Material Modelling
Exercises & Solutions
Lars Andersen
Scientific Publications at the Department of Civil Engineering

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Exercises

Exercise 1: Invariants and principal stresses

The following three-dimensional state of stress is given (measured in kPa):

\[
\sigma = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33}
\end{bmatrix} = \begin{bmatrix}
90 & -30 & 0 \\
-30 & 120 & -30 \\
0 & -30 & 90
\end{bmatrix}.
\] (1)

**Question 1:**

Use arrows on a cube with unit side length to illustrate the components of the stress tensor in the Cartesian \((x_1, x_2, x_3)\)-coordinate space.

**Question 2:**

Determine the stress invariants \(I_1, I_2\) and \(I_3\). Calculate the principal stresses \(\sigma_1, \sigma_2, \sigma_3\),

\[
\sigma_i^3 - \bar{I}_i \sigma_i^3 + \bar{I}_2 \sigma_i - \bar{I}_3 = 0, \quad i = 1, 2, 3.
\] (2)

**Question 3:**

The principal directions \(n_i\) are defined by the equation \((\sigma - \sigma_i I) n_i = 0, \ i = 1, 2, 3\). It may be shown that this provides the solution:

\[
n_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^T, \quad n_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T, \quad n_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T.
\]

Sketch the cube on which the principal stresses act in \((x_1, x_2, x_3)\)-space and establish the transformation matrix \(A\) which maps \(\sigma\) onto \((\sigma_1, \sigma_2, \sigma_3)\). Further, prove that \(A\) is in fact a valid transformation matrix. Finally, calculate the traction on each side of the cube by application of Cauchy’s law and determine the normal stress on each of sides.

**Question 4:**

Determine the maximal shear stress \(\tau_{\text{max}}\) and the unit outward normal \(n\) to one of the surfaces on which it acts. Also find the normal stress \(\sigma_n\) in direction \(n\).

**Hint:** Consider the problem in the principal stress space, where the stress tensor and the normal vector are denoted \(\sigma'\) and \(n'\), respectively. Exploit the fact that \(\tau_{\text{max}}\) is independent of the middle principal stress \(\sigma_2\) (why?) and that \(n'\) can therefore be written as \(n' = [a, b, c]^T\). Now \(n'\) may be determined by simultaneous solution of the
equations \((n')^T \sigma' n' = \sigma_n\) and \(a^2 + c^2 = 1\) (why?). Next, its counterpart \(n\) in the Cartesian \((x_1, x_2, x_3)\)-space is readily obtained by utilisation of the transformation matrix \(A\).

**Exercise 2: Relationship between elastic moduli**

The relationship between stresses and elastic deformation in an isotropic material is uniquely described by two material properties. The following parameters may be applied: Young’s modulus \(E\), the shear modulus \(G\), the bulk modulus \(K\), the Lamé constants \(\lambda\) and \(\mu = G\), and finally Poisson’s ratio \(\nu\).

**Question 1:**

Plot the relative magnitudes \(G/E\), \(K/E\) and \(\lambda/E\) as functions of \(\nu\). Discuss the physical meaning of each of the quantities and explain their variation with Poisson’s ratio.

**Exercise 3: Elastic properties based on triaxial tests**

A triaxial test is carried out on a linear elastic and isotropic material. Firstly, an isotropic step is performed, in which the specimen is subjected to the confining pressure \(p = 200\) kPa. Subsequently the axial stress, or piston pressure, \(\sigma_1\) is increased while the confining pressure, or chamber pressure, \(\sigma_2 = \sigma_3\) is kept constant at 200 kPa. During the test, the axial strain \(\varepsilon_1\) and the volume strain \(\varepsilon_v\) (i.e. the dilatation) are recorded and the following diagrams are made:

- for Step 1 the stress–strain curve is plotted in an \(\varepsilon_1–p\)-diagram,
- for Step 2 the stress–strain curve is plotted in an \(\varepsilon_1–q\)-diagram,
- the volume strain is illustrated for both steps in an \(\varepsilon_1–\varepsilon_v\)-diagram.

Note that \(q = \sigma_1 – \sigma_3\) denotes the stress difference due to the increase in \(\sigma_1\).

**Question 1:**

What is the relationship between the axial strain \(\varepsilon_1\) and the volume strain \(\varepsilon_v\) in the two parts of the triaxial test? Which one of the parameters \(\lambda, \mu, K, E\) and \(\nu\) can be determined from the first part of the test.

**Question 2:**

Provide the equations for the determination of \(\lambda, \mu, K, E\) and \(\nu\) based on the slope of the curves in the abovementioned diagrams. Discuss whether one or more alternative diagrams may be advantageous for the determination of the elastic parameters.
Exercise 4: A hyper-elastic model based on a triaxial test

On the basis of a triaxial test, a strain–stress curve is plotted in an \( \varepsilon_1-q \)-diagram (see the figure). The test results indicate that the stiffness of the material decreases with increasing stress difference, \( q = \sigma_1 - \sigma_3 \). When the stress difference reaches a certain value, \( q_u \), the stiffness is reduced to zero. Hence, a linear elastic model is completely unrealistic for the material. Instead, a nonlinear hyper-elastic model is sought.

**Question 1:**

Based on the test results in the figure, estimate the ultimate stress difference, \( q_u \), and the initial tangent elasticity modulus, \( E_0 \).

**Question 2:**

A hyper-elastic nonlinear model is proposed, in which the Young’s modulus, \( E \), depends on the state of stress according to the relation:

\[
E = E(I_1, J_2) = c_1 I_1 + c_2 \sqrt{3J_2}.
\]  

(1)

Determine the material properties \( c_1 \) and \( c_2 \). Subsequently, prove that \( E \) is independent of \( \sigma_3 \), provided that \( E_0 \) and \( q_u \) do not depend on the confining pressure \( \sigma_3 \).

**Hint:** It may be a good idea to express the invariant \( J_2 \) in terms of \( q \) and \( \sigma_3 \).

**Question 3:**

Make a Matlab program to plot the stress–strain curve.

**Hint:** Increment the strain \( \varepsilon_1 \) in fixed steps and determine \( q \) by means of, for example, the Euler method. Alternatively, try splitting the strain and stress into their mean and deviatoric parts. Evidently the same stress–strain curve should be obtained.
Exercise 5: Different ways of determining principal stresses

For an arbitrary state of plane stress \((\sigma_{11}, \sigma_{13}, \sigma_{33})\) the principal stresses \((\sigma_1, \sigma_3)\) may be determined in three different ways:

1. Graphically by means of Mohr’s circles,
2. Directly by means of the characteristic equation,
   \[ \det(\mathbf{\sigma} - \sigma_i \mathbf{I}) = 0, \quad i = 1, 3. \]  
   (1)
3. From an alternative characteristic equation expressed in the invariants \(\bar{I}_1, \bar{I}_2\).

Question 1:

Draw Mohr’s circle for an arbitrary state of plane stress \((\sigma_{11}, \sigma_{13}, \sigma_{33})\). Determine the principal stresses \(\sigma_1\) and \(\sigma_3\) by means of a graphical solution. Show that the same values of the principal stresses are obtained with the use of Equation (1).

Question 2:

Determine the two invariants \(\bar{I}_1\) and \(\bar{I}_2\), i.e. the Cauchy invariants, for plane stress. Subsequently, formulate the characteristic equation in terms of \(\bar{I}_1\) and \(\bar{I}_2\).

Exercise 6: Fitting of Drucker-Prager to Mohr-Coulomb

The Drucker-Prager criterion (D-P) is usually given in terms of the first invariant of total stresses, \(I_1\), and the second generic invariant of the deviatoric stresses, \(J_2\),
\[ F(I_1, J_2) = \sqrt{3J_2} + \alpha I_1 - \beta = 0. \]  
(1)
Here, \(\alpha\) [-] and \(\beta\) [Pa] are material properties. Coulomb’s criterion may be expressed in terms of the maximum and minimum principal stresses, \(\sigma_1\) and \(\sigma_3\), respectively,
\[ F(\sigma_1, \sigma_3) = (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3) \sin \phi - 2c \cos \phi = 0, \quad \sigma_1 \geq \sigma_2 \geq \sigma_3, \]  
(2)
where \(\phi\) is the internal angle of friction and \(c\) is the cohesion. In both failure criteria, stresses are defined as positive in tension. This is uncommon practice in geotechnical engineering, since tensile stresses cannot be sustained by most soils.

Generally the Coulomb criterion provides the better fit to the behaviour of granular materials, e.g. soil, and cemented materials like concrete and rock. However, in certain finite-element codes, only the D-P is available. In these circumstances, a calibration of the D-P has to be carried out in order to get results that are similar to those obtained with a Coulomb criterion. The problem is illustrated below.
Question 1:

Determine $\alpha$ and $\beta$ so that the D-P provides the same results as the Coulomb criterion, given in terms of $\phi$ and $c$, for a) triaxial tension and b) triaxial compression.

Question 2:

Calibrate $\alpha$ and $\beta$ so that the Drucker-Prager criterion in the octahedral plane forms a circle inscribed in the hexagon representing the Coulomb criterion.

Exercise 7: Tresca criterion with linear hardening

Undrained clay is assumed to be an isotropic and linear elastic material until the maximal plane shear stress becomes equal to the yield stress $\sigma_0 = ms_u$. Here $m$ denotes the degree of mobilisation, and $s_u$ is the ultimate strength.

Initially, the degree of mobilisation has the value $m_0$. Due to accumulated plastic strain, the material undergoes isotropic hardening until the strength is fully mobilised. Ultimate failure occurs when $m = 1$, i.e. when $\sigma_0 = s_u$.

Hence, yielding is governed by the Tresca criterion which is also applied as flow rule, i.e. associated plasticity is assumed. Thus, given in terms of principal stresses, the yield criterion becomes:

$$ f(\sigma_I,m) = 0, \quad f(\sigma_I,m) = \frac{1}{2}(\sigma_I - \sigma_{II}) - ms_u, \quad \sigma_I \geq \sigma_{II} \geq \sigma_{III}, \quad 0 \leq m \leq 1. \quad (1) $$

Note that stresses are defined as positive in compression, and $\sigma_I$ and $\sigma_{III}$ denote the largest and the smallest compressive principal stress, respectively.
Note: The Tresca criterion makes no sense for granular soil where the strength is highly dependent on the effective mean stress, i.e. the hydrostatic pressure in the grain skeleton. Likewise, inaccurate results are achieved for drained clay.

