

Bifurcations and Chaos in a Pulse Width Modulation Controlled Buck Converter

Łukasz Kocewiak, Claus Leth Bak, Stig Munk-Nielsen
 Institute of Energy Technology, Aalborg University,
 Pontoppidanstræde 101, 9220 Aalborg,
 Denmark

Abstract—Power electronic system with pulse width modulation (PWM) control is studied. Behaviour characteristic for a nonlinear dynamical system is observed and theoretically explained. A DC-DC buck converter controlled by a voltage feedback is taken as an example. The studied system is described by a system of piecewise-smooth nonautonomous differential equations. The research are focused on chaotic oscillations analysis and analytical search for bifurcations dependent on parameter. The most frequent route to chaos by the period doubling is observed in the second order DC-DC buck converter. Other bifurcations as a complex behaviour in power electronic system evidence are also described. In order to verify theoretical study the experimental DC-DC buck converter was build. The results obtained from three sources were presented and compared. A very good agreement between theory and experiment was observed.

I. INTRODUCTION

The dynamics of DC-DC buck system is studied. System of this type has a broad range of application in power control. There are a lot of cases where electrical energy is processed by power electronics before its final consumption. Power electronic technology is increasingly able to found in home and workplace: familiar examples are a domestic light dimmer, a switched mode power supplies of personal computers. Because of common presence in human life, to obtain more completely description of power electronic systems such phenomenons as bifurcations characteristic for nonlinear dynamics and chaos are studied [4].

The presence of switching elements, nonlinear components (e.g. the power diodes) and control methods (e.g. pulse-width modulation) implies that circuits are nonlinear, time varying dynamical systems. Anyone familiar with nonlinear dynamics will appreciate that power converters are difficult to analyse, and show strange and unable to observe basis of linear analysis methods behaviour.

II. STUDIED SYSTEM DESCRIPTION

DC-DC buck power converter is presented as an example [2]. The subject is one of the simplest but very useful power converters, a DC-DC buck converter, a circuit that converts a direct current (DC) input to a DC output. Many switched mode power supplies employ circuits closely related to it.

The experimental example is a second order DC-DC buck converter which output voltage is controlled by

a pulse width modulation (PWM) with a constant frequency, working in continuous conduction mode (CCM). A CCM operation exists when a inductor current is never zero. The switches in mathematical description are assumed to be ideal. In practise it is necessary to regulate low-pass filter output voltage v against changes in a input voltage and a load current, by adding a feedback control loop as in Fig. 1 [11].

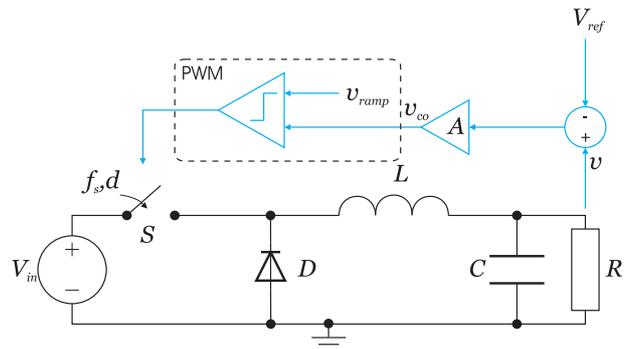


Fig. 1. DC-DC buck converter with feedback control loop.

The switched operating mode of converters implies a multi-topological model in which one particular circuit topology describes the system for a particular interval of time. For constant frequency PWM the operation is cyclic, implying that the topologies repeat themselves periodically. Thus, a natural way to model such kind of operation is to split the system into several subsystems, responsible for describing the system in one sub-interval [6].

The discontinuous conduction mode does not take place in considered buck converter, and can be represented by a piecewise linear vector field. Using the notation $\mathbf{x} = [v, i]^T$, (\mathbf{y}^T donates the transpose of \mathbf{y}) system description looks as follow.

