Determination of the Location of an Unknown Concentrated Force
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Determination of the Location of an Unknown Concentrated Force

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Abstract
In this paper a novel method for determination of location of a concentrated force acting on a structure is proposed. The method uses the relations between outputs for identification of the force location. Uniqueness of the predicted location of the force is discussed and a method for appropriate selection of a measurement set is introduced which guarantees the uniqueness of the predicted force location. Eventually three cases of simulation are carried out to exemplify various aspects of the discussion.

Keywords: Force Location, Force Determination, MIMO, Deconvolution.

Introduction
Measurement of forces applied on a structure can be solved using both direct and indirect methods [1]. In direct methods response of the structure on the place where force is applied must be available in measurements while indirect methods do not urge to have these responses. As a result indirect methods are more general albeit more complicated and ill-posed [2]. The privileges of direct methods over indirect ones is that they are more easier in use and their computation cost is lower while their great disadvantage initiates from the fact that measuring response of a structure on the location where the excitation force acts is not always possible due to the working limitations [3, 4] So indirect methods are preferred over direct ones.

Indirect force determination is an ill-conditioned problem [2, 5] as expected for inverse problems. It becomes more complicated when the location of the input force is not known so it should be determined to allow further works on determination of force pattern [6]. These cases pose two problems to be solved; first problem is about place of the force while second one tries to determine the force on the specified location. The problem with the first question is that uniqueness of estimated location for applied force is not guaranteed and consequently uniqueness of the estimated force itself.

In this work at the first step a method for determination of force location is proposed when a unique answer for force location problem is assumed. After that it is discussed that uniqueness of the location of the input force is related firstly to the places of the measured responses and secondly to the number of measurements. So then a method is introduced for determination of a measurement set that guarantees the uniqueness of force location and also a method for determination of force location in this case. Moreover concept of equivalent forces is addressed which are forces different form actual force which when are applied on the input locations different form actual input location, produce the responses of a measurement set which are identical to responses produced by the actual force in the actual input location.

At the next step we discuss about non-minimum phase systems which are problematic in this method and also the concept of non-proper transfer functions and both of these problems are solved. Two simulations are firstly carried out using two different chainlike structures to exemplify different aspects of the method and ultimately the third simulation shows Finite Element model of a 3D truss structure and more practical aspects of proposed method are investigated on it. In this case to simulate the real condition in the best way measured responses are contaminated with noise and the analysis is performed on noisy measurements. Finally Truncated Singular Value Decomposition technique (TSVD) is used for reconstruction of applied force from measurements.

Outline of the Method
In a linear nXr MIMO system, input-output relation can be represented by elements of a nXr transfer function matrix as below:

\[ y_i(\omega) = G_{ij}(\omega)u_j(\omega) \]  

(1)

Where \( y_i(\omega) \) is the frequency domain representation of the \( i^{th} \) output signal and \( u_j(\omega) \) is the frequency domain representation of the of \( j^{th} \) input signal. For defining relation between two outputs a three-dimensional \( \Delta(\omega) \) tensor is used as below:

\[
\Delta(\omega) = \begin{bmatrix}
\delta_{11}(\omega) & \cdots & \delta_{1n}(\omega) \\
\vdots & \ddots & \vdots \\
\delta_{m1}(\omega) & \cdots & \delta_{mn}(\omega)
\end{bmatrix}_{nxnxn}
\]  

(2)
Where
\[
\delta_k(\omega) = \begin{bmatrix} G_{i1}(\omega) & G_{i2}(\omega) & \cdots & G_{i\omega}(\omega) \end{bmatrix}^T 
\]  
\[ \quad \text{for } i=0,1,2,\cdots,\infty \tag{3} \]

As an explanation it should be noted that \( G_y(\omega)/G_{kj}(\omega) \) defines the relation between two outputs, \( y_1(\omega) \) and \( y_2(\omega) \), assuming force is applied on the \( j^{th} \) input location. So this can be represented with elements of \( \Delta(\omega) \) tensor in the following form:
\[
y_i(\omega) = \Delta_{ikj}(\omega)y_j(\omega) 
\]  
\[ \quad \text{for } i=1,2,3,\cdots,\infty \tag{4} \]