Question 1:

Illustrate the yield surface defined by Eq. (1) in Mohr’s diagram and the octahedral plane. Show how the yield locus (i.e. the yield surface) is changed during hardening. Indicate the direction of the plastic strain increments determined by associated plasticity. Discuss why the mathematical treatment of Tresca’s criterion may be more complicated than the numerical treatment of von Mises’ criterion.

Question 2:

Prove that the assumption about associated plasticity provides a realistic material model when the yield criterion (1) is employed. Show that this is also the case for a non-hardening Tresca model. Further, discuss whether associated plasticity makes any sense in a non-hardening Drucker-Prager model.

Hint: According to Drucker’s postulate, the deformation of an elastoplastic material may not result in negative work.

Question 3:

Formulate the Tresca criterion in the \((\sigma_{11}, \sigma_{13}, \sigma_{33})\)-stress space. Here it may be advantageous to start from Mohr’s circles.

Question 4:

The elastic behaviour of the material is described by the bulk modulus \(K\) and the shear modulus \(G\). Further, the following quantities are introduced:

\[
q = \frac{\sigma_{11} - \sigma_{33}}{2}, \quad \beta = \frac{G}{t^2(1 + H / G)}, \quad t = \sqrt{q^2 + \sigma_{13}^2}.
\]  

(2)

Here \(H\) is the hardening modulus. Show that the elastoplastic constitutive matrix for plane strain \((\varepsilon_{12} = \varepsilon_{22} = \varepsilon_{32} = 0)\) given in terms of these parameters becomes:

\[
C^p = \begin{bmatrix}
K + \frac{4}{3}G & -\beta q^2 & \left(K - \frac{2}{3}G\right) + \beta q^2 & -\beta q \sigma_{13} \\
\left(K - \frac{2}{3}G\right) + \beta q^2 & K + \frac{4}{3}G & -\beta q^2 & \beta q \sigma_{13} \\
-\beta q \sigma_{13} & \beta q \sigma_{13} & G - \beta \sigma_{13}^2 & \beta \sigma_{13}^2
\end{bmatrix}.
\]  

(3)
Question 5:

Hardening is assumed to arise due to accumulated plastic deformation, whence the hardening module achieves the form:

$$H = -\frac{\partial f}{\partial m} \frac{dm}{d\lambda}. \quad (4)$$

Here $d\lambda$ is the plastic multiplier. Based on this assumption, show that the hardening modulus becomes:

$$H = s_u \frac{dm}{d\varepsilon^p}, \quad d\varepsilon^p = \sqrt{(d\varepsilon_{11}^p - d\varepsilon_{33}^p)^2 + (d\gamma_{13}^p)^2}. \quad (4)$$

**Hint:** Firstly, determine $d\lambda$ from the principal strain increments $d\varepsilon_1$ and $d\varepsilon_3$. Secondly, find $H$ by means of Mohr’s circle for strain increments.

Question 6:

A simple compression test (i.e. a triaxial test with the chamber pressure $\sigma_2 = \sigma_3 = 0$) is performed on an undrained sample of clay. To begin with, elastic deformation of the specimen takes place. However, when the mobilised shear strength $\sigma_0 = m \sigma_u$ is reached, initial yielding occurs. The clay undergoes elastoplastic deformations until the ultimate shear strength $\sigma_u$ is reached. Based on the laboratory tests, the following material properties have been determined:

$$K = 20 \text{ MPa}, \quad G = 10 \text{ MPa}, \quad \sigma_u = 50 \text{ kPa}, \quad m = 0.6, \quad H = 8 \text{ MPa}.$$  

Draw the stress–strain curve for the simple compression test in a $(\varepsilon_1, \sigma_1 - \sigma_3)$-diagram and sketch the strain history in an $(\varepsilon_1, \varepsilon_v)$-diagram, where $\varepsilon_v$ is the volumetric strain.

**Hint:** At the compressive meridian defined by $\sigma_I = \sigma_1$, the plastic potential $g = f$ consists of two equal contributions: one for $\sigma_{III} = \sigma_2$ and one for $\sigma_{III} = \sigma_3$.

Exercise 8: **Interpretation of the Cam-clay model**

Consider the modified Cam-Clay model. The slope of the critical (or characteristic) line in a $(p', q)$ diagram is denoted $M$, whereas $\lambda$ and $\kappa$ are the slopes of the primary loading line (or normal consolidation line) and elastic unloading/reloading lines, respectively, in an $(\ln p', 1 - \varepsilon_v)$ diagram. Finally, the pre-consolidation pressure is coined $p_f$, and stresses and strains are defined as positive in compression.

Question 1:

Plot the critical line with $M = 1$ in the $(p', q)$ diagram and draw the corresponding modified Cam-Clay failure envelopes for $p_f = 100, 200$ and $300 \text{ kPa}$. Illustrate the plastic strain increment on the figure and discuss why the effective stresses, i.e. $p'$ and $q$, remain constant during plastic deformation for a stress point on the critical line.
**Question 2:**

Draw a \((p', 1 - \varepsilon_v)\) diagram directly under the \((p', q)\) diagram from Question 1. Plot the primary loading curve in this diagram and indicate the points corresponding to the pre-consolidation pressures \(p_f = 100, 200\) and \(300\) kPa.

**Question 3:**

Now, consider the case of elastic unloading from each of the three stress states \((p_f, 0)\), \(p_f = 100, 200\) and \(300\) kPa. Plot the stress–strain curves in the \((p', 1 - \varepsilon_v)\) diagram and identify the points on the curve corresponding to the critical state. Finally, sketch the critical state line in the \((p', 1 - \varepsilon_v)\) diagram.

**Exercise 9: Cam-clay model – drained and undrained tests**

Two triaxial tests are carried out on undisturbed clay with the initial pre-consolidation pressure \(p_f = 300\) kPa. Each of the two samples are consolidated under drained conditions in the triaxial cell to the effective isotropic stress \(p' = 200\) kPa. The clay is assumed to behave according to the modified Cam-Clay theory with the properties:

- \(M = 1\) = slope of critical line in \((p', q)\) diagram,
- \(\lambda = 0.25\) = slope of primary loading curve in an \((\ln p', 1 - \varepsilon_v)\) diagram,
- \(\kappa = 0.05\) = slope of elastic unloading/reloading curve in an \((\ln p', 1 - \varepsilon_v)\) diagram.

It may be shown that the increment \(\Delta \varepsilon_v\) in the volumetric strain, i.e. the dilatation, corresponding to the increment \(\Delta p'\) in the effective isotropic stress can be found as:

- \(\Delta \varepsilon_v = \lambda \{ \ln (p' + \Delta p') - \ln p' \}\) in primary loading along the \(p'\)-axis,
- \(\Delta \varepsilon_v = \kappa \{ \ln (p' + \Delta p') - \ln p' \}\) in elastic unloading/reloading.

The effective mean stress, \(p'\), and volumetric strain, \(\varepsilon_v\), are positive in compression.

**Question 1:**

After the consolidation phase, a **drained** triaxial compression test is carried out on the first sample of clay. Thus, the piston pressure \(q\) is increased from 0 to the ultimate failure stress \(q_{u,d}\), while the cell pressure \(\sigma_3\) is kept constant and pore water is allowed to leave the cell.

Plot the stress path and the stress–strain curve in a \((p', q)\) diagram and a \((p', 1 - \varepsilon_v)\) diagram, respectively. Next, determine the stress \(q_{u,d}\) at failure and the increment in volumetric strain, \(\Delta \varepsilon_v\), obtained during the triaxial compression step.

**Hint:** Exploit the fact that \(\Delta \varepsilon_v\) is known for primary loading along the \(p'\)-axis as well as elastic unloading/reloading.
Question 2:

After the consolidation phase, an undrained triaxial compression test is carried out on the second sample of clay. Thus, the piston pressure \( q \) is increased from 0 kPa to the ultimate failure stress \( q_{u,ud} \), while the cell pressure \( \sigma_3 \) is kept constant.

Plot the stress path and the stress–strain curve in a \((p', q)\) diagram and a \((p', 1 - \varepsilon_v)\) diagram, respectively. Next, determine the stress \( q_{u,ud} \) at failure.

Hint: Since pore water cannot leave the cell, the volume of the clay sample is constant during the undrained triaxial compression test, i.e. \( \Delta \varepsilon_v^p = - \Delta \varepsilon_p^v \). Hence, an increase of the plastic strain \( \Delta \varepsilon_v^p \) provides a decrease of the effective mean stress, \( p' \).

Exercise 10: Mohr-Coulomb model with linear hardening

A specimen of sand is modelled by the Mohr-Coulomb model. In principal stress space, the failure criterion is given as:

\[
f_u = \frac{1}{2} \sigma_2 - \frac{1}{2} (\sigma_1 + \sigma_3) \sin \varphi - c \cos \varphi \leq 0, \quad \sigma_1 \geq \sigma_2 \geq \sigma_3. \tag{1}
\]

Compressive stresses are defined as positive, and the parameters \( \varphi \) and \( c \) are the angle of friction and the cohesion, respectively. Failure occurs when \( f_u = 0 \).

An expression similar to Eq. (1) describes initial yielding:

\[
f = \frac{1}{2} \sigma_2 - \frac{1}{2} (\sigma_1 + \sigma_3) \sin \varrho - c \cos \varrho \leq 0, \quad \sigma_1 \geq \sigma_2 \geq \sigma_3. \tag{2}
\]

The material undergoes hardening from an initial value of the mobilised angle of friction \( \varrho = \varrho_0 \) and until \( \varrho = \varphi \), where the full strength is mobilised. Linear isotropic hardening with the hardening modulus \( H \) is assumed, and plastic deformation is governed by the plastic potential

\[
g = \frac{1}{2} \sigma_2 - \frac{1}{2} (\sigma_1 + \sigma_3) \sin \psi, \quad \sin \psi = \frac{d \varepsilon_v^p}{d \varepsilon_v^p - 2d \varepsilon_1^p}, \quad \sigma_1 \geq \sigma_2 \geq \sigma_3. \tag{3}
\]

Here \( \psi \) is the angle of dilatation, while \( d \varepsilon_1^p \) and \( d \varepsilon_v^p \) are the increments in the axial and volumetric plastic strain, respectively, in a triaxial state of stress.

