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Total domination in partitioned graphs
by
Allan Frendrup, Preben Dahl Vestergaard and Anders Yeo
Total domination in partitioned graphs

Allan Frendrup¹, Preben Dahl Vestergaard¹, Anders Yeo²

¹ Department of Mathematical Sciences, Aalborg University, Denmark
² Department of Computer Science, Royal Holloway, University of London

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Abstract. We present results on total domination in a partitioned graph \( G = (V, E) \). Let \( \gamma_t(G) \) denote the total dominating number of \( G \). For a partition \( V_1, V_2, \ldots, V_k, k \geq 2, \) of \( V \), let \( \gamma_t(G; V_i) \) be the cardinality of a smallest subset of \( V \) such that every vertex of \( V_i \) has a neighbour in it and define the following

\[
\begin{align*}
&f_t(G; V_1, V_2, \ldots, V_k) = \gamma_t(G) + \gamma_t(G; V_1) + \gamma_t(G; V_2) + \ldots + \gamma_t(G; V_k) \\
&f_t(G; k) = \max\{ f_t(G; V_1, V_2, \ldots, V_k) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V \} \\
g_t(G; k) = \max\{ \Sigma_{i=1}^k \gamma_t(G; V_i) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V \}
\end{align*}
\]

We summarize known bounds on \( \gamma_t(G) \) and for graphs with all degrees at least \( \delta \) we derive the following bounds for \( f_t(G; k) \) and \( g_t(G; k) \).

(i) For \( \delta \geq 2 \) and \( k \geq 3 \) we prove \( f_t(G; k) \leq 11|V|/7 \) and this inequality is best possible.

(ii) for \( \delta \geq 3 \) we prove that \( f_t(G; 2) \leq (5/4 - 1/372)|V| \). That inequality may not be best possible, but we conjecture that \( f_t(G; 2) \leq 7|V|/6 \) is.

(iii) for \( \delta \geq 3 \) we prove \( f_t(G; k) \leq 3|V|/2 \) and this inequality is best possible.

(iv) for \( \delta \geq 3 \) the inequality \( g_t(G; k) \leq 3|V|/4 \) holds and is best possible.

Key words. Total domination, Partitions and Hypergraphs.

1. Notation

By \( G = (V, E) \) we denote a graph \( G \) with vertex set \( V = V(G) \) and edge set \( E = E(G) \). The order of \( G \) is \( |V(G)| = n \). For \( x \in V(G) \) we denote by \( N_G(x) \) the set of neighbours to \( x \) and \( N_G[x] = \{ x \} \cup N_G(x) \). Indices may be omitted if clear from context. The degree of \( x \) is \( d_G(x) = |N_G(x)| \), the number of neighbours to \( x \). We let \( \delta(G) = \delta \) denote the minimum degree in \( G \) and \( \Delta(G) = \Delta \) the maximum degree. A hypergraph \( H = (V, E) \) has vertex set \( V = V(H) \) and its set of hyperedges, or edges for short, is \( E = E(H) \). Each hyperedge \( e \) is a subset of \( V, e \subseteq V(H) \). A vertex \( v \) is incident with an edge \( e \) if \( v \in e \), the degree of

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v is the number of hyperedges in H containing v. We let \( \delta(H) = \delta \) denote the minimum degree in \( H \) and \( \Delta(H) = \Delta \) the maximum degree. \( H \) is \( r \)-regular if each vertex has degree \( r \), i.e. \( d_H(x) = r \), or equivalently, \( x \) is contained in precisely \( r \) edges. \( H \) is \( k \)-uniform if each hyperedge contains exactly \( k \) vertices. Two edges \( e_1 \) and \( e_2 \) are said to be overlapping if \( |V(e_1) \cap V(e_2)| \geq 2 \). Let \( Y \subseteq V(H) \) then \( E(Y) \) denotes all hyperedges, \( e \), contained in \( Y \) (i.e. \( V(e) \subseteq Y \)).

For a hypergraph \( H \) a hitting set or a transversal \( T \) is a set of vertices \( T \subseteq V(H) \) such that \( e \cap T \neq \emptyset \) for each hyperedge \( e \) in \( E(H) \), i.e. each edge \( e \) contains at least one vertex from \( T \). \( T(H) \) denotes the minimum cardinality of a transversal for the hypergraph \( H \). For sets \( S, T \subseteq V \), in a graph \( G \) the set \( S \) totally dominates \( T \) if every vertex in \( T \) is adjacent to some vertex of \( S \). The minimum number of vertices needed to totally dominate \( V \) is the total domination number \( \gamma_t(G) \). For a subset \( S \) of \( V \) we let \( \gamma_t(G; S) \) denote the smallest number of vertices in \( G \) which totally dominates \( S \). A partition \( V = (V_1, V_2, \ldots, V_k) \) of \( V(G) \) into \( k \) disjoint sets, \( k \geq 2 \), has \( V = \bigcup_{i=1}^{k} V_i, V_i \cap V_j = \emptyset, 1 \leq i < j \leq k \). For a partition \( (V_1, V_2, \ldots, V_k) \) of \( V \), we define the following.

\[
\begin{align*}
 f_t(G; V_1, V_2, \ldots, V_k) &= \gamma_t(G) + \gamma_t(G; V_1) + \gamma_t(G; V_2) + \ldots + \gamma_t(G; V_k) \\
 g_t(G; V_1, V_2, \ldots, V_k) &= \gamma_t(G; V_1) + \gamma_t(G; V_2) + \ldots + \gamma_t(G; V_k)
\end{align*}
\]

We furthermore define \( f_t(G; k) \) and \( g_t(G; k) \) as follows.

\[
\begin{align*}
 f_t(G; k) &= \max\{ f_t(G; V_1, V_2, \ldots, V_k) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V \} \\
 g_t(G; k) &= \max\{ g_t(G; V_1, V_2, \ldots, V_k) \mid V_1, V_2, \ldots, V_k \text{ is a partition of } V \}
\end{align*}
\]

For further notation we refer to Chartrand and Lesniak [1].

2. Introduction

The theory of domination is outlined in two books by Haynes, Hedetniemi and Slater [5, 6]. A combination of domination and partitions is treated by Hartnell and Vester gaard [7], Seager [14], Tuza and Vester gaard [17], Henning and Vester gaard [11]. There has been an upsurge in the study of total domination. New results on total domination are given by Henning, Kang, Shan, Thomassé and Yeo in [10, 12, 15, 18]. In [9] Henning surveys recent results on total domination. Here we shall study total domination in partitioned graphs.

3. Bounds on \( \gamma_t \)

We summarize in Theorem 1 results found by Henning, Thomassé and Yeo. If \( C_{10} : v_1, v_2, \ldots, v_{10}, v_1 \) is the circuit with 10 vertices then let \( G_{10} \) denote the graph obtained from \( C_{10} \) by addition of the edge \( v_1v_6 \) and let \( H_{10} \) denote the graph obtained from \( C_{10} \) by addition of the edges \( v_1v_6 \) and \( v_2v_7 \).

