, the relative angular momentum \( h_G \) can be written as

\[ h_G = \begin{bmatrix} h_a \\ h_b \\ h_r \end{bmatrix} = \begin{bmatrix} I_{aa} & -I_{ab} & -I_{ar} \\ -I_{ba} & I_{bb} & -I_{br} \\ -I_{ca} & -I_{cb} & I_{cc} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = I_G \Omega \]  

where \( \Omega \) is a column vector consisted of the elements of \( \Omega \).

We need not only an equation of angular momentum about the center of gravity, but also an equation of angular momentum about any point in the rigid body.

**Theory of Mechanics:** the total dynamic action exerted to a rigid body from the external environment results in a force \( F \) and a moment \( M \) acting on a point (say \( G \), we have \( M_G \)). When moving the point to any other place within the body (say, \( A \), we have \( M_A \)), the resulted force is unchanged as \( F \), nevertheless, the resulted moment \( M_A \) is equal to \( M_G \) added by a force couple whose moment is \( \rho_G \times F \). Therefore

\[ M_A = M_G + \rho_G \times F = \dot{h}_G + \rho_G \times F \]  

Consider the **Parallel Axes Theorem** for the inertia momentum, we have [Yang, 1988]
where $I_A$ is the inertia tensor about an arbitrary origin $A$ and $I_{GA}$ is a variation of inertia tensor from center of gravity $CG$ to origin $A$.

$$I_A = I_G + I_{GA} \tag{4.18}$$

where $I_{x_Gx_G} = m(y_G^2 + z_G^2)$, $I_{y_Gy_G} = m(x_G^2 + z_G^2)$, $I_{z_Gz_G} = m(x_G^2 + y_G^2)$ are the inertia moments of $CG$ about the rigid body coordinate system $x, y, z$ axes; $I_{x_Gy_G} = mx_Gy_G = I_{y_Gx_G}$, $I_{x_Gz_G} = mx_Gz_G = I_{z_Gx_G}$, $I_{y_Gz_G} = my_Gz_G = I_{z_Gy_G}$ are the products of inertia of $CG$ about the same coordinate system.

From the equation (4.18), we see that

$$\Omega = (I_A - I_{GA}^{-1}) \Omega = I_A \Omega - I_{GA} \Omega \quad \tag{4.19}$$

Hence, we obtain from (4.16) and (4.19):

$$h_G = h_A - h_{GA} \quad \tag{4.20}$$

where

$$h_A = \sum \Delta_m \rho \times (\Omega \times \rho), \quad \rho$$

is a vector from the mass element to the original point and

$$h_{GA} = m \rho_G \times (\Omega \times \rho_G).$$

So that the following equation can be obtained

$$\dot{h}_A = \frac{d}{dt} \left[ (I_A \Omega)_x i + (I_A \Omega)_y j + (I_A \Omega)_z k \right]$$

$$= (I_A \dot{\Omega})_x i + (I_A \dot{\Omega})_y j + (I_A \dot{\Omega})_z k + (I_A \Omega)_x \dot{i} + (I_A \Omega)_y \dot{j} + (I_A \Omega)_z \dot{k}$$

$$= (I_A \dot{\Omega})_x i + (I_A \dot{\Omega})_y j + (I_A \dot{\Omega})_z k + (r \dot{j} - q \dot{k}) +$$

$$+ (I_A \Omega)_x (p \dot{k} - r \dot{i}) + (I_A \Omega)_y (q \dot{i} - p \dot{j}) \quad \tag{4.21}$$

From (4.20) and (4.17), we know that

$$M_A = \dot{h}_G + \rho_G \times F - \dot{h}_{GA} \quad \tag{4.22}$$

Considering that $\dot{h}_{GA} = m(\dot{\rho}_G \times (\Omega \times \rho_G) + \rho_G \times (\dot{\Omega} \times \rho_G) + \rho_G \times (\Omega \times \dot{\rho}_G))$, the last part of equation (4.22) can be expanded as

$$\rho_G \times F - \dot{h}_{GA}$$

$$= \rho_G \times m(\dot{U}_A + \Omega \times \rho_G + \Omega \times \dot{\rho}_G)$$

$$- m(\dot{\rho}_G \times (\Omega \times \rho_G) + \rho_G \times (\dot{\Omega} \times \rho_G) + \rho_G \times (\Omega \times \dot{\rho}_G))$$

$$= m(\rho_G \times \dot{U}_A - \dot{\rho}_G \times (\Omega \times \rho_G))$$

Because the last part of above equation $\dot{\rho}_G \times (\Omega \times \rho_G) = 0$, therefore

$$\rho_G \times F = \dot{h}_{GA}$$

$$= m(\rho_G \times (uii + vj + wk + \Omega \times U_A))$$

$$= m((x_G i + y_G j + z_G k) \times (uii + vj + wk) + (x_G i + y_G j + z_G k) \times (\Omega \times U_A))$$

$$= m((x_G v - y_G u)k + (z_G u - x_G w)j + (y_G w - z_G v)i +$$
Put (4.21) and (4.23) into equation (4.22) and then project it on the $Ax$, $Ay$ and $Az$ axes, the following equations are obtained

$$M_x = I_{xx} \dot{q} - (I_{yy} - I_{zz}) qr - I_{xy} (\dot{q} - rp) - I_{xz} (\dot{r} + pq)$$

$\quad - I_{yx} (r^2 - p^2) + m y_G (\dot{w} + pv - qu) - m z_G (\dot{z} + qw - rv)$

(4.24a)

$$M_y = I_{yy} \dot{q} - (I_{zz} - I_{xx}) rp - I_{xz} (\dot{r} - pq) - I_{xy} (\dot{p} + qr)$$

$\quad - I_{zy} (r^2 - p^2) + m z_G (\dot{u} + qw - rv) - m x_G (\dot{u} + pq - rv)$

(4.24b)

$$M_z = I_{zz} \dot{r} - (I_{xx} - I_{yy}) pq - I_{xz} (\dot{p} - qr) - I_{yx} (\dot{q} + rp)$$

$\quad - I_{zy} (p^2 - q^2) + m x_G (\dot{v} + ru - pw) - m y_G (\dot{v} + pw - ru)$

(4.24c)

These three equations represent the rotational motion. Equations (4.11) are combined with equations (4.24) to form ship dynamic equations. Using $X, Y$ and $Z$ to express the external forces acting on the body along axes $Axyz$ and $K, M$ and $N$ to express the moments of external forces about origin $A$:

$$[X, Y, Z]^T = [F_x, F_y, F_z]^T; \quad [K, M, N]^T = [M_x, M_y, M_z]^T;$$

the ship dynamic equations can be expressed in a more compact form as:

$$M_R \ddot{\mathbf{v}} + C_R \mathbf{v} = \tau$$

(4.25)

Here $\mathbf{v} = [u, v, w, p, q, r]^T$ is the body-fixed linear and angular velocity vector and

$$M_R = \begin{bmatrix}
      m & 0 & 0 & 0 & m z_G & -m y_G \\
      0 & m & 0 & -m z_G & 0 & m x_G \\
      0 & 0 & m & m y_G & -m x_G & 0 \\
      0 & -m z_G & m y_G & I_{xx} & -I_{xy} & -I_{xz} \\
      m z_G & 0 & -m x_G & -I_{yz} & I_{yy} & -I_{yz} \\
      -m y_G & m x_G & 0 & -I_{zx} & -I_{zy} & I_{zz}
\end{bmatrix}$$

is the rigid body inertia matrix, and

$$C_R(\mathbf{v}) = \begin{bmatrix}
      0 & 0 & 0 & 0 & 0 & 0 \\
      0 & 0 & 0 & 0 & 0 & 0 \\
      0 & 0 & 0 & 0 & 0 & 0 \\
      -m (y_G q + z_G r) & m (y_G p + w) & m (z_G p - v) \\
      m (x_G q - w) & -m (z_G r + x_G p) & m (z_G q + u) \\
      m (x_G r + v) & m (y_G r - u) & -m (x_G p + y_G q)
\end{bmatrix}$$
is the rigid body Coriolis and centripetal matrix
\[ \mathbf{\tau} = \begin{bmatrix} X, & Y, & Z, & K, & M, & N \end{bmatrix} \] is a generalized vector of external force and moment, these can be classified according to:

(i) Radiation-induced forces \( \mathbf{\tau}_R \) (added inertia; hydrodynamic damping; restoring forces)

(ii) Environmental forces \( \mathbf{\tau}_E \) (forces caused by ocean currents; waves; wind)

(iii) Propulsion forces \( \mathbf{\tau}_P \) (thruster/propeller forces; control surfaces/rudder forces)

We will restrict our treatment to Radiation-induced forces and Propulsion forces, the Environmental forces are considered as external disturbance, that is:

\[ \mathbf{M}_R \ddot{\mathbf{v}} + \mathbf{C}_R \dot{\mathbf{v}} = \mathbf{\tau} = \mathbf{\tau}_R + \mathbf{\tau}_P \quad (4.26) \]

The general expression for ship model can be considerably simplified by exploiting \( xy \)-plane of symmetry of ship body. In this condition, \( I_{xy} = I_{yz} = 0 \) and \( Y_G = 0 \). In addition to this, we will use the approximate relation of \( I_{xz} = 0 \) because the ship body is nearly symmetrical with respect to \( yz \)-plane. We further constrain our attention to ship steering and rolling problem, the motion in heave and pitch can be neglected, so that \( \dot{w} = \dot{q} = w = q = 0 \), this implies that the expression (4.26) for the ship dynamics reduces to:

\begin{align*}
\text{surge:} & \quad m(u - vr - z_gr^2) = X; \quad (4.27a) \\
\text{sway:} & \quad m(v + ur + x_gr) = Y; \quad (4.27b) \\
\text{yaw:} & \quad I_z \dot{\theta} + m(x_gr \dot{v} + ur) = N; \quad (4.27c) \\
\text{roll:} & \quad I_x \dot{\phi} - m(z_gr \dot{v} + ur) = K - \rho g \nabla GM \phi; \quad (4.27d)
\end{align*}

where we have added the metacentric restoring moment in roll to the right-hand side of the forth equation, which will be explained in following section; \( \nabla \) denotes the ship displacement, \( g \) the gravity constant, \( \rho \) the mass density of the water and \( GM \sin \varphi \approx GM \varphi \) the righting arm which is a known function of roll angle \( \varphi \) when \( \varphi \) is small.

### 4.2 Stability of Ship Motions

Stability of the uncontrolled ship can be defined as the ability of returning to an equilibrium state after a disturbance, without any corrective action of the control surfaces. There are two kinds of stability of ship motions which will be concerned about: Straight Line motion stability and Metacentric stability.

#### 4.2.1 Straight Line Stability

Consider an uncontrolled ship moving in a straight path, if the new path is straight after a disturbance in yaw, the ship is said to have straight line stability. The direction of the new path
will usually differ from the initial path because no restoring force are present. We use Nomoto’s 1st-order model to interpret that. Consider the model

\[ T\ddot{r}(t) + r(t) = K\delta(t) + w(t) \]  \hspace{1cm} (4.28)

where \( w(t) \) is the external disturbances, \( r(t) \) is the raw rate of the ship

\[ r(t) = \psi(t) \]  \hspace{1cm} (4.29)

Under the assumptions that the ship is moving with constant forward speed \( u_0 \), substituting (4.29) into (4.28), we have

\[ \dot{\psi}(t) = -\frac{1}{T}\psi(t) + \frac{K}{T}\delta(t) + \frac{1}{T}w(t) \]  \hspace{1cm} (4.30)

The eigenvalues \( \lambda_{1,2} \) of equation (4.30) are

\[ \lambda_1 = -\frac{1}{T}; \lambda_2 = 0. \]

When \( T>0 \), the ship is straight line stable, otherwise, it is unstable in straight-line.

### 4.2.2 Metacentric Stability

Normally, there are two forces acting on the ship when the ship stays in calm water (Figure 4.2). The weight on the ship acts vertically down through the center of gravity \( G \) while the upthrust acts through the center of buoyancy \( B \). Since the weight is equal to the upthrust, and the center of gravity and the center of buoyancy are in the same vertical line, the ship is in equilibrium.

When the ship is inclined by an external force to an angle \( \phi \), the waterline \( WL \) is changed to \( W_1L_1 \), which intersects the original waterline at \( S \). If \( \phi \) is small, the weight of ship is not changed after the ship’s inclination, the center of gravity remains in the same position. But a wedge of buoyancy \( WSW_1 \) has been moved across the ship to \( L_1SL \), causing the center of buoyancy to move from \( B \) to \( B_1 \) (Figure 4.2).

The buoyancy, therefore, acts up through \( B_1 \) while the weight still acts down through \( G \), creating a moment of \( \Delta \times \overline{GZ} \) which tends to turn the ship to the upright. \( M_R = \Delta \times \overline{GZ} \) is known as the righting moment and \( \overline{GZ} \) the righting lever. Since this moment tends to right the ship, the vessel is said to be metacentric stable or initial stable.

![Figure 4.2 Transverse metacentric stability](image-url)
The inclinations of a ship can be divided into two types, one type is that the inclination happens towards port/starboard called rolling, another type is that the inclination appears towards fore/aft called pitching. In this paper, we are only interested in rolling.

