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Abstract

A graph $G$ is $P_3$-equipackable if any sequence of successive removals of edge-disjoint copies of $P_3$ from $G$ always terminates with a graph having at most one edge. All $P_3$-equipackable graphs are characterised. They belong to a small number of families listed here.

Keywords: Packing, equipackable, randomly packable, covering, factor, decomposition, equiremovable

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1 Introduction

Let $H$ be a subgraph of a graph $G$. An $H$-packing in $G$ is a partition of the edges of $G$ into disjoint sets, each of which is the edge set of a subgraph of $G$ isomorphic to $H$, and possibly a remainder set. For short, $E(G)$ is partitioned into copies of $H$ and maybe a remainder set. We list some references to an extensive literature at the back. A graph is called $H$-packable if $G$ is the union of edge disjoint copies of $H$. An $H$-packing is maximal if the remainder set of edges is empty or contains no copy of $H$. An $H$-packing is maximum if $E(G)$ has been partitioned into a maximum number of sets isomorphic to $H$ and a possible remainder set. A graph is called $H$-equipackable if any maximal $H$-packing is also a maximum $H$-packing, i.e., a graph $G$ is $H$-equipackable if successive removals of copies of $H$ from $G$ can be done the same number of times regardless of the particular choices of edge sets for $H$ in each step. If every maximal $H$-packing of a graph $G$ uses all edges of $G$, then $G$ is called randomly $H$-packable. Equivalently, $G$ is randomly $H$-packable if each $H$-packing can be extended to an $H$-packing containing all edges of $G$, see e.g. [1, 2, 5, 6].

Zhang and Fan [9] have studied $H$-equipackable graphs for the case $H = 2K_2$. We shall consider path packing and in the following $H$ will always be assumed to be the graph $P_3$, the path of length two, and we may omit explicit reference to it. A graph $G$ is then ($P_3$-) equipackable if successive removals of two adjacent edges from $G$ can be done the same number of times.
regardless of the particular choices of edge pairs in each step. A component consisting of one vertex is called trivial, a non-trivial component contains an edge. A graph has order $|V(G)|$ and size $|E(G)|$. A graph of odd (even) size is called odd (even). A vertex of valency one is called a leaf. A star is called even if its size is even, and by $K_{1,2k}$ we denote the even star with $2k$ leaves.

**Observation 1** A graph is randomly $H$-packable if and only if it is $H$-packable and $H$-equipackable.


**Theorem 2** A connected graph $G$ is randomly packable if and only if $G \cong C_4$ or $G \cong K_{1,2k}$, $k \geq 1$.


**Lemma 3** A connected graph is packable if and only if it has even size.

This immediately implies Corollary 4 below.

**Corollary 4** If a connected graph is equipackable, a maximal packing either contains all edges or all but one edge of the graph.

From B.L. Hartnell, P.D. Vestergaard [4] and P.D. Vestergaard [8] we have the following observation.

**Observation 5** Let $G$ be an equipackable graph. Then any sequence of $P_3$-removals from $G$ will produce an equipackable graph.

From Corollary 4 and Observation 5 we obtain

**Corollary 6** Let $G$ be a connected graph. If there is a sequence of $P_3$-removals from $G$ that creates more than one component of odd size, then $G$ is not equipackable.

We now state our main result, a characterisation of all equipackable graphs with at most one non-trivial component:

**Theorem 7** Let $G = (V,E)$ be a graph with at most one non-trivial component. Then $G$ is equipackable if and only if its non-trivial component belongs to one of the thirteen families listed in Figure 1 or can be obtained by a sequence of $P_3$-removals from such a graph.

Clearly, we wish those thirteen families listed to be maximal w.r.t. $P_3$-removals, i.e., no graph from one of the families can be obtained as a subgraph of a larger equipackable graph by removing a $P_3$ from it.

In the figures below we indicate by an arrow from which family of graphs we may obtain the given graph by a sequence of $P_3$-deletions. The shaded vertex sets may vary in cardinality.
We will prove this characterisation in the following section.

2 Proof of Theorem 7

By Lemma 3 and Theorem 2 a graph with at most one non-trivial component, which has even size, is equipackable if and only if its non-trivial component is a 4-circuit or an even star (Figure 2). Thus it only remains to characterise equipackable graphs with at most one non-trivial component of odd size.

Figure 2: Connected $P_3$-equipackable graphs of even size (Ruiz graphs)
In [8] P.D. Vestergaard examined equipackable graph with all degrees $\geq 2$ and stated the following result.

**Theorem 8** A connected graph $G$ with all degrees $\geq 2$ is equipackable if and only if $G$ is one of the graphs listed in Figure 3.