**Theorem 1.** Let \( G \) be a connected graph with \( n \) vertices and minimum degree \( \delta(G) = \delta \). Then

- \( \delta \geq 2 \) implies \( \gamma_t(G) \leq 4n/7 \) for \( G \notin \{C_3, C_5, C_6, C_{10}, G_{10}, H_{10}\} \) ([8, Corollary 6], [9, Theorem 27]).
- \( \delta \geq 3 \) implies \( \gamma_t(G) \leq n/2. \) ([15]).
\( \delta \geq 4 \) implies \( \gamma_t(G) \leq 3n/7 \) ([15]) and there exists some \( \epsilon > 0 \) such that \( \gamma_t(G) \leq (3/7-\epsilon)n \) for \( G \neq G_{14} \), where \( G_{14} \) is an incidence bipartite graph of order 14 derived from the Fano plane ([19]).

It is a conjecture that \( \delta \geq 5 \) implies \( \gamma_t(G) \leq 4n/11 \).

Theorem 2 and Theorem 3 below, give conditions for equality in Theorem 1.

**Theorem 2.** ([9, Theorem 29]) Let \( G \) be a connected graph of order \( n > 14 \) with \( \delta \geq 2 \). Then \( \gamma_t(G) = 4n/7 \) if and only if \( G \) can be obtained from a connected graph \( F \) of order at least three by adding \( |V(F)| \) disjoint copies of \( C_6 \), one corresponding to each \( v \in V(F) \), such that either \( v \) is joined by a new edge to a vertex in its corresponding \( C_6 \) or by two new edges to two vertices at distance two apart in its corresponding \( C_6 \).

The family \( \mathcal{G} \cup \mathcal{H} \) is constructed in [3] as follows. Take two copies \( a_1b_1a_2b_2\ldots a_kb_k \) and \( c_1d_1c_2d_2\ldots c_ld_k \), of the path \( P_{2k}, k \geq 2 \), and add edges \( a_id_i, b_ic_i \) for \( i = 1, 2, \ldots, k \).

From this the graph of order \( 4k \) belonging to the infinite family \( \mathcal{G} \) is obtained by adding \( a_1c_1 \) and \( b_1d_1 \), while the graph of order \( 4k \) in \( \mathcal{H} \) is obtained by adding \( a_kb_k \) and \( c_1d_k \).

The generalized Petersen graph \( GP_{16} \) is obtained from two circuits \( u_1u_2u_3\ldots u_7u_8 \) and \( v_1v_2v_3\ldots v_7v_8 \) by addition of edges \( u_1v_1, u_2v_4, u_3v_7, u_4v_2, u_5v_5, u_6v_3, u_7v_3, u_8v_6 \).

**Theorem 3.** ([12, Theorem 5]) Let \( G \) be a connected graph with \( \delta(G) \geq 3 \). Then \( \gamma_t(G) = n/2 \) if and only if \( G \in \mathcal{G} \cup \mathcal{H} \) or \( G = GP_{16} \).

### 4. \( f_t \) for \( k \)-partitioned graphs with \( \delta \geq 2 \)

We have that \( f_t \) increases with the number of partition classes, i.e., \( f_t(G; k) \leq f_t(G; k+1) \). Therefore we get a weaker inequality if we partition \( V \) into more than two classes. That is demonstrated in Theorem 4 below.

**Theorem 4.** Let \( G \) be a connected graph of order \( n \) with \( \delta(G) \geq 2 \) and \( G \notin \{C_3, C_5, C_6, C_{10}\} \).

If \( k \geq 2 \) then \( f_t(G; k) \leq 11n/7 \).

If \( k = 2 \) then \( f_t(G; k) \leq 3n/2 \). Equality holds if and only if \( G \) is a circuit of length zero modulo four, \( G = C_4t, t \geq 1 \).

If \( k = 3 \) then \( f_t(G; k) \leq 11n/7 \). For \( n > 14 \) equality holds if and only if \( G \) can be obtained from a circuit or a path of order at least three by joining each of its vertices by one edge to disjoint copies of \( C_6 \).

If \( k \geq 4 \) then \( f_t(G; k) \leq 11n/7 \) and for \( n > 14 \) equality holds if and only if \( \Delta(G) \leq k \) and \( G \) can be obtained from a connected graph \( F \) having order at least three and \( g_t(F; k) = |V(F)| \) by adding disjoint copies of \( C_6 \), one corresponding to each \( v \in V(F) \), such that each \( v \) is joined by a new edge to one vertex in its corresponding \( C_6 \) or by two new edges to two vertices at distance two apart in its corresponding \( C_6 \).

**Proof.** By Theorem 1 we have \( \gamma_t(G) \leq 4n/7 \) and assigning to each vertex its own class dominator we have \( g_t(G; k) \leq n \). Therefore \( f_t(G; k) = \gamma_t(G) + g_t(G; k) \leq 11n/7 \). The result for \( k = 2 \) is proven by Frendrup, Henning and Vestergaard in [4, Theorem 2]. For \( k \geq 3 \) the equality \( f_t(G; k) = 11n/7 \) implies \( \gamma_t(G) = 4n/7 \) and \( g_t(G; k) = n \) and therefore \( G \) has the structure described in Theorem 2. Since \( g_t(G; k) = n \) each subgraph \( H \) of \( G \) must satisfy \( g_t(H; k) = |V(H)| \) and further \( \Delta(G) \leq k \). Let \( H_1 \) be the graph obtained from
a circuit $C_6 : v_1 v_2 \ldots v_6$ by adding a new vertex $x$ and the edge $xv_1$ and let $H_2 := H_1 + xv_3$. 
Observe for $k = 3$ that $g_3(H_1; k) = |V(H_1)|$ (obtainable from partitioning $x, v_1, v_2, \ldots, v_6$ into classes indexed $1122133$ or $1221133$) while $g_3(H_2; k) < |V(H_2)|$. For $k \geq 4$ we can easily show that $g_3(H_1; k) = |V(H_1)|$, $i = 1, 2$. This proves for $k \geq 3$ that $f_3(G; k) = 11n/7$ implies $G$ has the structure described in this theorem. Conversely, assume first that $k = 3$ and that $G$ is obtainable as a disjoint union of $H_1$’s with edges added between the vertices named $x$, so they span $F$, where $F$ is a path or circuit. We must exhibit a partition of $V(G)$ proving that $f_3(G; k) = 11n/7$, i.e. that $g_3(G; k) = |V(G)|$. It is easy to find a partition $V'_1, V'_2, V'_3$ of $V(F)$ such that $g_3(F; k) = |V(F)|$. If $k = 3$ we can extend this partition to all the $H_1$’s such that the following holds, which proves that $g_3(G; v_1, v_2, v_3) = n$.

- $N(x) = N_F(x) \cup \{v_1\}$ contains at most one vertex from each $V'_1, V'_2, V'_3$ (just put $v_1$ in the partition set which doesn’t contain any of the two vertices in $N_F(x)$).
- $N(v_1) = \{x, v_2, v_6\}$ contains one vertex from each $V'_1, V'_2, V'_3$ (just put $v_2$ and $v_6$ in the partition sets such that this holds).
- $N(v_3), N(v_5) \subseteq \{v_2, v_4, v_6\}$, which contains one vertex from each $V'_1, V'_2, V'_3$ (just put $v_4$ in the same set as $x$).
- $N(v_2), N(v_4), N(v_6) \subseteq \{v_1, v_3, v_5\}$, which contains one vertex from each $V'_1, V'_2, V'_3$ (just put $v_3$ and $v_5$ in the partition sets such that this holds).