For small angle $\varphi$ of heel (or the lower deck not inclines into water), the vertical through the new center of buoyancy $B_1$ intersects the centerline at $M$, the transverse metacenter. It may be seen from Figure 4.2 that

$$GZ = GM \sin(\varphi) \quad (4.31)$$

Thus for small angles of heel, $\overline{GM}$ is independent of $\varphi$ while $\overline{GZ}$ depends upon $\varphi$, it is usual to express the initial stability of a ship in terms of $GM$, the metacentric height. $GM$ is said to be **positive** when $G$ lies below $M$ and the vessel is stable. A ship with a small metacentric height will have a small righting lever at any angle and will roll easily. The ship is then said to be **tender**. A ship with large metacentric height will have a large righting lever at any angle and will have a considerable resistance to rolling. The ship is then said to be **stiff**. A stiff ship will be very uncomfortable, having a very small rolling period and in extreme cases may result in structural damage.

If the center of gravity lies above the transverse metacenter, the metacentric height being regarded as negative, the moment acts in the opposite direction, increasing the angle of heel. The vessel is then unstable and will not return to upright.

When the center of gravity and transverse metacenter coincide, there is no moment acting on the ship which will therefore remain inclined to angle $\varphi$. The vessel is then said to be in neutral equilibrium. Since any reduction in the height of $G$ will make the ship stable, and any rise in $G$ will make the ship unstable, this condition is regarded as the point at which a ship becomes either metacentric stable or unstable.

The buoyancy force $\Delta = \rho g \nabla$, here $\nabla$ is the volume of displacement, the righting moment in roll can be written as:

$$K_{\text{righting}} = \rho g \nabla GM \sin(\varphi) \quad (4.32)$$

where $GM\sin(\varphi)$ can be interpreted as the moment arms in roll. A commonly used formula for the metacentric height is obtained by defining the vertical distance between the center of gravity $G$ and the center of buoyancy $B$

$$\overline{BG} = z_G - z_B \quad (4.33)$$

From basic hydrostatics, we have

$$GM = BM - \overline{BG} \quad (4.34)$$

This relationship is seen directly from Figure 4.2 where $K$ is the keel line. For small inclinations, the radius of curvature can be approximated by:

$$BM = \frac{I_T}{\nabla} \quad (4.35)$$

here the moment of area about the waterplan is defined as

$$I_T = \int_A y^2 dA \quad (4.36)$$

For conventional ships this integral will satisfy the bound:

$$I_T < \frac{1}{12} B^3 L \quad (4.37)$$

here $B$ is the breadth of the ship and $L$ is the ship’s length.

The roll motion must satisfy $GM > 0.15m$ to guarantee a proper stability margin in roll.
When the ship is rolling in calm water, there are only two moments acting on it if the damping moment has been neglected. The inertial moment $I_{xx}\dot{\phi}$ and righting moment $\Delta GM\phi$ on the principle of dynamic equilibrium is equal

$$I_{xx}\dot{\phi} + \Delta GM\phi = 0 \quad (4.38)$$

The natural frequency of rolling can be obtained from above equation

$$\omega = \sqrt{\frac{\Delta GM}{I_{xx}}} = \sqrt{\frac{\rho g \sqrt{GM}}{I_{xx}}} \quad (4.39)$$

This in turn implies that the natural period is

$$T = 2\pi \sqrt{\frac{I_{xx}}{\rho g \sqrt{GM}}} \quad (4.40)$$

where $I_{xx} = I_{xx} + K_A$. $K_A$ is an added inertial moment. It is normally equal to 20% of the inertial moment $I_{xx}$ [Bhattacharyya, 1978] so it is written as $\delta I_{xx}$ in some papers.

When the damping moment has been considred, (4.38) should be

$$I_{xx}\dot{\phi} + K_p\phi + \Delta GM\phi = 0$$

### 4.3 Added Mass and Added Inertia

The concept of added mass is, from the name, a finite amount of water connected to the vessel such that the vessel and the fluid represents a new system with mass larger than the original system. The increased mass of the new system is called added mass. This is not true [Fos94] since the vessel motion will force the whole fluid to oscillate with different fluid particle amplitudes in phase with the forced harmonic motion of the vessel. However, the amplitudes will decay far away from the body and may therefore be negligible. Added mass should be pressure-induced forces and moments due to a forced harmonic motion of the body which are proportional to the acceleration of the body. Consequently, the added mass forces and the acceleration will be 180 degrees out of phase to the forced harmonic motion.

We will use the concept of fluid kinetic energy to derive the added mass terms. Any motion of the vessel will induce a motion in the otherwise stationary fluid. In order to allow the vessel to pass through the fluid, the fluid must move aside and then close behind the vessel. As a consequence, the fluid passage possesses kinetic energy that it would lack if the vessel was not in motion.

With the assumption that the added mass coefficients are constant and thus independent of the wave circular frequency, the expression for the fluid kinetic energy $T_A$ can be written as a quadratic form of the velocity vector components in the body reference frame, that is:

$$T_A = \frac{1}{2} V^T M_A V \quad (4.41)$$

where $V = [u, v, w, p, q, r]$, $M_A$ is a 6x6 added inertia matrix defined as:
4. Ship Dynamics Modeling

4.4 Ship Motion Models

4.4.1 Nonlinear Ship Motion Equations

Recall from (4.24) that the ship motion equation can be written as:

\[
\begin{align*}
\text{surge:} & \quad m(\ddot{u} - vr - x_Gr^2 + z_Gpr) = X; \\
\text{sway:} & \quad m(\ddot{v} + ur + x_Gr - z_Gp) = Y; \\
\text{yaw:} & \quad I_z \ddot{r} + m x_G (\ddot{v} + ur) = N; \\
\text{roll:} & \quad I_z \ddot{p} - mz_G (\ddot{v} + ur) = K - \rho g \sqrt{GM} \varphi;
\end{align*}
\]

The terms \( X, Y, N \) and \( K \) denote the hydrodynamic forces and moments. These terms are nonlinear functions of the motion variables and control variables as shown below:

\[
\begin{align*}
X &= f(\Delta u, \dot{u}, v, \dot{v}, r, \dot{r}, p, \dot{p}, \varphi, |p|, |u|, |r|, \delta, \cdots) \\
Y &= f(\Delta u, \dot{u}, v, \dot{v}, r, \dot{r}, p, \dot{p}, \varphi, |p|, |u|, |r|, \delta, \cdots) \\
N &= f(\Delta u, \dot{u}, v, \dot{v}, r, \dot{r}, p, \dot{p}, \varphi, |p|, |u|, |r|, \delta, \cdots) \\
K &= f(\Delta u, \dot{u}, v, \dot{v}, r, \dot{r}, p, \dot{p}, \varphi, |p|, |u|, |r|, \delta, \cdots)
\end{align*}
\]

The terms used in the series are deducted from physical and hydrodynamic considerations combined with experience from model testing [Blanke and Jensen, 1997]. They can be calculated by expanding to a 3rd-order truncated Taylor series representation in the neighborhood of \( u=u_0, \ v=0, \ p=0 \) and \( r=0 \):
where $\Delta u = u - u_0$ is the speed deviation from nominal speed. All the hydrodynamic coefficients are dependent on ship’s shape in structure and speed, they can be obtained by model test or identification. Consider following relations $\Phi = p$, $\Psi = r / \cos(\Phi) = r$, combing with equations (4.45), it will form six nonlinear equations about $u, v, r, p, \phi, \psi$. $\delta$ is the control variable. State vector $x$ is defined as follows:

$$x = [u, v, r, p, \phi, \psi]$$

these equations will be used to simulate the ship’s motion.

### 4.4.2 Linear Model

It is difficult to use the nonlinear model directly in robust controller design. The nonlinear model should be replaced in the calculation of controller by a linear model because most of the theorems are derived based on linear theory. It is easy to obtain a linear model if a nonlinear model exists. In this project, a linear model is obtained by omitting all the terms whose orders are higher than one from (4.45) and (4.46).

When we discuss the motion of roll and yaw, the surge equation is not considered due to the very weak coupling taking place between sway, roll, yaw and surge. Therefore, there are totally five states ($v, r, p, \phi, \psi$) in the linear model. The simplified linear model can be written as

$$\dot{x} = Ax + Bu = E^{-1}Fx + E^{-1}G\delta$$

where

$$E = \begin{bmatrix}
    m - Y_i & mx_G - Y_i & -mx_G - Y_p & 0 & 0 \\
    mx_G - N_i & I_z - N_i & -N_p & 0 & 0 \\
    -mz_G - K_i & -K_i & I_x - K_p & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 \\
    0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

$$F = \begin{bmatrix}
    Y_v + \Delta uY_{vy} & Y_r - mu & Y_p + \Delta uY_{pu} & Y_\phi & 0 \\
    N_v + \Delta uN_{vy} & N_r - mx_G u & N_p + \Delta uN_{pu} & N_\phi & 0 \\
    K_v + \Delta uK_{vy} & K_r + mz_G u & K_p + \Delta uK_{pu} & -mgGM & 0 \\
    0 & 0 & 1 & 0 & 0 \\
    0 & 0 & 0 & 0 & 0
\end{bmatrix}$$
where, $E$ is the inertia force coefficient matrix, $F$ is the viscous force coefficient matrix, $G$ is the rudder force coefficient vector. $A = E^{-1}F$ is the system matrix, $B = E^{-1}G$ is input matrix. $u = \delta$ is rudder angle. If considering $[\phi, \psi]^T$ as the output vector, it is easy to obtain $P_{\phi\delta}(s)$ and $P_{\psi\delta}(s)$ which are the transfer functions from $\delta \rightarrow \phi$ and $\delta \rightarrow \psi$. They are the basis of design of RRD controller.

The frequency characteristics of the open loop transfer functions $P_{\phi\delta}(s)$ and $P_{\psi\delta}(s)$ are shown in Figure 4.3. It is found that there is a resonant peak at about 0.23 rad/s for $P_{\phi\delta}(s)$, where, the ship’s roll angle will be significant. This frequency is known as the natural roll eigenfrequency.

![Frequency characteristic of roll loop of linear model](image-url)

**Figure 4.3** Frequency characteristic of open loop transfer function from rudder to roll angle. The solid line is $P_{\phi\delta}(s)$, the dash-dot line is $P_{\psi\delta}(s)$.

### 4.4.3 Model of The Steering Machine

A block diagram of the steering machine with its dynamics described by [Amerongen, 1982] is shown in Figure 4.4. The telemotor system is fast, compared with the main servo. In addition, the time constant of the main servo is of minor importance, compared with the influence of the limited rudder speed. Van Amerongen suggests using a simplified representation of the steering machine (see figure 4.5). This block diagram contain two limiters, one describing the limitation of the rudder angle and the other describing the limitation of the rudder speed. The rudder limit is either
determined by the rudder-angle constraints of the autopilot, or by the mechanical constraints. The maximum rudder speed is determined by the maximum valve opening and the pump capacity of the steering machine. The classification companies require that the rudder should be able to move from 35 degrees port to 35 degrees starboard within 30 seconds. A maximum rudder speed of as low as 2.5 degrees per second is sufficient to meet this requirement. But this does not satisfy the requirement of a RRD system. Many research work and experiments have shown that the rudder speed should be sufficiently high [Amerongen and Klugt, 1982, 1983; Klugt, 1987; Blanke et al., 1989], a rudder speed of 5-20 (deg/s) is usually required for a RRD system to work properly.

![Figure 4.4 A block diagram of a steering machine](image)

We know from Figure 4.5 that the steering machine model can be represented by a 1st-order time lag when it is working in the unsaturated area. It can be used for course keeping control. However, the nonlinear characteristics must be considered when it is used for RRD control.

![Figure 4.5 Simplified diagram of the rudder control loop](image)

4.5 Model Uncertainty

Model uncertainties can cause instability and poor performance of nominal stable systems, as it was mentioned for the sister ships in the previous section. Another example is the uncertainty on rudder force. The rudder force $Y$ depends on the average flow past the rudder, part of which is generated by the propeller. The rudder force is very much dependent on the instantaneous propeller thrust as well as on the ship speed squared [Blanke et al., 1994]. But the resistance of ship is known within the accuracy of 5 to 15%, because fouling of the hull increases the resistance gradually between dockings. A second effect is weather resistance which is 0 to 30% on nominal value at a certain speed. Finally, the flow pattern and the lift coefficient can easily differ from the nominal when the rudder is placed in the real aft of a ship. The uncertainty lies in the range of 0.7 to 1.3. The result is that the uncertainty of the rudder force coefficient $Y_\delta$ lies within the range of
0.5 to 2. Also change of trim angle on a container ship may change its dynamics from stable to unstable. A turn will always drive the ship towards a better stabilization in steering. The low frequency gain and phase of the transfer functions are therefore quite uncertain. At medium and high frequencies, compared to the ships natural time constant (the ratio of length/ship speed), steering gain is determined by the inertia and force produced by the rudder. If the rudder force is calibrated once, the uncertainty is fairly small in this range. The eigenfrequency in roll is fairly well determined, within some 5-10%, because a ship is always loaded to maintain GM within a very narrow range. This is mandatory to maintain the ship’s safety. Roll damping is more uncertain a priori and we can only assume a damping ratio between 0.1 and 0.25 for most ships.

There are several sources of uncertainty in constructing a ship model:

(i) The model is restricted in degrees of freedom to reduce complexity. This means that cross couplings are neglected and unmodelled dynamics exists.

(ii) Some parameters are known quite accurately, others are believed to be known within a range of +200% to -50% from model tests but may be much more uncertain in full scale.

(iii) Only linear terms and/or selected nonlinear terms are included. The dropped nonlinear terms occur as the model error that we need to interpret as uncertainty.