![Figure 3: All connected $P_3$-equipackable graphs $G$ without leaves](image)

Observe that this solution contributes to our characterisation five graphs ($F_6, F_3, F_4, F_5, F_9$) maximal with respect to $P_3$-removals. All other graphs of this solution are obtained by a sequence of $P_3$-removals from graphs of the thirteen graph families of our characterisation. Thus it now remains to characterise equipackable graphs $G$ which have only one non-trivial component, say $H$, where $H$ has odd size and contains a leaf.

Since $H$ has a leaf, it also has a bridge. Let $b = xy$ be a bridge of $H$. Throughout we shall denote the two components of $H - xy$ by $H_1$ and $H_2$ with $x \in V(H_1), y \in V(H_2)$. We shall first treat the case that $G$ has a non-leaf bridge, then the case that all bridges are leaf bridges.

**Case 1:** Assume $b = xy$ is a non-leaf bridge of $G$, i.e., $\deg(x) \geq 2$, $\deg(y) \geq 2$.

**Subcase 1.1:** Assume further that $H$ has a maximum $P_3$-packing $\mathcal{P}$ which does not contain $b$. Since $\mathcal{P}$ by Corollary 4 contains all but one edge of $G$ and $b \notin \mathcal{P}$, we have for $i = 1, 2$ that $\mathcal{P} \cap H_i$ is a $P_3$-packing of $H_i$ and therefore $H_i$ has even size $\geq 2$.

Let $z \in N(x) \setminus \{y\}$. By $P_3$-removal of $zxy$ we obtain an equipackable graph which has an odd component contained in $H_1 - xz$, and $H - \{zx, xy\}$ also has the even component $H_2$ which is connected, randomly packable and hence, by Observation 1, is either a 4-circuit or an even star. By symmetry also $H_1$ is a 4-circuit or an even star. Therefore $H$ belongs to one of the families of graphs depicted in Figure 4.
Figure 4: Connected, $P_3$-equipackable graphs in Case 1.1

Note that only three new graph families ($F_7, F_8, F_{10}$) maximal with respect to $P_3$-removals contribute in this case to our characterisation. All other graph families of this subcase are obtained by a sequence of $P_3$-removals from graphs of the thirteen graph families of our characterisation.

Subcase 1.2: Assume now that each non-leaf bridge of $H$ is contained in every maximum $P_3$-packing.

With notation as above let $b = xy$ be a non-leaf bridge of $H$, the components of $H - xy$ are $H_1, H_2$. Their sizes have the same parity since $H$ has odd size. If $H_1, H_2$ both had even size they would be $P_3$-packable and $H$ would have a maximum $P_3$-packing not containing $b$ in contradiction to assumption. Therefore $H_1, H_2$ both have odd size.

Claim: At least one of $H_1, H_2$ is an odd star.

Proof. $P_3$-removal from $H$ of $zxy$, $z \in N(x) \setminus \{y\}$, creates an odd size component, namely $H_2$. If $H_2$ is an odd star we are finished. Otherwise, we can isolate an odd component inside $H_2$: If $\deg_{H_2}(y)$ is even we $P_3$-remove all edges incident to $y$ in pairs and if $\deg_{H_2}$ is odd we $P_3$-remove all but one edge incident to $y$ in pairs and that remaining edge $yw, w \in N(y)$, together with $wr, r \in N(w) \setminus \{y\}$ (Since $H_2$ is not an odd star there has to exist at least one such vertex $w$).

Then $H_1 \cup \{xy\}$ is even, connected, randomly packable and hence is either a 4-circuit or an even star. Since $H_1 \cup \{xy\}$ contains a leaf, it is an even star and hence $H_1$ is an odd star. That proves the claim. □

Suppose $H_1$ and $H_2$ are both odd stars. Now assume that, say $x$, is not the center of $H_1$ and let $v$ be the center of $H_1$. Since $vx$ is a non-leaf bridge and there obviously exists a maximum $P_3$-packing $P$ which does not contain $vx$, we obtain a contradiction to the assumption of Subcase 1.2. Hence we find that $H$ is obtained by adding an edge between the centers of $H_1$ and $H_2$ (see Figure 5). Consequently $H$ can be obtained from one of the graphs of the family $F_{12}$ in our characterisation by $P_3$-deletions.
If, say, $H_2$ is an odd star and $H_1$ is not, then $P_3$-removal of $zxy$ from $H$, $z \in N(x) \setminus \{y\}$, gives that $H_1 - xz$ has an even size.

