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AN EXACT SUBSPACE METHOD FOR FUNDAMENTAL FREQUENCY ESTIMATION

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ABSTRACT

In this paper, an exact subspace method for fundamental frequency is presented. The method is based on the principles of the MUSIC algorithm, wherein the orthogonality between the signal and and noise subspace is exploited. Unlike the original MUSIC algorithm, the new method does not employ an approximate measure of the angles between the subspaces but rather uses an exact measure. This makes a difference, for example, when the fundamental frequency is low, for real signals, or when the number of samples is low. In Monte Carlo simulations, the performance of the new method is compared to a number of state-of-the-art methods and is demonstrated to lead to improvements in certain, critical cases.

***Index Terms*—** Spectral estimation, fundamental frequency estimation, subspace methods

1. INTRODUCTION

Many signals of interest to mankind are periodic or approximately periodic. This is, for example, the case for signals produced by the human speech production system, those produced by many musical instruments, bird songs, vital signs, rotating targets in sonar and radar, and boats and helicopters. Such periodic signals can be decomposed into sums of harmonically related sinusoids whose frequencies are integer multiples of a fundamental frequency and the problem of finding this fundamental frequency, sometimes also referred to as pitch estimation, is the topic of the present paper. A host of different methods have been proposed over the years for solving this important problem, e.g., [1–6], and we refer the interested reader to [7] for an overview. Many of these methods are, implicitly or explicit, based on asymptotic approximations and this causes trouble in certain situations. This is, for example, the case for a low number of observations, for low fundamental frequencies and for real signals.

In this paper, we propose a new method for dealing with these problems in the context of fundamental frequency estimation. It is a subspace method based on the orthogonality between the signal of noise subspaces, a principle known from the classical MUSIC algorithm [8]. Unlike the MUSIC algorithm, the proposed method method is based on an exact measure of the angles between subspaces [9–11]. It is generally not feasible to employ such exact measures in unconstrained frequency estimation with several nonlinear parameters. However, it is well-suited for the fundamental frequency estimation problem as it only involves one nonlinear parameter.

The remaining part of this paper is organized as follows: In Section 2, we introduce the basic signal model, the underlying assumptions and define the problem at hand. Then, in Section 3 the proposed method is presented. We then investigate the performance of the proposed method under various conditions and compare it to a number of state-of-the-art methods in Section 4. Finally, we conclude on our work in Section 5.

2. COVARIANCE MATRIX MODEL

We will now introduce the problem at hand and the signal model. The observed real signal $x(n)$ is comprised of L sinusoidal components having frequencies that are integer multiples of a fundamental frequency ω_0, real amplitude $A_l > 0$, and phases $\phi_l \in [0, 2\pi)$. Moreover, we assume that an additive noise source $e(n)$ is present, which is here assumed to be white with variance σ^2. In math, the signal model can be expressed for $n = 0, \ldots, N-1$ as

$$x(n) = \sum_{l=1}^{L} A_l \cos\left(\omega_0 l n + \phi_l\right) + e(n). \tag{1}$$

The problem at hand is then to estimate ω_0, which, for a given L, can be in the range $\omega_0 \in (0, \frac{\pi}{L})$. For a collection of samples $\{x(n)\}$, the model above can be expressed as

$$\mathbf{x}(n) = \mathbf{Z}\mathbf{a} + \mathbf{e}(n), \tag{2}$$

with the following definitions:

$$\mathbf{x}(n) = [\, x(n)\; x(n+1)\; \cdots\; x(n+N-1)\,]^T \tag{3}$$

$$\mathbf{Z} = [\, \mathbf{z}(\omega_0)\; \mathbf{z}^*(\omega_0)\; \cdots\; \mathbf{z}(\omega_0 L)\; \mathbf{z}^*(\omega_0 L)\,], \tag{4}$$

$$\mathbf{a} = \frac{1}{2}\left[A_1 e^{j\phi_1}\; A_1 e^{-j\phi_l}\; \cdots\; A_L e^{j\phi_L}\; A_L e^{-j\phi_L}\,\right]^T \tag{5}$$

$$\mathbf{z}(\omega_0 l) = \left[\, 1 \; e^{j\omega_0 l 1} \; \cdots \; e^{j\omega_0 l (N-1)} \,\right]^T \tag{6}$$