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) \quad (1)$$

$$\mathbf{f}(\mathbf{x}, t) = \begin{bmatrix} -1/(RC) & 1/C \\ -1/L & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ V_{in}/L \end{bmatrix} \mathbf{1}_s(t). \quad (2)$$

where

$$\mathbf{1}_s(t) = \begin{cases} 0 & \text{if } t \notin s \\ 1 & \text{if } t \in s \end{cases} \quad (3)$$

also where v is the capacitor voltage and i is the inductor current and

$$s = \{t \geq 0 : v_{co} < v_{ramp}(t)\}.$$

And the ramp voltage is given by

$$v_{ramp}(t) = V_l + (V_u - V_l)t/T$$

where V_l and V_u are respectively the lower and upper voltages of the ramp and T its period and

$$v_{co}(t) = A(v(t) - V_{ref}). \quad (4)$$

where A is the linear amplifier gain and V_{ref} the reference voltage.

III. METHODS OF THEORETICAL MODEL OF THE BUCK CONVERTER ANALYSIS

The nonlinear phenomena includes bifurcations (sudden changes in system operation), coexisting attractors (alternative stable operating modes), and chaos. If power converter is going to be designed, a knowledge about these issues existence and its investigation methods is desired. There should be emphasised that linear methods applied alone cannot give a wide spectrum of information of nonlinear phenomena and are insufficient in predicting and system analysing.

A. Poincaré map

When state vector evolution is known, there is a possibility to discretise it using mapping. Poincaré map is the most widely used discrete time model for DC-DC converters [5], [6]. This map can be obtained by sampling the system solution every T seconds, at the beginning of each ramp cycle. This nonlinear method of analysis gives a lot of information about system. In contrast to other types of mapping, Poincaré map is able to show aside from bifurcations and chaos also difference between quasiperiodic attractor and strange attractor (chaotic operation).

Consider a continuous-time dynamical system defined by

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \quad (5)$$

Assume, that (5) has a periodic orbit L_0 . Introduce a cross-section Σ to the cycle at point $\mathbf{x}_0 \in L_0$. The cross-section Σ is a smooth hypersurface of dimension $n - 1$, intersecting L_0 at a nonzero angle. The simplest choice of Σ is a hyperplane orthogonal to the cycle L_0 at \mathbf{x}_0 .

Consider now orbits of (5) near the cycle L_0 . The cycle itself is an orbit that starts at a point on Σ and returns to Σ at the same point. An orbit starting at a point $\mathbf{x} \in \Sigma$ sufficiently close to \mathbf{x}_0 also returns to Σ at some point $\tilde{\mathbf{x}} \in \Sigma$ near \mathbf{x}_0 . Moreover, nearby orbits will also intersect Σ transversely. Thus, a map $P : \Sigma \rightarrow \Sigma$,

$$\mathbf{x} \rightarrow \tilde{\mathbf{x}} = P(\mathbf{x}),$$

is constructed. The map P is called a Poincaré map associated with the cycle L_0 . The Poincaré map P is locally defined and smooth.

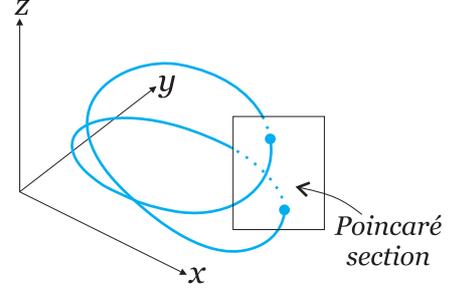


Fig. 2. Poincaré map of limit cycle in the Poincaré plane.

Let introduce local coordinates $\xi = (\xi_1, \xi_2, \dots, \xi_{n-1})$ on Σ such that $\xi = 0$ corresponds to \mathbf{x}_0 . Then the Poincaré map will be characterised by a locally defined map $P : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$, which transforms ξ corresponding to \mathbf{x} into $\tilde{\xi}$ corresponding to $\tilde{\mathbf{x}}$,

$$P(\xi) = \tilde{\xi}$$

The origin $\xi = 0$ of \mathbb{R}^{n-1} is a fixed point of the map $P : P(0) = 0$. The stability of the cycle L_0 is equivalent to the stability of the fixed point $\xi_0 = 0$ of the Poincaré map. Thus, the cycle is stable if all eigenvalues (multipliers) $\mu_1, \mu_2, \dots, \mu_{n-1}$ of the $(n - 1) \times (n - 1)$ Jacobian matrix of P are located inside the unit circle $|\mu| = 1$

B. Bifurcations

In the study of dynamical systems the appearance of a topologically nonequivalent phase portrait under variation of parameters is called a bifurcation. Thus, a bifurcation is a change of the topological type of the system as its parameters pass through a bifurcation value [5].