Assume next that $k \geq 4$. Then a vertex $x \in F$ may belong to a unit $H_1$ or $H_2$. Again there is a partition $V'_1, V'_2, \ldots, V'_k$ of $V(F)$ such that $g_k(F; k) = |V(F)|$ and similarly to above we can extend this partition to all of $G$, such that the neighbourhood of every vertex in $G$ contains at most one vertex from any partition set. The details are left to the reader. This proves that $g_k(G; k) = n$.

5. $g_t$ for two-partitioned graphs with $\delta \geq 3$

Chvátal and McDiarmid [2] and Tuza [16] independently established the following result about transversals in hypergraphs (see also Thomassé and Yeo [15] for a short proof of this result).

**Theorem 5.** ([2,16,15]) If $H$ is a hypergraph with all edges of size at least three, then $\mathcal{T}(H) \leq \lceil |V(H)| + |E(H)| \rceil / 4$.

**Theorem 6.** Let $G$ be a graph of order $n$ with $\delta \geq 3$. Then $g_t(G; 2) \leq 3n/4$.

*Proof.* From the two-partitioned graph $G$, we define for $i = 1, 2$, $H_i$ to be the hypergraph on $n$ vertices and $m_i$ edges where $V(H_i) = V(G)$ and the hyperedges of $H_i$ are the sets of neighbourhoods of class $i$ vertices. In other words, $e \in E(H_i)$ precisely if, for some vertex $v$ in $V_i$, $e = N_G(v)$. Each edge in $H_i$ has at least three vertices because $\delta(G) \geq 3$. In $G$ we see that a set $\mathcal{T}_i$ of vertices totally dominates $V_i$ if and only if $\mathcal{T}_i$ is a transversal of $H_i$. Applying Theorem 5 to $H_1$ and $H_2$ separately we obtain transversals $\mathcal{T}_i$ of $H_i$, $i = 1, 2$, satisfying

$$|\mathcal{T}_1| \leq \frac{m_1 + n}{4}, \quad |\mathcal{T}_2| \leq \frac{m_2 + n}{4}.$$ 

Since $m_1 + m_2 = n$ we obtain $|\mathcal{T}_1| + |\mathcal{T}_2| \leq \frac{m_1 + n}{4} + \frac{m_2 + n}{4} = \frac{3n}{4}$. This proves Theorem 6. \qed

An example of graphs with equality $g_6(G; 2) = 3n/4$ is given in the next section.
6. An infinite family of graphs extremal for Theorem 6

We have the following theorem.

**Theorem 7.** For each integer \( r \geq 1 \) there exists a connected bipartite graph \( G_r \) of order \( n = 16r \) with \( \delta(G_r) = 3 \) such that \( g_t(G_r; 2) = 3|V(G_r)|/4 \) and \( f_t(G_r; 2) \geq 9|V(G_r)|/8 \).

**Proof.** We define the graph \( G_r \) as follows. Define the vertex set of \( G_r \) to be \( V(G_r) = W_r \cup A_r \cup B_r \), where

\[
W_r = \{w_0, w_1, w_2, \ldots, w_{8r-1}\} \\
A_r = \{a_0, a_1, a_2, \ldots, a_{4r-1}\} \\
B_r = \{b_0, b_1, b_2, \ldots, b_{4r-1}\}
\]

We define the edge set of \( G_r \) such that the following holds, for all \( i \in \{0, 1, 2, \ldots, r-1\} \) (where \( b_{-1} = b_{4r-1} \) by definition):

\[
N(w_{4i}) = \{a_{4i}, a_{4i+1}, b_{4i}\} \\
N(w_{4i+1}) = \{a_{4i}, a_{4i+1}, b_{4i}\} \\
N(w_{4i+2}) = \{a_{4i}, a_{4i+2}, b_{4i}\} \\
N(w_{4i+3}) = \{a_{4i+1}, a_{4i+2}, b_{4i-1}\} \\
N(w_{4i+4}) = \{a_{4i+2}, b_{4i+1}, b_{4i+2}\} \\
N(w_{4i+5}) = \{a_{4i+3}, b_{4i+1}, b_{4i+2}\} \\
N(w_{4i+6}) = \{a_{4i+3}, b_{4i+1}, b_{4i+2}\} \\
N(w_{4i+7}) = \{a_{4i+3}, b_{4i+1}, b_{4i+2}\}
\]

We now assume \( r \geq 1 \) is fixed, and therefore omit the subscripts of the above sets and graph. Define \( V_1 \) and \( V_2 \) as follows.

\[
V_1 = A \cup \bigcup_{i=0}^{r-1} \{w_{8i+1}, w_{8i+2}, w_{8i+3}, w_{8i+5}\} \\
V_2 = B \cup \bigcup_{i=0}^{r-1} \{w_{8i}, w_{8i+4}, w_{8i+6}, w_{8i+7}\}
\]

We will now show that if \( S_i \) is a set such that every vertex in \( V_i \) has a neighbour in \( S_i \), then \( |S_i| \geq 3|V(G)|/8 \), for \( i = 1, 2 \). This would imply that \( f_t(G; 2) \geq 9|V(G)|/8 \) and \( g_t(G) \geq 6|V(G)|/8 \) when \( k = 2 \) (as clearly the above would also imply that \( \gamma_t(G) \geq 3|V(G)|/8 \)). From Theorem 6 follows that \( g_t(G) = 3|V(G)|/4 \).

Let \( S_1 \) be a set that totally dominates \( V_1 \) (i.e. every vertex in \( V_1 \) has a neighbour in \( S_1 \)). As \( w_{8i+5} \) has a neighbour in \( S_1 \) we note that \( |S_1 \cap \{a_{4i+3}, b_{4i+1}, b_{4i+2}\}| \geq 1 \), for all \( i = 0, 1, 2, \ldots, r-1 \). As \( w_{8i+1}, w_{8i+2} \) and \( w_{8i+3} \) all have a neighbour in \( S_1 \) we note that \( |S_1 \cap \{a_{4i}, a_{4i+1}, a_{4i+2}, b_{4i}, b_{4i-1}\}| \geq 2 \), for all \( i = 0, 1, 2, \ldots, r-1 \) (recall that \( b_{-1} = b_{4r-1} \)). As the above sets are all disjoint we note that \( |S_1 \cap (A \cup B)| \geq 3|A \cup B|/8 \).

As \( a_{4i+3} \) has a neighbour in \( S_1 \) we note that \( |S_1 \cap \{w_{8i+5}, w_{8i+6}, w_{8i+7}\}| \geq 1 \), for all \( i = 0, 1, 2, \ldots, r-1 \). As \( a_{4i}, a_{4i+1} \) and \( a_{4i+2} \) all have a neighbour in \( S_1 \) we note that \( |S_1 \cap \{w_{8i}, w_{8i+4}, w_{8i+6}, w_{8i+7}\}| \geq 2 \), for all \( i = 0, 1, 2, \ldots, r-1 \). As the above sets are all disjoint we note that \( |S_1 \cap W| \geq 3|W|/8 \). This implies the desired result for \( S_1 \).

The fact that if \( S_2 \) totally dominates \( V_2 \), then \( |S_2| \geq 3|V(G)|/8 \) is proved analogously to above. We now just need to show that \( G \) is connected. Let \( P_i = \{w_{8i}, w_{8i+1}, \ldots, w_{8i+7}\} \) and let \( Q_i = \{a_{4i}, a_{4i+1}, a_{4i+2}, a_{4i+3}, b_{4i}, b_{4i+1}, b_{4i+2}, b_{4i+3}\} \) for all \( i = 0, 1, 2, \ldots, r-1 \). Note that \( G[P_i \cup Q_i] \) is connected. As the edges \( w_{8i+3} b_{4i-1} \), for all \( i = 0, 1, 2, \ldots, r-1 \) connects \( P_i \) with \( Q_{i-1} \) (\( Q_{r-1} = Q_{r-1} \)) we are done.