$$\mathbf{e}(n) = [\, e(n) \; e(n+1) \; \cdots \; e(n+M-1) \,]^T. \tag{7}$$

The covariance matrix of $\mathbf{x}(n)$ this vector is given by

$$\mathbf{R} = \mathrm{E}\left\{\mathbf{x}(n)\mathbf{x}^H(n)\right\} = \mathbf{Z}\mathbf{P}\mathbf{Z}^H + \sigma^2\mathbf{I} \tag{8}$$

where the amplitude covariance matrix $\mathrm{E}\left\{\mathbf{a}\mathbf{a}^H\right\} = \mathbf{P}$ is given by

$$\mathbf{P} = \mathrm{E}\left\{\begin{bmatrix} a_1 a_1^* & a_1^* a_1^* & \dots & a_1 a_L^* & a_1^* a_L^* \\ a_1 a_1 & a_1^* a_1 & \dots & a_1 a_L & a_1^* a_L \\ \vdots & \vdots & & \vdots & \vdots \\ a_L a_1^* & a_L^* a_1^* & \dots & a_L a_L^* & a_L^* a_L^* \\ a_L a_1 & a_L^* a_1 & \dots & a_L a_L & a_L^* a_L \end{bmatrix}\right\}. \tag{9}$$

Assuming that the phases ϕ_l are uniformly distributed and independent over l we have that $\mathrm{E}\left\{\frac{A_k}{2}e^{j\phi_k}\right\} = 0$ and that $\mathrm{E}\left\{\frac{A_k}{2}e^{j\phi_k}\frac{A_l}{2}e^{-j\phi_l}\right\} = \frac{A_k}{2}\mathrm{E}\left\{e^{j\phi_k}\right\}\frac{A_l}{2}\mathrm{E}\left\{e^{-j\phi_l}\right\} = 0$ for $k \neq l$. For $k = l$ we get that $\mathrm{E}\left\{\frac{A_k}{2}e^{j\phi_k}\frac{A_k}{2}e^{-j\phi_k}\right\} = \frac{A_k^2}{4}$. Therefore, the amplitude covariance matrix $\mathbf{P}$ becomes

$$\mathbf{P} = \frac{1}{4}\mathrm{diag}\left(\left[\, A_1^2 \; A_1^2 \; \cdots \; A_L^2 \; A_L^2 \,\right]\right), \tag{10}$$

which means that the diagonal structure obtained for complex signals is retained for real signals, and the so-called covariance matrix model, therefore, still holds.

The eigenvalue decomposition (EVD) of the covariance matrix is $\mathbf{R} = \mathbf{U}\mathbf{\Gamma}\mathbf{U}^H$, where $\mathbf{\Gamma}$ is a diagonal matrix containing the positive eigenvalues, γ_k, ordered as $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_M$. Moreover, it can easily be seen that $\gamma_{2L+1} = \ldots = \gamma_M = \sigma^2$. $\mathbf{U}$ contains the M orthonormal eigenvectors of $\mathbf{R}$, i.e., $\mathbf{U} = \left[\, \mathbf{u}_1 \; \cdots \; \mathbf{u}_M \,\right]$. Let $\mathbf{S}$ be formed from a subset of the columns of this matrix as $\mathbf{S} = \left[\, \mathbf{u}_1 \; \cdots \; \mathbf{u}_{2L} \,\right]$. We denote the subspace spanned by the columns of $\mathbf{S}$ as $\mathcal{S} = \mathcal{R}(\mathbf{S})$ and refer to it as the signal subspace. Similarly, let $\mathbf{G}$ be formed from the remaining eigenvectors as $\mathbf{G} = \left[\, \mathbf{u}_{2L+1} \; \cdots \; \mathbf{u}_M \,\right]$. We refer to the space $\mathcal{G} = \mathcal{R}(\mathbf{G})$ as the noise subspace. Using these definitions, we now obtain $\mathbf{U}\left(\mathbf{\Gamma} - \sigma^2\mathbf{I}\right)\mathbf{U}^H = \mathbf{Z}\mathbf{P}\mathbf{Z}^H$. Introducing $\mathbf{\Psi} = \mathrm{diag}([\, \gamma_1 - \sigma^2 \; \cdots \; \gamma_{2L} - \sigma^2 \,])$, this leads to the following partitioning of the EVD:

