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Subdigraphs of almost Moore digraphs induced by fixpoints of an automorphism

by

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Abstract

The degree/diameter problem for directed graphs is the problem of determining the largest possible order for a digraph with given maximum out-degree $d$ and diameter $k$. An upper bound is given by the Moore bound $M(d,k) = \sum_{i=0}^{k} d^i$ and almost Moore digraphs are digraphs with maximum out-degree $d$, diameter $k$ and order $M(d,k) - 1$.

In this paper we will look at the structure of subdigraphs of almost Moore digraphs, which are induced by the vertices fixed by some automorphism $\varphi$. If the automorphism fixes at least three vertices, we prove that the induced subdigraph is either an almost Moore digraph or a diregular $k$-geodetic digraph of degree $d' \leq d - 2$, order $M(d',k) + 1$ and diameter $k + 1$.

As it is known that almost Moore digraphs have an automorphism $r$, these results can help us determine structural properties of almost Moore digraphs, such as how many vertices of each order there are with respect to $r$. We determine this for $d = 4$ and $d = 5$, where we prove that except in some special cases, all vertices will have the same order.

1. Introduction

Let $G$ be a digraph and $u$ be a vertex of maximum out-degree $d$ in $G$, and let $n_i$ denote the number of vertices in distance $i$ from $u$. Then we have $n_i \leq d^i$ for $i = 0, 1, \ldots, k$, and thus the order $n$ of $G$ is bounded by

$$n = \sum_{i=0}^{k} n_i \leq \sum_{i=0}^{k} d^i. \quad (1)$$
If equality is obtained in (1) we say that $G$ is a Moore digraph of degree $d$ and diameter $k$, and the right-hand side of (1) is called the Moore bound denoted by $M(d, k) = \sum_{i=0}^{k} d^i$. Moore digraphs are known to be diregular and exist only when $d = 1$ (cycles of length $(k + 1)$) or $k = 1$ (complete digraphs with order $d + 1$), see [1] or [2]. So we are interested in knowing how close the order can get to the Moore bound for $d > 1$ and $k > 1$. Let $G$ be a digraph of maximum out-degree $d$, diameter $k$ and order $M(d, k) - \delta$, then we say $G$ is a $(d, k, -\delta)$-digraph or alternatively a $(d, k)$-digraph of defect $\delta$. When $\delta < M(d, k - 1)$ we have out-regularity, see [3], whereas it in general is not known if we also have in-regularity. Of special interest is the case $\delta = 1$, and a $(d, k, -1)$-digraph is also denoted as an almost Moore digraph. Almost Moore digraphs do exist for $k = 2$ as the line digraphs of $K_{d+1}$ for any $d \geq 2$, see [4], whereas $(2, k, -1)$-digraphs for $k > 2$, $(3, k, -1)$-digraphs for $k > 2$, $(d, 3, -1)$-digraphs for $d > 1$ and $(d, 4, -1)$-digraphs for $d > 1$ do not exist, see [5], [6], [7] and [8]. We do know that almost Moore digraphs are diregular for $d > 1$ and $k > 1$, see [3].

In the last section of the paper, we will be needing the following theorem which summarises some of the above results.

**Theorem 1 ([5],[6]).** Almost Moore digraphs of degree 2 and 3 and diameter $k > 2$ do not exist.

Furthermore, almost Moore digraphs satisfies the following properties, where $a \leq k$-walk is a walk of length at most $k$.

**Lemma 1 ([9]).** Let $G$ be an almost Moore digraph, then

- for each pair of vertices $u, v \in V(G)$ there is at most one $k$-walk from $u$ to $v$,  
- for every vertex $u \in V(G)$ there exist a unique vertex $r(u)$ such that there are two $\leq k$-walks from $u$ to $r(u)$.

The mapping $r : V(G) \to V(G)$ is in fact an automorphism, see [9] and thus the two $\leq k$-walks from $u$ to $r(u)$ are internally disjoint. The vertex $r(u)$ is said to be the repeat of $u$. If we have $u = r(u)$, thus $u$ has order 1 with respect to $r$, $u$ is said to be a selfrepeat. If there is a selfrepeat in $G$, then there are exactly $k$ selfrepeats, which lie on a $k$-cycle, see [10].