Finally, for stress states inside the yield locus linear isotropic elastic behaviour with the Young’s modulus \( E \) and Poisson’s ratio \( \nu \) is assumed. The following parameters have been found: \( c = 14 \) kPa, \( \varphi = 39^\circ \), \( \varrho_0 = 31^\circ \), \( \nu = 0.33 \), \( E = 530 \) MPa, \( H = 65 \) Pa.

Question 1:

A triaxial compression test is carried out at the chamber pressure \( \sigma_2 = \sigma_3 = 80 \) kPa. Determine the piston pressures \( q_0 \) and \( q_u \) \( (q = \sigma_1 - \sigma_3) \) at initial yielding and ultimate failure, respectively.
**Question 2:**

Compute the total axial strain $\varepsilon_1$ and the total volumetric strain $\varepsilon_v$ at initial yielding. Linear (small) strain is considered, even if this is a crude approximation.

**Question 3:**

Determine the total axial strain $\varepsilon_1$ and the total volumetric strain $\varepsilon_v$ at ultimate failure for the constant angle of dilatation, $\psi = 17^\circ$. Sketch the stress–strain curve in an $(\varepsilon_1, q)$ diagram and the volumetric strain history in an $(\varepsilon_1, \varepsilon_v)$ diagram.
Example Solutions

Exercise 1: Invariants and principal stresses

The following three-dimensional state of stress is given (measured in kPa):

\[
\sigma = \begin{bmatrix}
\sigma_{11} & \sigma_{12} & \sigma_{13} \\
\sigma_{12} & \sigma_{22} & \sigma_{23} \\
\sigma_{13} & \sigma_{23} & \sigma_{33}
\end{bmatrix} = \begin{bmatrix}
90 & -30 & 0 \\
-30 & 120 & -30 \\
0 & -30 & 90
\end{bmatrix}.
\]

(1)

Question 1:

The state of stress is depicted in the figure below (100 kPa ~ 30 mm). Note that the stresses are defined as positive in directions of the Cartesian \((x_1, x_2, x_3)\)-coordinates. This explains the direction of the shear stress vectors.

Question 2:

The stress invariants become:

\[
\tilde{I}_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = 300 \text{ kPa}
\]

\[
\tilde{I}_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2 = 27900 \text{ (kPa)}^2
\]

\[
\tilde{I}_3 = \det(\sigma) = 810000 \text{ (kPa)}^3
\]

The principal stresses, \(\sigma_1, \sigma_2\) and \(\sigma_3\), are then determined as the roots to the characteristic equation.
\[
\sigma_i^3 - \bar{T}_i \sigma_i^2 + \bar{T}_2 \sigma_i - \bar{T}_3 = 0, \quad i = 1, 2, 3. \tag{2}
\]

This is done in MATLAB, and the following (sorted) values are obtained:

\[
\sigma_1 = 150 \text{ kPa}, \quad \sigma_1 = 90 \text{ kPa}, \quad \sigma_1 = 60 \text{ kPa}.
\]

It is easily checked by insertion into (2) that these values are indeed principal stresses.

**Question 3:**

The principal directions \( \mathbf{n}_i \) are defined as

\[
\mathbf{n}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}^T, \quad \mathbf{n}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}^T, \quad \mathbf{n}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T.
\]

The principal stresses act on a cube with an orientation as sketched below in the Cartesian \((x_1, x_2, x_3)\)-space. The hydrostatic stress is measured along the \(p\)-axis.

The transformation matrix, \( \mathbf{A} \), is simply obtained by stacking the direction vectors in the columns, i.e.

\[
\mathbf{A} = \begin{bmatrix}
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}}
\end{bmatrix}.
\]

To prove that \( \mathbf{A} \) is a valid transformation matrix we need to check that \( \mathbf{A}^T = \mathbf{A}^{-1} \). The inverse of matrix \( \mathbf{A} \) is computed in MAPLE, which provides the result:
Furthermore, we need to verify that the determinant of $A$ is positive. Otherwise we do not obtain a mapping from one right-handed coordinate system onto another. Again, using MAPLE we find:

$$\det(A) = 1 > 0.$$ 

The result $\det(A) = 1$ was expected as the vectors $n_i$ were all normalised to unit length.

Subsequently, the traction on each side of the cube is found by application of Cauchy’s law, $\mathbf{t} = \sigma \cdot \mathbf{n}$. The normal stress is then determined by further scalar multiplication with the unit normal vector, i.e.

$$\mathbf{t}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 90 & -30 & 0 \\ -30 & 120 & -30 \\ 0 & -30 & 90 \end{bmatrix} \mathbf{1} = \frac{1}{\sqrt{6}} \begin{bmatrix} 150 \\ -300 \\ 150 \end{bmatrix} \text{kPa}, \quad \sigma_1 = \frac{1}{6} \begin{bmatrix} 150 \\ -300 \\ 150 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 150 \text{kPa},$$

$$\mathbf{t}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 90 & -30 & 0 \\ -30 & 120 & -30 \\ 0 & -30 & 90 \end{bmatrix} \mathbf{1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 90 \\ 0 \\ -90 \end{bmatrix} \text{kPa}, \quad \sigma_2 = \frac{1}{2} \begin{bmatrix} 90 \\ 0 \\ -90 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 90 \text{kPa},$$

$$\mathbf{t}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 90 & -30 & 0 \\ -30 & 120 & -30 \\ 0 & -30 & 90 \end{bmatrix} \mathbf{1} = \frac{1}{\sqrt{3}} \begin{bmatrix} 60 \\ 60 \\ 60 \end{bmatrix} \text{kPa}, \quad \sigma_3 = \frac{1}{3} \begin{bmatrix} 60 \\ 60 \\ 60 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 60 \text{kPa}.$$

As expected, the principal stresses are reproduced.

**Question 4:**

The maximum shear stress $\tau_{\text{max}}$ and the normal stress $\sigma_n$ are determined as

$$\tau_{\text{max}} = \frac{\sigma_1 - \sigma_3}{2} = \frac{150 - 60}{2} = 45 \text{kPa}, \quad \sigma_n = \frac{\sigma_1 + \sigma_3}{2} = \frac{150 + 60}{2} = 105 \text{kPa}, \quad (3)$$

respectively. To find the outward unit normal $\mathbf{n}$ to a surface on which these stresses act, we consider the principal stress space and make use Cauchy’s equation:
The normal stress $\sigma_n$ can then be found by the expression

$$
\sigma_n = \begin{bmatrix} 150a \\ 90b \\ 60c \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 150a^2 + 90b^2 + 60c^2 = 105 \text{kPa.} \quad (4)
$$

Now, the maximum and minimum principal stresses are in directions 1 and 3, respectively, and according to (3) $\sigma_n$ is independent of $\sigma_2$. This independency must also be present in (4), which can only be true if $c = 0$. Further, since $\mathbf{n}^\prime$ must be a unit vector, it follows that $a^2 + c^2 = 1$. Then, according to equation (4), we find that

$$
\begin{align*}
150a^2 + 60c^2 &= 105 \\
a^2 + c^2 &= 1
\end{align*}
\implies \begin{cases} 90a^2 = 45 \\ c^2 = 1 - a^2 \end{cases} \implies a = \pm \frac{1}{\sqrt{2}} \quad \text{and} \quad c = \pm \frac{1}{\sqrt{2}}.
$$

Thus, one of the normal vectors in the Cartesian $(x_1, x_2, x_3)$-space is found as

$$
\mathbf{n} = \mathbf{A} \mathbf{n}^\prime = \begin{bmatrix} \frac{1}{\sqrt{6}} & 1 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{12}} + \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{12}} + \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{12}} + \frac{1}{\sqrt{6}} \end{bmatrix}.
$$

The remaining normal vectors to the surfaces on which $\tau_{\text{max}}$ acts are found by changing the sign on $a$ or $c$. Obviously, this produces a total of four normal vectors corresponding to the four sides of a cube. The middle principal stress acts on the two remaining sides of this cube. Finally, the validity of $\mathbf{n}$ is checked by the computation of the traction $\mathbf{t} = \mathbf{\sigma} \cdot \mathbf{n}$ and subsequent evaluation of the normal stress $\sigma_n = \mathbf{n} \cdot \mathbf{t}$. By numerical evaluation it is found that $\sigma_n$ is indeed 105 kPa.
Exercise 2: Relationship between elastic moduli

The relationship between stresses and elastic deformation in an isotropic material is uniquely described by two material properties. The following parameters may be applied: Young’s modulus $E$, the shear modulus $G$, the bulk modulus $K$, the Lamé constants $\lambda$ and $\mu = G$, and finally Poisson’s ratio $\nu$.

**Question 1:**

The shear modulus $\mu = G$, the bulk modulus $K$ and the Lamé constant $\lambda$ can be written in terms of Young’s modulus and Poisson’s ratio:

$$ G = \frac{E}{2(1+\nu)}, \quad K = \frac{E}{3(1-2\nu)}, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}. $$

The relative magnitudes $G/E$, $K/E$ and $\lambda/E$ are computed as functions of $\nu$ by the MATLAB program below and illustrated in the figure.

```matlab
% Exercise 2 - Solution
% Poisson's ratio
v = 0:0.0001:0.49999;
G = 1./(2+2*v); % shear modulus = mu
K = 1./(3*(1-2*v)); % bulk modulus
L = v./((1+v).*(1-2*v)); % lambda

% Plot the figure
semilogy(v,G,'g',v,K,'b',v,L,'r');
xlabel('
u');
ylabel('{\it G/E, K/E, \lambda/E}');
legend('{\it G/E}', '{\it K/E}', '{\it \lambda/E}', ...
'location', 'northwest')

% End of file
```

Young’s modulus provides the stiffness in uniaxial tension or compression and with no constraints of the material in the other directions, i.e. \( \sigma_1 = E \varepsilon_1 \) and likewise in coordinate directions 2 and 3.