Now consider that responses of \( j^{th} \) and \( k^{th} \) output locations are measured, therefore measurement set will be \( \{y^{me}_j(\omega), y^{me}_k(\omega)\} \). Using this measurement set and Equation (4), a SISO system is obtained with \( \Delta_{ikj}(\omega) \) as transfer function and \( y^{me}_i(\omega) \) as output. Now If:
\[
\forall \{m, j\} \subset \{1, \ldots, r\}, m \neq j \quad \Delta_{ikj} \neq \Delta_{ikm} 
\]  
\[ \quad \text{for } i=1,2,3,\cdots,\infty \tag{5} \]

Which means that elements of \( \delta_k(\omega) \) are not the same, then the relation between \( y^{me}_i(\omega) \) and \( y^{me}_j(\omega) \) is unique. According to this argument, assuming an incorrect location for excitation force, when \( y^{me}_i(\omega) \) is given to the SISO system of Equation (4) as input signal, output signal will differ from \( y^{me}_j(\omega) \). This property is used for determination of force location in this work.

**Uniqueness of the Force Location**

The point here is that uniqueness of elements of \( \delta_k(\omega) \) is not guaranteed. It implies that it is possible to have at least two locations for the force, consider them as \( j^th \) and \( m^th \) input locations, which their correspondent elements in \( \delta_j(\omega) \) matrix are equal to each other, \( \delta_{ikj}(\omega) = \delta_{ikm}(\omega) \). This condition means that if a force applied on the \( j^th \) input location, there is another force, name it Equivalent Force, which if is applied on the \( m^th \) input location, results the same outputs on \( j^th \) and \( k^th \) output locations. Under this circumstances location and consequently pattern of the applied force can not be defined uniquely using this measurement set.

So it seems that it would be reasonable to derive a method which can specify suitable output locations for measurement to avoid non-uniqueness of the answer. This can be done using \( \delta(\omega) \) vectors. With \( \{y^{me}_j(\omega), y^{me}_k(\omega)\} \) as measurement set, if elements of \( \delta_k(\omega) = [\delta_{k1}(\omega), \delta_{k2}(\omega), \cdots, \delta_{kr}(\omega)]^{me}_{me} \) are not the same then rank of \( \delta_k(\omega) \) will be equal to \( r \). If the rank of \( \delta_k(\omega) \) is less than \( r \), uniqueness of the answer will fail and there is possibility of existence of an Equivalent Force - with another pattern on another input location - that generates the same responses on the measured locations. The \( \delta_k(\omega) \) vector has also the ability to show the locations of the Equivalent Forces and their patterns. Moreover if the \( j^th \) and \( m^th \) elements of this vector turn out to be the same, although it results in failure of uniqueness, but if it is known that location of the force is neither \( j^th \) nor \( m^th \) input locations still there will be no problem to determine location of the force.

The whole discussion can be summarized as follows: Using \( y_i(\omega) \) and \( y_k(\omega) \) as measurements, location of the force can be uniquely determined on the input locations in which \( \Delta_{ikm}(\omega) \neq \Delta_{ikj}(\omega) \) where \( j, m = 1, \ldots, r \quad j \neq m \).

**More Measurements**

Now if output set is shown by \( \{Y\} \) and input set is shown by \( \{U\} \) and measurement set is \( \{y^{me}_j(\omega), y^{me}_k(\omega)\} \), and:
\[
\forall \{i, j\} \subset \{1, \ldots, r\} \quad \exists S_{i,k} = \{u_j, u_m\} \subset \{U\} 
\]
\[ \text{s.t. } \Delta_{ikj}(\omega) = \cdots = \Delta_{ikm}(\omega) \tag{6} \]

Then more than these two measurements are required to uniquely determine the location of the force. Locations of this (these) extra measurement(s) can be determined using what was summarized in previous Section: To choose next measurement(s), say \( q^{th} \) output location(s), it (they) must be chosen such that \( S_{i,k} \cap S_{i,q} \cap S_{q,k} \) is a set with maximum one member. In the case that \( S_{i,k} \cap S_{i,q} \cap S_{q,k} \) has one member, if estimated and measured responses are identical in all \( S_{i,k}, S_{i,q} \) and \( S_{q,k} \) then the member represents location of the force.

For better understanding of the reason of the above process we have to pay attention to the fact that actually any \( S \) set represents input locations on which force location cannot be uniquely determined using related measurement set. Therefore extra measurements must be chosen such that they don’t have any common problematic locations with the first measurement set because we have to cover whole range of possible input locations for investigation. A single common location is allowed because when in all measurement sets, just a single common location is predicted as one of the possible force locations and other possibilities in every measurement set are rejected by other sets, it can be distinguished that the single common location is the actual force location.