7. \( f_t(G) \) for two-partitioned graphs with \( \delta \geq 3 \)

Let \( G \) be a graph of order \( n \) with \( \delta(G) \geq 3 \).
From Theorems 1 and 6 it follows immediately that \( f_t(G; 2) = \gamma_t(G) + g_t(G; k) \leq \frac{n}{2} + \frac{3n}{4} = \frac{5n}{4} \) when \( \delta(G) \geq 3 \). We shall in Theorem 8 below prove a slightly stronger result and later pose an even stronger conjecture.

The following result is known (see for example [13]).

**Lemma 1.** ([13]) If \( G \) is a 3-regular graph, there exists a matching \( M \) in \( G \), such that \( |M| \geq \frac{7}{16} |V(G)| \).

**Lemma 2.** Let \( H \) be a 2-regular 3-uniform hypergraph with no two edges overlapping. Then \( T(H) \leq \frac{|V(H)| + |E(H)|}{4} - \frac{|V(H)|}{24} \).

**Proof.** Let \( H \) be a 2-regular 3-uniform hypergraph with no overlapping edges. Define the graph \( G_H \) as follows \( V(G_H) = E(H) \) and \( E(G_H) = \{ e_1 e_2 : |V(e_1) \cap V(e_2)| = 1 \} \). As there are no overlapping edges and \( H \) is 2-regular and 3-uniform, we note that \( G_H \) is a 3-regular graph. By Lemma 1, there exists a matching \( M \) in \( G_H \), such that \( |M| \geq \frac{7}{16} |V(G_H)| \).

If \( e_1 e_2 \in M \), then by the definition of \( G_H \) we note that \( V(e_3) \cap V(e_2) = \{ x_1 e_2 \} \) for some \( x \in V(H) \). Let \( X = \{ x_e \mid e \in M \} \) and note that \( 2|M| \) edges in \( H \) contain a vertex from \( X \) (as \( M \) was a matching). Let \( X' \) be a set of vertices of order \( |E(H)| - 2|M| \) containing a vertex from every edge in \( H \), which does not contain a vertex from \( X \). Note that \( X \cup X' \) is a transversal of \( H \) of order \( |M| + (|E(H)| - 2|M|) \). By the above bound on \( |M| \) we get the following, as \( 3|E(H)| = \sum_{x \in V(H)} d(x) = 2|V(H)| \).

\[
T(H) \leq |E(H)| - |M| \leq |E(H)| - |E(H)| - \frac{7}{16} |V(H)| = \frac{|V(H)|}{4} + \frac{5}{16} \times \frac{2|V(H)|}{3}
\]

\[
= \frac{|V(H)| + |E(H)|}{4} - \frac{|V(H)|}{24}
\]

\[\square\]

**Lemma 3.** Let \( H \) be a 3-uniform hypergraph, where multiple edges are allowed. For each edge and vertex in \( H \) we assign a non-empty subset of \( \{0, 1, 2\} \). Let this subset be denoted by \( \mathcal{L}(q) \) for all \( q \in V(H) \cup E(H) \). Let \( H_i \) be the 3-uniform hypergraph containing vertex-set \( V_i = \{ v : i \in L(v) \text{ and } v \in V(H) \} \) and edge-set \( E_i = \{ e : i \in L(e) \text{ and } e \in E(H) \} \), for \( i = 0, 1, 2 \). Let \( Y \subseteq V(H) \) be arbitrary and assume that the following holds.

(a): \( \Delta(H_0), \Delta(H_2) \leq 2 \)

(b): \( \Delta(H - E(Y)) \leq 4 \).

(c): There are no overlapping edges in \( H_i, i \in \{1, 2\} \).

(d): If \( e \in E(H) - E(Y) \), then \( 0 \in L(e) \text{ and } |L(e)| \geq 2 \).

This implies that the following holds.

\[
\sum_{i=0}^{2} T(H_i) \leq \left( \sum_{i=0}^{2} |V_i| + |E_i| \right) - \frac{|V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]|}{372}
\]

**Remark.** We assume here in Lemma 3 that the assignment of a set \( \mathcal{L}(q) \) to each \( q \) is done such that \( H_0, H_1, H_2 \) really are hypergraphs, i.e., such that each hyperedge in \( E_i \) consists of vertices from \( V_i, i = 0, 1, 2 \). This requirement will be satisfied in the proof of Theorem 8 where the lemma is applied.

**Proof.** Assume that the lemma is false, and that \( H \) is a counterexample with minimum \( |E_0| + |E_1| + |E_2| \). Clearly \( |E_0| + |E_1| + |E_2| > 0 \), as otherwise \( \sum_{i=0}^{2} T(H_i) = 0 \). For simplicity we will use the following notation:
\[ T^* = \sum_{i=0}^{2} T(H_i) \]
\[ S^* = \sum_{i=0}^{2} |V_i| + |E_i| \]
\[ V^* = V(H_0) \cap V(H_1) \cap V(H_2) \]

We recall that \( H \) was assumed to be a “minimal” counterexample to \( T^* \leq S^* - ([V^* \setminus N_H[Y]])/372 \). We will now prove a few claims, which end in a contradiction, thereby proving the lemma. For \( H \) the left hand side of the inequality, \( \ell \), and the right hand side of the inequality, \( r \), in Lemma 3 satisfies \( \ell > r \). We shall construct smaller \( H' \) which also satisfies (a)-(d) and which therefore has \( \ell' \leq r' \) by the minimality of \( H \). \( H' \) is to be constructed such that there exist \( \alpha \leq \beta \) for which \( \ell - \alpha \leq \ell' \) and \( r' \leq r - \beta \). Those inequalities combine to give the desired contradiction \( \ell \leq r \).

Claim A: If we add a vertex to \( Y \), then \( N[Y] \) does not increase by more than 9 vertices.

Proof of Claim A: This follows from the fact that \( H \) is 3-uniform and \( \Delta(H - E(Y)) \leq 4 \), by (b) in the statement of the lemma.

Claim B: There is no \( e = \{v_1, v_2, x\} \in E_i \), such that \( d_{H_i}(v_1) = d_{H_i}(v_2) = 1 \) and \( d_{H_i}(x) = 2 \), for \( i = 0, 1, 2 \).

Proof of Claim B: Assume that there is such an edge \( e = \{v_1, v_2, x\} \in E_i \). Let \( e' = \{w_1, w_2, x\} \) be the other edge in \( H_i \) containing \( x \). Now delete \( v_1, v_2, x, e \) and \( e' \) from \( H_i \) and add \( \{v_1, v_2, x, w_1, w_2\} \) to \( Y \). Note that (a)-(d) still hold and that \( T^* \) decreases by 1 as we simply add \( x \) to any transversal in the new \( H_i \) in order to get a transversal in the old \( H_i \). By Claim A the set \( N[Y] \) does not increase by more than 45 vertices. As \( V^* \) does not decrease by more than 3 vertices and \( S^* \) decreases by \( 5/4 \), we are done by the “minimality” of \( H \) (as \( \alpha = 1 \leq 5/4 - 48/372 = \beta \) in the argument above Claim A).

