Chapter on The history of latin squares

Andersen, Lars Døvling

Publication date:
2007

Document Version
Publisher's PDF, also known as Version of record

AALBORG UNIVERSITY

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The history of latin squares

Intended for publication in The History of Combinatorics

by

Lars Døvling Andersen

R-2007-32 December 2007

DEPARTMENT OF MATHEMATICAL SCIENCES
AALBORG UNIVERSITY
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 96 35 80 80 ▪ Telefax: +45 98 15 81 29
URL: http://www.math.aau.dk

ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

Lars Døvling Andersen

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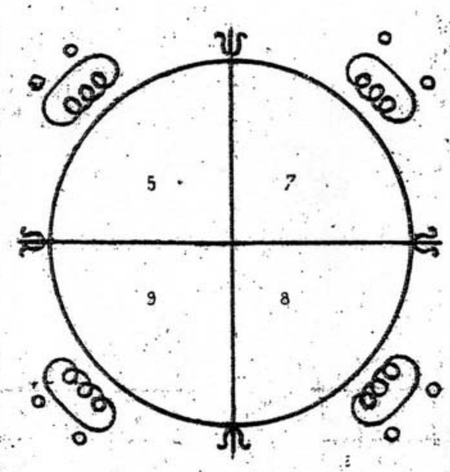
Intended for publication in

The History of Combinatorics

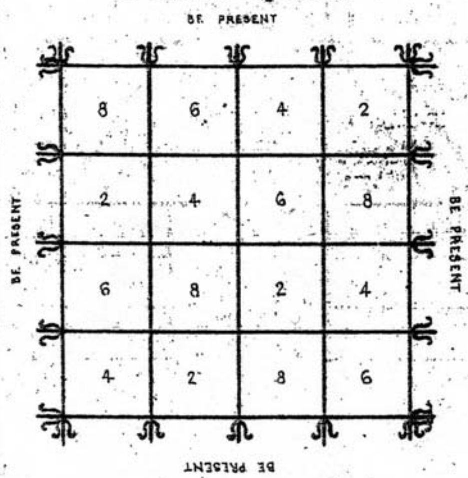
edited by Robin J. Wilson

Preliminary version

A MAGIC CIRCLE. No. 1. (to face Page 219.)



A MAGIC SQUARE.



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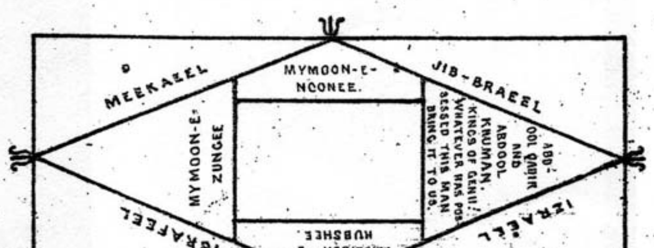


J. Higginbotham, Madras. A. Barren, Lithog

Latin Squares

Abstract

A latin square of order (or side) n is an $n \times n$ array with entries from a set of n symbols arranged in such a way that each symbol occurs exactly once in each row and exactly once in each column.
With this simplistic starting point, the theory of latin squares has developed to become an interesting discipline in its own right as well as a very important tool in design theory in general.

1. Introduction

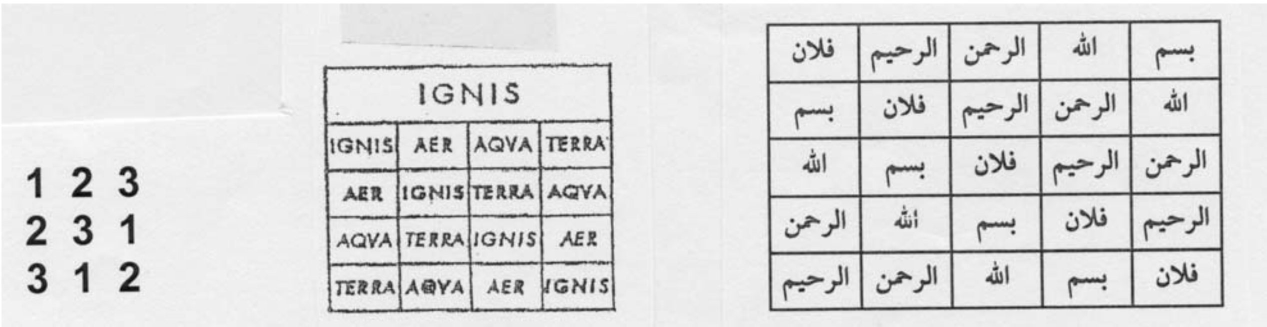


Figure 1. Latin squares of orders 3, 4 and 5[1]

The first known occurrences of latin squares seem to be their use on amulets and in rites in certain Arab and Indian communities from perhaps year 1000. The nature of the sources makes the dating difficult. Most similar amulets contain not latin, but *magic* squares (mathematically a magic square is an $n \times n$ array is filled with symbols 1, 2, …, n^2 – or possibly another sequence of consecutive numbers – such that the sum of the numbers in any row, column or main diagonal is the same), and the latin square amulets, like the magic square ones, were worn to fight evil spirits, show reverence for gods, celebrate the sun and the planets etc. Figure 2 (over page) shows an example. But the use of the squares was not restricted to amulets or talismans. The illustration to the left shows how a latin square – here called a magic square – was considered together with a magic circle to have powers for casting out devils; instructions for their use were given as:

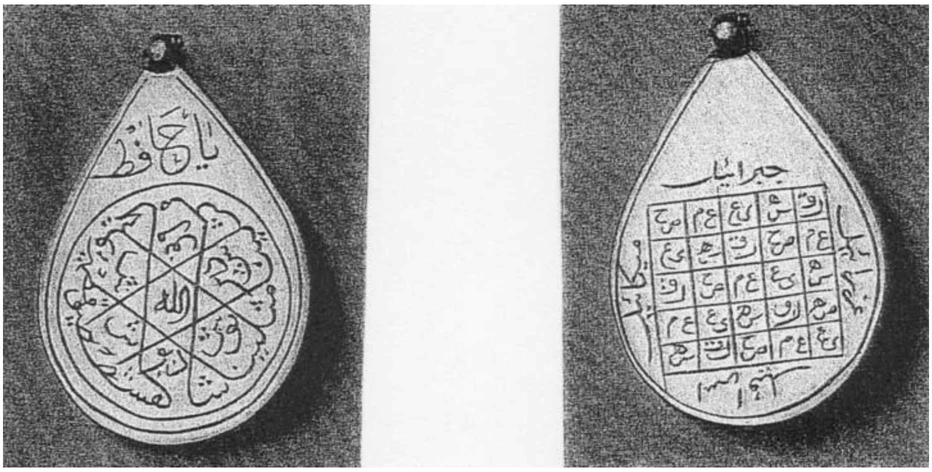


Figure 2. A silver amulet from Damascus. On one side a latin square, on the other the names of the Seven Sleepers who, according to legend, slept in a cave from app. year 250 to year 450[2].

Magic circles, squares, and figures, are sketched in the ground, or on a plank, with various coloured powders, bhubhoot (cowdung ashes), charcoal, or sundul; and the demoniac being seated in the centre of it, the afsoon (incantation) is read. Around these diagrams are placed various kinds of fruits, flowers, pan-sooparee, sheernee, sometimes sayndhee, taree, nariellee (intoxicating liquors), daroo (ardent spirits), &c. Some sacrifice a lamb in front of the circle, &c. sprinkle blood around it, set up the head in front, placing a lamb upon it, lighted up with a puleeta (charm-wick); or they merely slay a fowl, and sprinkle its blood around. Some give a rupee or two, according to their means, into the hands of the person possessed by the devil, to deposit therein[3].

The entries of the latin square, the numbers 2, 4, 6 and 8, were considered all over the Islamic orient to have magic powers when occurring together (in which case they could be associated with an often used magic word). It is significant that they also all occur on both diagonals and in the four corners of the square.

Tracing such occurrences of latin squares in writing back in history leads to a very famous book, *Shams al-Ma'arif al-Kubra*[4] (The sun of great knowledge), which was written by *Ahmad ibn 'Ali ibn Yusuf al-Buni*, an Arab Sufi believed to have died in 1225. This book contains many latin squares (in addition to many more magic squares), including the 5 × 5 square of Figure 1 (the first latin square of the book and thus the oldest known latin square?) and a description of a talisman containing 7 latin squares, each associated with both a weekday and a planet. Figure 3 shows one of these, the square related to Thursday and Jupiter.