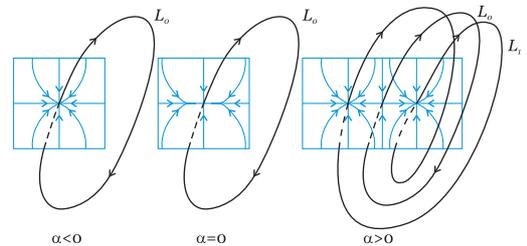


Fig. 3. Flip bifurcation of limit cycle.

If one varies bifurcation parameters the phase portrait may deform slightly without altering its qualitative (i.e., topological) features, or sometimes the dynamics may be modified significantly, producing a qualitative change in the phase portrait.

Consider a smooth continuous-time system that depends smoothly on a parameter:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \alpha), \quad \mathbf{x} \in \mathbb{R}^n, \alpha \in \mathbb{R}^1. \quad (6)$$

Let L_0 be a limit cycle of system (6) at $\alpha = 0$. Let P_α denote the associated Poincaré map for nearby α ; $P_\alpha : \Sigma \rightarrow \Sigma$, where Σ is a local cross-section to L_0 . If some coordinates $\xi = (\xi_1, \xi_2, \dots, \xi_{n-1})$ are introduced on Σ , then $\tilde{\xi} = P_\alpha(\xi)$ can be defined to be the point of the next intersection with Σ of the orbit of (6) having initial point with coordinates ξ on Σ . The intersection of Σ and L_0 gives a fixed point ξ_0 for $P_0 : P_0(\xi_0) = \xi_0$.

If an iterative map is used to model the system, the linearised system needs to be examined. Suppose the iterative map is

$$\mathbf{x} \mapsto \mathbf{f}(\mathbf{x}, \alpha) \quad \mathbf{x} \in \mathbb{R}^n, \alpha \in \mathbb{R}^1,$$

then the Jacobian $J_f = \partial \mathbf{f} / \partial \mathbf{x}^T$ characterising the linearised system is given by evaluated at the fixed point. The eigenvalues of system can be obtained by solving the characteristic equation $\det(\mu \mathbf{1} - J_f) = 0$

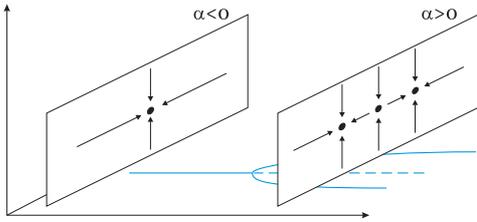


Fig. 4. An attracting fixed point loses stability at $\alpha = 0$ in a period-doubling bifurcation. For $\alpha > 0$ there is a saddle fixed point and a period-two attractor.

If one of the eigenvalues is observed to move out of the unit circle on the real line through the point -1 , then period doubling bifurcation appears. The bifurcation associated with the appearance of $\mu_1 = -1$ is called a period-doubling bifurcation presented schematically in Fig. 4.

Periodic orbits of periods greater than one can appear or disappear because of saddle-node bifurcations (see Fig. 5), and can undergo period doubling bifurcations. This kind of behaviour exists in analysed DC-DC buck converter.

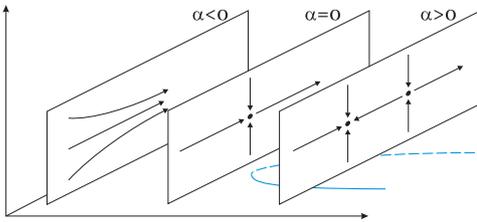


Fig. 5. A fixed point appears at parameter $\alpha = 0$ in a saddle-node bifurcation. For $\alpha > 0$ there is an attracting fixed point and a saddle fixed point.

At a period-doubling bifurcation from a period- k orbit, two branches of period- $2k$ points emanate from a path of period- k points. When the branches split off, the period- k points change stability.