Claim C: There is no \( e = \{x, v_1, v_2\} \in E_i \), such that \( d_{H_i}(v_1) = d_{H_i}(v_2) = 2 \) and \( d_{H_i}(x) = 1 \), for \( i = 1, 2 \).

Proof of Claim C: Assume that there is such an edge \( e = \{x, v_1, v_2\} \in E_i \). Let \( e_1 = \{w_1, w_2, v_1\} \) be the other edge in \( H_i \) containing \( v_1 \) and let \( e_2 = \{u_1, u_2, v_2\} \) be the other edge in \( H_i \) containing \( v_2 \). As there are no overlapping edges in \( H_i \) (by (c) in the statement of the lemma) we note that \( e_1 \neq e_2 \) and \( |\{w_1, v_2, u_1, u_2\}| \geq 3 \). Let \( S \) be any subset of \( \{w_1, w_2, u_1, u_2\} \) such that \(|S| = 3 \). We now separately consider the cases when addition of \( S \) as a new hyperedge to \( H_i \) causes overlapping edges in \( H_i \), and when it doesn’t.

Assume that adding \( S \) to \( E_i \) does not cause overlapping edges in \( H_i - e_1 - e_2 \). Now delete \( x, v_1, v_2, e, e_1 \) and \( e_2 \) from \( H_i \) and add the edge \( S \) to \( H_i \) (and \( H \)). Furthermore add \( \{x, v_1, v_2, w_1, w_2, u_1, u_2\} \) to \( Y \). Note that (a)-(d) still hold. If \( T' \) is a transversal in the new \( H_i \) then due to the edge \( S \) we either have \( \{u_1, u_2\} \cap T' \neq \emptyset \), in which case \( T' \cup \{v_1\} \) is a transversal in the old \( H_i \) or \( \{w_1, w_2\} \cap T' \neq \emptyset \), in which case \( T' \cup \{v_2\} \) is a transversal in the old \( H_i \). Therefore \( T^* \) decreases by at most one. By Claim A we have that \( N[Y] \) does not increase by more than 63 vertices. As \( V^* \) does not decrease by more than 3 and \( S^* \) decreases by \( 5/4 \), we are done by the “minimality” of \( H \) (as \( 1 \leq 5/4 - 66/372 \)).

So now assume that the above addition of \( S \) would cause overlapping edges in \( H_i - e_1 - e_2 \). This can only happen if there is an edge \( e' \in E_i \) such that \( |S \cap V(e')| \geq 2 \). Note that by (a) the degree in \( H_i \) is two for all vertices in \( S \cap V(e') \) (they only lie in \( S \) and \( e' \)). Now delete the vertices \( \{x, v_1, v_2\} \cup (S \cap V(e')) \) from \( H_i \) and delete the edges \( e_1, e_2 \) and \( e' \) from \( H_i \) (do not add the edge \( S \) to \( H_i \)). Furthermore add \( \{x, v_1, v_2, w_1, w_2, u_1, u_2\} \cup (V(e') - S) \) to \( Y \). Note that (a)-(d) still hold. By a similar argument to above we note that \( T^* \) decreases
by at most two. By Claim A we see that \( N[Y] \) does not increase by more than 72 vertices. As \( V^* \) does not decrease by more than 6 and \( S^* \) decreases by at least \( 9/4 \), we are done by the “minimality” of \( H \) (as \( 2 \leq 9/4 - 78/372 \)).

**Claim D:** There is no \( e = \{x, v_1, v_2\} \in E_0 \), such that \( d_{H_0}(v_1) = d_{H_0}(v_2) = 2 \) and \( d_{H_0}(x) = 1 \) and \( |N_{H_0}[V(e)]| \geq 6 \).

Proof of Claim D: Assume that there is such an edge \( e = \{x, v_1, v_2\} \in E_0 \). Let \( e_1 = \{w_1, w_2, v_1\} \) be the other edge in \( H_0 \) containing \( v_1 \) and let \( e_2 = \{u_1, u_2, v_2\} \) be the other edge in \( H_0 \) containing \( v_2 \). If \( e_1 = e_2 \), then \( |N_{H_0}[V(e)]| \leq 4 \), a contradiction. So assume that \( e_1 \neq e_2 \). As \( |N_{H_0}[V(e)]| \geq 6 \) we note that \(|\{w_1, w_2, u_1, u_2\}| \geq 3 \). We are now done analogously to Claim C.

**Claim E:** \( \Delta(H_1), \Delta(H_2) \leq 1 \).

Proof of Claim E: Assume that \( \Delta(H_1) \geq 2 \). By (a) we have \( \Delta(H_1) = 2 \). By Claim B and Claim C we note that there is a 2-regular component, \( R \), in \( H_1 \). There are no overlapping edges in \( R \) by (c). By Lemma 2 there is a transversal \( T_R \) in \( R \) of order at most \( (|V(R)| + |E(R)|)/4 - |V(R)|/24 \). So delete all edges and vertices in \( R \) and add all vertices in \( R \) to \( Y \). By Claim A we have that \( N[Y] \) increases by at most 9\(|V(R)|/24 \) vertices.

We now have a contradiction to the “minimality” of \( H \), as \( |V(R)|/24 \geq 9|V(R)|/372 \). Analogously we can show that \( \Delta(H_2) \leq 1 \).

**Claim F:** Assume \( e_1, e_2 \in E(H_0) \) overlap and \( e_i = (x_i, x_2, u_i) \) for \( i = 1, 2 \), where \( u_1 \neq u_2 \). If \( d_{H_0}(x_1) = d_{H_0}(x_2) = 2 \), then there is an edge \( e' \in E(H_0) \) such that \( \{u_1, u_2\} \subseteq V(e') \).

Proof of Claim F: Let \( e_1 \) and \( e_2 \) be defined as in the Claim, and assume that there is no edge \( e' \in E(H_0) \) such that \( \{u_1, u_2\} \subseteq V(e') \). Delete \( e_1, e_2, x_1, x_2 \) and \( u_1 \) from \( H_0 \). For every edge, \( e'' \), in \( H_0 \) that contains \( u_1 \), delete \( e'' \) and add the edge \( (e'' - \{u_1\}) \cup \{u_2\} \) instead. Furthermore add \( \{x_1, x_2, u_1, u_2\} \) and \( V(e'') \) from all transformed edges, to \( Y \). As there is at most 4 edges containing \( u_1 \) in \( H_0 - E(Y) \) we note that \( Y \) increases by at most 10 (the neighbours of \( u_1 \) in \( H_0 - E(Y) \) and \( \{u_1, u_2\} \)). Therefore \( V^* - N[Y] \) decreases by at most 3 + 90, by Claim A. We also note that \( S^* \) decreases by 5/4.

We now show that \( T^* \) decreases by at most one. If \( u_2 \in T^* \) then \( T^* \cup \{u_1\} \) is a transversal in the old \( H_0 \). If \( u_2 \notin T^* \) then \( T^* \cup \{x_1\} \) is a transversal in the old \( H_0 \). As (a)-(d) still holds after the above operations, we have a contradiction to the “minimality” of \( H \), as \( 1 \leq 5/4 - 93/372 \).

**Definition G:** Let \( x \in V^* - N[Y] \) be arbitrary. The vertex \( x \) exists since otherwise we would be done by Theorem 5.