حرف الظاء للمشتري وله يوم الخميس

ظ	ث	ج	ف	خ	ش	ظ
ج	ف	خ	ش	ظ	ز	ث
خ	ش	ظ	ز	ث	ج	ف
ظ	ز	ث	ج	ف	خ	ش
ث	ج	ف	خ	ش	ظ	ز
ف	خ	ش	ظ	ز	ث	ج
ش	ظ	ز	ث	ج	ف	خ

Figure 3. One of seven latin squares of a talisman from al-Buni's book. The first entry is in error, and though the structure of all the squares is obvious, only three of them are in fact correct.

The latin squares of Al-Buni seem to serve two types of purposes: first, they have certain magic powers in their own right, some of them, such as that of Figure 3, being related to a specific planet, and secondly, and mathematically very interesting, they seem to be crucial in constructing magic squares. As a matter of fact, there is strong evidence[5] that Al-Buni knew of methods for this that were taken up by mathematicians, among them Leonhard Euler, in the 18th century.

Likewise, a book from 1356 on Indian mathematics[6] contains latin squares with clear focus on their use in certain constructions of magic squares. Thus Hindu mathematics also anticipated what Euler later formalised and developed, as we shall see in the next section. In the next centuries, more texts contained such applications.

In the 13th century, Spanish mystic and philosopher *Ramón Lull* constructed latin squares in his efforts to explain the world by combinatorial means; the middle square of Figure 1 is taken from a larger drawing of Lull's. For more on Lull, including the full picture from which the above square is extracted, and his legacy, we refer to the chapter on Renaissance Combinatorics by Eberhard Knobloch in this volume[*].

[*] *The History of Combinatorics*, for which this preprint is intended as a chapter

An old card problem asks for the 16 court cards in an ordinary pack of playing cards to be arranged in a 4 × 4 array such that each row, each column, and each main diagonal contain an ace, a king, a queen and a jack, all of different suits. Thus both the suits and the values would form a latin square. At least from the early 18th century, this problem was a regular in collections of problems of recreational mathematical nature[7].

AS DE CŒUR	ROI DE TRÈFLE	DAME DE CARREAU	VALET DE PIQUE
VALET DE CARREAU	DAME DE PIQUE	ROI DE CŒUR	AS DE TRÈFLE
ROI DE PIQUE	AS DE CARREAU	VALET DE TRÈFLE	DAME DE CŒUR
DAME DE TRÈFLE	VALET DE CŒUR	AS DE PIQUE	ROI DE CARREAU

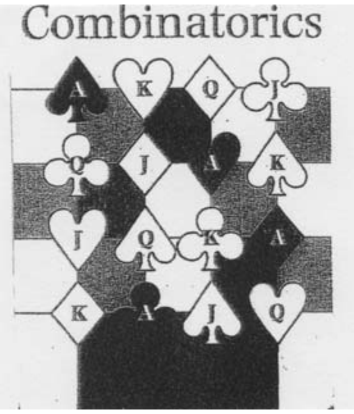


Figure 4. Two solutions to the card puzzle.

Like much of design theory, latin squares have applications in statistics, in experimental design. The earliest known example is by the French agricultural researcher *Francois Cretté de Palluel* who on 31 July 1788 presented a paper to the Royal Agricultural Society of Paris[8]. Its purpose was to show that one might just as well feed sheep on root vegetables during winter. This was much cheaper - and easier - than the normal diet of corn and hay. He described an experiment of feeding 16 sheep with different diets and comparing their weight gains

Memoires d'Agriculture, d'Economie rurale & Domestique, publiés par la Société Royale d'Agriculture à Paris, Année 1788. Trimestre d'été.

ON THE ADVANTAGE AND ECONOMY OF FEEDING SHEEP IN THE HOUSE WITH ROOTS.

By M. Cretté de Palluel, read July 31, 1788.

Figure 5. Heading of the English translation of Cretté de Palluel's paper.

Although there is no latin square in the published paper, the lay-out of his 16 sheep experiment amounted to a 4 × 4 latin square with four breeds of sheep as rows, four different diets as columns, and four different slaughtering times as symbols! His design is shown below.

EXPERIMENT UPON FATTING SHEEP, AND THEIR INCREASE FROM MONTH TO MONTH.

Sixteen ſheep, of the ſame age, of four different breeds, were picked out of my flock, viz. four the breed of the country, four of Beauce, four of Champagne, and four of Picardy; I weighed them alive, and marked each with a number; I divided them into four lots, and fed them on four different ſorts of food, as under:

Food.	No.	Breeds.	Weights at different Periods.—1788.					Increaſe each Month.				Total incr. which each food has produced upon four Sheep.
			Jan. 20.	Feb. 20.	Mar. 20.	April 20.	May 20.	1ſt M.	2d M.	3d M.	4th M.	
Potatoes,	1	Iſle de France,	69¾ lb.	79½ lb.	—	—	—	10 lb.	lb.	lb.	lb.	70 lb.
	2	Beauce,	70½	82½	90½ lb.	93 lb.	95 lb.	11½	7½	2¾	2	
	3	Champagne,	69¼	83	82½	84	—	13¾	loſs ½	1½	—	
	4	Picardy,	88	95	101	—	—	15	6	—	—	
								50½	13½	4¼	2	
Turnips,	5	Iſle de France,	69	86	87	—	—	17	1	—	—	67½
	6	Beauce,	71	86	—	—	—	15	—	—	—	
	7	Champagne,	68½	78½	82½	84	84½	10	4	1½	½	
	8	Picardy,	79	95½	97½	97½	—	16½	2	—	—	
								58½	7	1½	½	
Beets,	9	Iſle de France,	72	83¼	90½	94	—	11¼	7¼	3½	—	71
	10	Beauce,	70½	80½	86	—	—	10	5½	—	—	
	11	Champagne,	77¼	90½	—	—	—	13¼	—	—	—	
	12	Picardy,	80	93½	98½	100½	101	13½	5	1½	½	
								48	17½	5	½	
Oats, Barley, and grey peas.	13	Iſle de France,	74	91	95½	102	106	17	4½	6½	4	92½
	14	Beauce,	73½	84¼	91½	96	—	10¼	7¼	4½	—	
	15	Champagne,	71	86¼	93	—	—	15¼	6¾	—	—	
	16	Picardy,	71	87	—	—	—	16	—	—	—	
								59	18½	11	4	

OBSERVATION. The increaſe of theſe ſheep, during the firſt month, being ſo much more conſiderable than in the following months, muſt be attributed to this cauſe, that lean cattle put up to fatten, eat greedily until they are cloyed, which only fills them, without much increaſing their fleſh; but, on the contrary, the increaſe produced in the enſuing months, although apparently leſs, turns all to profit in fleſh and tallow.

Figure 6. The experimental design from Palluel's 1788 paper.

Around the same time, latin squares were introduced to the mathematical community by *Leonhard Euler.*

2. Euler and latin squares: orthogonality and transversals

Latin squares are among the mathematical concepts attributed to Euler: he gave them their name, and he seems to be the first to define them using mathematical terminology, and the first to investigate their properties mathematically. They had been known and used by Euler a little earlier, but he first published them in the paper beginning with the famous 36 officers problem (presented to the Academy of Sciences in St. Petersburg in 1779, published in 1782), and then he immediately launched a more complicated concept: *the orthogonality of Latin squares*.

1. Une question fort curieuse, qui a exercé pendant quelque temps la sagacité de bien du monde, m'a engagé à faire les recherches suivantes, qui semblent ouvrir une nouvelle carrière dans l'Analyse et en particulier dans la doctrine des combinaisons. Cette question rouloit sur une assemblée de 36 officiers, de six différens grades et tirés de six régimens différens, qu'il s'agissoit de ranger dans un quarré de manière que sur chaque ligne, tant horizontale que verticale, il se trouvât six officiers tant de différens caractères que de régimens différens. Or, après toutes les peines qu'on s'est données pour résoudre ce problème, on a été obligé de reconnoître qu'un tel arrangement est absolument impossible, quoiqu'on ne puisse pas en donner de démonstration rigoureuse.