Now assume that $xz$ is a leaf bridge of $H$ (and likewise of $H_1$), i.e., $\deg_H(z) = 1$.

Then $P_3$-removal of $zxy$ leaves the odd component $H_2$ and $H_1 - xz$ with one non-trivial even component. Thus the non-trivial even component of $H_1 - xz$ is either a 4-circuit or an even star. The former yields easily a non-equipackable graph, the latter gives that $H_1$ is an odd star, a contradiction to assumption on $H_1$.

Suppose now that $xz$ is a non-leaf bridge of $H$ (and likewise for $H_1$).

The two components of $H_1 - xz$ have sizes of same parity. That cannot be odd since $G - zxy$ would then have three odd components in contradiction to Corollary 6. It cannot be even either because then we could easily construct a maximum $P_3$-packing $P$ which does not contain the non-leaf bridge $xz$, a contradiction to the basic assumption of this subcase.

So we may for all $z \in N(x) \setminus \{y\}$ assume that $xz$ is not a bridge of $H$ (and $H_1$).

$P_3$-removal of $zxy$ for $z \in N(x) \setminus \{y\}$ produces the connected, even component $H_1 - xz$ which is then randomly $P_3$-packable and hence is either an even star or a 4-circuit. If $H_1 - xz$ is a 4-circuit we are immediately led to $H$ not being equipackable because, if $a, b, c, d$ are the edges of this 4-circuit (in cyclic order) then the packing $\{xy, a\}, \{xz, c\}$ cannot be extended to a maximum packing of $H$. Observe that we have $N(x) \setminus \{y\} = \{z_1, z_2, \ldots, z_p\}$ with $z_1 = z$ and $p \geq 2$. Thus for all $z_i \in N(x) \setminus \{y\}$ the connected graph $H_1 - xz_i$ is an even star.

It follows that $p = 2$ and $H_1 - xz_1$ must always be isomorphic to a $P_3 = K_{1,2}$ with a center vertex $z_{3-i}$ having neighbours $x$ and $z_i$ for $i = 1, 2$. Thus $H_1$ is a 3-circuit with vertices $x, z_1, z_2$ with $x$ joined to $y$, and $y$ has an odd number of leaves attached (see Figure 5).

![Figure 5: Connected $P_3$-equipackable graphs in Case 1.2](image)

Observe that none of these equipackable graph families are new families maximal with respect to $P_3$-removals for our characterisation. Both graph families of this subcase are obtained by a sequence of $P_3$-removals from graphs of the graph families $(F_{12}, F_{13})$ of our characterisation. We may now assume that there exist no non-leaf bridge of $H$.

**Case 2:** All bridges of $H$ are leaf bridges and there exists at least one bridge $b = xy$ of $H$, i.e. $H_2 = \{y\}$.

If all $xz, z \in N(x) \setminus \{y\}$, are bridges of $H$, then they are leaf bridges and $H$ is an odd star, derivable from a member of our characterisation by a sequence of $P_3$-removals. Thus we may assume that $x$ is contained in at least one cycle of $H_1$ and there exist at least two edges incident to $x$, which are not bridges.

If $x$ has an even number of neighbours in $H_1$ we can isolate $xy$ by pairing up and $P_3$-removing all $xz, z \in N(x) \setminus \{y\}$. If $x$ has an odd number of neighbours in $H_1$ we isolate $xy$ by $P_3$-removing
all \( xz, z \in N(x) \setminus \{y\} \), and one further edge \( zw, w \in N(z) \setminus \{x\} \) (observe that such an edge has to exist). For simplicity, let \( E' \) be the set of edges of all \( P_3 \)'s necessary to remove in order to isolate the bridge \( xy \) and \( H' = H - E' \). Since \( xy \) is isolated in \( H' \) and \( H' \) is equipackable, we obtain by Lemmas 3, Observation 5 and Corollaries 4, 6 that all non-trivial components \( D \) not containing \( x \) and \( y \) are randomly packable and therefore of even size \( \geq 2 \). Thus every such non-trivial component \( D \) is either a 4-circuit or an even star.

Assume that one of these components is a 4-circuit \( C \) with vertices \( \{e_0, e_1, e_2, e_3\} \) and edges \( \{e_i e_{(i+1) \mod 4} \mid 0 \leq i \leq 3\} \).

As all bridges of \( H \) are leaf bridges, with \( E_C = \{xc \mid c \in N_{H_1}(x) \cap V(C)\} \) we have \( |E_C| \geq 2 \). It is easy to see that we can remove two (if \( |E(C)| = 2 \)) or three \( P_3 \)'s from the subgraph of \( H \) induced by \( \{x\} \cup V(C) \) to produce two (if \( |E(C)| = 3 \)) or three isolated edges (including \( xy \)) in contradiction with Corollary 6.