**Claim H:** \( d_{H_1}(u) = d_{H_2}(u) = 1 \) for all \( u \in N_{H_0}[x] \), where \( x \) is defined in Definition G.

Proof of Claim H: Assume that \( u \in N_{H_0}[x] \) has \( d_{H_2}(u) = 0 \) or \( u \notin V(H_2) \), which are the only possibilities for \( u \), if \( d_{H_2}(u) \neq 1 \) (by Claim E). If \( u \in V(H_2) \) and \( d_{H_2}(u) = 0 \), then delete \( u \) from \( V(H_2) \). We are now done as \( T^* \) is unchanged, \( S^* \) decreases by 1/4 and \( V^* - N[Y] \) does not decrease by more than one. So we may assume that \( u \notin V(H_2) \). Since \( x \in V^* \) we note that \( x \in V(H_1) \) and \( x \in V(H_2) \), which by the above argument implies that \( d_{H_1}(x) = d_{H_2}(x) = 1 \) and \( u \neq x \). Let \( e_1 = \{x, u, q\} \) be the edge in \( H_1 \) (and \( H_0 \)) containing \( u \) and \( x \). Let \( e_2 \) be the edge in \( H_2 \) (and \( H_0 \)) that contains \( x \). Note that \( d_{H_1}(x) = 2 \) and \( d_{H_0}(u) = 1 \). If \( d_{H_0}(q) = 1 \) then we are done by Claim B. So \( d_{H_0}(q) \geq 2 \). However as any edge containing \( q \) must also lie in \( H_1 \) or \( H_2 \), as \( q \notin Y \), we note that
$d_{H_0}(q) = 2$. Let $e_q$ be the edge in $H_2$ that contains $q$. Note that $e_q \neq e_2$, by Claim F. As $e_q$ and $e_2$ do not intersect we note that $|N_{H_0}[V(e)]| = 7 \geq 6$, so we are done by Claim D.

Claim I: Let $e_1 \in E_1$ and $e_2 \in E_2$ be the edges containing $x$ (defined in Definition G). They exist by Claim H. Then $V(e_1) \cap V(e_2) = \{x\}$.

Proof of Claim I: Assume for the sake of contradiction that $|V(e_1) \cap V(e_2)| \geq 2$. If $|V(e_1) \cap V(e_2)| = 3$, then we delete $e_1$ from $H_0$ and add $V(e_1)$ to $Y$. This contradicts the "minimality" of $H$, as $T^*$ remains unchanged, $S^*$ decreases by $1/4$ and $N[Y]$ increases from Claim A by at most 27. Therefore assume that $|V(e_1) \cap V(e_2)| = 2$. Let $e_1 = \{x, v, w\}$ and let $e_2 = \{x, v, y\}$ where $w \neq y$. As $d_{H_0}(x) = d_{H_0}(y) = 2$, there is an edge, $e'$, in $H_0$ such that $\{w, y\} \subseteq V(e')$, by Claim F. However $e' \not\in E(H_1)$ and $e' \not\in E(H_2)$ by Claim E. This is however a contradiction to (d), as $w, y \notin Y$.

Claim J: We now obtain a contradiction.

Proof of Claim J: Let $e_1 = \in E_1$ and $e_2 \in E_2$ be the edges containing $x$ (defined in Definition G). They exist by Claim H and $V(e_1) \cap V(e_2) = \{x\}$, by Claim I. Let $e'_1 = \{x, v_1, v_2\}$ and let $e_2 = \{x, v_1, w_2\}$. Let $e''_1$ be the edge in $H_1$ containing $w_1$ and let $e''_2$ be the edge in $H_1$ containing $w_2$ (they exist by Claim H). Let $e'_2$ be the edge in $H_2$ containing $v_1$ and let $e''_2$ be the edge in $H_2$ containing $v_2$ (they exist by Claim H).

If $e'_1 = e''_1$, then $V(e'_1) \cap V(e_2) = \{w_1, w_2\}$ and $e'_1 = \{w_1, w_2, r\}$ for some $r \in V(H_0)$. By Claim F, there is an edge in $H_0$ that contains $x$ and $r$. But this is a contradiction, as neither $e_1$ or $e_2$ contain $r$, by Claim H. Therefore $e'_1 \neq e''_1$. Analogously we can show that $e'_2 \neq e''_2$.

We now delete $e_1, e'_1, e''_1$ from $H, H_0$ and $H_1$. Delete $e_2, e'_2, e''_2$ from $H, H_0$ and $H_2$. Delete $V(e_1) \cup V(e'_1) \cup V(e''_1)$ from $V(H_1)$ and delete $V(e_2) \cup V(e'_2) \cup V(e''_2)$ from $V(H_2)$. Delete $V(e_1) \cup V(e_2)$ from $H$ and $H_0$. Let $S_1$ be any subset of size three in $V(e'_1) \cup V(e''_1) - \{w_1, w_2\}$ and let $S_2$ be any subset of size three in $V(e'_2) \cup V(e''_2) - \{v_1, v_2\}$. Add the edges $S_1$ and $S_2$ to $H$ and $H_0$. Finally add all vertices in $V(e'_1) \cup V(e''_1) \cup V(e'_2) \cup V(e''_2) - \{w_1, w_2, v_1, v_2, x\}$ to $Y$.

We first show that $T^*$ decreases by at most 8. It is clear that the transversal size drops by three in both $H_1$ and $H_2$. So assume that $T^*$ is a transversal of the new $H_0$. As in the proof of Claim C we note that one of the three edges $e_1, e'_2, e''_2$ are already covered by a vertex in $T^*$ (due to $S_2$) and the other two edges can be covered by one additional vertex. Similarly by adding one more vertex to $T^*$ we can make sure that $e_2, e'_1, e''_1$ are all covered. Therefore the transversal size drops by at most two in $H_0$.

Note that $S^*$ drops by $33/4$ as we delete 9 vertices in each of $H_1$ and $H_2$ and we delete 5 vertices in $H_0$. We also delete three edges in each of $H_1$ and $H_2$ and six edges in $H_0$. But we also add two edges in $H_0$.

$N[Y]$ increases by at most 72 vertices by Claim A, as $|V(e'_1) \cup V(e''_1) \cup V(e'_2) \cup V(e''_2) - \{w_1, w_2, v_1, v_2, x\}| \leq 8$. As $V^*$ decreases by at most 13, we note that $V^* - N[Y]$ decreases by at most 85. We note that (a)-(d) still holds after the above operations. We therefore have a contradiction to the "minimality" of $H$, as $8 \leq 33/4 - 85/372$.

Theorem 8. If $G$ is a graph with $\delta(G) \geq 3$ then $f_t(G; 2) \leq (\frac{3}{4} - \frac{1}{372})|V(G)|$.