Figure 7. The introductory question in Euler's paper.

Thus Euler explained that the contents of the paper are inspired by a strange question about a collection of 36 officers, of six different grades and picked from six different regiments, who should line up in a square in such a way that in each line, both horizontal and vertical, was six officers both of different ranks and from different regiments. He added that he had had to realise that such an arrangement is impossible, although he could not prove it.

Euler clarified the Officers problem by denoting the regiments by Latin letters *a, b, c, d, e, f* and the ranks by Greek letters $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$. He explained that the task is to arrange the 36 pairs of a Latin and a Greek letter in a 6×6 array so that each row and column contains each Latin and Greek letter just once. He immediately stated that since he had not been able to solve this problem, he would generalise it to pairs from *n* Latin and *n* Greek letters for arbitrary *n*. He then worked with numbers 1, 2, …, *n* instead of both Latin and Greek letters and introduced the concept of a latin square. The first latin square presented employing this terminology is that of Figure 8.

1^1	2^6	3^4	4^3	5^7	6^5	7^2
2^2	3^7	1^5	5^4	4^1	7^6	6^3
3^3	6^1	5^6	7^5	1^2	4^7	2^4
4^4	5^2	6^7	1^6	7^3	2^1	3^5
5^5	1^3	7^1	2^7	6^4	3^2	4^6
6^6	7^4	4^2	3^1	2^5	5^3	1^7
7^7	4^5	2^3	6^2	3^6	1^4	5^1

1	2	3	4	5	6	7
2	3	1	5	4	7	6
3	6	5	7	1	4	2
4	5	6	1	7	2	3
5	1	7	2	6	3	4
6	7	4	3	2	5	1
7	4	2	6	3	1	5

Figure 8. The first latin square to go by this name. To the left is a square of pairs such that each number between 1 and 7 occurs exactly once in each row and in each column, both in line and as exponent, and such that each ordered pair occurs exactly once altogether. As a reference to the earlier usage of Latin and Greek letters, Euler called the in line numbers *Latin numbers* and the exponents *Greek numbers*. He then dropped the exponents obtaining the *latin* square to the right.

It is called a latin square because its numbers could be Latin letters in what might have a counterpart with Greek letters, satisfying the all pairs property!

Two latin squares of the same order are *orthogonal* if they have the property that whenever two places have the same entry in one of the squares, then they have distinct entries in the other. In other words, if the two squares are superimposed, then the n^2 cells contain each pair consisting of a symbol from the first square and a symbol from the second. Thus the Officers problem simply asks for two orthogonal latin squares of order 6. And the square to the left in Figure 8 shows two orthogonal latin squares of order 7. So Euler could solve the corresponding 49 officers problem.

In the same way, ignoring the condition on the diagonals, the card game of Figure 4 asks for two orthogonal latin squares of order 4, and the figure shows that such exist.

Euler's 1782 paper was called *Recherches sur une nouvelle espèce de quarrés magiques*, and it does contain a few references to magic squares, showing how orthogonal latin squares can be used for constructing these. However, there is more evidence that Euler came to consider latin squares through an interest in magic squares. These were well known at the time, and indeed he seems to have worked on them at a very early age and returned to them 50 years later. In his mathematical notebooks there is a brief piece on magic squares believed to be from 1726, and in 1776 he presented a long paper on the topic to the Academy of Sciences of St. Petersburg, both items published posthumously, both entitled *De quadratis magicis* (and both written in Latin). We find the evidence in the latter 1776 paper.

In this paper from 1776, the first construction is of a 3×3 magic square. Euler explained that he needed Latin letters a, b and c, and Greek letters α, β and γ, to be given numerical values. He then displayed the square

$$\begin{matrix} a & b & c \\ b & c & a \\ c & a & b \end{matrix}$$

with Latin letters and noted that since each row and column has all three letters occurring, and by the appearance of the diagonals, this square will have constant row, column and diagonal sums if and only *if* $a + b + c = 3c$, that is, if and only if $2c = a + b$. He wanted to obtain his magic square by adjoining to the above square one having Greek letters:

$$\begin{matrix} \gamma & \beta & \alpha \\ \alpha & \gamma & \beta \\ \beta & \alpha & \gamma \end{matrix}$$

So the magic square is:

$$\begin{matrix} a+\gamma & b+\beta & c+\alpha \\ b+\alpha & c+\gamma & b+\beta \\ c+\beta & a+\alpha & b+\gamma \end{matrix}$$

This method requires that the same two letters are not added twice, so the square obtained by combining the first two squares, also displayed by Euler,

$$\begin{matrix} a\gamma & b\beta & c\alpha \\ b\alpha & c\gamma & b\beta \\ c\beta & a\alpha & b\gamma \end{matrix}$$

must have the property that all nine pairs of letters occurring are distinct. Letting both a, c, b and α, γ, β be arithmetic progressions, one with difference 1 and the other with difference 3 (the size of the square), we readily produce a magic square. Euler put $(a, c, b) = (0, 3, 6)$ and $(\alpha, \gamma, \beta) = (1, 2, 3)$ and obtained the magic square

$$\begin{array}{ccc} 2 & 9 & 4 \\ 7 & 5 & 3 \\ 6 & 1 & 8 \end{array}$$

What Euler has done here, without using this terminology, is to find two *orthogonal* 3×3 Latin squares, and use them for constructing a 3×3 magic square.

Neither in the 1776 paper nor in the paper published in 1782 did Euler use the phrase 'orthogonal latin squares'. He never used the term *graeco-latin square* for a pair of orthogonal Latin squares either, a term later used quite frequently and obviously derived from his work. And certainly not the name *Euler square* or *Eulerian square*, also used later.

Euler then discussed the same trick for 4×4 squares. Here he was able to find two orthogonal Latin squares, each with the property that each diagonal also contains all symbols. The combined square that he displayed is

$$\begin{array}{cccc} a\alpha & b\delta & c\beta & d\gamma \\ d\beta & c\gamma & b\alpha & a\delta \\ b\gamma & a\beta & d\delta & c\alpha \\ c\delta & d\alpha & a\gamma & b\beta \end{array}$$

This has the sum property for a magic square, no matter what values are given to the letters. Euler noted that if a, b, c, d are given values 0, 4, 8, 12 in any order, and $\alpha, \beta, \gamma, \delta$ are assigned 1, 2, 3, 4 in any order, then all integers from 1 to 16 are obtained as pair sums, and a magic square is obtained. Thus he was able to obtain $4! \times 4! = 576$ distinct magic squares of side 4 in this way (in fact other assignments are possible, giving further magic squares). Euler also found orthogonal latin squares of side 5, each with distinct symbols on both diagonals, immediately obtaining 14400 distinct magic squares of side 5.

It is time to return to the old ways of making magic squares by means of latin squares mentioned in the introduction. The Hindu method for combining preliminary squares can be described in the following way[9] (Figure 9 illustrates it for 5×5 squares): Take a cyclic latin square *of odd order*, usually with the middle column having the numbers in natural order; place next to it a copy where all entries are multiplied by the order of the square; flip the second square on top of the first, as if closing a book, and add the numbers in corresponding cells; the result is a magic square on numbers beginning with the order of the square plus one, and subtracting the order from each entry gives an ordinary magic square.

4	5	1	2	3		20	25	5	10	15		19	15	6	27	23		14	10	1	22	18
5	1	2	3	4		25	5	10	15	20		25	16	12	8	29		20	11	7	3	24
1	2	3	4	5		5	10	15	20	25		26	22	18	14	10		21	17	13	9	5
2	3	4	5	1		10	15	20	25	5		7	28	24	20	11		2	23	19	15	6
3	4	5	1	2		15	20	25	5	10		13	9	30	21	17		8	4	25	16	12

Figure 9. The Hindu method of creating a magic square of odd order from a cyclic latin square

This construction works because the latin square obtained from a cyclic one of odd order by interchanging left and right is orthogonal to the original. Having obtained two orthogonal latin squares in this way, the method is exactly the same as that of Euler.