$$\mathbf{R} = \left[\, \mathbf{S} \quad \mathbf{G} \,\right]\left(\begin{bmatrix} \mathbf{\Psi} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \sigma^2\mathbf{I}\right)\begin{bmatrix} \mathbf{S}^H \\ \mathbf{G}^H \end{bmatrix}, \tag{11}$$

which shows that we may write $\mathbf{S}\mathbf{\Psi}\mathbf{S}^H = \mathbf{Z}\mathbf{P}\mathbf{Z}^H$. As the columns of $\mathbf{S}$ and $\mathbf{G}$ are orthogonal and $\mathcal{R}(\mathbf{Z}) = \mathcal{R}(\mathbf{S})$, it follows that $\mathbf{Z}^H\mathbf{G} = \mathbf{0}$, which is the subspace orthogonality principle used in the MUSIC algorithm [8].

3. PROPOSED METHOD

In practice, the estimated noise subspace eigenvectors will not be perfect due to the observation noise and finite observation lengths, and the above relation is, therefore, only approximate. A measure must then be introduced to determine how close a candidate model $\mathbf{Z}$ is to being orthogonal to $\mathbf{G}$. Traditionally, this has been done using the Frobenius norm [8]. However, this measure is only an accurate measure of the angles between the two spaces for orthogonal vectors in both $\mathbf{Z}$ and $\mathbf{G}$, and, the asymptotic orthogonality of the column of $\mathbf{Z}$ may not always be accurate. Instead, we propose to measure the orthogonality as follows. The principal angles $\{\xi_k\}$ between the two subspaces $\mathcal{Z}$ and $\mathcal{G}$ are defined recursively for $k = 1, \ldots, K$ as [10]

$$\cos(\xi_k) = \max_{\mathbf{u}\in\mathcal{Z}} \max_{\mathbf{v}\in\mathcal{G}} \frac{\mathbf{u}^H\mathbf{v}}{\|\mathbf{u}\|_2\|\mathbf{v}\|_2} \triangleq \mathbf{u}_k^H\mathbf{v}_k, \tag{12}$$

where K is the minimal dimension of the two subspaces, i.e., $K = \min\{2L, M - 2L\}$ and $\mathbf{u}^H\mathbf{u}_i = 0$ and $\mathbf{v}^H\mathbf{v}_i = 0$ for $i = 1, \ldots, k-1$. This results in a set of angles that are bounded and ordered, i.e., $0 \leq \xi_1 \leq \ldots \leq \xi_K \leq \frac{\pi}{2}$. Given the orthogonal projection matrices for $\mathcal{Z}$ and $\mathcal{G}$, denoted $\mathbf{\Pi}_Z$ and $\mathbf{\Pi}_G$, respectively, the expression in (12) can be written as

$$\cos(\xi_k) = \max_{\mathbf{y}} \max_{\mathbf{z}} \frac{\mathbf{y}^H\mathbf{\Pi}_Z\mathbf{\Pi}_G\mathbf{z}}{\|\mathbf{y}\|_2\|\mathbf{z}\|_2} \tag{13}$$

$$= \mathbf{y}_k^H\mathbf{\Pi}_Z\mathbf{\Pi}_G\mathbf{z}_k = \kappa_k. \tag{14}$$

As can be seen, $\{\kappa_k\}$ are the ordered singular values of the matrix product $\mathbf{\Pi}_Z\mathbf{\Pi}_G$, and the two sets of vectors $\{\mathbf{y}\}$ and $\{\mathbf{z}\}$ are the left and right singular vectors of the matrix product, respectively. The singular values are related to the Frobenius norm of $\mathbf{\Pi}_Z\mathbf{\Pi}_G$ and hence its trace, denoted with $\mathrm{Tr}\{\cdot\}$, as

$$\|\mathbf{\Pi}_Z\mathbf{\Pi}_G\|_F^2 = \mathrm{Tr}\{\mathbf{\Pi}_Z\mathbf{\Pi}_G\} = \sum_{k=1}^{K}\kappa_k^2. \tag{15}$$

If this Frobenius norm is zero, then the non-trivial angles are all $\frac{\pi}{2}$, i.e., the two subspaces are orthogonal. We can use this expression to find the fundamental frequency as

$$\hat{\omega}_0 = \arg\min_{\omega_0} \|\mathbf{\Pi}_Z\mathbf{\Pi}_G\|_F^2. \tag{16}$$