The shear modulus \( G \) relates the shear stress to the shear strain or angular strain, e.g. \( \sigma_{12} = 2G \varepsilon_{12} = G \gamma_{12} \). In a uniaxial state of stress, the Poisson effect introduces shear deformation by increasing the difference between the principal strains. Hence, a material with a high Poisson’s ratio acts stiffer than a material with the same shear modulus but with a low Poisson’s ratio when subjected to uniaxial tension or compression. Thus, the ratio \( G/E \) is expected to decrease with an increase of \( \nu \).

The bulk modulus provides the stiffness in hydrostatic loading, i.e. with the same pressure applied in all three coordinate directions. It relates the volumetric strain, \( \varepsilon_v \), to the mean stress, \( \sigma_m \), as \( \sigma_m = K \varepsilon_v \). When \( \nu = 0 \), uniaxial stress leads to uniaxial strain, i.e. the deformation in three orthogonal directions are independent of each other. Especially, for the uniaxial stress and strain in coordinate direction 1, the mean stress and the volumetric strain become, respectively: \( \sigma_m = (\sigma_1 + 0 + 0)/3 \) and \( \varepsilon_v = \varepsilon_v + 0 + 0 \). Hence, \( \sigma_m = K \varepsilon_v \) reduces to \( \sigma_1 = 3K \varepsilon_1 \), i.e. \( K = E/3 \). On the other hand, when Poisson’s ratio gets close to 0.5, the material becomes nearly incompressible and the resistance towards volume deformation increases dramatically. Consequently, the stiffness measured by the bulk modulus, and therefore the ratio \( K/E \), becomes infinite.

In uniaxial strain in a given direction, e.g. coordinate direction 1, the Lamé constant \( \lambda \) provides the additional normal stress required in the same direction to establish a contraction or expansion in the orthogonal directions when the specimen is confined. For \( \nu = 0 \), this contribution vanishes, whereas it becomes infinite for a incompressible material, i.e. when \( \nu = 0.5 \).
Exercise 3: Elastic properties based on triaxial tests

**Question 1:**

In the first step, the strain will be the same in all directions since both the material and the stress are isotropic. Hence, \( \Delta \varepsilon = \Delta \varepsilon_1 + \Delta \varepsilon_2 + \Delta \varepsilon_3 = 3 \Delta \varepsilon_1 \), where \( \Delta \) denotes an increment in the respective quantities. In step 2, the increase in stress is \( \Delta q \) in direction 1 and 0 in directions 2 and 3. This situation corresponds to compression of a bar, i.e. \( \Delta \varepsilon_2 = \Delta \varepsilon_3 = -v \Delta \varepsilon_1 \), where \( v \) signifies Poisson’s ratio. Hence, \( \Delta \varepsilon_v = (1 - 2v) \Delta \varepsilon_1 \). In step 1 the deviatoric stresses \( s_1 = s_2 = s_3 = 0 \) since the stress is applied as pure pressure. Hence, \( G \) is undefined. Further, since \( \Delta \varepsilon_v/\Delta \varepsilon_1 = 3 \), i.e. independent of \( v \), Poisson’s ratio cannot be determined. However, the bulk modulus may be found from the relation \( \Delta p = K \Delta \varepsilon_v \), i.e. \( K = \Delta p/3 \Delta \varepsilon_1 \).

**Question 2:**

Firstly, the four slopes of the lines in the two diagrams are determined:

- **Step 1:** \( b = \Delta \varepsilon_v/\Delta \varepsilon_1 = 3 \); \( a = \Delta p/\Delta \varepsilon_1 = 3 \Delta p/\Delta \varepsilon_v = 3K \)
- **Step 2:** \( d = \Delta \varepsilon_v/\Delta \varepsilon_1 = (1 - 2v) \); \( c = \Delta q/\Delta \varepsilon_1 = E \)

Hence, Poisson’s ratio and the Young’s modulus may be found in step 2 as:

\[ v = (1 - \Delta \varepsilon_v/\Delta \varepsilon_1)/2 = (1 - d)/2 \quad ; \quad E = \Delta q/\Delta \varepsilon_1 = c \]

The remaining properties may subsequently be determined as:

\[ G = \mu = \frac{E}{2(1+v)} \quad ; \quad \lambda = \frac{vE}{(1+v)(1-2v)} \quad ; \quad K = \frac{E}{3(1-2v)} = \lambda + \frac{2}{3} \mu \]

Alternatively, the bulk modulus is given as \( K = a/3 \).

It may of course be easier to determine the properties \( \lambda, K, \mu \) and \( v \) from diagrams, in which they are simply the slope of a curve. However, only an \( \varepsilon_v-p \)-diagram seems appropriate. Therefore, the suggested diagrams are a natural choice.
Exercise 4: A hyper-elastic model based on a triaxial test

The strain–stress curve is repeated in the figure below, where $\varepsilon_1$ is the axial strain and $q = \sigma_1 - \sigma_3$ is the stress difference between the piston pressure and the chamber pressure in the triaxial test.

![Strain–Stress Curve](image)

**Question 1:**

Based on the test results, the ultimate stress difference is estimated as $q_u \approx 220$ kPa, whereas the initial tangent elasticity modulus is $E_0 \approx 80/0.01 = 8000$ kPa.

**Question 2:**

Firstly, the constitutive model is rewritten as

$$ E = c_1 I_1 + c_2 \sqrt{3} J_2 \Rightarrow c_2 \sqrt{3} J_2 = E - c_1 I_1. $$

Before application of any deviatoric stress, i.e. when only the chamber pressure $\sigma_3$ is present, Young’s modulus has the value $E = E_0$ for $(I_1 = 3\sigma_3; J_2 = 0)$. Hence, the first constant is obtained as $c_1 = E_0/(3\sigma_3)$.

At high levels of strain, i.e. $\varepsilon_1 \to \infty$, the stiffness asymptotically approaches the value $E = 0$ for $(I_1 = q_u + 3\sigma_3; J_2 = q_u^2/3)$, which implies that $c_2 = -(E_0/(3\sigma_3) + E_0/q_u)$. Here, use has been made of the fact that

$$ \sigma_1 = \sigma_3 + q_u, \quad \sigma_2 = \sigma_3, \quad \begin{cases} p = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \sigma_3 + \frac{q_u}{3} \\ s_1 = \sigma_1 - p = \frac{2}{3} q \\ s_2 = s_2 = -\frac{q_u}{3} \end{cases} \Rightarrow J_2 = \frac{1}{2} (s_1^2 + s_2^2 + s_3^2) = -\frac{q_u^2}{3}. $$

Further, in the derivation of $c_2$, use has been made of the fact that $c_2 \sqrt{q_u^2} = 0 - E_0(q_u - 3\sigma_3)/(3\sigma_3)$ for $\varepsilon_1 \to \infty$. 
Finally, Young’s modulus may be written as

\[ E = \frac{E_0}{3\sigma_3} (3\sigma_3 + q) - \left( \frac{E_0}{3\sigma_3} + \frac{E_0}{q_u} \right) q = E_0 \left( 1 - \frac{q}{q_u} \right). \]

Clearly, \( E \) is independent of the confining pressure \( \sigma_3 \). The original nonlinear Young’s modulus may advantageously be written in terms of the stress difference \( q \), i.e. \( E = E(q) \), where \( q = \sqrt{3}J_2 \) in the case of triaxial compression (or tension).

**Question 3:**

The Matlab program utilized to plot the stress–strain curve in the exercise is listed below. A higher accuracy is obtained when a smaller strain increment is applied. Alternatively, the solution may be found analytically.

```matlab
% Exercise 4 - Solution

clear all; close all; clc

% Material properties and chamber pressure
qu = 220;
E0 = 8000;
p = 200;

gu = 0.0002; eps1 = 0;
sig3 = p;
q(1) = 0;

% Initial settings

deps1 = 0.0002; eps1 = 0;
sig3 = p;
q(1) = 0;

% Incremental solution

for i = 1:0.2/deps1-1
    eps1(i+1) = eps1(i) + deps1;
    E = E0*(1 - q(i)/qu);
    q(i+1) = q(i) + E*deps1;
end

% Plot the results

plot(eps1(1:30:end),q(1:30:end),'k.'); xlabel('\epsilon_1') ylabel('{\it q} [kPa]') axis([0 0.2 0 250])

% End of file
```
Exercise 5: Different ways of determining principal stresses

Question 1:

Firstly, Mohr’s circle for an arbitrary state of plane stress \((\sigma_1, \sigma_{13}, \sigma_{33})\) is plotted in Mohr’s diagram as illustrated in the figure below.