In this paper we will give some conditions for the existence of an almost Moore digraph $G$ with respect to some automorphism $\varphi : V(G) \to V(G)$. These results can then be used to investigate the orders of the vertices with respect to the automorphism $\varphi$. Before stating the core result of this paper,
we will introduce another type of digraph which shows to be important when characterizing induced subdigraphs of almost Moore digraphs.

Let $D$ be a digraph such that for each pair of vertices $u, v \in V(D)$ we have at most one $k$-walk from $u$ to $v$, then we say $D$ is $k$-geodetic. Let $u$ be a vertex of minimum out-degree $d$, and let $n_i$ be the number of vertices in distance $i$ from $u$ for $i = 0, 1, \ldots, k$. Then $n_i \geq d^i$ and the order $n$ of $D$ is bounded by

$$n \geq \sum_{i=0}^{k} n_i \geq \sum_{i=0}^{k} d^i. \quad (2)$$

Notice that the right-hand side is the Moore bound, $M(d, k)$ and that the diameter for a $k$-geodetic digraph is at least $k$. As we already know, Moore digraphs do only exist for $d = 1$ or $k = 1$, we wish to know how close the order of a $k$-geodetic digraph can get to the Moore bound. By a $(d, k, \epsilon)$-digraph we understand a $k$-geodetic digraph of minimum out-degree $d$ and order $M(d, k) + \epsilon$. Alternatively we say that we have a $(d, k)$-digraph of excess $\epsilon$. The first case which is interesting is when $\epsilon = 1$. A $(d, k, 1)$-digraph has diameter $k + 1$, and for each vertex $u$ there is exactly one vertex, the outlier $o(u)$ such that $\text{dist}(u, o(u)) = k + 1$, see [11].

A $(d, k, 1)$-digraph is diregular if and only the mapping $o: V(D) \mapsto V(D)$ is an automorphism, see [11]. From [11] we also have the following theorem.

**Theorem 2 ([11]).** No diregular $(2, k, 1)$-digraphs exist for $k > 1$.

2. Results

For simplicity, we will, in the remaining part of this paper, let a $(d, k, -1)$-digraph (almost Moore digraphs) denote any digraph which has degree $d > 0$, diameter $k > 0$ and order $M(d, k) - 1$, thus we will let $k$-cycles be included in this class. Similar, a $(d, k, 1)$-digraph will denote any $k$-geodetic digraph of minimum out-degree $d > 0$ and order $M(d, k) + 1$.

The scope of this paper is to prove the following theorem.

**Theorem 3.** Let $G$ be an almost Moore digraph of degree $d \geq 4$ and diameter $k \geq 3$ and let $H$ be a subdigraph induced by the vertices which are fixed by some automorphism $\varphi: V(G) \mapsto V(G)$. Then $H$ is either

- the empty digraph,
- two isolated vertices,
- an almost Moore digraph of degree $d' \leq d$ and diameter $k$ or
• a diregular \((d', k, 1)\)-digraph where \(d' \leq d - 2\).

In the remaining part of this paper we will assume \(G\) to be an almost
Moore digraph of degree \(d \geq 4\) and diameter \(k \geq 3\), and \(H\) to be a sub-
digraph of \(G\) induced by the fixpoints of some automorphism \(\varphi : V(G) \mapsto V(G)\).

We start by stating some properties of the fixpoints of \(G\).

**Lemma 2.** Let \(u\) and \(v\) be fixpoints of \(G\) with respect to the automorphism \(\varphi\), then

- \(r(u)\) is a fixpoint,
- if there is a \(\leq k\)-walk \(P\) from \(u\) to \(v\) and \(v \neq r(u)\), all vertices \(w \in P\) are fixpoints
- if \(v = r(u)\) and \(P\) and \(Q\) are the two \(\leq k\)-walks from \(u\) to \(v\), either all internal vertices on \(P\) and \(Q\) are fixpoints, or none of them are. Furthermore, if \(\text{dist}(u, r(u)) < k\), then all vertices on \(P\) and \(Q\) are fixpoints.