Similarly, an analysis of the magic squares in al-Buni's book from near year 1200 shows that these could well be formed by a similar method and thus in reality be based on orthogonal latin squares. As mentioned earlier, the method was also taken up in later writings about magic squares before Euler, and books by La Loubère (1691) and Poignard (1704), and a paper by La Hire (1705) led to Joseph Sauveur publishing, in 1710, a text[10] containing many latin squares, as he had had the same idea as Euler for making magic squares. Sauveur used capital Latin letters and small Latin letters where Euler later used Latin and Greek letters. Our Figure 11 in the next section shows an illustration from Sauveur's paper. So while Sauveur discussed orthogonal latin squares in an abstract way two thirds of a century before Euler, he did not extract the idea of a single latin square; nowhere in his paper is an ordinary latin square, with a single symbol in each cell, displayed.

Coming back to Euler, in the final section of the 1776 paper he considered magic squares of order 6, but for the first time there is no pair of orthogonal latin squares displayed! Instead, he presented a similar construction (which he also introduced for side 4) where the entries are still sums of a Latin and a Greek letter, but where the squares formed by each type individually are no longer latin squares (in fact there is an error in his construction). Such methods, for constructing magic squares by adding the entries of two auxiliary squares that are not both latin squares, were also known by the earlier authors mentioned above. However, it seems obvious that Euler must have tried to find a pair of orthogonal latin squares of side 6, but failed.

Certainly, in 1779 he publicly announced such considerations in his lecture on March 8 to the Academy, in the form of the Officers problem. In the corresponding 1782 paper he proceeded to consider orthogonal latin squares in general. In discussing these, Euler defined another important concept for latin squares: that of a *transversal*, in Euler's paper *une formule directrice*. A transversal in a latin square of side n is a set of n distinct entries occurring in distinct rows and distinct columns. He explained that for a latin square of side n to have a latin square orthogonal to it (in modern terminology an *orthogonal mate*), it must have n mutually disjoint transversals, each corresponding to the n occurrences of a particular symbol in the other square. Euler actually stated that the search for transversals was the main object of the paper, but added that he had no method for finding them.

First, he looked for transversals in cyclic latin squares, and he proved that for even n, the cyclic latin square of side n has no transversal. More precisely, he proved only that there is no transversal containing the first entry of the cyclic square, but this generalises easily. For the proof, he assumed that there is such a transversal, and that its entries are 1 in the first column, a in the second, b in the third and so on. If these occur, say, in rows 1, α, β, etc., then, by the definition of the cyclic square, $a = \alpha + 1$, $b = \beta + 2$, $c = \gamma + 3$, etc., calculating modulo n. (Note the ease with which Euler let the distinction between Latin and Greek letters serve a completely different purpose from that at the outset of the paper). Since $\{a, b, c, \ldots\} = \{\alpha, \beta, \gamma, \ldots\} = \{2, 3, \ldots, n\}$, he obtained, still modulo n:

$$S := a + b + \ldots = \alpha + 1 + \beta + 2 + \ldots = S + \tfrac{1}{2} n (n-1),$$

which is true if and only if n is odd. A consequence is that cyclic latin squares of even order do not have orthogonal mates.

For odd n, the main diagonal is a transversal, and by the cyclic property it is easy to find a set of n disjoint transversals. Thus, *cyclic latin squares of odd order have orthogonal mates*, and consequently *orthogonal latin squares exist of all odd orders*. Euler noted this, and also gave rules for finding many other transversals from a given one.

In the next three sections of this long paper, Euler investigated the existence of transversals in three other types of latin squares: those built cyclically from cyclic 2×2, 3×3 and 4×4 squares, respectively. Along the way, he found new examples of latin squares without transversals. He also proved, during his investigation of the squares of the first kind, that *there are orthogonal latin squares of all sides n divisible by 4.*

The orthogonal latin squares of odd order based on the diagonal transversals were exactly those important for the Hindu method of Figure 9, and the Indians were also aware of the existence of orthogonal latin squares of orders divisible by 4.

In conclusion, Euler offered the now famous *Euler conjecture* on orthogonal latin squares: *For n congruent to 2 (modulo 4), there is no pair of orthogonal Latin squares of side n.*

Euler knew this to be true for $n = 2$. As we shall see in the next section, it took more than a century before it was proved true for $n = 6$ by Tarry, and almost two centuries before Bose, Parker and Shrikhande proved that it is false for all other values of n.

The final pages of this remarkable paper contained some pointers to future developments. Euler observed that a pair of orthogonal latin squares of side n can be described by a list of n^2 quadruples, each consisting of a row number, a column number, the number in this position in the first square, and the number in the same position in the second square; this anticipated the later notion of an *orthogonal array.* Euler realised that the meaning of the positions in the quadruples is interchangeable, so that given one pair of orthogonal latin squares of a given side, there could be 24 such pairs – though he knew that these are not all distinct. He also stated that he considered the problem of enumerating latin squares very important, but also very difficult.

As can be seen from the above, magic squares do not play a very prominent role in the 1782 paper, and yet Euler chose to put the concept in the title. Whatever the reason for this – to attract readers, perhaps, because magic squares were well known – it seems clear that the then unpublished paper from the 1776 lecture contains important preliminary work. It was, however, the concept of orthogonality itself that was to prove fruitful for future combinatorics.

3. Mutually orthogonal latin squares

Euler's Officers problem and his more general conjecture remained unsolved for a very long time. It seems to have been well known, at least late in the 19th century, where several papers on the topic appeared. Euler's paper was reissued in 1849, and this may have helped to arouse interest. In 1900, Tarry[11] proved the truth of the conjecture in the officers case, $n = 6$ – the earliest preserved proof known, although possibly not the first to be conceived. (Among some famous mathematicians publishing false proofs was Julius Petersen - his paper appeared after Tarry's proof.) Tarry partitioned the 6×6 Latin squares into 17 classes and did a case-by-case analysis of these. Remarkably, there is a letter[12] from 1842 from Heinrich Schumacher to Carl Friedrich Gauss, where Schumacher informed Gauss that the astronomer Thomas Clausen had solved the question by reducing it to 17 cases, but there is no known written work from Clausen on this. Schumacher and Gauss discussed Euler's full conjecture without mentioning Euler; Scumacher wrote "Clausen vermuthet, dass es für jede Zahl von der Form $4n + 2$ unmöglich sei …" Since Tarry's case-by-case proof people have kept looking for simpler proofs of the officers case, and a number of such have been found. Notable examples are Fisher and Yates in 1934, Bruck and Ryser in 1949, Betten in 1983, Stinson in 1984, and Dougherty in 1994.

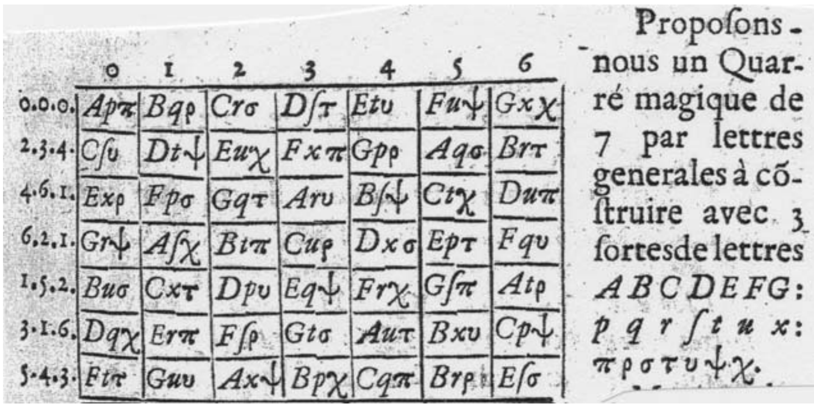
More than two latin squares of the same order may be mutually orthogonal. Mutually orthogonal latin squares are now referred to as MOLS. Before further progress on the Euler conjecture was made, MacNeish[13] in 1922 mentioned a possible generalisation of Euler's conjecture to larger sets of MOLS.

$$
\begin{array}{cccc} 1&2&3&4\\ 2&1&4&3\\ 3&4&1&2\\ 4&3&2&1 \end{array}
\qquad
\begin{array}{cccc} 1&2&3&4\\ 3&4&1&2\\ 4&3&2&1\\ 2&1&4&3 \end{array}
\qquad
\begin{array}{cccc} 1&2&3&4\\ 4&3&2&1\\ 2&1&4&3\\ 3&4&1&2 \end{array}
$$

Figure 10. 3 MOLS of order 4

For any positive integer *n,* let N(*n*) denote the maximum number of MOLS of order *n*. It is not hard to see that N(*n*) ≤ *n* – 1 for all *n* > 1 (as in Figure 10, all first rows can be assumed to be 1, 2, …, *n,* and then all squares must have different symbols in the first cell of the second row, and these cannot be 1).