Bifurcation can sometimes be catastrophic. For example, a converter may operate nicely when a certain

parameter is kept below a certain threshold. Beyond this threshold, a chaotic attractor may suddenly take over, with its trajectory extended to a much wider voltage and current ranges causing damage to the devices. Thus, the study of bifurcation in an engineering system is relevant not only to its functionality but also to reliability and safety [10], [12].

IV. COMPUTER SIMULATION

Assuming the notation used previously, the parameters of the circuit are: R , C , and L , the resistance, the capacitance and the inductance of the circuit respectively. V_l and V_u , the lower and upper voltages of the ramp in feedback control loop and T its period, A is the gain of the amplifier and V_{ref} the reference voltage. V_{in} is the input voltage and established as the bifurcation parameter varied in interval [20 V, 35 V]. The buck converter is investigated using the following parameter values: $L=20$ mH, $C=47$ μ F, $R=22$ Ω , $A=8.2$, $V_{ref}=11.3$ V, $V_l=3.8$ V, $V_u=8.2$ V, ramp frequency $f=2.5$ kHz.

A. Theoretical model simulation

One of the routes to chaos observed in studied DC-DC buck converter is by period doubling [9], [10], which continues until there are no further stable states. At the beginning of simulation when input voltage is 20 V circuit exhibits periodic behaviour. During system bifurcation parameter changes, periodic state becomes unstable because of period doubling bifurcation.

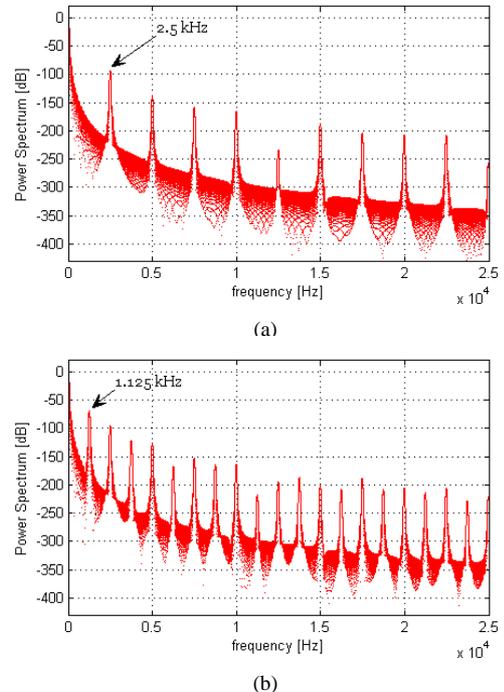


Fig. 6. Spectral analysis of inductor current, (a) 1T-periodic inductor current power spectrum density, (b) 2T-periodic inductor current power spectrum density

In spectral analysis shown in Fig. 6 and Fig. 7 it is observed as second frequency appearing at half the driving frequency [7]. Further increase in input voltage results in splitting of two periods, giving quadrupling

and finally chaos. In periodic system, only one harmonic peak occurs, associated with driving frequency. During bifurcation parameter variations the system changes its behaviour, more peaks occur, associated with harmonics and subharmonics of the system. This is called the period doubling cascade route to chaos [13].

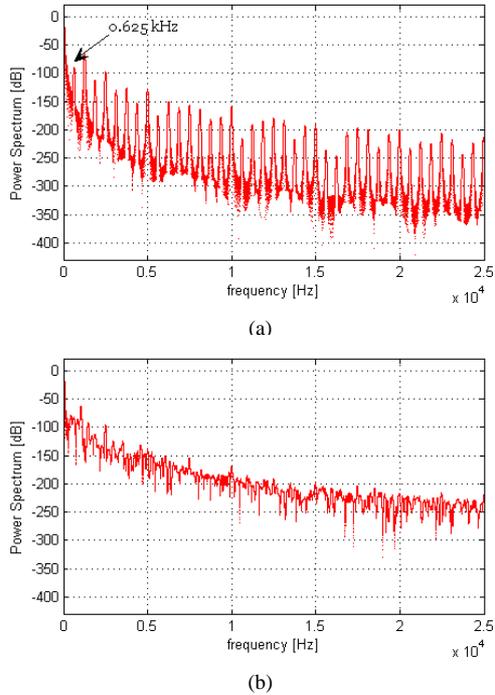


Fig. 7. Spectral analysis, (a) 4T-periodic inductor current power spectrum density, (b) Chaotic inductor current power spectrum density