Proof. Let $G$ be any graph with $\delta(G) \geq 3$ and let $(W_1, W_2)$ be a partition of $V(G)$. Define the hypergraph $H_G$, such that $V(H_G) = V(G)$ and $E(H_G)$ is obtained by selecting for each $v \in V(G)$ one set of three vertices from $N_G(v)$ to form a hyperedge. $E(H_G) =$
\( \{ e_v : v \in V(G) \}, e_v = \{ x_v, y_v, z_v \} \subseteq N_G(v) \). Furthermore for every hyperedge, \( e \in E(H_G) \) let \( L(v) \) be the set \( \{ 0, i \} \) if \( v \in W_i \). For reasons which will be clear later we let \( L(v) = \{ 0, 1, 2 \} \) for every \( v \in V(H_G) \). Let \( H_i \) be the 3-uniform hypergraph containing vertex-set \( V_i = \{ v : i \in L(v) \text{ and } v \in V(H) \} \) and edge-set \( E_i = \{ e : i \in L(e) \text{ and } e \in E(H) \} \), for \( i = 0, 1, 2 \). Note that a transversal of \( H_0 \) corresponds to a total dominating set in \( G \) and a transversal of \( H_i \ (i \in \{ 1, 2 \}) \) corresponds to a total dominating set in \( G \) of the set \( W_i \). Therefore we would be done if we could show that \( T(H_0) + T(H_1) + T(H_2) \leq (\frac{3}{2} - \frac{1}{372})|V(G)| \). Let \( Y \) be an empty set. We note that \( |E_1| + |E_2| = |E_0| = |V_0| = |V_1| = |V_2| = |V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]| = |V(G)| \) and therefore the inequality above is equivalent to
\[
(*) \quad \sum_{i=0}^{2} T(H_i) \leq \left( \sum_{i=0}^{2} |V_i| + |E_i| \right) - \frac{|V(H_0) \cap V(H_1) \cap V(H_2) \setminus N_H[Y]|}{372}
\]

For simplicity we will use the following notation:
\( T^* = \sum_{i=0}^{2} T(H_i) \)
\( S^* = \sum_{i=0}^{2} \frac{|V_i| + |E_i|}{4} \)
\( V^* = V(H_0) \cap V(H_1) \cap V(H_2) \)

We will now do a few transformations on \( H, H_0, H_1, H_2 \).

Transformation 1: While there is some vertex \( x \in V(H) \) with \( d_{H_0}(x) \geq 5 \) (or equivalently \( d_H(x) \geq 5 \)), delete \( x \) and all edges incident with \( x \) from \( H \) (and therefore also from \( H_0, H_1 \) and \( H_2 \)).

Claim A: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

Proof of Claim A: We note that \( T^* \) drops by at most three, as we may place \( x \) in the transversal of the new \( H_i \)'s in order to get transversals in the old \( H_i \)'s. We note that \( S^* \) decreases by at least 13/4, as we delete \( x \) from \( H_0, H_1, H_2 \) and 5 edges from \( H_0 \) plus a total of 5 edges from \( H_1 \) and \( H_2 \). As \( V^* \) decreases by one and \( N_H[Y] = \emptyset \) remains unchanged, we are done.

Transformation 2: While there is a vertex \( x \in V(H) \) with \( d_{H_1}(x) \geq 3 \), delete \( x \) and all edges incident to \( x \) from \( H_0 \) and \( H_1 \). Also delete these edges from \( H \) (but do not delete \( x \) or any edges incident to \( x \) in \( H_2 \)). If \( d_{H_2}(x) = 0 \) then delete \( x \) from \( H_2 \) (i.e. delete 2 from \( L(x) \)). If \( d_{H_2}(x) > 0 \) then note that \( d_{H_2}(x) = 1 \) (as we have performed transformation 1 as long as we could) and put \( N_{H_2}[x] \) in \( Y \).

Claim B: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

Proof of Claim B: We note that \( T^* \) drops by at most two, as we may place \( x \) in the transversal of the new \( H_0 \) and \( H_1 \) in order to get transversals in the old \( H_0 \) and \( H_1 \). We note that \( S^* \) decreases by at least 9/4, as we delete 3 edges and 1 vertex from \( H_0 \) and \( H_1 \) and we either delete a vertex in \( H_2 \) or 4 edges from \( H_0 \). As \( V^* \) decreases by one and \( N_H[Y] \) increases by at most 21 (as \( \Delta(H) \leq 4 \), after Transformation 1), we are done.

Transformation 3: While there is a vertex \( x \in V(H) \) with \( d_{H_2}(x) \geq 3 \), then do the following. Delete \( x \) and all edges incident to \( x \) from \( H_0 \) and \( H_2 \). Also delete these edges from \( H \) (but do not delete \( x \) or any edges incident to \( x \) in \( H_1 \)). Furthermore delete any
vertices in $H_2$, which get degree zero by the above transformation. If $d_{H_1}(x) = 0$ then delete $x$ from $H_1$. If $d_{H_1}(x) > 0$, then we put $N_{H_1}[x]$ in $Y$.

Claim C: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

Proof of Claim C: We note that $T^*$ drops by at most two, as we may place $x$ in the transversal of the new $H_0$ and $H_2$ in order to get transversals in the old $H_0$ and $H_2$. Lets count any edge, $e$, in $H_1$, which does not lie in $H_0$ as contributing $1 + |V(e) ∩ V(H_0)|/3$ to the sum $S^*$. We note that there are no such edges when we start the transformation 3’s.

We note that $S^*$ now decreases by at least 25/12, because of the following. For every edge containing $x$ in $H_2$, which does not lie in $H_0$ there is a vertex of degree one in the edge, due to the above transformations. Therefore we either delete an edge in $H_0$ or a vertex in $H_2$ for each of the edges containing $x$ in $H_2$. As we also delete the edges in $H_2$ and the vertex $x$ in $H_0$ and $H_2$ we note that $S^*$ drops by at least 8/4. So if $d_{H_1}(x) = 0$ then $S^*$ decreases by at least 9/4 as claimed. If $d_{H_1}(x) > 0$ and the edge, $e$, containing $x$ in $H_1$ also lies in $H_0$, then we are done as we delete an extra edge in $H_0$ and the edge left in $H_1$ is counted as at most $1 + 2/3$. If $d_{H_1}(x) > 0$ and the edge, $e$, containing $x$ in $H_1$ does not lie in $H_0$, then we decrease the value of $e$ by $1/3$ as $1 + |V(e) ∩ V(H_0)|/3$ decreases. This shows that $S^*$ decreases by at least 25/12.

As $V^*$ decreases by one and $N[Y]$ increases by at most 21 (as $Δ(H) ≤ 4$, after Transformation 1), we are done.

Transformation 4: If $e_1, e_2 ∈ E(H_i)$ and $|V(e_1) ∩ V(e_2)| ≥ 2$ for some $i ∈ \{1, 2\}$, then we do the following.

If $|V(e_1) ∩ V(e_2)| = 3$, then if $e_1, e_2 ∈ E_0$ we delete $e_2$ from both $H_0$ and $H_i$. If $e_j ⊆ E_0$ ($j ∈ \{1, 2\}$) then we delete $e_j$ from $H_i$ (in this case $V(e_j) ⊆ Y$). So now assume that $|V(e_1) ∩ V(e_2)| = 2$ and $e_1 = (u_1, x, y)$ and $e_2 = (u_2, x, y)$, where $u_1 ≠ u_2$.

If $d_{H_1}(u_1) = d_{H_1}(u_2) = 2$, then by the above transformations we note that $e_1, e_2 ∈ E_0$. We now add a new vertex $q$ to $H$, $H_0$ and $H_i$. We delete $e_1$ and $e_2$ from $H$, $H_1$ and $H_0$ and add the edges $\{q, x, y\}$ to $H, H_1$ and $H_0$.

If $d_{H_1}(u_j) = 1$, for some $j ∈ \{1, 2\}$, then do the following. Delete $e_1$, $e_2$ and the vertices $\{u_j, x, y\}$ from $H_i$. Add the vertices $\{u_1, u_2, x, y\}$ to $Y$.