Already in his 1710 paper, Sauveur had published three MOLS of order 7, without using this terminology. The relevant extract from his paper is shown in Figure 11.



	0	1	2	3	4	5	6
0.0.0.	Apπ	Bqρ	Crσ	Dſτ	Etυ	Fuψ	Gxχ
2.3.4.	Cſυ	Dtψ	Euχ	Fxπ	Gpρ	Aqσ	Brτ
4.6.1.	Exρ	Fpσ	Gqτ	Arυ	Bſψ	Ctχ	Duπ
6.2.1.	Grψ	Aſχ	Btπ	Cuρ	Dxσ	Epτ	Fqυ
1.5.2.	Buσ	Cxτ	Dpυ	Eqψ	Frχ	Gſπ	Atρ
3.1.6.	Dqχ	Erπ	Fſρ	Gtσ	Auτ	Bxυ	Cpψ
5.4.3.	Ftπ	Guυ	Axψ	Bpχ	Cqπ	Brρ	Eſσ

Propoſons-nous un Quarré magique de 7 par lettres generales à cõſtruire avec 3 ſortesde lettres ABCDEFG: p q r ſ t u x: π ρ σ τ υ ψ χ.

Figure 11. Three mutually orthogonal latin squares of order 7, from Joseph Sauveur 1710.

MacNeish defined an *Euler square* of order *n* and degree *k* as a square array of n^2 *k*-tuples with properties corresponding to the coordinates defining *k* MOLS; he also said that such an Euler square has *index* (*n,k*). Thus Figure 11 shows an Euler square of index (7,3). He required *k* < *n,* so he was obviously aware of this necessary condition. The first of the two basic results stated by MacNeish is that if *q* is a prime power, then there exists an Euler square of index (*q,q* – 1) (that is, there are *q* – 1 MOLS of order *q*). The second is, in present day language, that there is a direct product construction showing that *k* MOLS of order *m* and *k* MOLS of order *n* can be combined to give *k* MOLS of order *mn*. He mentioned that this construction is an extension of a method used by Tarry for index 2, which is similar to 'the method for combining two magic squares'. MacNeish's arguments were not always rigid (his paper also included an erroneous proof of the Euler conjecture), and, as Parker later wrote, the construction was 'put on an algebraic foundation by H B. Mann' (although Moore had done it before, and Bose did it before Mann). Still, MacNeish is usually credited with the following theorem, which follows from the above statements:

If the prime factorization of n is $p_1^{r_1}p_2^{r_2}\ldots p_t^{r_t}$ then $N(n) \geq \min(p_1^{r_1}, p_2^{r_2}, \ldots, p_t^{r_t}) - 1$.

MacNeish mentioned the possibility that the bound in this statement is actually the true value of N(n). For $n \equiv 2$ (modulo 4) one of the prime powers in the product is 2, so the suggested value of N(n) is 1, in agreement with Euler's conjecture. He did not explicitly state this as a conjecture in the paper, but wrote that the proof of this 'is a generalisation of the Euler problem of the 36 officers which has not been proved'.

For a third of a century both Euler's conjecture and MacNeish's generalisation of it were open, and there were even published results lending support to both conjectures. But over the course of less than a year, in 1958-59, both conjectures fell. First Parker discovered four MOLS of order 21, thereby disproving MacNeish's conjecture (his construction yielded other counterexamples as well). Then Bose and Shrikhande constructed two MOLS of order 22 and five MOLS of order 50, and after that Parker two MOLS of order 10, and the fact that there are infinitely numbers n for which Euler's conjecture does not hold was contained in these papers. Finally, Bose, Parker and Shrikhande[14] disproved the conjecture for all $n > 6$.

By then, the problem was so famous that its solution was announced on the front page of the *New York Times.* This happened on April 26, 1959. In our last section there is an artistic representation of two MOLS of order 10 (Figure 13).

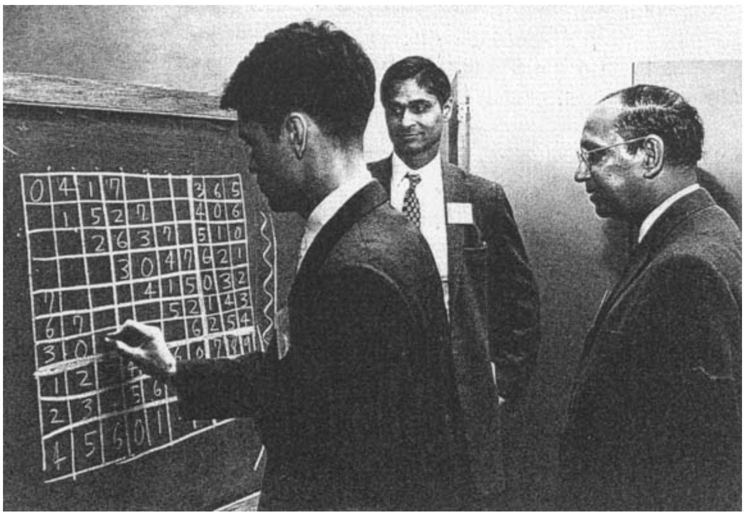


Figure12. From the *New York Times*, April 26, 1959. Illustration to the news story: "Major Mathematical Conjecture Propounded 177 Years Ago Is Disproved"

The methods used by these so-called Euler's spoilers (a phrase coined by Martin Gardner) had the general feature that is so often useful: that of creating new designs from old in clever ways.

In 1960, the *Canadian Journal of Mathematics* contained several contributions to the theory of MOLS; four papers in a row deal with the subject. In particular, the third of these was the joint paper by Bose, Shrikhande and Parker where they showed the falsity of Euler's conjecture for all $n > 6$. And the fourth paper was quite remarkable as well: Chowla, Erdös and Straus showed that N(n) tends to infinity as n tends to infinity. This disproved Euler's conjecture with a vengeance, yet it was based only on number theory, the MacNeish results and a single recursive construction due to Bose and Shrikhande. Since then, much more has been revealed about the function N(n)[15].

A simple direct construction for a pair of orthogonal latin squares of order $3k+1$ (such as 10, 22, …) was given by Menon in 1961. This was basically a reformulation of a construction in the Bose, Shrikhande and Parker paper, but kept in very simple terms.

It is interesting to note that some of the results that could be used for the disproof of the Euler and MacNeish conjectures had been available for a very long time, as described in the next section.

4. MOLS and projective planes

In 1896, *E. H. Moore* published[16] the paper "Tactical Memoranda I-III". This contained several results about MOLS, although they were not formulated in this language. Hence these results were sometimes rediscovered by later authors unaware of their existence or the scope of their contents. One such result was that if n is a prime power then there exists a set of $n - 1$ MOLS of order n. This precedes MacNeish's publication of the same result by 26 years. The product construction used by MacNeish was also in Moore's paper. Curiously, at the end of MacNeish's paper, there is a reference to Moore's paper as a place to find information about the application of Euler squares to contests between k teams of n members each. But there are no references to Moore in the part of the paper where MacNeish presented, in the language of Euler squares, the results also found in Moore's paper with a different terminology! As a matter of fact, the proofs and constructions of Moore were clearer than those in MacNeish's paper. For a long while, however, it was MacNeish that was referred to in later papers. Such *complete sets* (N(n) attaining its maximum $n-1$) of MOLS are particularly important as they are equivalent to *projective planes* of order n.

A finite projective plane is a geometry of a finite number of points and lines, with the property that each pair of points is on exactly one line, and each pair of lines meet in exactly one point (hence *projective* – even parallel lines meet, if projected into infinity); furthermore, there must be four points for which no three are on a line. It can be shown that in a finite projective plane, each line contains the same number k of points; such a plane is said to be of *order* $k-1$. A projective plane of order n is known to exist when n is a prime power. This can be proved using finite fields. Moore used finite fields in his construction of complete sets of MOLS, but projective planes over finite fields were first treated by von Staudt in 1856. Already in 1850, Kirkman had proved the existence of $(r^2+r+1,r+1,1)$-designs for primes r (and r equal to 4 or 8); these are projective planes of order r.