All of these uncertainties can be considered in one of two forms: additive or multiplicative model uncertainty. The uncertainties mentioned above can be considered as output multiplicative model uncertainty. The relationship between the multiplicative model uncertainty and the nominal process $P(s)$ has been given in (3.3) and is rewritten as follows

$$P_p(s) = (1 + \Delta_p(s))P(s)$$

where $P_p(s)$ is the perturbed process, i.e. the “real” process. The system block diagram is shown in Figure 3.1(c). We redraw it in Figure 4.6. In this figure $\Delta_p(s) = E_0(s)$ in equation (3.3).

Suppose that the system in Figure 4.6 is stable for $\Delta_p(j\omega) = 0$, the size of the smallest stable $\Delta_p(s)$ for which the system become unstable is

$$\sigma(\Delta_p(j\omega)) = \frac{1}{\sigma(T(j\omega))}$$

(4.53)

The smaller is $\sigma(T(j\omega))$, the greater will be the size of the smallest destabilizing multiplicative perturbation $\sigma$, and hence the greater will the stability margin be [Chiang and Safonovm, 1992].

For the ship roll damping, the following uncertainty model was found for a multi-variable naval vessel. Since there are no general results of the uncertainty for container ships, we use this uncertainty model here [Blanke, 1996].

$$\Delta_p(s) = \frac{2\zeta_0\omega_0(1 + \alpha \frac{\nabla}{\nabla_0} - \alpha(\frac{\nabla}{\nabla_0})^2)s^2 + \omega_0^2(1 - (\frac{\nabla}{\nabla_0})^2)}{s^2 + 2\zeta_0\omega_0^2s + \omega_0^2}$$

(4.54)

where $\alpha$ is a constant with a value of 0.5, $\nabla$ is the volume displacement and $\nabla_0$ is the nominal displacement. This equation is indeed simple, but it is only valid for displacement smaller than nominal. The terms $\omega_0$ and $\zeta_0$ are natural roll frequency and damping, respectively. Typical values for damping are 0.15 to 0.25.

For the yaw rate motion, a frequency independent uncertainty of 0.1 is suggested by the same paper. According to that, the following uncertainty description can be obtained for yaw angle from [Blanke, 1996].

$$\Delta_{p_0}(s) = \frac{0.42\zeta_0\omega_0^2s + 0.21\omega_0^2}{s^2 + 2\zeta_0\omega_0^2s + \omega_0^2}$$

(4.55)
Wave, wind and current are the principal factors causing disturbances to the ship on the sea. By nature both wave and wind are random processes, and their influence on ship motions can be adequately characterized through appropriate measures of their spectral densities. In terms of ship rolling, wave is the most important disturbance. The objective of this section is devoted to the description of wave spectra and ship motion in waves.

The response of a ship to waves is quite complex. Having a certain velocity of advance, a ship experiences the wave excitation at an encounter frequency. This frequency is not related linearly to the wave frequency, as seen from a fixed point, but varies with ship speed and angle of attack from wave through a nonlinear mapping. Furthermore, forces and moments on the hull are determined by the wavelength of incident waves through a square root function of the wave-frequency.

The mathematical description of the motion of regular gravity waves over a free surface is classical and is detailed in [Price and Bishop, 1974]. A two dimensional wave progressing at an angle $\chi$ with respect to inertial axis, is described by its elevation $\zeta$ at a certain position $x_0, y_0$ at time $t$:

$$\zeta(x_0, y_0, t) = \zeta_0 \cos(kx_0 \cos \chi + ky_0 \sin \chi - \omega_w t + \theta)$$  (4.56)

where $k$ = the wave number
$\omega_w$ = wave frequency seen from a fixed position
$\zeta_0$ = wave amplitude;
$\theta$ = an arbitrary phase angle

The phase velocity of the wave, $c$, is the velocity with which the wave crest move relative to ground. Assuming a gravity wave and infinite depth of water, following relations hold

$$c = \sqrt{\frac{g\lambda}{2\pi}}; \quad k = \frac{2\pi}{\lambda}; \quad \omega_w = \sqrt{gk} = \frac{g}{c}$$

here $g$ is acceleration of gravity, $\lambda$ is the wavelength.

This expression is known as the dispersion of gravity waves. The phase velocity is inversely proportional to its frequency. In other words, long waves propagate faster than short ones. This phenomenon is crucial for simulation of wave motion. A ship advancing in a seaway in following seas will overtake some short waves, while it will be overtaken by some long ones. A motion of the ship at a certain encounter frequency can, therefore, be caused by up to three harmonic waves with three different wavelengths.
When the wave amplitude and frequency become random variables the simple waves are extended to an irregular sea. In general, both will be functions of time and position at the surface, and a characterization of wave spectral densities must employ two-dimensional spectra. A commonly accepted simplification is to consider the sea as a superposition of long crested waves from different directions, and then compute the ship motion as a weighted sum of the individual responses.

A long crested irregular sea is described by a one-dimensional amplitude spectrum. The main parameters are the significant wave height, $h_{1/3}$ (the average height of the largest third of the waves) and the average zero-crossings wave period $T_z$. The ITTC one-parameter spectrum uses the wave height $h_{1/3}$. Figure 4.7 shows wave spectra for different significant wave heights.

![ITTC one-parameter power spectrum](image)

**Figure 4.7** ITTC one-parameter wave spectra when significant wave height is 3m, 4m, 5m, 6m and 7m respectively.

From the figure, we know that the bulk of energy in the spectra is shift to lower frequency as the wave height increases.

If another parameter, average wave period, is available, the ITTC two-parameter spectrum can be used. Sea spectra differ significantly in different waters, and with different degree of development. The ITTC two-parameter spectrum is therefore commonly used for more detailed analysis.

Using SI units, the Modified Pierson-Moskowitz (MPM) spectrum recommended by ITTC is

$$G_{ss}(\omega_w) = \frac{4\pi^3}{\omega_w^4} \frac{h_{1/3}^2}{T_w^4} e^{-\frac{16\pi^4}{\omega_w^2 \omega_{ws}^2}} \quad [m^2/s]$$  \hspace{1cm} (4.57)

where $T_z = 0.9205 T_w$, and $T_w$ is average wave period. On the other hand, when $T_w$ is used ITTC spectrum can be written as:

$$G_{ss}(\omega_w) = \frac{173 h_{1/3}^2}{\omega_w^4} e^{-\frac{691}{\omega_w^2 \omega_{ws}^2}} \quad [m^2/s]$$  \hspace{1cm} (4.58)

The relation between the ship motion response and the wave height as function of the wave frequency is commonly named a receptance function. This term is adopted here also to distinguish an ordinary transfer function from the wave response function. The latter uses encounter frequency transformed from wave frequency to calculate motion spectra [Blanke, 1981].

When the speed of ship is zero, the response spectrum to the sea wave is expressed as
\[ G_{zz}(\omega_e, \chi, U)_{U=0} = G_{zz}(\omega_w, \chi, U)_{U=0} = E\left\{ z(t, U)z^T(t, U) \right\}_{U=0} = \left| R_{zz}(\omega_w, \chi, U) \right|_{U=0}^2 G_{zz}(\omega_w, \chi) \] (4.59)

where \( R_{zz} \) are the Response Amplitude Operators (RAO).

For the ship moving with forward speed \( U \), the wave frequency \( \omega_w \) should be changed to the encounter frequency \( \omega_e \) in the equations. The relation between wave frequency and encounter frequency is

\[ \omega_e = \omega_w - \frac{\omega_w^2 U \cos \chi}{g} \] (4.60)

where \( \omega_w^2 = k g \) (assuming deep water) and \( \chi \) is the encounter angle, the angle between the ship heading and the direction of the wave propagation. Notice that the encounter frequency can be negative for large values of \( U \) when \(|\chi| < 90^\circ\). It means that the wave does not move from fore to aft but from aft to fore for the observer on the ship. The definition of the encounter angle \( \chi \) is shown in Figure 4.8

Figure 4.8 Definition of the encounter angle of ship with waves

Because wave energy in a frequency interval is unaffected by the speed of observation platform, we have the equation [Blanke, 1981]

\[ G_{zz}(\omega_w, \chi, U) d\omega_w = G_{zz}(\omega_e, \chi, U) d\omega_e ; \] (4.61)

Further, from equation (4.60), the following relation is obtained

\[ \frac{d\omega_e}{d\omega_w} = 1 - 2\omega_w \frac{U}{g} \cos \chi \] (4.62)

Note that, in following sea, i.e. \( 0^\circ < \chi < 90^\circ \), the denominator (in (4.63)) \( 1 - 2\omega_w \frac{U}{g} \cos \chi \) would become zero or negative when \( \omega_w \geq \frac{g}{2U \cos (\chi)} \), therefore the function of \( G_{zz}(\omega, \chi) \) must be divided into two segments. Note also that, no minus power spectrum exists, therefore
\[ G_{zz}(\omega_e, \chi) = \frac{G_{zz}(\omega_w, \chi, U)}{1 - 2\omega_w \frac{U}{g} \cos(\chi)} ; \quad \omega_w \neq \frac{g}{2U \cos(\chi)}. \] (4.63)

However, since there is a possibility that the denominator becomes zero in this expression, this approach is not feasible. Another approach is taking an approximation of a sea spectrum by a finite sum of sinusoids with random initial phases

\[ z(t) = \sum_{i=1}^{n} a_i \sin(\omega_{e,i} t + \varphi_i + \varphi_{init}) ; \] (4.64)

For different response operators, \( z(t) \) will be \( u_{w}(t), v_{w}(t), \varphi_{w}(t) \) and \( \psi_{w}(t) \), that depends on which one of \( R_{zz}, R_{zv}, R_{z\varphi}, R_{z\psi} \) replaces \( R_{zz} \) in (4.59). The individual amplitudes of the sines are conveniently taken as median points between frequencies where the response operators are known (tabulated by Danish Maritime Institute). Frequencies \( \omega_{e,i} \) and phase angles \( \varphi_i \) are tabular values from the response operator tables. The initial phase \( \varphi_{init} \) is a random number used for initialization. The amplitude \( a_i \) is calculated from

\[ \frac{1}{2} a_i^2 = \int_{\omega_{w,i}}^{\omega_{w,i+1}} G_{zz}(\omega_e, \chi, U) d\omega_e \]

\[ = \int_{\omega_{w,i}}^{\omega_{w,i+1}} |R_{zz}(\omega_w, \chi, U)|^2 G_{ls}(\omega_w, \chi) d\omega_w. \] (4.65)

Let

\[ \sigma^2_n(\omega_{1,j}, \omega_{2,j}) = \int_{\omega_{w,i}}^{\omega_{w,i+1}} G_{ls}(\omega_w) d\omega_w \] (4.66)

When the tabular wave frequencies are spaced by an amount \( \Delta_i \), i.e. \( \omega_{w,i+1} = \omega_{w,i} + \Delta_i \), we get the following equation

\[ a_i = \sqrt{2} \left| R_{zz}(\omega_{w,i}, \chi, U) \right| \sqrt{\sigma^2_n(\omega_{1,i}, \omega_{2,i})} \] (4.67)

where \( \sigma^2_n(\omega_{1,i}, \omega_{2,i}) \) represent the power of the \( n \)th sine wave

\[ \sigma^2_n(\omega_{1,i}, \omega_{2,i}) = \frac{h^2}{16} \left( e^{\frac{-691}{\omega_{1,i}^2 f_e^2}} - e^{\frac{-691}{\omega_{2,i}^2 f_e^2}} \right) \] (4.68)

\[ \omega_{1,i} = (\omega_{w,i-1} + \omega_{w,i}) / 2 \]

\[ \omega_{2,i} = (\omega_{w,i} + \omega_{w,i+1}) / 2 , \quad i = 1, \ldots, n \]

Because \( \omega_{w,0} \) and \( \omega_{w,n+1} \) can not be found from the mentioned reference, we expand the ranges of \( \omega \) by adding two extra nodal points, i.e. \( \omega_{w,0} = \omega_{w,1} - (\omega_{w,2} - \omega_{w,1}) \) , \( \omega_{w,n+1} = \omega_{w,n} + (\omega_{w,n} - \omega_{w,n-1}) \), or including the rest of the wave energy, i.e. \( \omega_{w,0} = 0, \omega_{w,n+1} = \infty. \)

The responses of the ship motion to the sea wave are calculated from equation (4.64), (4.67) and (4.68). The following are the responses of ship displacement to waves for four degree of freedom respectively

\[ X_{\text{wave}} = \sum_{i=1}^{n} \sqrt{2} \left| R_{zz}(\omega_{w,i}, \chi, U) \right| \sqrt{\sigma^2_n(\omega_{1,i}, \omega_{2,i})} \sin(\omega_{e,i} t + \varphi_{e,i}) \] (4.69a)

\[ Y_{\text{wave}} = \sum_{i=1}^{n} \sqrt{2} \left| R_{zz}(\omega_{w,i}, \chi, U) \right| \sqrt{\sigma^2_n(\omega_{1,i}, \omega_{2,i})} \sin(\omega_{e,i} t + \varphi_{y,i}) \] (4.69b)
\[ \phi_{\text{wave}} = \sum_{i=1}^{n} \sqrt{2} R_{\phi x} (\omega_{w,i}, \chi, U) \sqrt{\sigma^2 (\omega_{1,i}, \omega_{2,i})} \sin(\omega_{e,i} t + \varphi_{x,i}) \] (4.69c)

\[ \psi_{\text{wave}} = \sum_{i=1}^{n} \sqrt{2} R_{\psi y} (\omega_{w,i}, \chi, U) \sqrt{\sigma^2 (\omega_{1,i}, \omega_{2,i})} \sin(\omega_{e,i} t + \varphi_{y,i}) \] (4.69d)