Since a plane state of stress is considered, the mean stress and the first invariant are:

\[
p = \frac{1}{2} (\sigma_1 + \sigma_3) = \frac{1}{2} (\sigma_{11} + \sigma_{33}) = \frac{1}{2} I_1. \tag{2}
\]

Further, the diameter of the circle can be expressed in terms of the principal stresses \((\sigma_1, \sigma_3)\) or the Cartesian stress components \((\sigma_{11}, \sigma_{13}, \sigma_{33})\). Hence, we find that the

\[
(\sigma_1 - \sigma_3)^2 = (\sigma_{11} - \sigma_{33})^2 + 4\sigma_{13}^2 \quad \Rightarrow \quad \frac{1}{2} (\sigma_1 - \sigma_3) = \frac{1}{2} \sqrt{(\sigma_{11} - \sigma_{33})^2 + 4\sigma_{13}^2}. \tag{3}
\]

By combination of Eqs. (2) and (3) it then follows that

Eq. (2) + Eq. (3):

\[
\sigma_1 = \frac{1}{2} (\sigma_{11} + \sigma_{33}) + \frac{1}{2} \sqrt{(\sigma_{11} - \sigma_{33})^2 + 4\sigma_{13}^2},
\]

Eq. (2) – Eq. (3):

\[
\sigma_3 = \frac{1}{2} (\sigma_{11} + \sigma_{33}) - \frac{1}{2} \sqrt{(\sigma_{11} - \sigma_{33})^2 + 4\sigma_{13}^2}.
\]

Next, the characteristic equation is obtained from Eq. (1):

\[
\det(\sigma - \sigma_i \mathbf{I})^2 = \sigma_i^2 - (\sigma_{11} + \sigma_{33})\sigma_i + \sigma_{11}\sigma_{33}^2 - \sigma_{13}^2 = 0, \quad i = 1, 3. \tag{4}
\]

Equation (4) has two solutions or roots, \(\sigma_1\) and \(\sigma_3\), defined as

\[
\begin{bmatrix} \sigma_1 \\ \sigma_3 \end{bmatrix} = \frac{1}{2} (\sigma_{11} + \sigma_{33}) \pm \frac{1}{2} \sqrt{(\sigma_{11} - \sigma_{33})^2 + 4\sigma_{13}^2}.
\]
Question 2:

In the three-dimensional case, the characteristic equation reads

\[ \sigma_i^3 - \tilde{I}_i \sigma_i^2 + \tilde{I}_2 \sigma_i - \tilde{I}_3 = 0, \quad i = 1, 2, 3, \tag{5} \]

where the Cauchy invariants of the stress are defined as

\[ \tilde{I}_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}, \]

\[ \tilde{I}_2 = \sigma_{11} \sigma_{22} + \sigma_{22} \sigma_{33} + \sigma_{33} \sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2, \]

\[ \tilde{I}_3 = \det(\sigma). \]

However, by examination of the characteristic equation for the state of plane stress, i.e. Eq. (4), it becomes clear that Eq. (5) should be replaced by

\[ \sigma_i^2 - \tilde{I}_i \sigma_i + \tilde{I}_2 = 0, \quad i = 1, 3, \tag{6} \]

where the two Cauchy invariants for plane stress take the form

\[ \tilde{I}_1 = \sigma_{11} + \sigma_{33} = \sigma_1 + \sigma_3, \]

\[ \tilde{I}_2 = \sigma_{11} \sigma_{33} - \sigma_{13}^2 = \sigma_1 \sigma_3 = \det(\sigma). \]

It is noted that there should obviously only be two stress invariants in the case of plane stress, since the third (or in this case actually the second) principal stress is identically equal to zero.
Exercise 6: Fitting of Drucker-Prager to Mohr-Coulomb

The Drucker-Prager criterion is given as

\[ F(I_1, J_2) = \sqrt{3J_2 + \alpha I_1 - \beta} = 0, \tag{1} \]

where \( I_1 \) is the first invariant of total stresses and \( J_2 \) is the second generic invariant of the deviatoric stresses. Similarly, defining \( \sigma_1 \) and \( \sigma_3 \) as the minimum and maximum principal stresses, respectively, the Coulomb criterion reads

\[ F(\sigma_1, \sigma_3) = (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3) \sin \phi - 2c \cos \phi = 0, \quad \sigma_1 \geq \sigma_2 \geq \sigma_3, \tag{2} \]

This interpretation of the criterion holds for both triaxial tension (\( \sigma_1 > \sigma_2 = \sigma_3 \)) and triaxial compression (\( \sigma_1 = \sigma_2 > \sigma_3 \)).

Question 1:

Firstly, \( \sqrt{3J_2} \) and \( I_1 \) are determined from \( \sigma_1, \sigma_2 \) and \( \sigma_3 \):

\[
I_1 = \sigma_1 + \sigma_2 + \sigma_3 = \begin{cases} \sigma_1 + 2\sigma_3 & \text{for } \sigma_1 > \sigma_2 = \sigma_3 \quad \text{(triaxial tension)} \\ 2\sigma_1 + \sigma_3 & \text{for } \sigma_1 = \sigma_2 > \sigma_3 \quad \text{(triaxial compression)} \end{cases} \]

and by application of the definition \( p = I_1/3 \),

\[
J_2 = \frac{1}{2} (s_1^2 + s_2^2 + s_3^2) = \frac{1}{2} (\sigma_1 - p)^2 + (\sigma_2 - p)^2 + (\sigma_3 - p)^2 \\
= \frac{1}{2} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3)^2 - \frac{2}{3} (\sigma_1 + \sigma_2 + \sigma_3)(\sigma_1 + \sigma_2 + \sigma_3) \\
= \frac{1}{2} \left( \frac{2}{3} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{2}{3} (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1) \right) \\
= \frac{1}{3} (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2, \]

which provides the definition

\[ \sqrt{3J_2} = \sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \sigma_1 - \sigma_3. \]

This definition of \( \sqrt{3J_2} \) is valid in triaxial tension as well as triaxial compression, since the principal stresses are order according to \( \sigma_1 > \sigma_3 \). Thus, the Drucker-Prager criterion may be expressed in terms of the principal stresses:

\[
\begin{cases} (\sigma_1 - \sigma_3) + \alpha_c (\sigma_1 + 2\sigma_3) - \beta_c = 0 & \text{for } \sigma_1 > \sigma_2 = \sigma_3 \quad \text{(triaxial tension)} \\ (\sigma_1 - \sigma_3) + \alpha_c (2\sigma_1 + \sigma_3) - \beta_c = 0 & \text{for } \sigma_1 = \sigma_2 > \sigma_3 \quad \text{(triaxial compression)} \end{cases} \]
which implies that

\[
\begin{cases}
(1 + \alpha_t) \sigma_i = (1 - 2 \alpha_t) \sigma_3 + \beta_t \quad \text{for} \quad \sigma_i > \sigma_2 = \sigma_3 & \text{(tensile meridian)} \\
(1 + 2 \alpha_t) \sigma_i = (1 - \alpha_t) \sigma_3 + \beta_t \quad \text{for} \quad \sigma_i = \sigma_2 > \sigma_3 & \text{(compressive meridian)}
\end{cases}
\]

Similarly, the Coulomb criterion may be expressed as

\[
(1 + \sin \phi) \sigma_i = (1 - \sin \phi) \sigma_3 + 2c \cos \phi, \quad \sigma_i \geq \sigma_2 \geq \sigma_3,
\]

which applies to triaxial tension as well as triaxial compression.

Comparing the Drucker-Prager criterion with the Coulomb criterion for the case of triaxial tension, we then observe that

\[
\frac{1 + \alpha_t}{1 + \sin \phi} = \frac{1 - 2 \alpha_t}{1 - \sin \phi} \Rightarrow (1 - \sin \phi + 2 \sin \phi) \alpha_t = 1 + \sin \phi - 1 + \sin \phi \Rightarrow \alpha_t = \frac{2 \sin \phi}{3 + \sin \phi},
\]

and by similar argumentation we find that for triaxial compression

\[
\frac{1 + 2 \alpha_c}{1 + \sin \phi} = \frac{1 - \alpha_c}{1 - \sin \phi} \Rightarrow \alpha_c = \frac{2 \sin \phi}{3 - \sin \phi}.
\]

Subsequently, the values of \(\beta\) are determined by insertion of \(\sigma_3 = 0\):

**Drucker-Prager:**

\[
\begin{cases}
(1 + \alpha_t) \sigma_i = \beta_t & \text{(tensile meridian)} \\
(1 + 2 \alpha_t) \sigma_i = \beta_t & \text{(compressive meridian)}
\end{cases}
\]

**Coulomb:**

\[
(1 + \sin \phi) \sigma_i = 2c \cos \phi \Rightarrow \sigma_i = \frac{2c \cos \phi}{1 + \sin \phi}
\]

\[
\Rightarrow \begin{cases}
\beta_t = \frac{3 + \sin \phi + 2 \sin \phi}{1 + \sin \phi} \quad \frac{2c \cos \phi}{3 + \sin \phi} = \frac{6c \cos \phi}{3 + \sin \phi} & \text{(tensile meridian)} \\
\beta_c = \frac{3 - \sin \phi + 4 \sin \phi}{1 + \sin \phi} \quad \frac{2c \cos \phi}{3 - \sin \phi} = \frac{6c \cos \phi}{3 - \sin \phi} & \text{(compressive meridian)}
\end{cases}
\]

Thus, to summarise, the following results are obtained:

**Triaxial tension:** \(\alpha_t = \frac{2 \sin \phi}{3 + \sin \phi}; \quad \beta_t = \frac{6c \cos \phi}{3 + \sin \phi}\)

**Triaxial compression:** \(\alpha_c = \frac{2 \sin \phi}{3 - \sin \phi}; \quad \beta_c = \frac{6c \cos \phi}{3 - \sin \phi}\)

Note that the same result is obtained if compression is defined as positive.
Question 2:

In the octahedral plane, a number of geometrical quantities are defined as illustrated in the figure below.

![Octahedral Plane Diagram]

By the formulation of the Drucker-Prager criterion in the answer to the previous question, the values of $\sqrt{3J_2}$ leading to failure in triaxial tension and compression are found as

**Triaxial tension:**

$$a = \sqrt{3J_{2t}} = \beta_t - \alpha_t I_1 = \frac{6c\cos\phi}{3 + \sin\phi} - \frac{2\sin\phi}{3 + \sin\phi} I_1$$

**Triaxial compression:**

$$b = \sqrt{3J_{2c}} = \beta_c - \alpha_c I_1 = \frac{6c\cos\phi}{3 - \sin\phi} - \frac{2\sin\phi}{3 - \sin\phi} I_1$$

It is then evident that

$$\frac{b}{a} = \sqrt{\frac{J_{2c}}{J_{2t}}} = \frac{\rho_c}{\rho_t} = \frac{3 + \sin\phi}{3 - \sin\phi},$$

where $\rho$ denotes the second Haigh-Westergaard coordinate. Subsequently, by application of the cosine relation, the distance $d$ in the figure is found as

$$d^2 = a^2 + b^2 - 2ab \cos 60^\circ = a^2 + b^2 - ab.$$  \hspace{1cm} (4)

The area, $A$, of the triangle with side lengths $a$, $b$ and $d$ can be found as

$$A = \frac{1}{2} dh = \frac{1}{2} ab \sin 60^\circ = \frac{\sqrt{3}}{4} ab.$$  \hspace{1cm} (5)
Then, by a comparison of Eqs. (4) and (5) we find:

\[ d^2h^2 = (a^2 + b^2 - ab)h^2 = \frac{3}{4}a^2b^2 \quad \Rightarrow \quad \left( \frac{a + b - 1}{a} \right)h^2 = \frac{3}{4}ab \]  