Because it is easy to vary, the input voltage V_{in} was chosen as the bifurcation parameter. The i_L , v_C and v_{co} were sampled at the start of every ramp cycle and plotted as the bifurcation diagram shown in Fig. 8. A period doubling route to chaos is visible. This process is repeated for every discrete value of the bifurcation parameter in the interval $V_{in} = [20, 35]$ V.

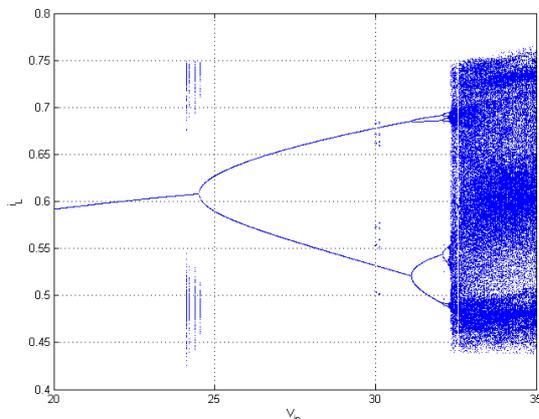


Fig. 8. Bifurcation diagram with V_{in} as the bifurcation parameter

There was calculated that stable 1T-periodic limit cycle is found at the beginning of simulation and continued until some value near 24.5 V. Then, a first period doubling bifurcation occurs, and the stability of the 1T-periodic

orbit is lost in favour of the 2T-periodic orbit. This 2T-periodic orbit also loses stability in a period doubling bifurcation near 31.15 V and 4T-periodic appears. Near the last period doubling bifurcation at approximately 32.4 V, there is a large chaotic behaviour.

Coexisting attractors are also able to detect in studied buck converter. When V_{in} is about 24 V, unstable chaotic orbits coexist with the periodic attractor, giving rise to a long transient chaotic behavior before the converter settles to the stable periodic orbit. A parallel branches of 6T-periodic orbit are detected in a neighbourhood of $V_{in}=30.000$ V after saddle-node bifurcation. This undergoes its own period-doubling cascade which ends in a six-piece chaotic attractor coexisting with the main 2T-periodic stable orbit.

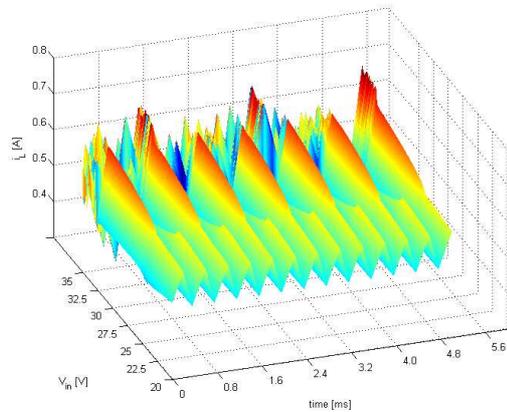


Fig. 9. Inductor current changes during period doubling cascade

Fig. 9 shows inductor current changes during bifurcation parameter variation. It is able to observe how the amplitude of current in inductor can increase from 1T-periodic waveform to chaotic. The maximum value was measured in chaotic operating mode. Strange attractor with non-integer dimension characteristic for chaotic behaviour is shown in Fig. 10.

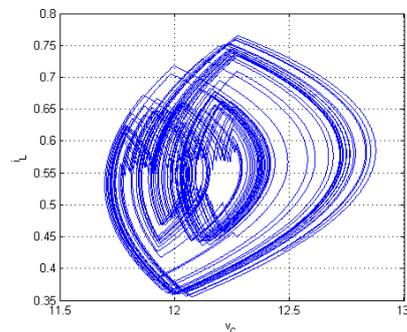


Fig. 10. Strange attractor in DC-DC buck converter, inductor current against capacitor voltage

The Poincaré section diagram come into being as a result of simulated waveforms sampling synchronised with the ramp voltage, one sample of the current and voltage variables at the beginning of the ramp. Then the representation in the state space of the points obtained with this procedure gives us the discrete evolution of the system. Period doubling route to chaos shown in Poincaré section is presented in Fig. 11.