Also we can obtain the responses of ship velocities in the four degrees of freedom to waves by the derivatives of above equations

\[ u_{\text{wave}} = \dot{x}_{\text{wave}} = \sum_{i=1}^{n} \sqrt{2} R_{x x} (\omega_{w,i}, \chi, U) \sqrt{\sigma^2 (\omega_{1,i}, \omega_{2,i})} \omega_e \cos(\omega_{e,i} t + \varphi_{x,i}) \] (4.70a)

\[ v_{\text{wave}} = \dot{y}_{\text{wave}} = \sum_{i=1}^{n} \sqrt{2} R_{y y} (\omega_{w,i}, \chi, U) \sqrt{\sigma^2 (\omega_{1,i}, \omega_{2,i})} \omega_e \cos(\omega_{e,i} t + \varphi_{y,i}) \] (4.70b)

\[ r_{\text{wave}} = \psi_{\text{wave}} = \sum_{i=1}^{n} \sqrt{2} R_{\psi x} (\omega_{w,i}, \chi, U) \sqrt{\sigma^2 (\omega_{1,i}, \omega_{2,i})} \omega_e \cos(\omega_{e,i} t + \varphi_{x,i}) \] (4.70c)

\[ p_{\text{wave}} = \dot{\phi}_{\text{wave}} = \sum_{i=1}^{n} \sqrt{2} R_{\phi y} (\omega_{w,i}, \chi, U) \sqrt{\sigma^2 (\omega_{1,i}, \omega_{2,i})} \omega_e \cos(\omega_{e,i} t + \varphi_{y,i}) \] (4.70d)

The roll RAO from measurement were shown in Figure 4.9 for a container ship. The wave environment conditions are \( h_{1/3} = 3 \text{m}, T_w = 8 \text{s} \) and \( \chi = 45^\circ, 90^\circ, 135^\circ \) respectively. From Figure 4.9 we see that the frequency (encounter frequency) of most of RAOs concentrated on the range from 0.25 to 0.5 rad/s. all the data of response operator tables come from Blanke and Jensen (1997).
Similar to the roll motion, all motions can be calculated to provide an output disturbance vector to the ship. The first order interpolation is used for calculating the response operators in tabular values of response operators for different $U$ and $\chi$.

The equations (4.) and (4.) will be used in ship controller simulation.

### 4.7 Rolling Angle vs. Encounter Angle

The equation of motion for rolling in calm water is [Bhattacharyya, 1978]

$$a \frac{d^2 \varphi}{dt^2} + b \frac{d \varphi}{dt} + c \varphi = 0$$

(4.71)

or

$$\frac{d^2 \varphi}{dt^2} + 2 \zeta \varphi \omega_\varphi \frac{d \varphi}{dt} + \omega_\varphi^2 \varphi = 0$$

(4.72)

where $a$, $b$ and $c$ are the coefficient of the $2^{nd}$-order equation and

$a = I_{xx}$

$b = -K_p$

$c = \Delta GM$

$$\omega_\varphi = \sqrt{\frac{c}{a}} = \sqrt{\frac{\Delta GM}{I_{xx}}}$$ is the natural frequency for rolling

$$\zeta = \frac{b}{2a\omega_\varphi} = \frac{-K_p}{2\sqrt{\Delta GM I_{xx}}}$$ is the damping coefficient for rolling

$\Delta = \rho g \nabla$ is the buoyancy force of the ship

The largest rolling motion occurs when resonance takes place, i.e. when $\omega_\varphi = \omega_w$ in the still water. For the ship in motion, $\omega_w$ should be replaced by $\omega_e$, namely:

$$\omega_e = \omega_\varphi$$

From the encounter angle representation (4.60) we know that when the largest rolling motion occurs

$$\omega_\varphi = \omega_e = \omega_w - \frac{\omega_w^2 U}{g} \cos \chi$$

thus

$$\cos \chi = \frac{(\omega_w - \omega_\varphi) g}{\omega_w U}$$

(4.73)

For waves in deep water

$$\omega_w = \sqrt{\frac{2\pi g}{L_w}} = \frac{2\pi}{T_w}$$

where $L_w$ is the wavelength and $T_w$ is the wave period.

If we know the natural rolling frequency of a ship, the ship speed and the wavelength, we will know which encounter angle will induce the largest rolling motion of the ship. The rolling motion
in a seaway can be reduced by changing the ship’s heading. The encounter frequency is thereby altered, and the resonance, which is normally the main reason for the heavy rolling in a seaway, will be removed.

4.8. Linear Wave Model

In order to simplify the design of a control system, a linear wave model is used to replace the nonlinear complex wave model.

A linear approximation to ITTC spectral density function can be obtained as follows

\[ y(s) = h(s)w(s) \] (4.74)

where \( w(s) \) is a zero mean Gaussian white noise process with power spectrum

\[ G_{ww}(\omega) = 1.0 \]

\( h(s) \) is a 2nd order wave transfer function

\[ h(s) = \frac{K_w s}{s^2 + 2\zeta \omega_0 s + \omega_0^2} \] (4.75)

where \( K_w \) is defined as

\[ K_w = 2\zeta \omega_0 \sigma_w \] (4.76)

here \( \sigma_w \) is a constant describing the wave intensity, \( \zeta \) is a damping coefficient while \( \omega_0 \) is the dominating wave frequency. Hence, substituting \( s = j\omega \) into (4.75), we have

\[ h(j\omega) = \frac{j2\zeta \omega_0 \sigma_w \omega}{(\omega_0^2 - \omega^2) + j2\zeta \omega_0 \omega} \] (4.77)

The amplitude of \( h(j\omega) \) is

\[ |h(j\omega)| = \frac{2(\zeta \omega_0 \sigma_w)\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4(\zeta \omega_0 \omega)^2}}. \] (4.78)

The power spectral density function for \( y(s) \) can be calculated as

\[ G_{yy}(\omega) = |h(j\omega)|^2 G_{ww}(\omega) = |h(j\omega)|^2 \]

\[ = \frac{4(\zeta \omega_0 \sigma_w)^2 \omega^2}{(\omega_0^2 - \omega^2)^2 + 4(\zeta \omega_0 \omega)^2} \] (4.79)

When \( \omega \) is equal to the dominating wave frequency \( \omega_0 \), the maximum value of \( G_{yy}(\omega) \) can be obtained as

\[ \max_{\omega} G_{yy}(\omega) = G_{yy}(\omega_0) = \sigma_w^2 \] (4.80)

The dominating frequency in the linear model is equal to the modal frequency of the ITTC spectrum for the same waves. In ITTC spectrum, the modal frequency is

\[ \omega_0 = \sqrt{\frac{4 \cdot \frac{691}{5} \cdot \frac{4.85}{5}}{T_w}}. \]

Therefore the value of ITTC power spectrum at \( \omega_0 \) is
\[ G_{yy}(\omega_0) = \frac{173h_{1/3}^2}{(4.85T_w)^3T_w^4} e^{-\frac{3.81}{T_w^4}} \omega_0^{4.85T_w} \omega_0 = 0.0185T_w h_{1/3}^2. \]

The maximum values at the same \( \omega_0 \) of both spectra ought to be the same, hence

\[ \sigma_w^2 = 0.0185T_w h_{1/3}^2. \]

The value of \( \sigma_w \) is obtained

\[ \sigma_w = \sqrt{0.0185T_w h_{1/3}}. \]

The value of \( \zeta \) determines the slope of curve of \( G_{yy}(\omega) \). When \( T_w = 8 \) (Sec), \( h_{1/3} = 3 \) (m) and \( \zeta = 0.3 \), the curve of \( G_{yy}(\omega) \) is shown in Figure 4.10, which is the approximation of \( G_{yy}(\omega) \) given in (4.58).

![Figure 4.10 The Power spectra of linear wave model](image)

Real line is measured wave spectrum, dash line is modeled spectrum

### 4.9 Observer-Based Wave Filter

Wave filters are used to treat the problem of removing the oscillatory motion (1st-order wave disturbances) from the measurements. The 1st-order wave disturbances are usually around 0.1 Hz, which is close to the control bandwidth of the vessel. This is usually inside the bandwidth of the rudder servo of the ship. Nevertheless, we do not want the rudder to compensate for the oscillatory wave induced motion since this causes too much control action.

The heading angle of ship in the rough sea can be considered as the combination of the low-frequency ship yaw motion \( \psi_L \) and the high-frequency wave disturbance \( \psi_H \):
\[ \psi = \psi_H + \psi_L \]  

The oscillatory motion of the waves is described by a linear wave model:

\[ \psi_H(s) = \frac{K_n s}{s^2 + 2\zeta \omega_n s + \omega_n^2} w_H(s) \]  

where \( w_H \) is a zero-mean Gaussian white noise process and the oscillatory frequency \( \omega_n \) is an estimate of the frequency of encounter \( \omega_e \).

The ship yaw motion can be represented by a low-frequency model (here we use Nomoto 1\textsuperscript{st}-order model):

\[ \psi_L(s) = \frac{K}{s(1 + T s)} \delta(s) \]  

Making the following definition and the corresponding deductions with respect to (4.82) and (4.83):

\[ \dot{\psi}_L = r_L \]  

\[ \ddot{r}_L = -\frac{1}{T} r_L + \frac{K}{T} \delta + w_L \]  

\[ \xi_H = \psi_H + K_n w_H \]  

\[ \ddot{\xi}_H = -2\zeta \omega_n \psi_H - \omega_n^2 \xi_H + K_n w_H \]  

An observer is obtained from equation (4.84)~(4.87), here \( w_L \) is modeled as a zero-mean Gaussian white noise process. Rewrite them into a state space form:

\[ \hat{x} = A\hat{x} + Bu + k(y - C \hat{x}) \]  

where

\[ \hat{x} = \begin{bmatrix} \dot{\psi}_L & \dot{r}_L & \dot{\xi}_H & \dot{\psi}_H \end{bmatrix}^T ; \quad u = \delta ; \]

\[ A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{1}{T} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\omega_n^2 & -2\zeta \omega_n \end{bmatrix} ; \quad B = \begin{bmatrix} 0 & \frac{K}{T} & 0 & 0 \end{bmatrix}^T \]

\[ C = [1 \ 0 \ 0 \ 1] ; \quad k = [K_1 \ K_2 \ K_3 \ K_4]^T \]

The observer also can be written as:

\[ \hat{x} = \tilde{A}\hat{x} + Bu + ky \]  

where

\[ \tilde{A} = A - kC = \begin{bmatrix} -K_1 & 1 & 0 & -K_1 \\ -K_2 & -\frac{1}{T} & 0 & -K_2 \\ -K_3 & 0 & 0 & 1 - K_3 \\ -K_4 & 0 & -\omega_n^2 & -2\zeta \omega_n - K_4 \end{bmatrix} \]  

The characteristic equation of (4.90) can be written as:

\[ \pi(s) = \det(sI - \tilde{A}) = s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \]
where

\[ a_3 = K_1 + K_4 + 2\zeta\omega_n + \frac{1}{T} \]
\[ a_2 = \left(1 + 2\zeta\omega_n\right)K_1 + K_2 - \omega_n^2K_3 + \frac{1}{T}K_4 + \left(\omega_n^2 + 2\zeta\omega_n\frac{1}{T}\right) \]
\[ a_1 = (2\zeta\omega_n\frac{1}{T} + \omega_n^2)K_1 + 2\zeta\omega_nK_2 - \omega_n^2\frac{1}{T}K_3 + \omega_n^2\frac{1}{T} \]
\[ a_0 = \omega_n^2\frac{1}{T}K_1 + \omega_n^2K_2 \]

The required observer dynamics can be satisfied by assigning eigenvalues \( p_i \):

\[ \prod_{i=1}^{4}(s - p_i) = \pi(s) \]  \hspace{1cm} (4.92)  

where \( p_i \) (\( i = 1, \ldots, 4 \)) are real values specifying the desired poles of the error dynamics. The solution can be written in an abbreviated form as:

\[ k = \Gamma^{-1}\Lambda \]  \hspace{1cm} (4.93)  

where \( k = (K_1, K_2, K_3, K_4)^T \) is the estimator gain vector and:

\[
\Gamma = \begin{bmatrix}
\omega_n^2 & \omega_n^2 & 0 & 0 \\
\frac{2\zeta\omega_n}{T} + \omega_n^2 & 2\zeta\omega_n & -\omega_n^2 & 0 \\
\frac{1}{T} + 2\zeta\omega_n & 1 & -\omega_n^2 & \frac{1}{T} \\
1 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\Lambda = \begin{bmatrix}
p_1p_2p_3\frac{1}{T} \\
p_1p_2p_4 - p_1p_2p_3 - p_2p_3p_4 - p_1p_3p_4 - \frac{\omega_n^2}{T} \\
p_1p_2 + p_2p_4 + p_2p_3 + p_1p_4 + p_1p_3 + p_3p_4 - \left(4\zeta\omega_n + \frac{1}{T}\right) \\
-p_1 - p_2 - p_3 - p_4 - \left(4\zeta\omega_n + \frac{1}{T}\right) \\
\end{bmatrix}
\]

Using the Pole-placement method [Fossen, 1993]

\[ p_1 < -1/T; \]
\[ p_2 < 0; \]
\[ p_3 = p_4 < -\zeta\omega_n; \]

where \( \zeta = 0.01-0.1 \), then \( K_1, K_2, K_3, K_4 \) can be solved from (4.93).