(6)

Insertion of Eq. (3) into Eq. (6) provides

\[ \left( \frac{3 - \sin \phi + 3 \sin \phi}{3 - \sin \phi} \right)h^2 = \frac{3}{4}ab \quad \Rightarrow \quad h^2 = \frac{9 - \sin^2 \phi}{4(3 + \sin^2 \phi)} ab. \]  

(7)

Insertion of \( a \) and \( b \) leads to the following definition of the distance \( h \):

\[ h^2 = \frac{9 - \sin^2 \phi}{4(3 + \sin^2 \phi)} \left( 6c\cos \phi - \frac{2 \sin \phi}{3 + \sin \phi} I_1 \right) \left( 6c\cos \phi - \frac{2 \sin \phi}{3 - \sin \phi} I_1 \right). \]

Utilising that \((3 + \sin \phi)(3 - \sin \phi) = 9 - \sin^2 \phi\), this equation simplifies to

\[ h^2 = \frac{6c\cos \phi - 2I_1 \sin \phi}{2} \]  

\[ \Rightarrow \quad h = \frac{3c\cos \phi}{\sqrt{3 + \sin^2 \phi}} - \frac{\sin \phi}{\sqrt{3 + \sin^2 \phi}} I_1 = \sqrt{3J_{2\text{min}}} . \]

Here, \( \sqrt{3J_{2\text{min}}} \) is identified as the value of \( \sqrt{3J_2} \) leading to failure for the inscribed Drucker-Prager criterion. Since this criterion may be expressed as

\[ F_{\text{min}}(I_1, J_2) = \sqrt{3J_{2\text{min}}} + \alpha_{\text{min}} I_1 - \beta_{\text{min}} = 0 , \]

it should be evident that we may summarise the results as

- **Triaxial tension**: \( \alpha_i = \frac{2 \sin \phi}{3 + \sin \phi}; \quad \beta_i = \frac{6c\cos \phi}{3 + \sin \phi} \)
- **Triaxial compression**: \( \alpha_c = \frac{2 \sin \phi}{3 - \sin \phi}; \quad \beta_c = \frac{6c\cos \phi}{3 - \sin \phi} \)
- **Inscribed D - P**: \( \alpha_{\text{min}} = \frac{\sin \phi}{\sqrt{3 + \sin^2 \phi}}; \quad \beta_{\text{min}} = \frac{3c\cos \phi}{\sqrt{3 + \sin^2 \phi}} \)

Comparing the results for the three different cases, it is observed that a general relationship exists between the two parameters \( \alpha \) and \( \beta \) of the Drucker-Prager criterion:

\[ \beta = \frac{3\alpha}{\tan \phi} . \]

Finally it is noted that the inscribed Drucker-Prager with the properties \( \alpha_{\text{min}} \) and \( \beta_{\text{min}} \) does actually correspond to plane strain. However, it is not possible to define a Drucker-Prager criterion that leads to the correct bearing capacity of a circular foundation subject to vertical loading, since some parts of the soil will fail in triaxial tension while other parts of the soil fail in triaxial compression.
Exercise 7: Tresca criterion with linear hardening

Question 1:

The Tresca criterion is illustrated in the figure below in both the Mohr’s diagram and the octahedral plane.

![Mohr’s diagram and Octahedral plane](image)

The plastic strain increment is orthogonal to the yield surface since associated flow is assumed. However, at a corner in the octahedral plane, the normal to the yield surface is not uniquely defined. This implies a problem with the mathematical and numerical treatment of the Tresca criterion, which is not present in the von Mises model.

Question 2:

For a stress point on the yield surface, plastic deformation occurs when \( f = 0 \) and \( \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} \geq 0 \). Now, since the plastic strain increment is \( d\varepsilon_{ij}^p = d\lambda \frac{\partial f}{\partial \sigma_{ij}} \) for associated plasticity, it follows that \( d\varepsilon_{ij}^p d\sigma_{ij} \geq 0 \). Here use has been made of the fact that \( d\lambda \) is a positive scalar. Hence, non-negative work is produced by the stress increments and the plastic strain increments, both in neutral loading \( (d\sigma_{ij} = 0) \) and during hardening. Finally, if \( f = 0 \) and \( \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} < 0 \), the material undergoes elastic unloading. Here \( d\sigma_{ij} = C_{ijkl} d\varepsilon_{kl} \), and since the elastic tensor \( C \) with the components \( C_{ijkl} \) is positive definite it follows that \( d\varepsilon_{ij} d\sigma_{ij} \geq 0 \) in the elastic case.
Question 3:

The yield criterion is given in terms of \((\sigma_I, \sigma_{III})\), but a formulation in terms of the Cartesian stresses \((\sigma_{11}, \sigma_{33}, \sigma_{13})\) is wanted. By Mohr’s circles we get:

\[
(\sigma_I - \sigma_{III})^2 = (\sigma_{11} - \sigma_{33})^2 + 4\sigma_{13}^2 \Rightarrow \frac{1}{2}(\sigma_I - \sigma_{III}) = \sqrt{\left(\frac{\sigma_{11} - \sigma_{33}}{2}\right)^2 + \sigma_{13}^2}.
\]

Thus, the Tresca criterion enters the form

\[
f(\sigma, m) = \sqrt{\left(\frac{\sigma_{11} - \sigma_{33}}{2}\right)^2 + \sigma_{13}^2} - m s_u = 0.
\]

Question 4:

Alternatively, the Tresca criterion may be expressed as

\[
f(q, \sigma_{13}, m) = \sqrt{q^2 + \sigma_{13}^2} - m s_u = 0, \quad q = \frac{\sigma_{11} - \sigma_{33}}{2},
\]

or, by a further substitution, in the simple form

\[
f(t, m) = t - m s_u = 0, \quad t = \sqrt{q^2 + \sigma_{13}^2}.
\]

The elastoplastic constitutive tensor may now be derived. Since we consider the case of plane strain, we shall do this in a matrix–vector format with the three stress components \((\sigma_{11}, \sigma_{33}, \sigma_{13})\) and the conjugate strain components \((\varepsilon_{11}, \varepsilon_{33}, \gamma_{13})\):

\[
C^p = C - \frac{C v v^T C}{H + v^T C v}, \quad v = \left(\frac{\partial f}{\partial \sigma_{11}}, \frac{\partial f}{\partial \sigma_{33}}, \frac{\partial f}{\partial \sigma_{13}}\right)^T,
\]

where the hardening modulus is given as \(H = -\frac{\partial f}{\partial m} \frac{dm}{d\lambda}\) and the elastic stiffness is

\[
C = \begin{bmatrix}
K + \frac{4}{3}G & K - \frac{2}{3}G & 0 \\
K - \frac{2}{3}G & K + \frac{4}{3}G & 0 \\
0 & 0 & G
\end{bmatrix}.
\]

Subsequently, the numerator and the denominator in the plastic reduction term are determined. After a few manipulations we get:
\[
v = \frac{1}{2t} \left( q - q_{{13}} \right)^{T} \Rightarrow \quad \mathbf{ Cv} = \frac{G}{t} \left( q - q_{{13}} \right)^{T},
\]
which implies that
\[
\mathbf{v}^{T} \mathbf{ Cv} = \frac{G}{2t^{2}} \left( q^{2} + q^{2} + \sigma_{{13}}^{2} \right) = \frac{G}{2t^{2}} \left( 2t^{2} \right) = G.
\]
Similarly, the following expression is obtained:
\[
\mathbf{Cvv}^{T} \mathbf{C} = \mathbf{C} \left( \mathbf{C} \right)^{T} = \frac{G^{2}}{t^{2}} \mathbf{A}, \quad \mathbf{A} = \begin{bmatrix}
q^{2} & -q^{2} & q_{13} \\
-q^{2} & q^{2} & -q_{13} \\
q_{13} & -q_{13} & \sigma_{{13}}^{2}
\end{bmatrix}.
\]
Thus, the elastoplastic matrix for plane strain may be written as
\[
\mathbf{C}^{\text{ep}} = \mathbf{C} - \frac{G^{2}}{t^{2} \left( H + G \right)} \mathbf{A} = \mathbf{C} - \frac{G}{t^{2} \left( 1 + H / G \right)} \mathbf{A} = \mathbf{C} - \beta \mathbf{A}, \quad \beta = \frac{G}{t^{2} \left( 1 + H / G \right)}.
\]
By insertion of \( \mathbf{C} \) and \( \mathbf{A} \), the full matrix becomes:
\[
\mathbf{C}^{\text{ep}} = \begin{bmatrix}
\left( K + \frac{4}{3} G \right) - \beta q^{2} & \left( K - \frac{2}{3} G \right) + \beta q^{2} & -\beta q_{13} \\
\left( K - \frac{2}{3} G \right) + \beta q^{2} & \left( K + \frac{4}{3} G \right) - \beta q^{2} & \beta q_{13} \\
-\beta q_{13} & \beta q_{13} & G - \beta \sigma_{{13}}^{2}
\end{bmatrix}.
\]

**Question 5:**

From Equation (1) we know that
\[
f \left( \sigma_{1}, \sigma_{II}, \sigma_{III}, m \right) = \frac{1}{2} \left( \sigma_{1} - \sigma_{III} \right) - m s_{u}, \quad \sigma_{1} \geq \sigma_{II} \geq \sigma_{III}.
\]
We then obtain the result
\[
d\mathbf{e}_{p} = \begin{bmatrix} d\varepsilon_{1}^{p} \\ d\varepsilon_{II}^{p} \\ d\varepsilon_{III}^{p} \end{bmatrix}^{T} = d\lambda \begin{bmatrix} \frac{\partial f}{\partial \sigma_{1}} \\ \frac{\partial f}{\partial \sigma_{II}} \\ \frac{\partial f}{\partial \sigma_{III}} \end{bmatrix}^{T} = d\lambda \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}^{T},
\]
which implies that \( d\varepsilon_{1}^{p} = \frac{1}{2} d\lambda \) and \( d\varepsilon_{III}^{p} = -\frac{1}{2} d\lambda \), so that \( d\lambda = d\varepsilon_{III}^{p} - d\varepsilon_{II}^{p} \). It is noted that \( d\varepsilon_{v}^{p} = d\varepsilon_{1}^{p} + d\varepsilon_{II}^{p} + d\varepsilon_{III}^{p} = 0 \). Hence, there is no volumetric plastic strain.
Finally, by Mohr’s circle we may determine the strain increments in Cartesian coordinates:

\[
\left( d\varepsilon_i^p - d\varepsilon_{ii}^p \right)^2 = \left( d\varepsilon_{11}^p - d\varepsilon_{33}^p \right)^2 + \left( 2d\varepsilon_{13}^p \right)^2 = \left( d\gamma_{13}^p \right)^2
\]