If \( |E_C| = 2 \) there exist \( i, j, k, \ell = \{0, 1, 2, 3\} \) such that \( xe_i, xe_j \in E_C \) and \( xe_k e_\ell \) are \( P_3 \)'s of \( H \) that isolate the two independent edges \( e_0 e_k, e_1 e_\ell \) remaining in \( C \). By Corollary 6 then \( H \) is not equipackable, a contradiction. If \( |E_C| = 3 \), without loss of generality we may assume that \( E_C = \{xe_0, xe_1, xe_2\} \) and in that case \( E' \cup \{e_0 e_3, e_1 e_2\} \) is an edge set of even size, which can paired up in \( P_3 \)'s whose removal isolate two remaining edges \( e_0 e_1 \) and \( e_2 e_3 \) on \( C \), by Corollary 6 that contradicts \( H \) being equipackable. If \( |E_C| = 4 \), again \( E' \cup \{e_0 e_3, e_1 e_2\} \) has even size and can be paired up and \( P_3 \)-removed to leave two independent edges \( e_0 e_1 \) and \( e_2 e_3 \) on \( C \), giving a contradiction to \( H \) being equipackable.

Hence every such non-trivial component \( D \) not containing \( x \) and \( y \) is an even star.

Now suppose there exist two different components \( R_1 \) and \( R_2 \) of this kind. Analogously to the previous argumentation let \( E_{R_i} \) be the subset of \( E' \) of edges incident to the vertices of \( R_i \) for \( i = 1, 2 \). Since \( H \) is connected, and all bridges of \( H \) are leaf bridges there has to exist for each \( i = 1, 2 \) at least two edges \( f_i', f_i'' \) of \( E_{R_i} \) adjacent to an edge of \( R_i \). Pairing up \( f_i' \) with one edge of \( E(R_i) \), say \( f_i \), \( i = 1, 2 \), and \( P_3 \)-removing all remaining edges of \( E' \) (their number is even, recall that \( f_i \notin E' \)) will isolate two odd stars \( E_{R_1} - f_1 \) and \( E_{R_2} - f_2 \), a contradiction to Corollary 6. Thus there exists only one non-trivial component \( R \) of \( H' \) not containing \( x \) and \( y \), and that is an even star.

We now distinguish between two cases depending on the parity of \( \deg_{H_1}(x) \). Assume \( \deg_{H_1}(x) \) is even. Then obviously \( H \) is, regardless of whether the centre \( r \) of \( R \) is adjacent to \( x \) or not, a member of the graph family \( F_{12} \) or can be obtained by a sequence of \( P_3 \)-removals from a member of \( F_{12} \).

Now it remains to consider that \( \deg_{H_1}(x) \) is odd, i.e. \( d_H(x) \) is even. As already noted at the beginning of Case 2 the vertex \( x \) must be contained in at least one cycle of \( H_1 \) and there exist at least two edges incident to \( x \), which are not bridges. Since \( R \) is an even star \( K_{1,2l} \) with \( l \geq 1 \) it is not difficult to deduce that the cycle has length \( \leq 5 \). First let \( R \) be a star with at least four branches. Recall that \( E' \) is the set of edges of all \( P_3 \)'s necessary to remove in order to isolate the bridge \( xy \) and let \( H' = H - E' \). Moreover, since \( x \) has an odd number of neighbours in \( H_1 \) we isolate \( xy \) by \( P_3 \)-removing all \( xz, z \in N(x) \setminus \{y\} \), and one further edge \( zw, w \in N(z) \setminus \{x\} \). Regardless of the choice of this additional edge \( zw \) the remainder will be an even star with at least four edges. Concatenation of all ingredients builds up a member of \( F_{12} \) or a graph that can be obtained by a sequence of \( P_3 \)-removals from a member of \( F_{12} \). Therefore we conclude that \( R \) is always an even star with two branches regardless of the choice of the additional edge \( zw \). By
inspection we obtain that $H$ is either the graph $F_{11}$ or $F_{13}$ depicted in Figure 6.

\[
\begin{align*}
F_{11} & \quad F_{12} & \quad F_{13} \\
\text{# even} & \quad \text{# odd} & \quad \text{# odd}
\end{align*}
\]

Figure 6: Connected $P_3$-equipackable graphs in Case 2

This completes the proof of our main result. \(\square\)

The proof can also be done by induction on $|E(G)|$, but the arguments are not shorter.

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**References**