**Proof.**
- We know there are two \(\leq k\)-walks, \(P\) and \(Q\), from \(u\) to \(r(u)\).
  Now, \(\varphi(P)\) and \(\varphi(Q)\) are two \(\leq k\)-walks from \(u\) to \(\varphi(r(u))\), and hence \(\varphi(r(u))\) is a repeat of \(u\). As \(u\) only has one repeat, the statement follows.
- Let \(P\) be the unique \(\leq k\)-walk from \(u\) to \(v\). Then \(\varphi(P)\) will also be a \(\leq k\)-walk from \(u\) to \(v\), and hence \(P = \varphi(P)\).
- Assume not all vertices on the \(\leq k\)-walk \(P\) are fixpoints, hence there exist a vertex \(w \in P\) such that \(w \neq \varphi(w)\) and thus \(\varphi(P) \neq P\) is also a \(\leq k\)-walk from \(u\) to \(v = r(u)\). As there are only two \(\leq k\)-walks from \(u\) to \(v = r(u)\), we must have \(\varphi(P) = Q\) and thus none of the internal vertices of \(P\) are fixpoints, as \(P\) and \(Q\) are internally disjoint. Now if \(\text{dist}(u, r(u)) < k\), then \(P\) and \(Q\) are obviously of different length, so we must have all vertices on \(P\) and \(Q\) as fixpoints.

**Corollary 1.** Let \(\varphi\) be an automorphism of \(G\), then all \(\leq k\)-walks among the fixpoints of \(\varphi\) in \(G\) are preserved to \(H\), except for possibly the \(k\)-walks from a vertex to its repeat.
Notice, that if \( u \) and \( v \) are selfrepeats fixed by \( \varphi \), then there are exactly \( d \) internally disjoint \((k+1)\)-walks from \( u \) to \( v \), \((u,u_i,\ldots,v_i,v)\) for \( i = 1,2,\ldots,d \). Hence if the order of \( u_i \) with respect to \( \varphi \) is \( p \), and the order of \( v_i \) with respect to \( \varphi \) is \( q \), then \((u,u_i = \varphi^p(u_i),\ldots,\varphi^q(v_i),v)\) and \((u,u = \varphi^q(u_i),\ldots,v_i = \varphi^q(v_i),v)\) are both \((k+1)\)-walks, and thus we must have \( p = q \). Said in another way, the permutation cycles with respect to some automorphism \( \varphi \) of the vertices in \( N^+(u) \) and \( N^-(v) \) are the same when \( u \) and \( v \) are selfrepeats.

The following lemma is a more general result than that of [12].

**Lemma 3.** If \( G \) has a selfrepeat which is fixed by \( \varphi \), then \( H \) is an almost Moore digraph with selfrepeats of degree \( d' \leq d \) and diameter \( k \).

**Proof.** Let \( z = r(z) = \varphi(z) \), then according to Lemma 2 we must have all vertices on the two \( \leq k \)-walks from \( z \) to \( r(z) \) as fixpoints, and all the selfrepeats lie on the non-trivial walk from \( z \) to \( z \), so \( H \) contains a \( k \)-cycle.

Notice that \( d_H^1(z) = d_H^r(z) = d' \leq d \) for all \( z = r(z) \in V(H) \), as the permutation cycles in \( N^+(z) \) and \( N^-(z) \) are the same. Now, if we have a vertex \( u = \varphi(u) \neq r(u) \), then we can pick a selfrepeat \( z \) such that \( r(u) \notin N^-(z) \), as otherwise we would have \( r(u) \in N^-(z') \) for all selfrepeats \( z' \) of \( G \), and therefore \( r(r(u)) \) would be a selfrepeat, a contradiction as \( u \) is not a selfrepeat. Thus for this \( u \) and \( z \) we have \( d \) internally disjoint \( \leq (k+1)\)-walks \((u,u_i,\ldots,z_i,z)\) in \( G \). Then \( d' \) of the internally disjoint \( \leq (k+1)\)-walks from \( u \) to \( z \) will also be in \( H \), due to Lemma 2, and thus \( d^+(u) \geq d' \). Assume that \( d^+(u) > d' \), then there exists a \( j \in \{1,2,\ldots,d\} \) such that \( u_j = \varphi(u_j) \) and \( z_j \neq \varphi(z_j) \). But then \((u_j,\ldots,z_j,z)\) and \((u_j,\ldots,\varphi(z_j),z)\) are two distinct \( \leq k \)-walks from \( u_j \) to \( z \), a contradiction as \( z \) is a selfrepeat.