⇒

\[
d\lambda = d\varepsilon_i^p - d\varepsilon_{ii}^p = d\varepsilon_p, \quad d\varepsilon_p = \sqrt{\left( d\varepsilon_{11}^p - d\varepsilon_{33}^p \right)^2 + \left( 2d\varepsilon_{13}^p \right)^2}.
\]

Hence, the hardening modulus achieves the form:

\[
H = \frac{\partial f}{\partial m} \frac{dm}{d\lambda} = \frac{ds_u}{d\varepsilon_p}.
\]

**Question 6:**

Based on the laboratory tests, the following material properties have been determined:

\[
K = 20 \text{ MPa}, \quad G = 10 \text{ MPa}, \quad s_u = 50 \text{ kPa}, \quad m = 0.6, \quad H = 8 \text{ MPa}.
\]

Initial yielding occurs when

\[
f(\sigma_i, \sigma_{ii}, \sigma_{iii}, m) = \frac{1}{2}(\sigma_i - \sigma_{iii}) - ms_u = \frac{1}{2}\sigma_i - 0.6 \cdot 50 = 0 \quad \Rightarrow \quad \sigma_i = 60 \text{ kPa}.
\]

In order to find the axial strain \( \varepsilon_i \) and the volumetric strain (dilation) \( \varepsilon_v \), we determine Young’s modulus, \( E \), and Poisson’s ratio, \( \nu \):

\[
\begin{align*}
G &= \frac{E}{2(1+\nu)} = 10 \text{ MPa} \\
K &= \frac{E}{3(1-2\nu)} = 20 \text{ MPa}
\end{align*}
\]

\[
\Rightarrow \left\{ \begin{array}{l}
E = \frac{9KG}{3K + G} = 25.71 \text{ MPa} \\
\nu = \frac{3K - 2G}{6K + 2G} = 0.2857
\end{array} \right.
\]

Then, in the uniaxial state of stress, we get the axial and volumetric strain:

\[
\varepsilon_i = \frac{\sigma_i}{E} = \frac{60}{25.71 \cdot 10^3} = 0.2333\% \quad \text{and} \quad \varepsilon_v = \varepsilon_i(1-2\nu) = 0.1\%,
\]

Where it has been utilised that \( \varepsilon_{ii} = \varepsilon_{iii} = -\nu\varepsilon_i \) and \( \varepsilon_v = \varepsilon_i + \varepsilon_{ii} + \varepsilon_{iii} \) in the state of triaxial compression.

Next, ultimate failure occurs when

\[
f(\sigma_i, \sigma_{ii}, \sigma_{iii}, m) = \frac{1}{2}(\sigma_i - \sigma_{iii}) - ms_v = \frac{1}{2}\sigma_i - 1 \cdot 50 = 0 \quad \Rightarrow \quad \sigma_i = 100 \text{ kPa}.
\]

Here, the axial strain is found as the sum of two contributions, i.e. a plastic and an elastic part. Firstly, since the hardening modules is assumed constant, we get:
\[ H = s_u \frac{dm}{d\lambda} \Rightarrow d\lambda = \frac{s_u}{H} dm \Rightarrow \Delta \lambda = \frac{s_u}{H} \Delta m = \frac{50}{8 \cdot 10^3} (1 - 0.6) = 0.25 \cdot 10^{-2}. \]

Next, the plastic and elastic contributions to the axial strain are obtained as

\[ \varepsilon_{\lambda}^p = \frac{1}{2} \Delta \lambda = 0.125\% \quad \text{and} \quad \varepsilon_{\lambda}^e = \frac{\sigma_{\lambda}}{E} = \frac{100}{25.71 \cdot 10^3} = 0.3889\%. \]

Finally, the total axial and volumetric strain at ultimate failure

\[ \varepsilon_\lambda = \varepsilon_{\lambda}^e + \varepsilon_{\lambda}^p = 0.3889\% + 0.125\% = 0.5139\% \quad \text{and} \quad \varepsilon_v = \varepsilon_{\lambda} (1 - 2\nu) = 0.1667\%. \]

Thus, with \( \sigma_\lambda = \sigma_1 \) and \( \sigma_{III} = \sigma_{III} = 0 \) we get the stress–strain curve and the volumetric strain curve in the figure below.

It is noted that there is no plastic contribution to the volumetric strain. This may not be obvious from the definition of the yield function that is also applied as plastic potential. Firstly, it is observed that \( \Delta \varepsilon_{III}^p = -\Delta \varepsilon_{\lambda}^p \), which follows immediately by partial differentiation of \( f \) with respect to \( \sigma_1 \) and \( \sigma_{III} \). However, in the state of triaxial compression we are on the compressive meridian defined by \( \sigma_1 > \sigma_{II} = \sigma_{III} \). Intuitively, since the yield criterion is isotropic and \( \sigma_{II} = \sigma_{III} \), we must have \( \Delta \varepsilon_{II}^p = \Delta \varepsilon_{III}^p \). Hence, in accordance with the figure in the answer to Question 1, the plastic strain increment \( d\varepsilon_p \) points in the \( s_1 \)-direction. This means that two yield criteria are active at the same time and with the same weight. If, for example, \( \sigma_1 = \sigma_1 \), we have

\[ f_2(\sigma_1, \sigma_2, \sigma_3, m) = \frac{1}{2} (\sigma_1 - \sigma_2) - m s_u \quad \text{and} \quad f_3(\sigma_1, \sigma_2, \sigma_3, m) = \frac{1}{2} (\sigma_1 - \sigma_3) - m s_u \]

Then, according to the discussion above, the active plastic potential is \( g = (f_2 + f_3)/2 \), and it is easily proved that in this case \( \Delta \varepsilon_{II}^p = \Delta \varepsilon_{III}^p = -\Delta \varepsilon_{\lambda}^p / 2 \) such that the plastic contribution to the volumetric strain becomes \( \Delta \varepsilon_v^p = \Delta \varepsilon_{\lambda}^p + \Delta \varepsilon_{II}^p + \Delta \varepsilon_{III}^p = 0 \).
Exercise 8: Interpretation of the Cam-clay model

Question 1:

In the \((p', q)\) diagram below, the critical state line is presented as the dashed line. Stresses on the \(p'\) and \(q\) axes are given in kPa. The modified Cam-Clay failure envelopes are plotted at the pre-consolidation pressure \(p_f = 100, 200\) and \(300\) kPa. Since \(M = 1\), the failure envelopes are circular.

Associate flow is assumed in the modified Cam-Clay theory, i.e. \(g = f\). Therefore, the plastic strain increments, \(de_p\), are orthogonal to the yield surface defined by \(f = 0\). In particular, at the stress point on the intersection between the failure envelope and the critical state line, the plastic strain increment is in the \(q\)-direction. Hence, at the critical line there is no plastic volumetric strain, i.e. \(de_v^p = 0\). Further, it follows by the definition of the critical state, that there is no change of the volume, i.e. \(de_v = 0\). Thus, according to the definition of the strain in elastoplasticity,

\[
de_v = de_v^e + de_v^p \quad \Rightarrow \quad de_v^e = de_v - de_v^p = 0 \quad \Rightarrow \quad dp' = 0.
\]
Use has been made of the fact that, in the modified Cam-Clay theory, \( dp' = (p'/\lambda) \, d\varepsilon_v \) in primary loading and \( dp' = (p'/\kappa) \, d\varepsilon_v \) in unloading/reloading. Hence, the effective mean stress does no change. Further, since the stress point stays on the critical line, the effective stress difference \( q \) will not change either.

**Question 2:**

Below the \((p', q)\) diagram, the primary loading line (or normal consolidation line) has been plotted in a \((p', 1 - \varepsilon_v)\) diagram. The slope of the primary loading line is defined by the dimensionless flexibility parameter \( \lambda \) and the points on the curve for \( p_f = 100, 200 \) and \( 300 \) kPa have been identified in the figure.

**Question 3:**

The slope of the elastic unloading/reloading lines in the \((p', 1 - \varepsilon_v)\) diagram is defined by the dimensionless flexibility parameter \( \kappa \). The three unloading/reloading lines corresponding to \( p_f = 100, 200 \) and \( 300 \) kPa have been drawn in the figure.

For any given value of the pre-consolidation pressure, \( p_f \), the critical state is reached for \( p = p_f / 2 \). Thus, the unloading/reloading lines intersect the representation of the critical state line in the \((p', 1 - \varepsilon_v)\) diagram as illustrated in the figure.
Exercise 9: Cam-clay model – drained and undrained tests

Question 1:

In a drained test, the pore water is allowed to drain away freely. Accordingly, the additional pressure applied by the piston in a triaxial compression test is carried by the soil skeleton as effective stresses. Thus the stress path indicated as $ABC$ with the slope 3:1 in the $(p', q)$ diagram below is obtained. The ultimate value of the piston pressure is recorded at the intersection between the stress path $ABC$ and the critical state line with slope $M = 1$, i.e. at the stress point $C$:

$$(p_u', q_u) = (300 \text{ kPa}, 300 \text{ kPa}).$$

In the first part of this stress path, $AB$, the response is elastic reloading. However, in the last part of the stress path, $BC$, the response is elastoplastic. Hardening occurs during the deformation from point $B$ to point $C$.