In 1936, in the second of two papers introducing balanced incomplete block designs[17], Yates explained how to obtain affine and projective planes from complete sets of MOLS – which he referred to as '*completely orthogonalized Latin squares* (hyper-Graeco-Latin squares)'; Yates noted that these exist for prime orders and for orders 4, 8 and 9 'but higher non-primes have not been investigated'. Unaware of the results due to Moore and MacNeish, Yates referred to R. A. Fisher's book *Design of Experiments*, the 2nd edition from the same year, for this information. In later editions, Fisher noted that Stevens had proved the existence for prime powers in 1939.

There is a record of Fisher's delight in this result. In a report in *Nature* in 1938 about a British Association meeting in Cambridge, with talks also by Norton, Youden and Yates, Fisher wrote the following. 'Mr. W. L. Stevens had a surprise in store, in the form of a demonstration of the fact that for any power of a prime a completely orthogonal square exists'.

But it was a 1938 paper by Bose that was to prove the equivalence between projective planes of order n and sets of $n-1$ MOLS of order n:

N(n) = $n-1$ *if and only if there exists a projective plane of order n.*

Since projective planes of prime power orders were well known, the result implied the existence of complete sets of MOLS of such orders. But Bose also gave a direct construction of complete sets of MOLS from Galois fields. Bose's proof of this theorem and his version of Moore's and MacNeish's prime power construction are quite similar to those used in most textbooks nowadays. Bose began his paper by stating that it was a conjecture of Fisher that a complete set of MOLS would exist of every prime power order, and he wrote that the constructions of affine and projective planes from such sets corresponded exactly to those of Yates.

In 1949 Bruck and Ryser proved that if a projective plane of order n exists, where $n \equiv 1$ or $n \equiv 2$ (modulo 4), then n is the sum of two squares. This gives non-existence for $n = 14, 21, 22, \ldots$ (and for $n = 6$, but the latter was of course known from Tarry's result). Their result was generalised the next year to become the celebrated Bruck-Ryser-Chowla theorem[18].

The connection between MOLS and projective planes makes it possible to relate a number of further results about projective planes, such as non-existence results, to MOLS. Let us just note that so far no projective plane not of prime power order is known, so it could well be that $N(n) = n - 1$ if and only if n is a prime power. After a long computer search by Lam et al, based on coding theoretic considerations, these authors concluded that no projective plane of order 10 exists. The value $n = 12$ is now the smallest for which this statement is in doubt.

5. The influence from experimental design

In spite of Cretté de Palluel's 1788 application not much use was made of latin squares in experimental design before *R. A. Fisher*'s boost to this area in the 1920s. Fisher was statistician at Rothamsted Experimental Station from 1919 to 1933 and later professor in London. He wrote numerous papers on various aspects of statistics. He also conducted many experiments, often in agriculture, and took a large interest in the design of experiments, writing a classic book, *Design of Experiments,* about it in 1935 (many subjects from the area, including latin squares in field experiments, also appear in his first book from 1925). And in 1926, he specifically wrote about using latin squares and mutually orthogonal such in experiments[19]. And Yates began his 1936 paper mentioned earlier with: 'Most biological workers are probably by now familiar with the methods of experimental design known as randomized blocks and the Latin square. These were originally developed by Prof. R. A. Fisher, when Chief Statistician at Rothamsted Experimental Station, for use in agricultural field trials'.

In *Design of Experiments* however, Fisher mentioned earlier uses of latin squares in design. In particular, in a subsection of the latin square chapter on randomisation he warned against the use of systematic squares, that is a preferred lay-out (latin square) used repeatedly. He displayed a square of order 5 which has constant diagonals, so that if this square was used (and apparently it had been) for elimination of soil differences in an agricultural experiment, if would fail as far as diagonal fertility differences are concerned. He noted that this shortcoming was realised by many and went on to mention the square below 'known in Denmark since about 1872'.

A B C D E
C D E A B
E A B C D
B C D E A
D E A B C

Though this square was certainly published in Denmark several times before that, it was named after a 1924 paper by Knut Vik (who was Norwegian). Its apparent advantages notwithstanding, Fisher was not happy with the systematic use of this square either. While it is safe to say that a reference to *the* Knut Vik square will be to the above square, there is confusion about the generalisation of this terminology. All diagonals (including "broken" diagonals) of the square are transversals, and squares with this property are now called Knut Vik *designs*; they exist if and only if n is not a multiple of 2 or 3. Focussing on another property of the square, it is also an example of a *knight's move* square, because all cells with the same symbols can be visited by a chess knight with allowable moves.

It seems, however, that the theory of latin squares has been more influenced by statisticians than by statistics. Certainly, some of the pioneers in the area of statistical design of experiments, though naturally interested in the practical uses of designs and in the enumeration of them, also contributed significantly to the theory. Some statisticians have actually warned against too much emphasis on latin squares in practice! Donald Preece has written that the prominent feature of latin squares in textbooks has 'led to their uncritical use', and even Fisher had an interesting remark: 'This experimental principle is best illustrated by the arrangement known as the Latin square, a method which is singularly reliable in giving precise comparisons', adding '*when the number of treatments to be compared is from 4 to 8'*. Still, in a standard work on statistical tables first published in 1938, Fisher and Yates provided tables of pairs of orthogonal Latin squares of orders 3 to 9 (except 6). And Bose wrote on the topic of complete sets of MOLS, that 'The work of Fisher and Yates has shown that such squares are of fundamental importance in experimental design'.

6. Other results

Euler's paper from 1782 introducing latin squares left several immediate challenges open. In addition to his conjecture itself, he introduced such topics as enumeration and transversals. Naturally, new questions have come to light since then, and research in latin squares has both broadened and deepened. Some of the more prominent developments are described in the following.

Enumeration.

A latin square of order n on symbols 1, …, n is said to be *reduced* if both the first row and the first column are 1, 2, …., n in order. A much used enumeration function[20] is $l(n)$, the number of reduced latin squares of order n (then the total number of *distinct* latin squares on these symbols amounts to $n!\,(n-1)!\,l(n)$). Euler determined $l(n)$ for n at most 5 (the values are 1, 1, 1, 4 and 56).

A *latin rectangle* is a rectangular array filled with symbols in such a way that no symbol occurs twice in any row or column. We let $l(k,n)$ denote the number of $k \times n$ latin rectangles on n symbols with first row 1, 2, …, n and first column 1, 2, …, k. It is a result of Marshall Hall that any latin rectangle of size $k \times n$ on n symbols can be completed to a latin square of order n. Thus it is interesting to examine $l(k,n)$ for all k with a view to determining the specific value $l(n,n) = l(n)$.

For $k = 2$, Euler found the recursion $l(2,n) = (n-2)\,l(2,n-1) + (n-3)\,l(2,n-2)$. He wrote down the first 10 values of $l(2,n)$ but apparently did not recognise the *problem of derangements* here. This Cayley did in 1890, referring to the "well-known problem" and stating the solution

$$l(2,n) = (n-2)!\,n\,(1 - 1/1! + 1/2! - \ldots + (-1)^n/n!\,).$$

In the same paper, Cayley recalculated the values of $l(n)$ for small n also found by Euler.

Also in 1890, Frolov correctly stated that $l(6) = 9408$ but gave a value for $l(7)$ which was about 13 times too large. In a series of papers from 1949-1951, Sade determined that $l(7) = 16{,}942{,}080$, at the same time correcting earlier mistakes due to Jacob and Norton. In 1969, Wells determined the value of $l(8)$ – it is 535,281,401,856. Then $l(9)$ was determined in 1975 by Bammel and Rothstein, and in 1995 McKay and Rogoyski found the value of $l(10)$. In 2005, also $l(11)$ and the corresponding numbers of rectangles were determined by McKay and Wanless; their paper[21] also lists $l(k,n)$ for all $k \leq n \leq 11$. Obviously, the parameters being determined in later years show the possibilities and limitations of computers for these enumeration problems.

Attempts at finding algebraic descriptions for the enumeration functions for latin squares and rectangles have resulted in formulae impractical for calculation. Asymptotic results exist, however. Ryser noted in 1969, before the Van der Waerden conjecture was proved, that the truth of it would imply that

$$l(n) \geq (n!)^{2n-2}/n^{n^2}.$$

In 1992, it was proved by van Lint and Wilson in their textbook (whose front page is featured in Figure 4) that this bound is asymptotically of the right order of magnitude (in a sense involving comparing logarithms of the expressions).