Claim D: If (*) holds for the resulting hypergraphs, then it also holds for our original hypergraphs.

Proof of Claim D: In the case when $|V(e_1) ∩ V(e_2)| = 3$ we note that $T^*$ remains unchanged, $S^*$ decreases by 1/4 and $V^* − N[Y]$ remains unchanged. We are now done with this case.

In the case when $d_{H_1}(u_1) = d_{H_1}(u_2) = 2$, we note that $T^*$, $S^*$ and $V^*$ remain unchanged and $N[Y]$ can only grow by adding $q$ to it, but $q ∉ V^*$. We also note that the above transformation decreases the number of edges in $H_i$, so it cannot continue indefinitely. We are now done with this case.

In the case when $d_{H_1}(u_j) = 1$, we note that $T^*$ decreases by at most one, $S^*$ decreases by 5/4, $V^*$ decreases by at most three and $N[Y]$ increases by at most 24 (In $H − e_1 − e − 2$ we note that $u_1$ and $u_2$ have degree at most 3 while $x$ and $y$ have degree at most 2). As $1/4 ≥ 27/372$ we are done with this case.
Claim E: \( \Delta(H_1), \Delta(H_2) \leq 2 \) and \( \Delta(H - E(Y)) \leq 4 \) and there are no overlapping edges in \( H_i, \ i \in \{1, 2\} \).

Proof of Claim E: The fact that \( \Delta(H_1), \Delta(H_2) \leq 2 \) follow from Transformations 2 and 3. As \( \Delta(H) \leq 4 \) after Transformation 1 and no other transformation increases \( \Delta(H) \), we note that \( \Delta(H - E(Y)) \leq \Delta(H) \leq 4 \). There are no overlapping edges in \( H_i, \ i \in \{1, 2\} \) due to Transformation 4.

Claim F: If \( e \in E(H) - E(Y) \), then \( 0 \in L(e) \) and \( |L(e)| \geq 2 \).

Proof of Claim F: This was true before Transformation 1 as it was true for all edges. Transformation 1 clearly does not change this property. In Transformation 2, we only keep an edge, \( e \), in \( H_i \), where \( i \in \{1, 2\} \) but delete it in \( H_0 \) if we put \( V(e) \) in \( Y \). So the above still holds after Transformation 2. Analogously it also holds after Transformation 3. It is not difficult to check that it also holds after Transformation 4 (note that the above property holds for the edge we might add to \( H \) in Transformation 4).

We now see that (*) holds due to Lemma 3. That implies the theorem.

8. Possible strengthening of Theorem 8

No graph extremal for Theorem 8 is known and probably an inequality \( f_t(G; 2) \leq \alpha|V(G)| \) can be obtained for some \( \alpha \) smaller than \( \frac{5}{3} - \frac{1}{372} \). Certainly \( \alpha \) must be at least \( 9/8 \), that is demonstrated by the graphs of section 6.

There is a graph of order 12 having \( f_t(H_{12}; 2) = 7n/6 \), namely \( H_{12} \) from the family \( \mathcal{H} \) defined after Theorem 2, with the two \( P_6 \)'s as its partition classes. Unless we, e.g., demand that the order of the graphs be large, \( H_{12} \) shows that we cannot get a better inequality than the following conjecture.

Conjecture 1. Let \( G \) be a graph of order \( n \) with \( \delta \geq 3 \) then \( f_t(G; k) \leq 7n/6 \).

9. Three partition classes

Theorem 9. Let \( G \) be a graph of order \( n \) with \( \delta \geq 3 \) then \( f_t(G; 3) \leq 3n/2 \).

For arbitrarily large \( n \), \( n \equiv 0 \pmod{6} \), there exist graphs \( G_n \) with \( g_t(G_n; 3) = n \), \( \gamma_t(G_n) = n/3 \), \( f_t(G; 3) = 4n/3 \).

Proof. By Theorem 1 we have that \( \gamma_t(G) \leq n/2 \), and \( g_t(G; 3) \leq n \) holds trivially, so by addition we get \( f_t(G; 3) \leq 3n/2 \) as desired.

Assume a graph \( G \) has \( g_t(G; 3) = n \). Then \( \Delta(G) \leq 3 \) and as \( \delta(G) \geq 3 \), \( G \) is cubic. Since each vertex has three neighbours, one in each partition class, we see for each \( i = 1, 2, 3 \), that vertices in class \( V_i \) span a matching in \( G \).

Listing the 3 neighbours to each \( V_i \)-vertex we count each vertex of \( G \) once, so \( 3|V_i| = n \) giving \(|V_1| = |V_2| = |V_3| = n/3| \).

Each \( V_1 \)-vertex is adjacent to precisely one \( V_2 \)-vertex and that has no other \( V_1 \)-neighbour, so there is a perfect matching of \( V_1V_2 \)-edges and analogously \( G \) contains perfect matchings of \( V_1V_3 \)- and \( V_2V_3 \)-edges.

One partition class \( V_i \) totally dominates \( G \) so \( \gamma_t(G) \leq n/3 \). In fact, \( \gamma_t(G) = n/3 \) because each vertex in \( G \) can totally dominate at most its three neighbours.
Following the steps above, it is now easy for \( n \equiv 0 \pmod{3} \) to construct a graph \( G_n \) with \( g_t(G_n; 3) = n \). This graph has \( f_t(G_n; 3) = \gamma_t(G_n) + g_t(G_n; 3) = 4n/3 \).

We do not know if there, for \( \delta \geq 3 \), are graphs \( G \) with \( 4n/3 < f_t(G; 3) \leq 3n/2 \), but we pose the following conjecture.

**Conjecture 2.** There exists some positive \( \epsilon \) such that the following holds. If \( G \) is a graph with \( \delta(G) \geq 3 \), then \( f_t(G; 3) \leq (3/2 - \epsilon)|V(G)| \).

**Theorem 10.** Let \( G \) be a graph of order \( n \) with \( \delta \geq 3 \) and let \( k \geq 4 \). \( f_t(G; k) \leq 3n/2 \) and there exists an infinite family of graphs with \( f_t(G; k) = 3n/2 \).

**Proof.** The inequality is proven as in Theorem 9. For a graph with \( f_t(H; k) = 3n/2 \) take \( H \in \mathcal{H} \) (\( \mathcal{H} \) is defined after Theorem 2). Let \( v_1, v_2, \ldots, v_{n/2} \) and \( u_1, u_2, \ldots, u_{n/2} \) be two disjoint paths in \( H \) such that \( \{v_1u_2, v_2u_1, v_1v_{n/2}, u_1u_{n/2}\} \subseteq E(H) \). Let \( V_1, V_2, V_3, V_4 \) be a partition of \( H \) such that \( l(v_1), l(v_2), \ldots, l(v_{n/2}) = 1, 2, 3, 4, 1, 2, 3, 4, \ldots \) and \( l(u_1), l(u_2), \ldots, l(u_{n/2}) = 4, 3, 2, 1, 4, 3, 2, 1, \ldots \) where \( l(x) = i \) if \( x \in V_i \), then \( f_t(H; V_1, V_2, V_3, V_4) = 3n/2 \).

**References**


19. A. Yeo, Excluding one graph significantly improves bounds on total domination in connected graphs of minimum degree four. *In preparation.*

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