![Diagram showing stress paths and critical state line](image)

Now, in order to plot the stress–strain curve in a $(p', 1 - \varepsilon_v)$ diagram use is made of the fact that the final state represented by point $C$ may as well be reached by going through the stress path $ADEC$. Elastic unloading occurs along $AD$, while primary loading occurs along $DE$. Finally, along the stress path $EC$ there is elastic unloading.
The total volumetric strain can now be determined in the following manner:

\[ \Delta \varepsilon_{v}^{AD} = \kappa \left\{ \ln (p' + \Delta p') - \ln p' \right\} = 0.05 \left\{ \ln 300 - \ln 200 \right\} = 0.0203 \]

\[ \Delta \varepsilon_{v}^{DE} = \lambda \left\{ \ln (p' + \Delta p') - \ln p' \right\} = 0.25 \left\{ \ln 600 - \ln 300 \right\} = 0.1732 \]

\[ \Delta \varepsilon_{v}^{EC} = \kappa \left\{ \ln (p' + \Delta p') - \ln p' \right\} = 0.05 \left\{ \ln 300 - \ln 600 \right\} = -0.0347 \]

Total: \[ \Delta \varepsilon_{v}^{AC} = \Delta \varepsilon_{v}^{AD} + \Delta \varepsilon_{v}^{DE} + \Delta \varepsilon_{v}^{EC} = 0.0203 + 0.1732 - 0.0347 = 0.1589 \]

Thus, the total dilatation during the drained triaxial compression step is \( \Delta \varepsilon_{v} = 0.1589 \). Since compression is defined as positive, this corresponds to a decrease in volume.

**Question 2:**

In an undrained test, the pore water cannot drain away. Therefore the volume of the specimen is constant. In the first part of the triaxial compression test, the response is elastic reloading where \( d \varepsilon_{v} = d \varepsilon_{v}^{e} = 0 \). Accordingly, \( dp' = \frac{p'}{\kappa} d \varepsilon_{v}^{e} = 0 \). This provides the stress path \( AB \) plotted in the \( (p', q) \) diagram below, whereas state \( A \) and \( B \) coincide in the corresponding \( (p', 1 - \varepsilon_{v}) \) diagram.
Next, when \( q \) is increased beyond the failure envelope for \( p_f = 300 \) kPa, hardening occurs so that the stress point stays on the yield surface. At stress point \( A \) in the \((p', q)\) diagram, the plastic volumetric strain increment is positive, i.e. \( d\varepsilon^p_v > 0 \). However, due to undrained behaviour the volume of the cell cannot change. Therefore,

\[
d\varepsilon_v = d\varepsilon^e_v + d\varepsilon^p_v = 0 \quad \Rightarrow \quad d\varepsilon^e_v = -d\varepsilon^p_v < 0 \quad \Rightarrow \quad dp' < 0.
\]

Thus, during hardening the effective mean pressure \( dp' \) is decreased. This continues until the critical state is reached, leading to the line \( BC \) in the \((p', 1 - \varepsilon_v)\) diagram and the corresponding stress path \( BC \) in the \((p', q)\) diagram. As indicated in the figure, the final value of the pre-consolidation pressure is unknown. In order to determine the value of the stress difference \( q_u \) at ultimate failure, firstly \( p_f = x \) has to be found.

In order to compute \( x \) we shall exploit the fact that state \( C \) may alternatively be reached by following the stress path \( ADEC \) in the \((p', q)\) diagram and the curve \( ADEC \) in the \((p', 1 - \varepsilon_v)\) diagram. Along each part of this curve, the volumetric strain can be evaluated as:

\[
\begin{align*}
AD: \quad \Delta\varepsilon^D_v &= \kappa \{\ln (p' + \Delta p') – \ln p'\} = 0.05 \{\ln 300 – \ln 200\} = 0.0203 \\
DE: \quad \Delta\varepsilon^E_v &= \lambda \{\ln (p' + \Delta p') – \ln p'\} = 0.25 \{\ln x - \ln 300\} \\
EC: \quad \Delta\varepsilon^C_v &= \kappa \{\ln (p' + \Delta p') – \ln p'\} = 0.05 \{\ln x/2 – \ln x\} = -0.0347 \\
Total: \quad \Delta\varepsilon^C_v &= \Delta\varepsilon^D_v + \Delta\varepsilon^E_v + \Delta\varepsilon^C_v = 0.0203 + \Delta\varepsilon^E_v - 0.0347 = 0
\end{align*}
\]

Hence, it follows that

\[
\Delta\varepsilon^E_v = 0.25 \{\ln x - \ln 300\} = 0.0347 - 0.0203 = 0.0143 \quad \Rightarrow \quad x = 317.8.
\]

The value of the effective mean stress at the intersection of the final yield surface with the critical state line is \( p_u' = x/2 = 158.9 \) kPa. Since \( M = 1 \), ultimate failure occurs at the stress point

\[
(p_u', q_u) = (158.9 \text{ kPa}, 158.9 \text{ kPa}).
\]

Evidently, the curves in the figure are a bit disturbed, since they suggest a value of \( q_u \) which is somewhat higher.

**A final remark:** The maximum shear stress in the triaxial compression test is computed as \( \tau_{\text{max}} = (\sigma_1 - \sigma_3)/2 = q/2 \), where \( \sigma_1 \) and \( \sigma_3 \) are the larger and the smaller principal stress corresponding to the piston pressure and the chamber pressure, respectively. Hence, the shear strength of the material is different in drained and undrained conditions. In the present case, the following values are obtained:

- **Drained triaxial compression:** \( \tau_{\text{max}} = s_d = 150.0 \text{ kPa} \),
- **Undrained triaxial compression:** \( \tau_{\text{max}} = s_u = 79.5 \text{ kPa} \).

Thus a significantly greater strength is observed in drained conditions than is obtained in undrained conditions. Therefore, short-term loading is critical.
Exercise 10: Mohr-Coulomb model with linear hardening

The solution is given by the following MATLAB code.

```matlab
% Exercise 10 - Solution

clear all; close all; clc

%% Initial setup

c = 14E3; % cohesion
phi = deg2rad(39); % angle of friction
r_0 = deg2rad(31); % mobilised angle of friction
psi = deg2rad(17); % angle of dilatation
nu = 0.33; % Poisson's ratio
E = 530E6; % Young's modulus
H = 65; % hardening modulus
s_3 = 80E3; % chamber pressure

%% Question 1

% The yield criterion provides:

0.5*(s_1-s_2) - 0.5*(s_1+s_2)*sin(r_0) - c*cos(r_0) = 0 <=>

(1-sin(r_0))*s_1 - (1+sin(r_0))*s_2 - 2*c*cos(r_0) = 0 <=>

s_1 = ((1+sin(r_0))*s_3 + 2*c*cos(r_0))/(1-sin(r_0)) ;

q_0 = s_1 - s_3 ; % piston pressure at initial yielding

disp(['1: Piston pressure at initial yielding: q_0 = ' num2str(q_0)])

% The failure criterion gives:

0.5*(s_1-s_2) - 0.5*(s_1+s_2)*sin(phi) - c*cos(phi) = 0 <=>

(1-sin(phi))*s_1 - (1+sin(phi))*s_2 - 2*c*cos(phi) = 0 <=>

s_1 = ((1+sin(phi))*s_3 + 2*c*cos(phi))/(1-sin(phi)) ;

q_u = s_1 - s_3 ; % piston pressure at ultimate failure

disp(['   Piston pressure at ultimate failure: q_u = ' num2str(q_u)])

%% Question 2

e_1_0 = q_0/E ; % axial strain at initial yielding

disp(['2: Axial strain at initial yielding: e_1_0 = ' num2str(e_1_0)])

% The elastic strains for a uniaxial stress increment are related as

% e_2 = e_3 = - nu*e_1
% e_v = e_1 + e_2 + e_3 = (1-2*nu)*e_1

e_v_0 = (1-2*nu)*e_1_0 ; % volumetric strain at initial yielding

disp([' Volu. strain at initial yielding: e_v_0 = ' num2str(e_v_0)])
```

---

% Exercise 10 - Solution

"Prepare and serve a mathematical meal that is both appetizing and nourishing."
Question 3
Firstly, the elastic contribution to the deformation is computed:

\[ e_{1e} = \frac{q_u}{E} ; \quad \text{elastic axial strain at failure} \]
\[ e_{ve} = (1-2\nu)e_{1e} ; \quad \text{elastic volumetric strain at failure} \]

Next, the increase in the mobilised angle of friction is:

\[ dsinr = \sin(\phi) - \sin(\theta_0) ; \]

The increment in the plastic multiplier, \( dL \), is then determined by

\[ H = \frac{d(\sin(\gamma))}{dL} \Rightarrow dL = \frac{dsinr}{H} ; \]

Now, the plastic increment in volumetric strain is determined as

\[ e_{vp} = -dL \cdot \sin(psi) ; \quad \text{plastic volumetric strain at failure} \]

The plastic increment in axial strain is found by the relation

\[ \sin(psi) = \frac{de_{vp}}{de_{vp} - 2*de_{1p}} \Rightarrow 2*de_{1p} = de_{vp} \cdot (\sin(psi) - 1)/\sin(psi) \Rightarrow e_{1p} = e_{vp} \cdot (\sin(psi) - 1)/(2*\sin(psi)) ; \quad \text{plastic axial strain at failure} \]

Finally, the total strain at ultimate failure is computed:

\[ e_{1u} = e_{1e} + e_{1p} ; \quad \text{total axial strain at failure} \]
\[ e_{ve} = e_{ve} + e_{vp} ; \quad \text{total volumetric strain at failure} \]

```
% Based on this program, the following values are obtained:

Question 1: \( q_0 = 219.4 \text{ kPa}, \quad q_u = 330.3 \text{ kPa} \)

Question 2: \( \varepsilon_{1,0} = 0.041399 \%, \quad \varepsilon_{v,0} = 0.014076 \% \)

Question 3: \( \varepsilon_{1,0} = 0.012454 \%, \quad \varepsilon_{v,0} = -0.030213 \% \)
```
Finally, the stress difference, $q$, and the volumetric strain, $\varepsilon_v$, are plotted as functions of the axial strain, $\varepsilon_1$, in the figure below. Note that beyond the range of strains included in the figure, the $(\varepsilon_1, \varepsilon_v)$-curve has a slope of zero.