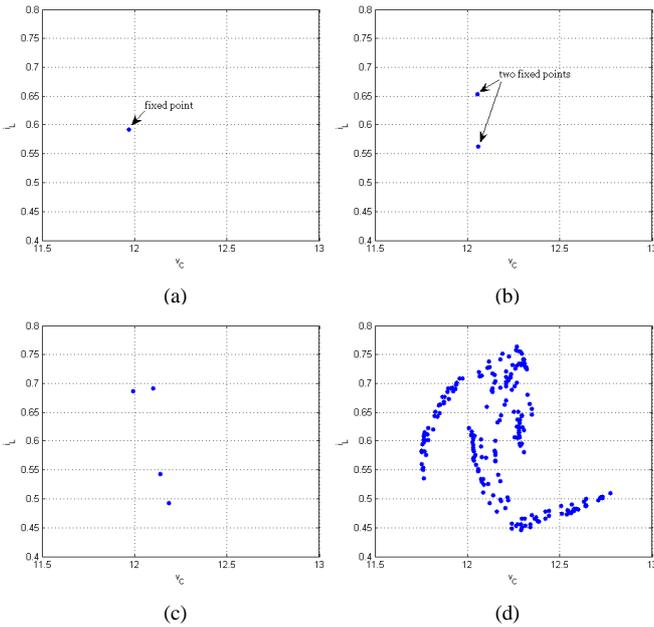


Fig. 11. Poincaré section diagram, (a) 1T-periodic operation, (b) 2T-periodic operation, (c) 4T-periodic operation, (d) Chaotic operation

B. Electrical circuit computer simulation

Before DC-DC buck converter physical realisation the designed circuit was simulated in PSpice. The observation of effects characteristic for nonlinear dynamics like bifurcations and chaos in circuit with voltage feedback was carried out.

In the designed circuit the power circuit uses a power metal-oxide-semiconductor field-effect transistor (MOSFET) IRF9640 and power standard recovery diode 1N4001. The DC input voltage was varied from 20 to 35 V as in mathematical model, control circuit was supplied from a voltage regulator LM7812. The ramp generator, based on a 555 timer, produces a sawtooth waveform. A bandwidth dual operational amplifier TL082 is used as a comparator and a difference amplifier.

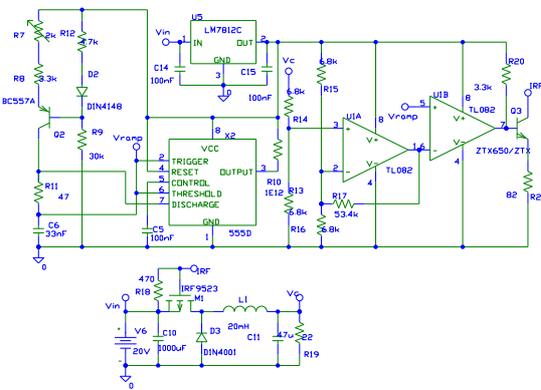


Fig. 12. Circuit diagram of the experimental buck converter.

The suggestion of practical controlled DC-DC buck converter is shown in Fig. 12. The circuit is closely related to those used in switch-mode power supplies. The coil current plotted against the capacitor voltage constitute strange attractor in chaotic operating mode which is shown in Fig. 13.

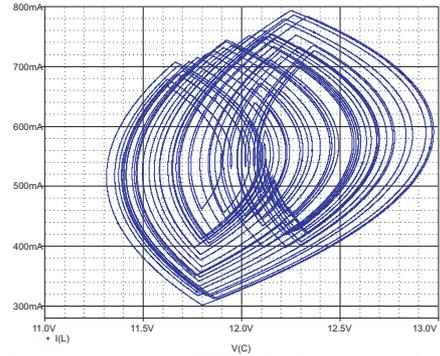


Fig. 13. Strange attractor in DC-DC buck converter obtained from PSpice

V. PRACTICAL VERIFICATION

In order to verify the theoretical model an experimental buck converter was built. The main aim was to make its operation close to ideal piecewise linear model. Built converter was very similar to simulated in PSpice. As distinct from PSpice simulation as the comparator is applied LM311 and as the difference amplifier is used a complementary metal-oxide-semiconductor (CMOS) operational amplifier LMC662.