Figure 4.11(a) shows the estimated yaw angle by the observer-based filter; dot line is estimated yaw angle and the solid line is the measurement yaw angle.

The real part of the first two poles \( p_1 \) and \( p_2 \) affect the bandwidth of the wave filter. The farther in the left-half plane the poles are, the faster the response of estimated states is. This means that the estimated error is smaller, but the effect of high frequency is more evident. Figure 4.11~4.12 show the simulation results with different values of \( p_1 \) and \( p_2 \).
Figure 4.11 The simulation results of wave filter when $p_1=1.3/T$, $p_2=0.01$
(a). Yaw angle and estimated yaw angle (Low frequency)

(b). The error of estimation

Figure 4.12 The simulation results of wave filter when $p_1=0.3/T, p_2=0.0001$

In the simulation, the parameters are chosen as

\[ p_1 = 0.3 / T; \]
\[ p_2 = -0.0001; \]
\[ p_3 = p_4 = -15\zeta\omega_n; \]

The simulation results for linear model and nonlinear model of a container ship are shown in following figures.

Figure 4.13 shows the simulation of a linear model for course keeping. In the first 400 second, the ship model is controlled by PD controller without wave observer. The amplitude of rudder motion is very large. In the period of 400 to 450 second, the rudder command is set to a constant to make a disturbance for course keeping. After 450 second, the PD controller is reworking and the wave observer turns on. The amplitude of rudder motion is small.

Figure 4.14 shows the simulations of nonlinear model controlled by a PID controller with a wave observer that is the same as used in linear model simulation. Since yaw_L has a constant value at the beginning, which is caused by the effect of the wave drift force, the integration is necessary to eliminate the residual error of the heading.
Figure 4.14 Simulation of nonlinear model controlled by PID controller with wave observer
Chapter 5

Robust Controller Design with Mixed Sensitivity Approach

The force of sea-wave on ships can be considered as external disturbance acting on the system output. Based on the concept of the output disturbance uncertainty and ship model uncertainty, the mixed-sensitivity approach was used to design the $\mathcal{H}_\infty$ controller for the purpose of RRD. In order to improve the system performance, we need to study the system sensitivity function $S = (1 + PK)^{-1}$; also for the robust stability of the system, the complementary sensitivity function $T = G(K(1 + PK))^{-1}$ that relate to stability margin should be investigated. In the mixed sensitivity approach, it is usual to combine the weighted disturbance attenuation ($W_1S$) and weighted stability margin ($W_3T$) together into a simple $\mathcal{H}_\infty$ norm. The needed controller must satisfy (3.64), i.e.:

$$\frac{W_1S}{W_3T}_\infty < 1 \quad (5.1)$$

The $\gamma$ in (3.64) is chosen as $1$ here. The problem of finding a controller under the object (5.1) can be changed to the $\mathcal{H}_\infty$ control standard problem, it's block diagram is shown in Figure 3.16. We redraw it here.

![Figure 5.1 The structure of mixed sensitivity standard problem](image)

In this figure, $w$ is the exogenous input, typically consisting of command signal and/or disturbances $d$; $u$ is control signal; $z$ is output to be controlled; $y$ is the measured output. Here the system output $z$ is combined by two components: the weighted error signal $z_1 = W_1\tilde{e}$, and the weighted model output, $z_3 = W_3\tilde{y}$. Figure 5.1 shows a standard structure of $G-K$, $K$ is the controller and $G$ is the general target except the $K$, or called general controlled object.
A closed loop control system is consisting of the general object and the controller. From (3.63) it is known that $G$ is a transfer matrix from the input vector $[w, u]^T$ to the output vector $[z, y]^T$:

$$G(s) = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} W_1 & -W_1 P \\ 0 & W_2 P \\ I & -P \end{bmatrix}$$ (5.2)

The time domain state space implementation is

$$G = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$$ (5.3)

It is easy to verify that $\begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix}$ is the closed loop transfer function of $G-K$ system from input $w$ to output $z$, so the mixed sensitivity optimization problem in (5.1) is equivalent to the following criterion

$$\left\| G_{zw} \right\|_\infty < 1 ;$$ (5.4)

The first step in designing a controller is to select the weight functions $W_1(s)$ for sensitivity function and $W_2(s)$ for complementary sensitivity function respectively, i.e. to decide what shapes of $S(s)$ and $T(s)$ are the best for roll damping of a ship.

After that, placing $P, W_1, W_2$ into the generalized plant matrix $G$ in (5.2), then the controller can be solved by the method given by Chiang and Safonov (1992).

Since the ship is a Single-Input Multi-Output (SIMO) plant, many design schemes may be used for calculating RRD controller. One scheme is to design two controllers for course keeping and roll damping respectively. This method will make the problem simpler by which we first consider heading control only, and then regard the heading loop as an inner loop to design roll damping controller. There are two ways to realize the design, one way is to put the outputs of the two controllers together, and the two controllers are linked with parallel connection, which is called parallel controller. The other way is to connect the output of roll damping controller to the input of heading controller, these two controllers are in the form of series connection, which is called cascade controller.

Considering $\mathcal{H}_\infty$ optimization is suitable for multivariable control, we will treat the course keeping and roll damping problems simultaneously and design one controller other than two of them. The SIMO plant controller designed by the direct design mode is called MISO controller because it has multi inputs and single output, this will be presented in section 5.4.

5.1 Parallel Controller

In this section, we will discuss the design of a parallel controller. The structure of parallel control is shown in Figure 5.2. We will design a course keeping controller first (the inside of dash line in Figure 5.2) and then design a roll damping controller (the outside of dash line in Figure 5.2). The outputs of these two controller are added together to control the rudder of the ship. Note that, the model of steering machine is included in the ship model, as a 1st-order dynamic block. So, when we say a ship linear model, it is a 6th-order state equation and it does include the steering machine. If we say a ship nonlinear model, it includes not only the 1st-order dynamic block, but also the two limiters.
5.1.1 Course Keeping Controller

Actually, the first step is to design an autopilot in the common sense (see the inside of dash line in Figure 5.2). Then we will consider how to select the weight functions $W_1(s)$ and $W_3(s)$.

![Diagram of parallel control](image_url)

**Figure 5.2 The structure of parallel control**

In order to reduce the high frequency disturbance of wave on the ship heading, and satisfy the requirement of no steady state error (integral action), it is necessary to specify $W_1(s)$ with at least one pole located near zero frequency to obtain the desired integral action. Two poles are placed near zero frequency to obtain a more abrupt increase of the sensitivity function. In order to balance the two poles and make the bandwidth wide enough, two zeros of $W_1(s)$ are needed. Therefore $W_1(s)$ for heading controller is selected as

$$W_1 = 0.78 \frac{(s + 0.02)(s + 0.32)}{(s + 0.00004)(s + 0.000055)};$$

$W_3(s)$ is the upper bound of the norm of multiplicative model uncertainty. It should have high-pass property [Wu and Xie, 1997; Zames, 1966]. Therefore $W_3(s)$ is chosen as

$$W_3 = 26 \frac{s + 0.1}{s + 3.8}$$

Using $\gamma$-iteration of MATLAB toolbox, a 9th-order heading controller was obtained. By using model order reduction method in the same toolbox, this controller could be reduced to 5-order. The Zeros and Poles of the course keeping controller are shown in Table 5.1

<table>
<thead>
<tr>
<th>Zeros/Poles of the course keeping controller</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poles</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>-62.356</td>
</tr>
<tr>
<td>-3.8011</td>
</tr>
<tr>
<td>-0.73845±0.15735</td>
</tr>
<tr>
<td>-0.02763±j0.23318</td>
</tr>
<tr>
<td>-0.04187</td>
</tr>
<tr>
<td>-0.00004</td>
</tr>
<tr>
<td>-0.000055</td>
</tr>
</tbody>
</table>

5.1.2 Roll Damping Controller

Closed the heading loop including in the obtained heading controller, the inside of dash line in Figure 5.2 can be considered as a general ship model. We can obtain an open loop transfer function from rudder command to roll angle. The structure plot to design roll damping controller is shown in Figure 5.3.
Figure 5.3 The structure plot of General ship model for RRD

Figure 5.4. The Bode plot of system from rudder angle to roll angle with closed course control loop but open RRD loop.

The Bode plot of the general ship transfer function is shown in Figure 5.4. From the figure it is known that there is a resonant peak at $\omega = 0.23$ rad/sec. The ship will amplify roll significantly around this natural roll eigenfrequency. Considering that the range of frequencies is 0.25 to 0.4 for roll RAO, the ship roll motion will be seriously amplified in this range, roll damping performance in this frequency region is, therefore, particularly important. In order to make the ship insensitive to the disturbance in the mentioned frequency range, two pairs of complex poles are specified in this range to get resonant peaks to eliminate the disturbance. The weight function $W_1(s)$ for roll controller is specified as

$$W_1 = 0.7 \frac{(s + 0.26)(s + 0.2925)(s + 0.442)(s + 0.975)}{(s + 0.13 \pm j0.26)(s + 0.13 \pm j0.2925)};$$

With the same reasons as in the design of course keeping controller, the selection of weight function $W_3(s)$ for the roll damping controller should satisfy the high frequency property. So $W_3(s)$ is specified as
Based on this weighting strategy, a 16th-order roll damping controller is obtained by the $\gamma$-iteration with an optimal result $\gamma = 0.27$. The Zeros/Poles of the roll damping controller are given in Table 5.2. The roll disturbance sensitivity was obtained (see Figure 5.5). From this figure we know that a large attenuation on roll motion is obtained by this controller in the range of wave disturbance. A small roll amplification (1.2~1.3) exists out of the range, however we know that the values of power spectral density are very small in the ranges of lower than 0.1 rad/s and higher than 0.5 rad/s from wave power spectrum analysis. Therefore the disturbance force and moment are small in those ranges, the small amplification brings no problem for roll damping performance.

The roll damping controller is a 16th-order controller. The order of the controller is too high to be used for on-line control. A model reduction function in MATLAB toolbox by balanced truncation was used to reduce the controller order from 16 to 9, without remarkable change in performance (see Figure 5.6).

### 5.1.3 Simulation Results

The performance of the controller in the sea-way is simulated with both linear and nonlinear models. The ship model including rudder is built for a navy multipurpose vessel. The model parameters are from [Blanke and Jensen 1997]. The principal particulars are: ship length = 230 m, displacement = 46000m$^3$. The block diagram of simulation is shown in Figure 5.2. The simulation conditions are $T_w = 8$ s, $h_{1/3} = 3$ m. The angle between ship and wave (encounter angle) is $\chi = 45^\circ$, ship speed is $U = 12$ m/sec.

In order to have a criterion for roll damping, the definition of roll reduction is given first:

$$\text{Roll reduction} = \frac{\varphi \text{ without damping} - \varphi \text{ with damping}}{\varphi \text{ without damping}}$$
The simulation results are given in Figure 5.7. The real line is the result of roll damping and dot line is the result without roll damping. From the figure we see that the roll reduction are nearly 70% for linear model. The roll reduction is 45% for nonlinear model. It is obvious lower than linear model. The reason is that the sensitivity of nonlinear model to the wave disturbance is larger than that of linear model. The large roll angle makes both rudder angle and rudder speed saturation (the maximum rudder angle is 35 deg. and the maximum rudder speed is 7 deg/sec in this model), so that the effectiveness of RRD controller is lower in nonlinear model.

### Table 5.2 Zeros/Poles of the roll damping controller and the reduced order controller

<table>
<thead>
<tr>
<th>Roll damping controller</th>
<th>Reduced order controller</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poles</td>
<td>Zeros</td>
</tr>
<tr>
<td>-62.233</td>
<td>-62.233</td>
</tr>
<tr>
<td>-21.556</td>
<td>-4.2</td>
</tr>
<tr>
<td>-4.2</td>
<td>-1.071</td>
</tr>
<tr>
<td>-0.9774±j0.6363</td>
<td>-0.2183±j0.2478</td>
</tr>
<tr>
<td>-0.44863</td>
<td>-0.2549±j0.1431</td>
</tr>
<tr>
<td>-0.13±j0.2925</td>
<td>-0.0369±j0.2287</td>
</tr>
<tr>
<td>-0.13±j0.26</td>
<td>-0.0472±j0.0246</td>
</tr>
<tr>
<td>-0.24565</td>
<td>0.0128</td>
</tr>
<tr>
<td>-0.03258</td>
<td>-0.0050±j0.0081</td>
</tr>
<tr>
<td>0.0007±j0.0069</td>
<td>-0.0047</td>
</tr>
<tr>
<td>0.0019±j0.0015</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5.6 The Bode plots of controller and reduced order controller
Also we can see from the Figures that RRD makes the course keeping performance deteriorate. It is even worse for nonlinear model. This is obviously because there is only one actuator to receive two inputs to the controlled object, the course keeping and RRD only are traded-off by the rudder, and the large rudder angle makes the course error increase inevitably.
5.2 Parallel Controller with Wave Filter

In Figure 5.6~5.8, large rudder angles appear even if the course keeping controller works only. This is not desired since more rudder action will increase the consume of fuel and the wearing of the rudder mechanism. An observer-based wave filter is added to eliminate the undesired rudder motion.