So \( H \) is a diregular digraph of degree \( d' \). Now, assume \( H \) has diameter \( k+1 \), this implies that there exists a vertex \( v \) such that \( dist_H(v,r(v)) = k+1 \) thus the order of \( H \) is \( n = 1 + d' + d'^2 + \ldots + d'^k + 1 = M(d',k)+1 \), according to Corollary 1. However, looking at a selfrepeat \( z \in H \), we get the order as \( n = 1 + d' + d'^2 + \ldots + d'^k - 1 = M(d',k) - 1 \), a contradiction.

So \( H \) must be diregular with degree \( d' \leq d \), diameter \( k \) and its order must be \( M(d,k) - 1 \), hence it is an almost Moore digraph with selfrepeats, as the girth of \( H \) is \( k \). \( \square \)

**Lemma 4.** Let \( \varphi \) fix at least three vertices, then \( H \) is diregular of degree \( d' \) and either

- \( H \) is an almost Moore digraph of degree \( d' \leq d \) and diameter \( k \), or
- \( H \) is a \((d',k,1)\)-digraph of degree \( d' \leq d - 2 \).
Proof. If $\varphi$ fixes a selfrepeat, then we have the first case of the statement according to Lemma 3. Thus we can assume $\varphi$ does not fix any selfrepeats.

Let $u$ and $v$ be any two fixed vertices in $G$, thus they are not selfrepeats, and let $N^+(u) = \{u_1, u_2, \ldots, u_d\}$ and $N^-(v) = \{v_1, v_2, \ldots, v_d\}$. Assume $r(u) \neq v_j$ for $j = 1, 2, \ldots, d$. Then in $G$ we have internally disjoint $(k+1)$-walks $(u, u_1, \ldots, v_i, v)$ for $i = 1, 2, \ldots, d$. As $r$ is an automorphism, we get $r(u_i) \neq v$ for $i = 1, 2, \ldots, d$. Now, we have $u_i = \varphi(u_i)$ if and only if $v_i = \varphi(v_i)$ due to Lemma 2, hence $d^+_H(u) = d^-_H(v)$. As we could have $v = r(u)$, we see that each vertex in $H$ is balanced, as $d^+(u) = d^+(r(u))$ and $d^-(u) = d^-(r(u))$.

Now, assume $H$ is not diregular, thus for each vertex $u \in V(H)$ we must have a vertex $v \in N^+(r(u)) \cap V(H)$ such that $d^+_H(u) \neq d^-_H(v)$. Let $u \in V(G)$ be a vertex of minimum degree $d_1 \leq d$ in $H$, and let $v \in V(H)$ be a vertex with $d^+_H(v) > d_1$. Then $d^-_H(v) = d_1 + 2$ as we must have $v \in N^+(r(u))$ with $\text{dist}_H(u, r(u)) = k + 1$ and $\text{dist}_H(r^-(v), v) \leq k$. But then there must be at most $d_1$ vertices of degree different from $d_1$ in $H$ and at most $d_1 + 2$ vertices of degree different from $d_1 + 2$, hence $|V(H)| \geq d_1 + d_2 + \ldots + d_k$ as the diameter of $H$ is at least $k \geq 3$. So, obviously $H$ is diregular. If $\text{dist}(u, r(u)) = k + 1$, then each vertex in $H$ must have at least two out-neighbours of order two with respect to $\varphi$ and thus the statement follows.

Theorem 3 now follows directly from Lemmas 3 and 4.

3. Almost Moore digraphs of degree 4 and 5

In this section we will look at almost Moore digraphs of degree 4 and 5 and specify the order of the vertices with respect to the automorphism $r$.

Lemma 5. Let $u \in V(G)$ be a vertex with $\varphi(u) = u \neq r(u)$, then if $H$ is two isolated vertices or has diameter $(k + 1)$ we must have two vertices in $N^+_G(u)$ which have order 2 with respect to $\varphi$.

Proof. In $G$ we have two $\leq k$-paths, $P$ and $Q$ from $u$ to $r(u)$. If $H$ is either two isolated vertices or has diameter $k + 1$, we must have that the internal vertices on $P$ and $Q$ are not in $H$. Thus $\varphi(P) = Q$ and $\varphi(Q) = P$, and hence $\varphi^2(v) = v$ and $\varphi(v) \neq v$ for all internal vertices $v$ on $P$ and $Q$. 

The following theorem is a more general result than that of [13] and [12].