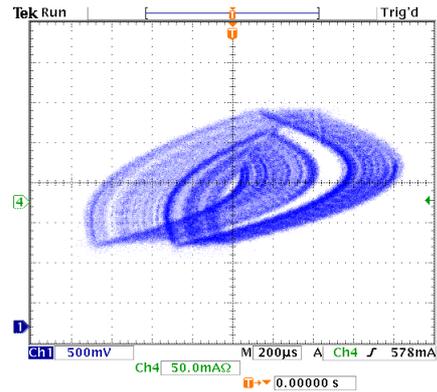


Fig. 14. Strange attractor in DC-DC buck converter measured in a laboratory

A very good premise of chaotic behaviour presence is non-periodic attractor shown in Fig. 14. Likewise in computer simulation there is a possibility to observe chaotic phenomena in laboratory experiment. In Fig. 15 there is a Poincaré section measured in a laboratory similar to obtained from the mathematical model computer simulation in Fig. 11(d).

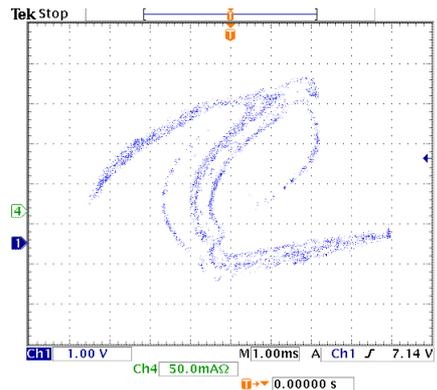


Fig. 15. Poincaré section characteristic for chaotic behaviour

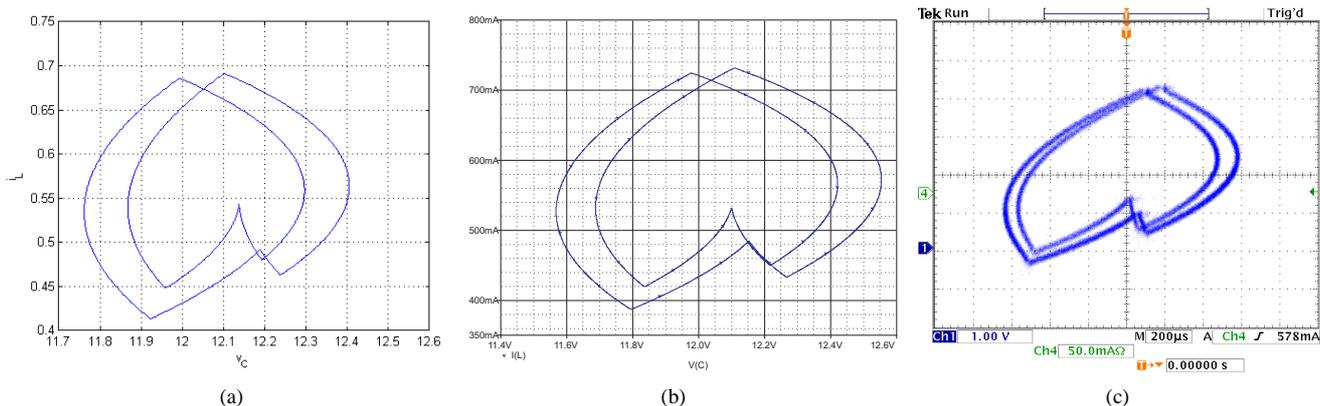


Fig. 16. 4T periodic limit cycle comparison (a) 4T periodic orbit from Matlab (b) 4T periodic orbit from PSpice (c) 4T periodic orbit obtained in a laboratory

VI. COMPARISON OF RESOLUTES

In order to verify obtained results from computer simulations and laboratory experiment there will be a comparison presented. The outcomes of mathematical model from Matlab in comparison with the results from PSpice simulated circuit and the physical laboratory experiment will be shown.