5.2.1 Course Keeping Controller

The block diagram of a heading controller with a wave filter is shown in Figure 5.9.
The filter is obtained by using the Pole-placement method mentioned in section 4.8. The controller may be of PID, LQG or $\mathcal{H}_\infty$ type. Because the attention is paid to RRD, the course keeping control should be made as simple as possible, so we use PD control for course keeping here and choose $K_p = 0.2, K_d = 5.0$.

### 5.2.2 Roll Damping Controller

In order to guarantee the course keeping when roll damping is active, the course loop must be closed when designing the roll damping controller. This course loop includes the wave filter and the course keeping controller. An open loop transfer function from rudder angle to roll angle can be obtained that is an 11th-order system and its Bode plot is shown in Figure 5.10. It is seen that there are two resonant peaks at $\omega = 0.02$ and $\omega = 0.24$ rad/sec. The later resonant peak is in the range of wave power spectrum. It is necessary to make the closed loop system of the ship insensitive to this frequency. Therefore, the performance weight function $W_1(s)$ was specified as

$$W_1(s) = \frac{(s + 0.11)(s + 0.385)(s + 0.605)(s + 0.825)}{(s + 0.137 \pm j0.22)(s + 0.192 \pm j0.2475)};$$

The weight function $W_3$ is specified being

$$W_3(s) = 0.6 \frac{s + 1.1}{s + 4.2}$$

---

**Figure 5.10** Bode plot of system from rudder to roll angle with closed heading control loop but open RRD loop.
Figure 5.11 Output sensitivity plot of closed loop roll damping control

The reason of taking this choice for these weight functions is the same as stated in the above section. Based on these weight functions, the roll damping controller was obtained by the \( \gamma \) iteration. Table 5.3 shows Zeros/Poles of the roll damping controller and its reduced order controller. The output sensitivity plot by this controller is shown in Figure 5.11. From the figure, it is easy to know that this controller will attenuate the wave disturbance effectively in the main range of wave power spectrum, but the roll reduction of this controller will be slightly worse than that of the controller without wave filter.

Table 5.3  Zeros/Poles of the roll damping controller and its reduced order controller

<table>
<thead>
<tr>
<th>Roll damping controller</th>
<th>Reduced order controller</th>
</tr>
</thead>
<tbody>
<tr>
<td>Poles</td>
<td>Zeros</td>
</tr>
<tr>
<td>-165.11</td>
<td>-4.2</td>
</tr>
<tr>
<td>-4.2</td>
<td>-1.0551</td>
</tr>
<tr>
<td>-0.9609±j0.6093</td>
<td>-0.7027±j0.1582</td>
</tr>
<tr>
<td>-0.79151</td>
<td>-0.2688±j0.1991</td>
</tr>
<tr>
<td>-0.65807</td>
<td>-0.0347±j0.2329</td>
</tr>
<tr>
<td>-0.1375±j0.22</td>
<td>-0.05262</td>
</tr>
<tr>
<td>-0.05138</td>
<td>-0.00488</td>
</tr>
<tr>
<td>-0.00007</td>
<td>-0.00007</td>
</tr>
</tbody>
</table>
5.2.3 Simulation Results

The performance of the controller in sea-way is simulated with both linear and nonlinear models. The block diagram of simulation is shown in Figure 5.2. The simulation conditions are the same with section 5.1. The simulation results are given in Figure 5.12.

Figure 5.12 Simulation results: $T_s=8s$, $h_{1/3}=3m$ $\chi=30^\circ$ $U=12$ m/s
In the Figures, the real lines are the results of roll damping and dot lines are the results without roll damping. From the Figures we know that the roll reduction are above 50% for both linear model and nonlinear model.

(a). $T_w=8s$, $h_{1/3}=3m$ $\chi=45^\circ$ $U=12 \text{ m/s}$

(b). $T_w=6s$, $h_{1/3}=2m$ $\chi=45^\circ$ $U=12 \text{ m/s}$

Figure 5.13. Simulation results for nonlinear model when simulation condition changed
From Figure 5.12 we also know that the rudder motion is very small (no more than 1°) when RRD is inactive. This is due to the wave observer; otherwise it may be larger than 5°. Another observation is that RRD makes the course keeping performance deteriorate and the situation is even worse for nonlinear model simulation (see Figure 5.12 (b)). This is because of the rudder saturation in nonlinear model.

From section 4.7 we know that this ship will have the largest roll angle at $\chi = 47^\circ$ when the speed is $U = 12\text{m/s}$ and $T_w = 8\text{s}$, $h_{1/3} = 3\text{m}$. Figure 5.13 shows the simulation results when the encounter angle is 45°. The wave condition are unchanged in (a) and changed in (b). Although the nonlinear phenomena of rudder are severe in Figure 5.13 (a), the roll reduction is still about 45% but the yaw angle is large by the rudder saturation. If the wave height and period are changed to smaller values, the roll reduction will rise to 50% because the roll angle is not large enough to make the rudder motion saturate. Another result in Figure 5.13 (b) is obviously that the yaw angle is small and similar to the case of no roll damping.

### 5.3 Cascade Controller

The block diagram of the cascade controller design is shown in Figure 5.14. Similar to the parallel scheme, we first consider the course keeping controller only and then close the course keeping loop to design the roll damping controller. This architecture chosen in the design is not quite conventional. Usually, as in the case of the parallel controller, the command rudder angle given by an RRD controller is superposed to the command from the course keeping autopilot, subject to limits in available rudder angle and rate. However, in the present scheme, the output of the roll damping controller is a part of the setting course of the autopilot, so it is possible to reduce the restriction. The disadvantage is that a disturbance has been added to the setting command for course keeping.

From a practical point of view, the proposed architecture is equivalent to adding an RRD control loop onto an existing course keeping controller. If desirable for practical reasons, the RRD controller could easily be converted to give rudder angle command instead of altering the autopilot’s heading reference. But if it is done like that, it will be better to design the controller with parallel scheme directly.

![Figure 5.14 Cascade system design block diagram](image)

### 5.3.1 Course Keeping Controller

We still use mixed sensitivity approach to design the heading controller. The steps of design are the same as in the parallel controller design. Similar to the parallel controller design, $W_i(s)$ for heading controller is specified as
The weight function $W_3(s)$ is chosen as

$$W_3(s) = 26 \frac{s + 0.1}{s + 3.8} \quad (5.6)$$

Using $\gamma$ iteration subroutine of MATLAB, an 8th-order heading controller was obtained. By using model order reduction method, this controller can be reduced to 5-order (see figure 5.15).

5.3.2 Roll Damping Controller

Close the course keeping loop including the course keeping controller, an open loop transfer function from rudder command to roll angle is obtained. The former can be considered as a general ship model. The structure of the general ship model is shown in figure 5.16.
There is a resonant peak at $\omega = 0.23$ rad/sec. The weight function $W_i(s)$ for roll controller was specified as

$$W_i(s) = \frac{(s + 0.36)^2 (s + 0.3816)^2}{(s + 0.072 \pm j0.3348)(s + 0.054 \pm j0.216)};$$

The weight function $W_3(s)$ for robust stability is specified as

$$W_3(s) = 0.6 \frac{s + 1.1}{s + 4.2}$$

Based on this weighting strategy, the roll disturbance sensitivity was obtained (see Figure 5.18) by the $\gamma$ iteration with an optimal result $\gamma = 0.519$. The roll damping controller is a 16th-order controller. A 7th-order reduced order controller, by balanced truncation is reached without
remarkable change in performance (see figure 5.19). Table 5.4 shows Zeros/Poles of the roll damping controller and its reduced order controller.

![Figure 5.18 Output sensitivity with roll loop closed (from $\phi$ to $\delta_c$)](image)

![Figure 5.19 The frequency characteristics of roll damping controller and order reduced roll damping controller](image)

5.3.3 Simulation Results

The performance of the controller in a sea-way is simulated with both linear and nonlinear models. The block diagram of simulation is shown in Figure 5.14. The wave conditions are $T_w =$
8 sec, $h_1/3 = 3$ m. The angle between the ship and the wave is $\chi = 30$ deg, corresponding to a following sea. The ship speed is $U=12$ m/s. Simulation results are given in Figure 5.20 and 5.21. The roll reduction of 70% was obtained approximately for the linear model whereas about 45% for the nonlinear model. The reason is the well known fact that the rudder angle and rudder rate saturate in the later case so the effectiveness of RRD is lower. Another observation is that RRD naturally makes the course keeping performance deteriorate. The explanation has been given in the previous section.

Figure 5.20. Simulation with linear model

Figure 5.21. Simulation with nonlinear model
5. Robust Controller Design with Mixed Sensitivity Approach

The above result for the non-linear simulation clearly demonstrates that the design of rudder roll damping controllers must be done with due regard to non-linear phenomena, the predominant one being rudder rate saturation. This has been observed in all the previous studies in the RRD field, and the importance is emphasized by the high gain controller produced by the $\mathcal{H}_\infty$ optimization with the chosen weight functions.

5.4 MISO Controller

Considering $\mathcal{H}_\infty$ optimization is suitable for multivariable control, we will handle the course keeping and roll damping problem in a unified fashion to get one controller other than two of them.

5.4.1 MISO Controller Design

Ship control by rudder for course keeping and roll damping simultaneously forms a single input multi output SIMO plant. Figure 5.22 shows the MIMO control system structure, in which the controller is a MISO subsystem. $\mathcal{H}_\infty$ control theory is directly applicable to design this kind of controller. Nevertheless, differing from previous sections, trade off between roll damping sensitivity and course keeping ability has to be made. Fortunately, the two outputs of the ship are controlled in different frequency regions, resulting in few interactions between the two sensitivities.

![Figure 5.22 MIMO system control structure](image)

The desired disturbance attenuation is assumed in a diagonal matrix form for this SIMO plant. The weight matrix is represented as

$$W_1(s) = \begin{bmatrix} W_{r_1}(s) & 0 \\ 0 & W_{h_1}(s) \end{bmatrix}; \quad W_3(s) = \begin{bmatrix} W_{r_3}(s) & 0 \\ 0 & W_{h_3}(s) \end{bmatrix}$$

(5.7)

where $W_{r_1}(s)$ is the weight function associated with the heading angle and $W_{r_3}(s)$ corresponds to the roll angle.

According to the knowledge from the cascade controllers, $W_{h_1}(s)$ is specified directly from the heading controller of section (5.3), and $W_{r_1}(s)$ is similar to that of the roll controller.

$$W_{r_1}(s) = \frac{0.4(s + 0.845)^2(s + 0.975)^2}{(s + 0.0455 \pm j0.39)(s + 0.052 \pm j0.26)};$$

$$W_{h_1}(s) = \frac{0.78(s + 0.3)(s + 0.32)}{(s + 0.00004)(s + 0.000055)};$$
The obtained roll sensitivity function and heading sensitivity function are depicted in figure 5.23. Roll damping reduction is about 40% in the specified range of frequency, and about -30% out of this range. However the disturbance for heading will be amplified about 2 times in the frequency range from 0.04 to 0.14 rad/sec. Compare it with the results of the cascade controllers mentioned in previous sections, the control results are worse than the former ones. This is due to the competing objectives, trying to fulfill specifications in roll and heading simultaneously.

Table 5.5 Zeros/Poles of the MISO order reduced controller

<table>
<thead>
<tr>
<th>Poles</th>
<th>Zeros of roll damping controller</th>
<th>Zeros of course keeping controller</th>
</tr>
</thead>
<tbody>
<tr>
<td>-12.612</td>
<td>-0.0837±j0.3278</td>
<td>-0.0498±j0.3932</td>
</tr>
<tr>
<td>-0.3207±j0.2222</td>
<td>0.03176</td>
<td>-0.0530±j0.2362</td>
</tr>
<tr>
<td>-0.00004</td>
<td>0.00951</td>
<td>-0.0054±j0.0044</td>
</tr>
<tr>
<td>-0.00006</td>
<td>0.00048</td>
<td></td>
</tr>
<tr>
<td>-0.0466±j0.3889</td>
<td>0.000005</td>
<td></td>
</tr>
</tbody>
</table>

Figure 5.23 Roll sensitivity $\phi_w$ to $\phi$ and heading sensitivity $\psi_w$ to $\psi$

Compare with the original 24th-order controller, the order reduced controller is a 12th-order one. If the model order reduction technique is used again for this controller, it could further reduce the order to 7. Figure 5.24 shows the frequency characteristics of the MISO controller.
5.4.2 Simulation of MISO Controller

With the same simulation conditions and parameters as used in 5.3, the simulation results are shown in figure 5.25 (linear model) and figure 5.26 (nonlinear model). For the linear model, the roll reduction is about 40%. For the nonlinear model, the same roll reduction has been obtained. At the same time, the yaw angle is nearly the same for both simulation models with roll damping and without roll damping. Compare with cascade controllers, the roll reduction is less, but the yaw angle is better.

Figure 5.24 The frequency characteristic of MISO controller and order reduced controller
5.5 Summary

It is known from simulation results of the preview sections, the $\mathcal{H}_\infty$ controllers designed by mixed sensitivity approach have good robust performance. Though there is an obvious difference
between the results of the leaner model and the nonlinear model, the satisfactory roll reduction has been obtained by these controllers. All of the three kinds of controller, discussed in this chapter, can get the roll reduction higher than 40% in the simulation with nonlinear model. Since the design frameworks of the three controllers are different, the effectiveness of RRD is different too, although the theory in the design is the same.