Theorem 4. Let $G$ be an almost Moore digraph of degree 4, then the vertices of $G$ have orders with respect to the automorphism $r$ according to one of the following:
• there are \( k \) vertices of order 1 and \( M(4, k) - 1 - k \) of order 3 or

• all vertices are of the same order \( p \geq 2 \).

Proof. Assume throughout that not all vertices are of the same order. Let \( u \) be a vertex of \( G \) of the smallest order \( p \) with respect to \( r \) in \( G \). Let \( N^+(u) = \{u_1, u_2, u_3, u_4\} \), then we can split \( N^+(u) \) into permutation cycles with respect to \( r^p \) in one of the following ways: \((u_1)(u_2)(u_3, u_4)\), \((u_1)(u_2, u_3, u_4)\), \((u_1, u_2, u_3, u_4)\) or \((u_1, u_2)(u_3, u_4)\). Notice however that the splitting \((u_1)(u_2)(u_3, u_4)\) is not possible, as there according to Theorem 3 where \( \varphi = r^p \) would exist a \((2, k, -1)\)- or \((2, k, 1)\)-digraph as an induced subdigraph of \( G \), a contradiction to Theorems 1 and 2.

First assume there is a vertex \( u \) of order 1, thus \( u \) is a selfrepeat and hence there are exactly \( k \) vertices of order 1 inducing a \( k \)-cycle in \( G \). Thus among the above ways of having permutation cycles, the only possibility is \((u_1)(u_2, u_3, u_4)\). Then all vertices which are not selfrepeats must have order \( 3 \) according to Lemma 3 by letting \( \varphi = r^3 \).

Now assume \( u \in V(G) \) has the smallest possible order \( p \geq 2 \), then according to Lemma 5 the only possible permutation cycles are \((u_1, u_2)(u_3, u_4)\). In turn, this is only possible if \( p = 2 \), as there will always be at least \( p \) vertices of order \( p \) in \( G \).

Thus \( G \) will contain \( M(4, k) - 3 \) vertices of order 4, thus 4 should divide \( M(4, k) - 3 \). But in fact

\[
M(4, k) - 3 \equiv 2 + 4^2 + \ldots 4^k \equiv 2 \mod 4,
\]

a contradiction. \( \square \)

Theorem 5. Let \( G \) be an almost Moore digraph of degree 5, then one of the following is true regarding the orders with respect to the automorphism \( r \) of the vertices in \( G \):

• there are \( M(3, k) + 1 \) vertices of order \( p \geq 2 \) and \( M(5, k) - M(3, k) - 2 \) of order \( 2p \)

• there are \( k + 2 \) vertices of order \( p \geq 2 \) and \( M(5, k) - 3 - k \) of order \( 2p \)

• there are \( k \) vertices of order 1 and either \( M(5, k) - 1 - k \) of order 2 or \( M(5, k) - 1 - k \) of order 4

• all vertices are of the same order \( p \geq 2 \).
Proof. Assume throughout that not all vertices are of the same order. Let $u$ be a vertex of $G$ of the smallest order $p$. Let $N^+(u) = \{u_1, u_2, u_3, u_4, u_5\}$, then we can split $N^+(u)$ into permutation cycles with respect to $r^p$ in one of the following ways: $(u_1)(u_2, u_3, u_4, u_5)$, $(u_1)(u_2)(u_3)(u_4, u_5)$ or $(u_1)(u_2, u_3)(u_4, u_5)$ due to Lemma 5 and Theorems 1 and 2.

If the permutation cycles are $(u_1)(u_2, u_3, u_4, u_5)$, then due to Lemma 5 we must have $u$ is a selfrepeat, hence there is $k$ vertices of order 1 and $M(5, k) - k - 1$ of order 4. If instead the permutation cycles are $(u_1)(u_2, u_3)(u_4, u_5)$, then we could have $k$ vertices of order 1 and $M(5, k) - k - 1$ of order 2 or $k + 2$ vertices of order $p \geq 2$ and $M(5, k) - k - 3$ of order $2p$.

Finally, if the permutation cycles are $(u_1)(u_2)(u_3)(u_4, u_5)$, then if $\varphi = r^p$, we would have $H$ to be either a $(3, k, -1)$-digraph or a $(3, k, 1)$-digraph. But $(3, k, -1)$-digraphs do not exist according to Theorem 1, thus we must have $M(3, k) + 1$ vertices of order $p \geq 2$ and $M(5, k) - M(3, k) - 2$ of order $2p$. 

References