The results were obtained from three sources:

- 1) Mathematical model numerical calculations in Matlab.
- 2) Simulation using PSpice, with consideration of additional effects present in practical realisation.
- 3) Results from the laboratory experiment.

The 4T-orbit was chosen for the sake of sufficient complexity and is presented in Fig.16. Such a presentation enables clear observation of behaviour present in the system. This attractor appears after second flip bifurcation and is a premise of chaotic phenomena existed in the DC-DC second-order buck converter with the voltage control.

VII. CONCLUSION

The main aim was to present nonlinear system analysing methods and its application in power electronics. The DC-DC second-order buck converter with the voltage control was taken as an example. The main objective was to build a converter which is able to work in chaotic operating mode basis of the mathematical model. Simultaneously there were shown analytical methods helpful in detecting, analysing and classifying this kind of nonlinear behaviour.

Additionally it was presented that nonlinear analysis describes analysed system more accurately and explains phenomenons such as subharmonics, bifurcations and chaos, which cannot be detected by using linear approach of analysis. It proves that it could be useful in circuits study, specially in a field where high reliability is essential, like in spacecraft power systems or terrestrial power systems.

The investigation was carried out in three different ways and the results were compared. There were considered three independent cases: the mathematical model simulated in Matlab, the circuit builded from components

exist in reality and simulated in PSpice and the laboratory experiment. All of cases give satisfactory results and they were described in relevant sections. A very good agreement between theory and experiment was reached.

REFERENCES

- [1] L.H. Kocewiak: *Bifurcations and Chaos in Automatic Control Systems*, 2007.
- [2] Zh.T. Zhusubaliyev, E. Soukhoterin, E. Mosekilde: *Quasiperiodicity and Torus Breakdown in a Power Electronic DC-DC Converter*, Mathematics & Computers in Simulation, 73:364-377, February 2007.
- [3] N. Mohan, T.M. Undeland, W.P. Robbins: *Power Electronics: Converters, Applications and Design*, John Wiley & Sons Inc., January 2003.
- [4] F. Angulo, C. Ocampo, G. Olivar, R. Ramos: *Nonlinear and Non-smooth Dynamics in a DC-DC Buck Converter: Two experimental set-ups*, Nonlinear Dynamics, 46:239-257, 2006.
- [5] Y.A. Kuznetsov, *Elements of Applied Bifurcation Theory*, Springer-Verlag, 1998.
- [6] Chi Kong Tse, *Complex Behaviour of Switching Power Converters*, CRC Press, 2000.
- [7] D.G. Manolakis, V.K. Ingle, S.M. Kogon, *Statistical and Adaptive Signal Processing: Spectral Estimation, Signal Modelling, Adaptive Filtering and Array Processing*, McGraw-Hill, 2000.
- [8] F. Angulo, J.E. Burgos, G. Olivar: *Chaos Stabilization with TDAS and FPIC in a Buck Converter Controlled by Lateral PWM and ZAD*, 15th Mediterranean Conference on Control & Automation, July 2007.
- [9] S. Mazumder, M. Alfayyoumami, A.H. Nayfeh, D. Borojovic: *A Theoretical and Experimental Investigation of the Nonlinear Dynamics of DC-DC Converters*, IEEE, 2000.
- [10] H.H.C. Iu, C.K. Tse, O. Dranga: *Comparative Study of Bifurcation in Single and Parallel-Connected Buck Converters Under Current-Mode Control: Disappearance of Period-Doubling*, Circuits Systems Signal Processing, 24:201-219, 2005.
- [11] Z.P. Li, Y.F. Zhou, J.N. Chen: *Complex Intermittency in Voltage-Mode Controlled Buck Converter*, IEEE, 2006.
- [12] G. Olivar, M. di Bernardo, F. Angulo: *Discontinuous Bifurcations in DC-DC Converters*, IEEE, 2003.
- [13] L. Premalatha, P. Vanajaranjan: *Spectral Analysis of DC-DC Buck Converter with Chaotic Dynamics*, IEEE Indicon Conference, December 2005.