In terms of roll reduction, the best control framework is the cascade mode, and then, is the parallel mode, this is because that they only consider the roll damping when designing the roll damping controller, it is thought of that the course keeping has been guaranteed by the inner loop. Therefore the roll reduction is 70% for the cascade controller. However for MISO controller, since the trade off must be made between the performance of RRD and course keeping control during the process of design, the roll reduction of it is lower than that of the former two. It is only 45%.

When the roll damping is inactive, only the course keeping is considered, the performance of the former two controllers are the same. The control effect of them is obviously better than that of the MISO controller.

However, if both roll damping and course keeping are considered simultaneously, the results and effects of the MISO controller are better than the former. The course keeping ability deteriorates for both cascade controller and parallel controller when their roll damping controller are turned on. But for the MISO controller, there is no effect upon the course-keeping whether the roll damping is active or not.

In conclusion, if only the roll reduction is emphasized, the separated design mode, cascade control and parallel control, are suggested. Since it is difficult in practice for cascade controller to be implemented, the parallel mode is the best one for this usage. It will obtain the largest roll reduction and the realization is easy to perform, although it will cause the course keeping deteriorate.
Chapter 6

Robust Controller Design with $\mu$ Synthesis

It has been shown in chapter 5 that the robust controllers, especially the cascade controller designed by mixed sensitivity approach for roll damping works effectively. The roll reduction of all controllers is about 35% for the nonlinear model. However, all of them are not fully satisfactory. The cascade controller has good roll reduction but worse course keeping performance, so does the parallel controller. The MISO controller has a good course keeping characteristics, but the roll damping performance is worse, the roll reduction is only about 40%. Remember that all of the controllers discussed above did not make use of the information of the model uncertainty $\Delta$ explicitly in the design process. The robust performance problem was only solved conservatively. If the model uncertainty is structured, however, the controller performance will be improved using $\mu$ synthesis with the aid of D-K iteration.

6.1 Framework and Parameters of Ship for D-K Iteration

When the model uncertainty and disturbances are considered, the system structure is described in Figure 6.1, where $u$ is the system input and $y$ the system error output of the perturbed plant.

We lump all of the model uncertainty effects (for instance, variations in speed, loading; modifications of rudder and bilge keel layout) together into a full-block weighted uncertainty $\Delta W_{del}$.

For a given nominal ship model $Ship_{nom}$, we specify a stable 2x2 transfer matrix $W_{del}(s)$, called uncertainty weight function. This transfer matrix parametrizes an entire set of the uncertain plant, $Ship$, which must be suitably controlled by the robust controller $K$.

$$Ship = \{Ship_{nom} \left( I + \Delta_s W_{del} \right) | \Delta_s \text{ stable, } \|\Delta_s\|_\infty \leq 1 \}$$ (6.1)
The unknown transfer function $\Delta(s)$ is used to parametrize the potential differences between the nominal model $Ship_{nom}(s)$, and the actual behavior of the real ship, denoted by $Ship$. The uncertainty weight $w_{del}$ as a function of $s$ indicates that the level of uncertainty in the ship’s behavior depends on frequency.

Therefore it is assumed that (refer to section 3.8 and section 4.3) the uncertainty has a diagonal structure:

\[
W_{del} = \begin{bmatrix}
\Delta_{p_r}(s) \\
\Delta_{p_y}(s)
\end{bmatrix};
\]

where

\[
\Delta_{p_r}(s) = \frac{2\zeta_0\omega_0(1 + \alpha \frac{\nabla}{\nabla_0} - \alpha (\frac{\nabla}{\nabla_0})^2)s + \omega_0^2(1 - (\frac{\nabla}{\nabla_0})^2)}{s^2 + 2\zeta_0\omega_0s + \omega_0^2}
\]

\[
\Delta_{p_y}(s) = 0.21 \frac{2\zeta_0\omega_0s + \omega_0^2}{s^2 + 2\zeta_0\omega_0s + \omega_0^2}
\]

For this project, we select $\gamma_{\zeta_0} = 0.7$, $\alpha = 0.5$. $\zeta_0$ and $\omega_0$ can be obtained from ship roll equations directly.

At any frequency $\omega_0$ the value of $|W_{del}(j\omega)|$ can be interpreted as the relative variation of uncertainty expressed in percentage in the model at that frequency.

$W_p$ is a stable, rational 2x2 transfer matrix called performance weight matrix, which is specified as:

\[
W_p = \begin{bmatrix}
W_{p1} & 0 \\
0 & W_{p2}
\end{bmatrix}
\]

Comparing Figure 6.1 with Figure 5.1, we conclude that when the uncertainty $W_{del}$ goes to zero, i.e. the structure of uncertainty is unknown, then $W_p$ is just the sensitivity weight function $W_1$ in the mixed sensitivity approach. So, consulting $W_{r1}$ and $W_{h1}$, the structure and the parameters of $W_{p1}$, $W_{p2}$ are chosen as

\[
W_{p1} = 0.16 \frac{(s + 0.06)(s + 0.085)(s + 0.9)(s + 1.0)}{(s + 0.15 \pm j0.14)(s + 0.22 \pm j0.21)}
\]

\[
W_{p2} = 0.2 \frac{(s + 0.02)(s + 0.032)}{(s + 0.00004)(s + 0.000055)}
\]

When $W_{del}$ and $W_p$ are specified, the system open-loop interconnection is structured that is illustrated in Figure 6.2. If we define the interconnection structure as $Ship_O$, Figure 6.1 can be redrawn into a framework of standard optimization shown in Figure 6.3.

It is seen that there are 5 inputs and 7 outputs in this system from Figure 6.3. Input 1 consists of 2 perturbation inputs $p(1:2)$ and 2 disturbance inputs $d(1:2)$. Input 2 is a control signal $u(1)$. 
Output 1 consists of 2 outputs of model uncertainty weights $W_{del}(w(1:2))$ and 3 error outputs signals $e(1:3)$ (2 outputs of performance weights $w_p(e(1:2))$ and 1 auxiliary output $e(3)$). Output 2 has two components $y(1:2)$ (roll angle and yaw angle).

Figure 6.2 The system open loop interconnection structure $Ship\_O$

Figure 6.3 Closed loop Linear Fractional Transformation

The auxiliary output $\beta$ in figure 6.1 is interpreted as follows:

D-K iteration is a process of $\mathcal{H}_\infty$ optimization. The requirements mentioned in section 3.6 must be satisfied. However for the ship model, $D_{12}=[0 \ 0 \ 0 \ 0]^T$, (R2) can not be satisfied. If a small value $\varepsilon$ is added into $D_{12}$, it will have full column rank. $D_{12}$ is a scaling gain from the input 2 to the output 1 directly, therefore an auxiliary block $\beta$ which includes $\varepsilon$ appears in Figure 6.1 to connect the input 2 (control) to the output 1 (error).

From (4.48) we know that matrix $A$ of the state space equation of the ship model is not full rank. The requirement (R3) is not satisfied either. The reason is that $A$ has a pole on the imaginary axis. If the imaginary axis is moved a little distance $\varepsilon$ to the left half plane, this problem could be avoided. In this way, we build an auxiliary $\hat{s}$-plane, where $\hat{s} = s + \varepsilon$, and we obtain

$$\hat{s}I + A = (s + \varepsilon)I + A = sI + \varepsilon I + A = sI + \hat{A}$$

where $\hat{A} = A + \varepsilon I$.

$\hat{A}$ is full rank and (R3) is satisfied in this auxiliary plane. Using $\hat{A}$, a controller $\hat{K}$ can be obtained in this system. It is easy to recover the controller from $\hat{K}(\hat{s})$ in the auxiliary plane to $K(s)$ in the original system.
\[ K(s) = \hat{K}(\hat{s} - \varepsilon) \] (6.9)

In case \( \varepsilon \) is positive and sufficiently small, the system stability and performance will not be affected practically in this procedure.

### 6.2 Iteration Processes

D-K iteration is proceeded by performing two parameter minimization in sequence.

1. Making an open loop interconnection \( N \) with the generalized plant.
2. Selecting an initial stable rational transfer matrix \( D(s) \) with an appropriate structure. An identity matrix for \( D(s) \) is often a good choice provided that the system has been reasonably scaled for performance.
3. Designing an \( \mathcal{H}_\infty \) suboptimal controller for the scaled problem

   \[ \min_{K} \|DN(K)D^{-1}\|_\infty \quad \text{with fixed} \quad D(s); \] (6.10)

   to produce a controller \( K \).
4. Closing the feedback loop by \( K \) to form a closed loop system.
5. Since \( \mu(N) \leq \min_{D\in A} \sigma(DND^{-1}) \), a structured singular value calculation is performed on the frequency response of the closed loop system.
6. Minimizing \( \sigma(DND^{-1}(j\omega)) \) pointwisely across the desired frequency range with fixed \( N \) to find \( D(j\omega) \).
7. Fitting the magnitude of each element of \( D(j\omega) \) to a stable and minimum phase transfer function \( D(s) \) and go to step 3.

The iteration may continue until satisfactory performance is achieved, such that \( \|DND^{-1}\|_\infty < 1 \) or until the \( \mathcal{H}_\infty \) norm no longer decreases.

For this RRD project, the iteration continues until the 4\(^{th} \) loop. Table 6.1 shows the parameters obtained after the 4 iterations.

<table>
<thead>
<tr>
<th>Iteration #</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Controller Order</td>
<td>15</td>
<td>23</td>
<td>23</td>
<td>31</td>
</tr>
<tr>
<td>Total D-Scale Order</td>
<td>0</td>
<td>8</td>
<td>8</td>
<td>16</td>
</tr>
<tr>
<td>( \gamma ) Achieved</td>
<td>1.012</td>
<td>1.012</td>
<td>0.999</td>
<td>0.982</td>
</tr>
<tr>
<td>Peak ( \mu )-Value</td>
<td>0.975</td>
<td>0.977</td>
<td>0.967</td>
<td>0.953</td>
</tr>
</tbody>
</table>

From table 6.1, we know that we have achieved the robust performance objective since the peak-\( \mu \) is smaller than 1. However the controller order is 31\(^{st} \). It is too high to be used practically. A lower order controller may be obtained by using a MATLAB subroutine hankmr.m (optimal Hankel norm approximation). Here we get a 9\(^{th} \)-order controller using this method. The comparison of the reduced order controller with 31\(^{st} \)-order controller is shown in the Figure 6.4. The characteristics of the reduced order controller are the same with the original one in the high and middle frequency range, only a slightly higher gain than the original one is present at lower frequencies, which has no significant influence on the roll damping performance.

The zeros/poles of the controller are listed in Table 6.2.
The sensitivity functions of the reduced order controller are shown in Figure 6.5. The roll reduction at 0.23 rad/sec should be 65% from the figure. This controller is better than the MISO $\mathcal{H}_\infty$ controller.

Table 6.2  Zeros/Poles of the reduced $\mu$ controller

<table>
<thead>
<tr>
<th>Poles</th>
<th>Zeros of roll damping controller</th>
<th>Zeros of course keeping controller</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3.001</td>
<td>4648.8</td>
<td>5289.3</td>
</tr>
<tr>
<td>-1.231±2.246</td>
<td>-1.3295</td>
<td>-1.3161</td>
</tr>
<tr>
<td>-0.521</td>
<td>-0.6352</td>
<td>-0.1622±j0.2475</td>
</tr>
<tr>
<td>-0.157±0.242</td>
<td>-0.0565±j0.2613</td>
<td>-0.0359±j0.2394</td>
</tr>
<tr>
<td>-0.0647</td>
<td>-0.0199</td>
<td>-0.0249</td>
</tr>
<tr>
<td>0.00002±j0.00003</td>
<td>0.0052</td>
<td>-0.00008</td>
</tr>
<tr>
<td></td>
<td>-0.00004±j0.00002</td>
<td>-0.000003</td>
</tr>
</tbody>
</table>

Using the 4th iteration $\mu$ controller here, the $\mu$ value is 0.986. It means that for all the uncertainty block $\Delta \in \|\Delta\|_{\infty} < \frac{1}{\mu}$, system stability is guaranteed by this $\mu$ controller. We can make $\mu$ much smaller than 1 by choosing $W_p$ to get even better robust stability, but it will then lose some performance.

For example, when the gain in $W_p$ is changed from 0.16 to 0.1 in equation (6.6), the iteration results in Table 6.3 are obtained. We see that here $\mu$ has reached much smaller value than the original ones given in Table 6.1.

If we use the controllers from the second iteration and the fifth iteration respectively for the Gain in $W_p$ being 0.1, different sensitivity functions will result in (see Figure 6.6 and 6.7), which show that only 30% and 20% roll reductions are obtained respectively. So when the number of iteration is increased from 2 to 5, $\mu$ is reduced by 2.2%, and the roll reduction is not increased, instead it is reduced by 50%. However, even the result of roll reduction for the case of 2 iterations is worse than that of the controller for which the gain of $W_p$ is set to 0.16.

These comparisons indicate that the selection of $W_p$ is very important and must be careful. One may think that to get a better performance, a controller corresponding to a smaller $\mu$ should be chosen, the analysis made above shows that this is not always the case.