In 1981, Smetaniuk proved that the number of latin squares is strictly increasing.

Transversals and partial transversals

As we have seen, in Euler's paper introducing the concept of orthogonality, transversals were important – according to himself, the first and foremost object of the paper. He found the first examples of latin squares without transversals. This line of investigation was continued, often in connection with research on orthogonality, as a latin square without a transversal cannot have an orthogonal mate. On the other hand, there are situations, for example if the latin square is the multiplication table of a group, where existence of just one transversal is enough to imply a decomposition into transversals and thus an orthogonal mate. Euler had already exploited this.

In 1894, Maillet generalised Euler's findings that within a certain class of latin squares those of order $n \equiv 2$ (modulo 4) have no transversal. A latin square of order mq is said to be of q-step type if it can be obtained by replacing each entry in a cyclic latin square of order m by a latin square of order q in such a way that these are on the same set of symbols if they replace the same symbol, and on disjoint sets of symbols otherwise. Maillet then proved that a latin square of order mq of q-step type has no transversal when m is even and q is odd. His proof is similar to the one given by Euler for $q = 1$ (only more complicated).

While it has been known since Euler's time that there are latin squares without a transversal, two related questions are still unsolved. In 1967, Ryser conjectured that every latin square of odd order has a transversal. This is true for latin squares coming from groups and for symmetric latin squares, where the main diagonal is a transversal. And it has been suggested (by R. A. Brualdi, S. K. Stein and probably others) that perhaps every latin square of order n has a *partial transversal* of length at least $n - 1$. A partial transversal is a set of cells in distinct rows, in distinct columns and containing distinct symbols. This conjecture prompted people to search for long partial transversals. Since Koksma in 1969 proved that a latin square of order $n \geq 7$ contains a transversal having length at least $(2n + 1)/3$ this has seen a number of improvements. In 1982, Shor proved the existence of a partial transversal of length at least $n - 5.53 (\log n)^2$.

Quasigroups, completion and critical sets

The so-called Cayley table for any finite group (its multiplication table) is obviously a latin square, so when group theory began flourishing in the second half of the 19th century there were plenty of examples of latin squares to study. It seems, however, that only a few of the leading group theorists were interested in this aspect, Cayley himself being one of them. Not every latin square bordered by a permutation of its symbols is the Cayley table of a group. Instead, such a table gives rise to the concept of a quasigroup, an algebraic structure with a binary operation such that the equation $x * y = z$ has a unique solution in x, given y and z, and a unique solution in y, given x and z. Quasigroups were essentially introduced by Schröder in the 1870s but did not achieve general attention until the 1930s. Since then, many results on latin squares have been first formulated in terms of quasigroups.

One particular area that has often been discussed in terms of quasigroups is that of completion of partial structures. For latin squares, the typical question is whether a *partial latin square,* i.e., a square array possibly having empty cells but for which the entries do not violate the latin square conditions, can be completed to a latin square. A conjecture that was open from 1960 to 1981, known as the Evans conjecture, and now known to be true, is that a partial latin square of order n with at most $n - 1$ non-empty cells can always be completed. In quasigroup terminology, such problems are often formulated with extra conditions such as idempotency, commutativity etc.

A partial latin square is called a *critical set* if it can be completed in exactly one way to a latin square of the same order, and the deletion of any entry destroys the uniqueness. The size of critical sets is a topic with many open questions. Perhaps this problem has become especially popular because of its connection with the unsolved problem of whether sudoku puzzles with only 16 givens exist which are uniquely soluble, which is a variant of the general problem (we return to sudokus in the next section).

7. Latin squares in arts and recreation

Latin squares seem to have inspired artists throughout history. The silver amulet of Figure 2 shows the decorative value of a latin square. In the medieval books on magic and latin squares, these are often framed by fanciful ornamental structures.

Figure 13. Two orthogonal latin squares of order 10 as a needlepoint rug.

When two orthogonal latin squares of order 10 were found, the magazine *Scientific American* put on its cover a reproduction of a painting by its staff artist Emi Kasai illustrating the squares with colours (November 1959). In 1960, Mrs. Karl Wihtol made the rug of Figure 13 from the painting.

In Caius College, Cambridge, England, there is a stained-glass window commemorating the Caian R. A. Fisher. It is a colourful 7 × 7 latin square made by Artist Maria McClafferty. It is seen in Figure 14; the other window, made by the same artist, commemorates John Venn.

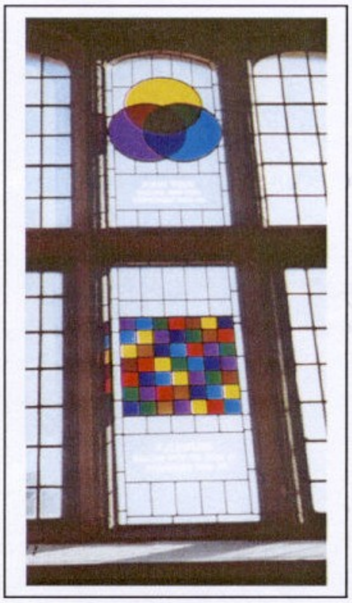


Figure 14. Window from Caius College.

When the French author Georges Perec (1936-82) wrote his masterpiece *la vie mode d'emploi*, he thrust upon himself a number of restrictions of mathematical nature. One of them was that the 100 chapters should correspond to a knight's move tour through a 10 × 10 array which was itself two superimposed orthogonal latin squares of order 10. He achieved this by letting the physical locations of the chapters be different parts of a 9-storey building with a basement. The front of the book, Figure 15, gives a hint of this. (The actual book only has 99 chapters, and the knight's move tour is not perfect.)

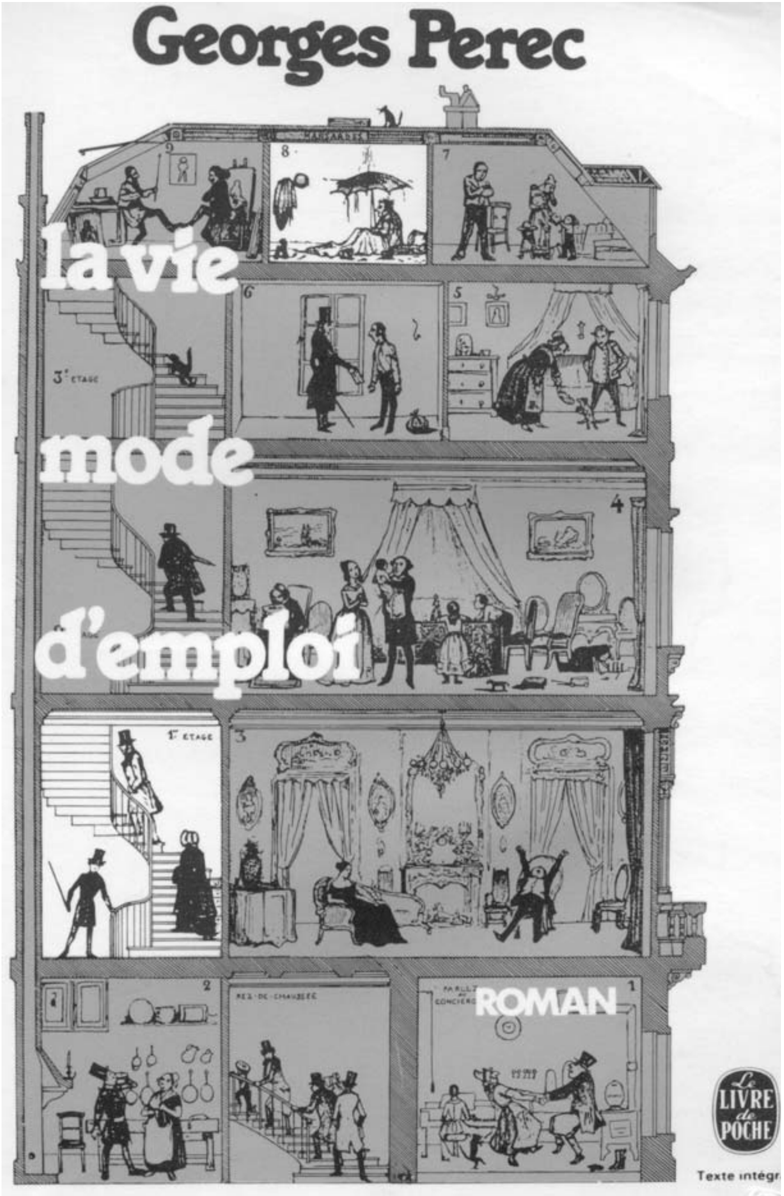


Figure 15. Front page of paperback edition of *la vie mode d'emploi*.