**Non-Minimum Phase Systems**

In the non-minimum phase systems elements of \( \Delta(\omega) \) become unstable transfer functions so can not be used as appropriate transfer functions for prediction of the location of the force. To overcome this problem the
following procedure is taken into account. Equation (4) can be reformulated as below:

\[
\int_0^t y_i(\tau)g_{ij}(t-\tau)\,d\tau = \int_0^t y_k(\tau)g_{ij}(t-\tau)\,d\tau
\]

(7)

Where \( g_{ij}(t) \) is the response of \( i^{th} \) output to impulse excitation on \( j^{th} \) input. As Equation (7) is just a reformulated form of Equation (4) therefore the previous criterion which was based on the likeness of actual and predicted responses can be changed as a criterion which suggests that with the actual input location, RHS and LHS of Equation (7) should yield the same results. Obviously this new criterion will not be of problem when system is a non-minimum phase one. It is appropriate to emphasize that as these two criterions are actually equal in essence, whole discussion which was made based on the first one is also valid for the second one.

**Simulation**

In this section firstly two simulations have been carried out to exemplify outlines of the method. A third simulation has also been represented next, focusing on application for real cases.

First system is a chainlike structure shown in Figure 1 with \( M = \text{diag}[9.1, 9.2, 9.3, 9.4, 9.5] \) and \( k_i = 10^5 N/m, i = 1,\ldots,5 \). Damping matrix is considered to be proportional to the mass matrix with coefficient of proportionality \( \alpha = 0.2 \). MIMO representation of this system will have a 10x10 transfer function matrix and as a result \( \Delta(\omega) \) will be a 10x10x10 tensor.

![Figure 1: Chainlike system for simulation](image)

But as it can be concluded form state space representation of the system that no force is applied on the first 5 states if states are chosen as below:

\[
X = [x_1 \cdots x_5]^T
\]

\[
\frac{d}{dt} [X] = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} [X] + \begin{bmatrix} 0 \\ [M^{-1}]F \end{bmatrix}
\]

(8)

So \( \Delta(\omega) \) would be reduced to a 5x5x5 tensor. As \( \Delta(\omega) \) contains relatively complex transfer functions, a pictorial representation of this tensor is shown in Figure 2. The cell in \( i^{th} \) row and \( k^{th} \) column in Figure 2 contains curves which are elements of \( \delta_{ik}(\omega) \) when \( \omega \) varies from 0 to 1000. If some of the elements of \( \delta_{ik}(\omega) \) happen to be the same then the cell in \( i^{th} \) row and \( k^{th} \) column will contain less than 5 curves. Therefore each cell in Figure 2 shows if a specific pair of output locations as measurements can yield a unique answer of force determination or not. Using this figure also the input locations which force location can not be uniquely determined on them, using a specific measurement set, can be distinguished. It should be noted that cells on the diagonal line does not represent a pair of measurements, they have been shown just for convenience.

![Figure 2: Pictorial representation of \( \Delta(\omega) \) for the first system](image)

From the above explanations and comparison of the plots in Figure 2 it can be deduced that just by measuring the responses of the 1\(^{st}\) and 5\(^{th}\) masses there is the possibility of determining the location and pattern of input excitation force uniquely. A multi-sine input consisting of three harmonics with circular frequencies of 13Hz, 20Hz and 26Hz and equal amplitudes of 1 has been applied to the system on the 4\(^{th}\) mass. With this input, velocities of 1\(^{st}\) an 5\(^{th}\) mass are measured. Now using what was discussed before, relation of these two velocities is defied uniquely by one of the elements of \( \delta_{55}(\omega) \) so when one of these velocities is considered as input for transfer functions in \( \delta_{55}(\omega) \), just the transfer function which representing the true input location results an output identical to the other measured velocity. Outputs of 5 transfer functions of \( \delta_{55}(\omega) \) are shown in Figure 3, compared with measured outputs.

![Figure 3: Predicted versus measured responses based on different force locations](image)
As can be seen in Figure 3, predicted response is the same as measured response just in the case that input location is considered to be on the 4th mass, which is the true location of the force. This phenomenon describes the outline of the method.