Table 6.3  The summary after the 5 iterations for different $W_p$

<table>
<thead>
<tr>
<th>Iteration #</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Controller Order</td>
<td>15</td>
<td>23</td>
<td>23</td>
<td>19</td>
<td>19</td>
</tr>
<tr>
<td>Total D-Scale Order</td>
<td>0</td>
<td>8</td>
<td>8</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$\gamma$ Achieved</td>
<td>1.012</td>
<td>0.784</td>
<td>0.773</td>
<td>0.765</td>
<td>0.763</td>
</tr>
<tr>
<td>Peak $\mu$-Value</td>
<td>0.801</td>
<td>0.778</td>
<td>0.771</td>
<td>0.763</td>
<td>0.761</td>
</tr>
</tbody>
</table>
Figure 6.4 Frequency characteristics of controller and reduced order controller

Figure 6.5 Output sensitivity of system
Now we assess the controller designed with $\mu$ synthesis approach.

First, based on Theorem 3.5 the robust stability can be assessed with $\mu$. The upper and lower bounds from $\mu$ calculation are shown in Figure 6.8. These two curves are nearly coincident with each other. The peak is about 0.59 in Figure 6.8, which implies that for perturbations $\Delta(s)$ with
correct structure, the stability is preserved so long as $\|\Delta(s)\|_\infty < \frac{1}{0.95}$, and for the perturbation $\Delta_{pert} \in \Delta$ with correct structure, condition $\|\Delta_{pert}(s)\|_\infty \geq \frac{1}{0.39}$ will cause instability.

From equation 3.37 and Definition 3.2 we know that a system is NP if the controller $K$ exists to satisfy $\|N_{22}\|_\infty < 1$. In Figure 6.9 we have $\|N_{22}\|_\infty = 0.78$ so the system is of nominal performance.

At last, we estimate the system robust performance by examining $F_u(N,\Delta)$. From Theorem 3.6, we know that we can use $\mu$ to evaluate it. Figure 6.10 shows that the peak of $\mu_{N}(N(j\omega))$ is equal to 0.955, so that the condition of RP is satisfied. Theorem 3.6 further implies that for any structured $\Delta(s)$ with $\|\Delta(s)\|_\infty < \frac{1}{0.955}$, the perturbed closed loop system remains robust performance, and the $\|\cdot\|_\infty$ norm of $F_u(N,\Delta)$ is guaranteed to be $\leq 0.955$. The converse of the theorem shows that there is a perturbation $\Delta$, whose $\|\cdot\|_\infty$ is larger or equal to $\frac{1}{0.955}$ that causes $\|F_u(N,\Delta)\|_\infty > 0.955$.
Figure 6.9 Nominal Performance plot

Figure 6.10 Robust Performance $\mu$ plot
6.3 Simulation Results

The 4th iteration controller, whose $\mu$ is 0.953, is used in simulation to test the performance of the closed loop system. Simulation conditions and parameters are the same as in Chapter 5. The simulation results with the linear model and nonlinear models are shown in Figure 6.11 and Figure 6.12 respectively. The roll reduction is 45% with the linear model simulation and also 45% for the nonlinear model simulation. In the simulation with the nonlinear model, the $\mu$ controller is better than the MISO controller in roll reduction and is better than the cascade and parallel controllers in course keeping.

The advantage of $\mu$ controller is that it has excellent robust stability and robust performance. Not only it fits the change in the simulation models from linear to nonlinear without effect to its performance of roll reduction, but also it can handle great variation in model parameters. Figure 6.13 shows the simulation results with a set of parameters of a RPMM container ship model\(^1\). The RPMM model is unable to be controlled by all of the controllers designed by mixed sensitivity method (cascade controller, parallel controller and MISO controller). Even PID controller can not get roll reduction more than 10% for it. However, this model is still controllable by $\mu$ controller and, most of time, has a good roll reduction in the simulation. This is not universality for $\mu$ controller, just a particular case, but we can know how about the robust stability of a $\mu$ controller.

\(\mu\) controller and course keeping.

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\(^1\) The parameters of the RPMM container ship model come directly from towing tank test results with the aid of the Roll Planer Motion Mechanism (RPMM). The RPMM model is straight line unstable because there is a RHP pole in the transfer function from rudder angle $\delta$ to heading angle $\psi$ [Blanke and Jensen, 1997].

The parameters of the ship model used in this thesis throughout the previous chapters are modified according to the identification results from the full scale sea trails, this model does not have any RHP poles.
Figure 6.12 Simulation of $\mu$ controller with nonlinear model

Figure 6.13 Simulation of $\mu$ controller with RPMM nonlinear model
Chapter 7

Conclusions

The development of both control theory and application of computer technique have been very fast during last twenty years. How can the modern control theory be used to solve practical problems is the main concern in the industrial control field, especially for complicated plants on which classical control theory can not work well. A practical and interesting problem considered in this thesis is to realize roll damping and course keeping simultaneously by only using the rudder. The control object is a SIMO system including model uncertainty and disturbance. This control problem is highly sensitive to model uncertainty, so robust controllers were considered in this thesis to deal with it.

In this thesis, the necessity and the general schemes of roll damping are discussed first. Then the development of rudder roll damping recently developed in China and abroad are surveyed, problems for which research should be done are pointed out. The basic concepts of space, norm and robust control have been studied in chapter 2 and chapter 3. Two kinds of $\infty$ controller design methods have been discussed. They are the mixed sensitivity approach ($\gamma$ iteration) and the structured singular value $\mu$ synthesis (D-K iteration).

Derivation of a ship mathematical model was done and the model is simplified from nonlinear to linear so that it can be used in analysis of the characteristics of the closed loop and in the design of the controller. Ship model uncertainty is also discussed. Wave is a main reason causing the ship to roll. ISSC wave model has been described and the relationship between wave and ship roll is discussed in chapter 4.

The RAO for ship roll are concentrated on 0.25~0.5 rad/s of encounter frequency. Unfortunately, in the Bode plot there is a resonant peak of ship transfer function from rudder to roll angle in this frequency range. It will make ship roll seriously. Bearing this situation in mind, four kinds of robust controllers have been designed. All of them can solve this problem effectively.

The first controller is a parallel controller that is a combination of two controllers, heading controller and roll damping controller, both designed by mixed sensitivity approach. The outputs of the two controllers are added together to form a rudder command signal applied to the rudder servo mechanism.

The wave filter is applied to this controller to reduce the rudder motion when the course keeping controller works only.

The second controller is a cascade controller that is consisting of the heading controller and the roll damping controller, both designed by mixed sensitivity approach. The output of the roll damping controller is an input of the course keeping controller. The two controllers are connected in series.

The third controller is a MISO one. Both course keeping and roll damping are designed in an unified framework, mixed sensitivity approach is used too.

In the process of designing the first three controllers, the ship model uncertainty is assumed being unstructured, only the upper bond has been estimated roughly.

The last controller is also a MISO one. It is designed by $\mu$ synthesis where the model uncertainty is considered as structured.

The simulations in a sea-way have been made with these controllers for the linear model and the nonlinear model of the ship.
When the four controllers designed are applied to the nonlinear model, the cascade controller has the best roll reduction performance and the worst course keeping performance, due to the fact that course keeping is not considered when the roll damping controller is designed. And then is the parallel controller. The effectiveness of roll damping and course keeping are nearly the same as the results controlled by the cascade controller. This similarity between the two controllers is because they are designed by same methodology and in the same order: separately, and first, course keeping then roll damping, although the design architectures are different. The MISO controller has the best course keeping and the worst roll damping in the four controllers. This is because two objects for course keeping and roll damping are in competition, they can not be fulfilled equally perfect. The performance in roll reduction of $\mu$ controller is in the middle between the MISO controller and the former two controllers.

The simulations show that all of the four controllers have good robust stability and robust RRD performance when the ship model is changed from linear to nonlinear. The $\mu$ controller has the best robust characteristics with respect to model uncertainty. Even when the ship model change to RPMM model which is unstable in straight line motion, the $\mu$ controller is still working, however, all other controllers fail in the same condition. This is important since the roll damping is highly sensitive to model uncertainty as mentioned in the introduction.

The selection of $W_p$ is very important. A proper $W_p$ may result in the best performance and good robust stability simultaneously. So it must be careful in selection.

A lot of work is still needed to be done for the RRD project after the Ph. D. study. For example:
1. To find the structure of ship model uncertainties. If we know the structure of the model uncertainty of a ship correctly, better robust performance can be achieved because of less conservativity in design. In this thesis, the uncertainty model comes from the naval multi-variable vessel. If the structure uncertainty model of the container ship has been used in this thesis, better control results would have been obtained. The method to obtain the uncertainty structure model may be by the computation of the ship structure parameters and/or the identification from a full scale test.
2. Actual sea-trial. Anything is said good and useful only after it is put into operation practically, it is the case for a new algorithm or a new technique. For the robust RRD controller, it is necessary to make a sea-trial. So the farther work will be done around this.
3. Rudder nonlinear. This is a key point in popularly using of RRD. It includes rudder rate saturation and rudder angle saturation. The new algorithm and new electric circuit will be developed to make the requirement of renewing rudder machine for RRD no longer need.
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Appendix

Notation

$m$  Ship mass [kg]
$u, v$  Surge velocity, sway velocity [m/s]
$u_w, v_w$  Response operator of wave on surge and sway [m/s]
$p, \varphi$  Roll rate [rad/s], roll angle [rad]
$r, \psi$  Yaw rate [rad/s], yaw angle [rad]
$\phi_w, \psi_w$  Response operator of wave on roll and yaw [rad/s]
$\delta$  Rudder angle [rad]
$U, u_0$  Ship actual and nominal speed [m/s]
$\Delta u = \frac{u - u_0}{U}$  Non-dimensional relative surge velocity
$x_G, z_G$  Ship center of gravity coordinates [m]
$I_x, I_z$  Ship moments of inertia in roll and yaw [kg m^2]
$\rho$  Sea water density [kg/m^3]
$g$  Gravity constant [m/s^2]
$\nabla, \nabla_0$  Ship displacement and nominal displacement [m^3]
$GM$  Metacentric height [m]
$h_{1/3}$  Significant wave height [m]
$T_w$  Average wave period [s]
$G_{xz}(\omega, \chi, U)$  Motion response spectrum [deg^2 s] or [(m/s)^3]
$G_{z\zeta}(\omega)$  One sided amplitude spectrum [m^2 s]
$R_{xz}(\omega)$  Response function from $\zeta$ to $x$ [deg/m] or [(m/s)/m]
$\chi$  Encounter angle [rad]
$\zeta_0$  Natural roll damping of ship
$\omega_0$  Natural roll frequency of ship
$\omega$  Angular frequency [rad/s]
$\omega_x, \omega_\theta$  Frequency of encounter and frequency of wave [rad/s]
$G(s)$  Nominal process
$G_p(s)$  Perturbed process
$D(s)$  Scaling used in $D-K$ iteration
$K(s)$  Controller transfer function matrix
$\gamma$  A small constant value used for $\gamma$-iteration
$W_{p1}(s)$  Disturbance weight matrix
$W_{p2}(s)$  Control error weight matrix
$W_{u1}(s)$  Perturbation input weight matrix
$W_{u2}(s)$  Perturbation output weight matrix
$\Delta, \Delta_w(s)$  Perturbation
$\Delta_p(s)$  Performance block
$F_l(P(s), K(s)) = P_{11} + P_{12}K(I - P_{21}K)^{-1}P_{21}$  Lower LFT
$F_u(N(s), \Delta(s)) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$  Upper LFT
$S(s), T(s)$  Sensitivity and complementary sensitivity function
\begin{align*}
W_1(s) & \text{ Output sensitivity weight} \\
W_3(s) & \text{ Output complementary sensitivity weight} \\
W_{rr}(s) & \text{ Output sensitivity weight for roll angle} \\
W_{hh}(s) & \text{ Output sensitivity weight for yaw angle} \\
T_{zw} & \text{ Closed loop system transfer matrix} \\
\det(A) & \text{ Determinant of matrix } A \\
\bar{\sigma}(A) & \text{ Maximum singular value of matrix } A \\
\rho(A) & \text{ The spectral radius of } A \\
\mu(A) & \text{ Structured singular value of } A \\
\Lambda & \text{ Block diagonal perturbation structure used with } \mu \\
\Theta, \Gamma & \text{ Sets of scaling matrices used for } \mu \text{ upper and lower bounds} \\
\varepsilon & \text{ A small constant value in } \beta \\
\beta & \text{ Auxiliary block used for system interconnection structure} \\
\varepsilon & \text{ Control error} \\
W_{\text{del}} & \text{ Uncertainty weight matrix} \\
W_p & \text{ Performance weight matrix} \\
\Delta_{\text{gr}} & \text{ Roll model uncertainty} \\
\Delta_{\text{gy}} & \text{ Yaw model uncertainty} \\
Y_v = \frac{\partial Y}{\partial v} & \text{ Hydro-dynamic coefficient} \\
K_{v\mu} = \frac{\partial^2 K}{\partial v \partial \mu} & \text{ Hydro-dynamic coefficient}
\end{align*}

\textbf{Abbreviation}

\begin{align*}
\text{CG} & \text{ Center of Gravity} \\
\text{NP} & \text{ Nominal performance} \\
\text{NS} & \text{ Nominal stability} \\
\text{RP} & \text{ Robust performance} \\
\text{RS} & \text{ Robust stability} \\
\text{RAO} & \text{ Response amplitude operator} \\
\text{RPMM} & \text{ Roll planar motion mechanism} \\
\text{RRD} & \text{ Rudder roll damping} \\
\text{SIMO} & \text{ Single-input multi-output} \\
\text{MISO} & \text{ Multi-input single-output}
\end{align*}
Acknowledgments

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