On the more recreational side, latin squares are behind the so-called *sudoku* puzzles that are very popular at the time of writing. A completed sudoku is a latin square of order 9 with the additional condition that the 9 natural 3 × 3 subsquares each contain all 9 symbols. The task is to complete the square from a given set of entries; this set must be a critical set for the sudoku, that is, there is a unique solution, and for any proper subset of the givens there would be more than one solution.

Sudokus have their name from Japanese, but the were invented in the United States in the 1970s, when Howard Garns began publishing such puzzles in Dell Magazines. The puzzles were then called ‘number place’, but when they became popular in Japan in the mid 1980s, it was under the name *Suuji wa dokushin ni kagiru* which means ‘number is limited only single’. This became abbreviated to sudoku. In 1986, when the interest exploded, the Japanese publisher Nikoli introduced the rule that the cells filled with given entries should have 180 degrees rotational symmetry. Since then, mathematicians have become interested, and generalisations of the classical type of sudoku have appeared. As mentioned in the previous section, there are open mathematical problems related to sudokus.

Finally, we have come to the end of a section on arts in a contribution on history. What is more fitting than poem from a brass plate in St. Mawgan Church, England, commemorating Hanniball Basset who died in 1709:

Shall wee all dye
Wee shall dye all
All dye shall wee
Dye all wee shall

NOTES

1. The 3×3 latin square is a standard cyclic latin square which exists for all orders n and can be defined by the entry in cell (i,j) being $i + j - 1$ (modulo n), and the two other squares are original examples of very early latin squares whose sources are introduced later in the first section.
2. Illustration from *Duncan B. MacDonald, Description of a silver amulet, Zeitschrift für Assyriologie und Verwandte Gebiete* 26, 1912, pp. 267-269. Also in *S. Seligmann, Das Siebenschläfer-Amulett, der Islam* 5, 1914, pp. 370-388. Note that the penultimate entry in the last column is wrong.
3. The illustration and the instruction are taken from *Jaffur Shurreef, Qanoon-E-Islam*, 1832, “composed under the direction of”, and translated into English by *G. A. Herklots*; English

title and subtitle *Customs of the Moosulmans of India – full and exact account of their various rites and ceremonies from the moment of birth till the hour of death.* The same latin square, except that the last 3 rows are permuted, is also shown in the book of Al-Buni mentioned later (with Arab numbers) and (in both versions) in *Edmond Doutté, Magie et Religion dans l'Afrique du Nord*, 1908, which also gives instructions on how to use this square to obtain the consent of a lover who has previously declined a marriage proposal. Doutté's book contains many such examples of early latin squares.

4. The book has been reissued in the Arab countries until the present time. I know of no translation into any other language. As exemplified by Figure 3, many illustrations are quite defective – a fact that can actually be used when trying to look through the constructions behind the squares presented.
5. As argued in *W. Ahrens, Die "magischen Quadrate" al-Buni's, der Islam* 12, 1922, 157-177.
6. *Narayana Pandit, Ganita-kaumudi*, 1356. Republished in The Princess of Wales Saravati Bhavana Sanskrit Series, No. 57, Part II.
7. The card problem is sometimes called Bachet's Square due to its appearance in *Claude Gaspar Bachet de Meziriac, Problèmes Plaisans et Délectables qui se font par les nombres* which first came out in 1612 but underwent several changes through later editions, and in fact the card puzzle may first have occurred as late as the 1874 revision. It occurred in the 1723 edition of *Jacques Ozanam, Récréations mathématiques et physiques*, where the solutions also were enumerated (wrongly). Figure 4 shows a solution from a later edition of Bachet, and a (different) solution from the front page of a modern textbook.
8. *Francois Cretté de Palluel, Sur les avantages et l'economie que procurent les raciness employees a l'engrais des moutons a l'étable, Mémoires d'Agriculture, trimester d'éte*, pp. 17-23. English translation, *On the advantage and economy of feeding sheep in the house with roots, Annals of Agriculture* 14, 1790, pp. 51-55.
9. This appears already in *Narayana Pandit*'s book from 1356, note 6. Here taken from *Schuyler Cammann, Islamic and Indian Magic Squares. Part II, History of Religions* 8, 1969, pp. 271-299. Cyclic latin squares are defined in note 1.
10. *Joseph Sauveur, Construction générale des quarrés magiques, Mémoires de l'Academie Royales des Sciences*, 1710, pp. 92-138.
11. *G. Tarry, Le problème des 36 officiers, C. R. Assoc. Franc. Av. Sci.* **29,** 1900, 170-203.

12. The letter is discussed *in Harald Gropp, "Gaußsche Quadrate" or Knut Vik designs – the history of a combinatorial structure, Proceedings of the 2nd Gauss Symposium. Conference A: Mathematics and Theoretical Physics, Munich, August 2-7 1993 (ed. Behara, Minaketan et al), Berlin: Walter de Gruyter, Symposia Gaussiana,* 1995, 121-134.
13. *H. F. MacNeish, Euler Squares, Annals of Mathematics* **23** (1922), 221-227.
14. *R. C. Bose, S. S. Shrikhande and E. T. Parker, Further results on the construction of mutually orthogonal latin squares and the falsity of a Euler's conjecture, Canad. J. Math.* **12,** 1960, 189-203.
15. The 1960 paper by Chowla, Erdös, and Straus actually proved that $N(n) > 1/3\ n^{1/91}$ for large n. In 1973, Wilson published the bound $N(n) > n^{1/17} - 2$ (for large n). As in the above paper, number theory plays a big part, but whereas Chowla, Erdös and Straus could write 'Our proof involves no new combinatorial insights', Wilson's paper contains the first form of his important theorem on constructions of transversal designs. Later improvements on this bound includes one by Lu, who in 1985 proved that $N(n) \geq n^{10/143} - 2$. Upper bounds are rarer, but it is known that arbitrarily large gaps between $N(n)$ and $n - 1$ can occur.
16. *American J. Mathematics* **18** (1896), 264-303.
17. *F. Yates, Incomplete randomized blocks*, *Annals of Eugenics* **vii** (1936), 121-140
18. H. *Bruck and H. J. Ryser, The non-existence of certain finite projective planes*, *Canad. J. Mathematics* **1** (1949), 88-93, and *S.A. Chowla and H.J. Ryser, Combinatorial problems, Canad. J. Math*. 2, 1950, 93-99.
19. *R. A. Fisher, The arrangement of field experiments*, *Journal of Ministry of Agriculture* **33** (1926), 503-513.
20. Other possibilities are to try to establish the number of isomorphism classes, the number of isotopy classes or the number of main class isotopy classes of latin squares of order n. Two latin squares of order n are *isotopic*, if one can be obtained from the other by a permutation of rows, a permutation of columns and a permutation of symbols. They are *isomorphic*, if these three permutations are the same. A *conjugate* (or *parastrophe*) of a latin square is a latin square obtained by a permutation of the triple (rows, columns, symbols). Two latin squares of order n are *main class isotopic* if one is isotopic to a conjugate of the other. The 17 types of latin squares of order 6 found by Tarry in his solution of the Officers problem can be identified as coming from the 22 isotopy classes for order 6 by merging 10 of them into 5 pairs.

21. B. D. MacKay and I. M. Wanless, On the Number of Latin squares, Ann. Combin. 9 (2005). See also B. D. Mackay, A. Meynert and W. Myrvold, Small Latin Squares, Quasigroups , and Loops, J. Combin. Designs 15 (2007), 98-119.
22. M. Gardner, Euler's Spoilers: The Discovery of an Order-10 Graeco-Latin Square, *Martin Gardner's New Mathematical Diversions From Scientific American*, George Allen and Unwin Ltd., London, 1966, 162-172.