Something important which have to be mentioned here is the fact that above procedure urges us to use elements of \( \delta_{3k}(\omega) \) which are not guaranteed to be proper transfer functions. Since there is no limitation on the measured signals that which one should be considered as input, simply by changing the input and output signals non-proper transfer functions change to proper ones. This procedure for solving non-proper cases is used in the shown example and transfer functions of 1st and 2nd plots counting from above in Figure 3 have been changed so they are \( \Delta_{151}(\omega), \Delta_{152}(\omega) \).

In the second part of the first simulation system is excited by the same input force on the 4th mass but this time the measurements include responses of the 1st and 3rd masses. As Figure 2 predicts, these measurements should not have the ability to determine the authentic location of the input force uniquely because these set of measurements can not distinguish whether the input is applied on the 3rd, 4th or 5th masses. The same procedure applied for velocities of the 1st and 5th mass is now measured signals that which one should be considered as input, simply by changing the input and output signals non-proper transfer functions change to proper ones. This procedure for solving non-proper cases is used in the shown example and transfer functions of 1st and 2nd plots counting from above in Figure 3 have been changed so they are \( \Delta_{151}(\omega), \Delta_{152}(\omega) \).

As can be seen in Figure 3, predicted response is the same as measured response just in the case that input location is considered to be on the 4th mass, which is the true location of the force. This phenomenon describes the outline of the method.

Table 1 - Stiffness and mass of the second system

<table>
<thead>
<tr>
<th>Stiffness or mass index</th>
<th>Stiffness (N/m)</th>
<th>Mass (kg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 9 \times 10^4 )</td>
<td>9.1</td>
</tr>
<tr>
<td>2</td>
<td>( 1.1 \times 10^4 )</td>
<td>9.2</td>
</tr>
<tr>
<td>3</td>
<td>( 1 \times 10^4 )</td>
<td>9.3</td>
</tr>
<tr>
<td>4</td>
<td>( 1 \times 10^4 )</td>
<td>9.3</td>
</tr>
<tr>
<td>5</td>
<td>( 1.2 \times 10^4 )</td>
<td>9.5</td>
</tr>
<tr>
<td>6</td>
<td>( 8 \times 10^4 )</td>
<td>-</td>
</tr>
</tbody>
</table>

According to the discussion in the pervious case, \( 5 \times 5 \times 5 \Delta(\omega) \) tensor of this system is shown in Figure 6. This plot does not have 5 curves in any of its cells (actually not more than 3 curves in each cell) which induces that in each \( \delta_{ik}(\omega) \) there are at least three similar elements.

As the first measurement set velocities of 1st and 4th mass (DOFs 6 and 9) are measured. Now it can be seen in the plotted responses in Figure 7 that based on this measurement set, there are three possibilities for force location which are masses number 2, 3 and 5. As it was...
discussed before, to determine force location another set of measurements has to be made such as DOFs 7 and 10 (velocities of 2\textsuperscript{nd} an 5\textsuperscript{th} mass) to distinguish true location of the force among these locations.

![Figure 7: Predicted versus measured responses based on different force locations, \{y_6, y_9\} measured](image)

Results of the process for the second measurement set are depicted in Figure 8. It is seen that the only input location for which measured and predicted responses are the same in both measurement sets is on the 3\textsuperscript{rd} mass which is the actual location of the force.

![Figure 8: Predicted versus measured responses based on different force locations, \{y_7, y_{10}\} measured](image)

Last system is a 21-DOF, 3D structure shown in Figure 9. With the discussed method of choosing appropriate output locations for measurement, DOFs 6 and 15 can be measured to yield unique answer of the force location. In this case, regarding to the vast number of possible input locations curve-form plots for each cell in Figures 2, 6 are not appropriate so bar charts are used to demonstrate the possibility of rank deficiency of elements of \( \omega \). Figure 10 represents the cell related to DOFs 6 and 15. Bars represent the \( \log(\Delta_{6,15,j}(\omega) \|_2) \) when \( \omega \) varies from 1 to 1000. Any rank deficiency in the elements of \( \delta(\omega) \) can be observed with two or more bars with equal amplitudes. As Figure 10 represents, there is no such a condition in the cell related to the measured DOFs so these are appropriate measurements.

![Figure 9: Representation of 3D structure](image)

![Figure 10: Bar representation of \( \Delta_{6,15,j}(\omega) \) at different input locations](image)

In this case the convolution criterion is used to predict the force location. For a comprehensive survey on the possibility of the unique determination of the force location on every input location, all of the 21 DOFs are considered once as the actual input location. Excitation force has the following equation:

\[
 f(t) = 10^5 (\sin(120\pi t) - 1.5\sin(66.5\pi t))
\]

Measurements are contaminated with 8.5% of white noise to simulate real conditions. In each case error norm of the difference between RHS and LHS of the Equation (7) have been calculated for every 21 DOFs as predicated input locations. A graphical representation of the process can be seen in Figure 11. Color maps in any column show rank of the norm of difference between RHS and LHS of the Equation (7) for each predicated input location on the vertical axis, while actual input location is represented on horizontal axis. As expected, diagonal line is where the minimum difference occurs, which means that in any desired location for the actual force, correct location is predicted.
Force Determination

Determination of the force is done using Truncated Singular Value Decomposition (TSVD). This is a deconvolution technique based on discretization of the convolution integral. The convolution integral of Equation (9) can be discretized into algebraic Equation (10) in time domain.

\[ y(t) = \int_0^t g(t-\tau)f(\tau)d\tau \]  

(9)

In which \( f \) is the force vector and \( G \) is a matrix constructed from impulse response of the structure.

\[ \{y(t)\}_{\text{act}} = [G(t)]_{\text{nor}} \{f(t)\}_{\text{act}} \]  

(10)

Here \( n \) is the number of samples. Above algebraic equation can be solved inversely to yield the applied force on the structure through inversion of the \( G \) matrix and pre-multiplication by the measurement. But measurements are contaminated with noise which decreases their quality and can lead to corruption or instability of the reconstructed force so these low quality parts of the signal should be disassembled from the original part that can be done through truncation of the singular values of the \( G \) matrix or some more sophisticated methods such as using Tikhonov regularization [7-8] on singular values of \( G \) matrix. Some other methods for this purpose are available which are addressed by H. Inoue and J. J Harrigan and S. R Reid [1]. Further discussions on the TSVD can be found in literature [1, 5].

For the simulation case it should be noted that for the force determination case excitation force has been applied on the 21st DOF. It doesn’t have to be mentioned that the represented method was used to determine force location this condition. Noise contaminated responses of DOFs 6 and 15 are shown in Figure 12. Predicted force is shown in Figure 13. It can be seen that there is a good correlation between predicted and actual force.

Although force has been predicted for whole time duration, it has been compared with actual force in just half of time duration just as a matter of convenience in observation.

So the fact can be seen that combining force location identification and a force determination method like TSVD, excitation force can be defined without any prior knowledge of its location.

Conclusion

In this paper a method is proposed for identification of the location of a concentrated force using output measurements regardless of any information about force characteristics or pattern. It also includes the algorithm of choosing appropriate output locations as measurements which have the ability to guarantee the uniqueness of the answer for location of excitation force. Problems of non-proper and non-minimum phase transfer functions are also discussed for application of the method. The proposed method is investigated via different simulations to represent different theoretical and also real cases in which contribution of noise in measurements is taken into account by adding white noise to measured responses to simulate the realistic conditions.
### List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{ij}(\omega)$</td>
<td>Transfer function between $i^{th}$ output and $j^{th}$ input</td>
</tr>
<tr>
<td>$g_{ij}(t)$</td>
<td>Impulse response of transfer function between $i^{th}$ output and $j^{th}$ input locations</td>
</tr>
<tr>
<td>$S_{i,j}$</td>
<td>Set of undetectable force locations using $i^{th}$ and $j^{th}$ outputs as measurement set</td>
</tr>
<tr>
<td>${U}$</td>
<td>Set of locations of the excitation force</td>
</tr>
<tr>
<td>$u_j(\omega)$</td>
<td>Frequency domain representation of the input signal on the $i^{th}$ input location</td>
</tr>
<tr>
<td>${\hat{y}}$</td>
<td>Set of output locations</td>
</tr>
<tr>
<td>$y_i(\omega)$</td>
<td>Frequency domain representation of the output signal on the $i^{th}$ output location</td>
</tr>
<tr>
<td>$y_{im}(\omega)$</td>
<td>Frequency domain representation of the measured output signal on the $i^{th}$ input location</td>
</tr>
<tr>
<td>$\Delta(\omega)$</td>
<td>Tensor of all possible transfer functions between all two output locations in the output set</td>
</tr>
<tr>
<td>$\delta_{ij}(\omega)$</td>
<td>Vector of the all possible transfer functions between $i^{th}$ and $j^{th}$ output locations</td>
</tr>
</tbody>
</table>

### References