As can be seen the estimate can be seen to be the value for which the sum of cosine to the angles squared is the least. Finally, (16) can be expressed as

$$\hat{\omega}_0 = \arg\min_{\omega_0} \mathrm{Tr}\left\{\mathbf{Z}\left(\mathbf{Z}^H\mathbf{Z}\right)^{-1}\mathbf{Z}^H\mathbf{G}\mathbf{G}^H\right\}. \tag{17}$$

We henceforth refer to this estimator as the angles between subspaces (ABS) method. Interestingly, it is asymptotically equivalent to the estimator proposed in [12] but is different for finite M and N in that it takes the possible non-orthogonality of the sinusoids into account. The estimator requires that a number of quantities are computed as initialization, i.e., only once, namely the EVD of $\mathbf{R}$ and the projection matrix for the noise subspace, which results in a complexity

of $\mathcal{O}((M-L)M^2+M^3)$ with $L < M$. For each candidate fundamental frequency, operations having complexity $\mathcal{O}(L^2M+M^2L+L^3)$ are computed. Regarding the covariance matrix, we use the sample covariance matrix and note that it is not required for this method that its estimate has full rank. It must, however, allow for estimation of a basis for the signal subspace, which requires that $M \leq N-2L+1$. Furthermore, we requre that $M \geq 2L+1$ for the orthogonal complement to the signal subspace to be non-empty, which means that we obtain that $2L+1 \leq M \leq N-2L+1$. Moreover, M should be chosen proportionally to N for the method to be consistent.

4. EXPERIMENTAL RESULTS

The proposed method is compared to a number of other fundamental frequency estimators using Monte Carlo simulations by generating signals according to the model in (1) and then applying various estimators to those signals. The so-obtained parameter estimates are then compared to the true parameters and the estimation error is measured in terms of the mean square error (MSE). We compare the proposed method (which, as mentioned, is referred to as ABS) to the weighted least-squares (WLS) method of [3], the approximate nonlinear least-squares (ANLS) method [1,2,7], and the optimal filtering method (OPTFILT) [13] and the MUSIC-based method of [12]. Regarding the MUSIC-based method, the proposed method should outperform it under adverse conditions and they should perform the same for high N and M as the methods will become identical then. For each set of experimental conditions, 100 realizations are used and the CRLB shown in the figures to follow is the average over the exact CRLB. The signals were generated with the following parameters, except for the parameters that are varied: a fundamental frequency with $\omega_0 = 0.3129$ is used with five harmonics each having unit amplitude and phases uniformly distributed between $-\pi$ and π. Segments of $N = 100$ samples were used with $M = 50$ and white Gaussian noise added at a signal-to-noise ratio (SNR) of 40 dB. Note that this is the SNR for the fundamental frequency estimation problem as defined in [7]. The high SNR is used so that the noise will not be the limiting factor but rather the asymptotic approximations.

The results are shown in Figures 1(a), 1(b), 1(c), and 1(d) in terms of the MSE as a function of N, ω_0, the SNR and M. From the figures, a number of interesting observations can be made. Firstly, it can be seen from Figure 1(a) that all methods perform well for a high number of observations, N, except the ANLS method which does not perform well at all. It can also be observed that the methods exhibit different threshold behavior, but the ABS and MUSIC methods perform similarly here. This is, however, not the case when the MSE is observed as a function of the fundamental frequency, as shown in Figure 1(b). From this Figure, it can be seen that the MUSIC method is indeed improved by avoiding the approximate measure of orthogonality as is done in the ABS method. In fact, the ABS method now performs as well as any of the other methods. This clearly shows that, as claimed, the exact measure is preferable when dealing with non-orthogonal sinusoids. In Figure 1(c), the MSE is depicted as a function of the SNR. This figure shows that the subspace methods appear to hold an advantage over the WLS and OPTFILT methods in terms of being robust towards noise. In this case, it does, though, not appear to matter whether the exact measure of the ABS method or the approximate one of MUSIC is used. Finally, the performance is assessed as a function of M, the covariance matrix size, with the results being shown in Figure 1(d). It can be seen that as long as M is chosen not to low or too high, it does appear to be all that critical to the performance of the estimator, although this may be different for different fundamental frequencies.

5. CONCLUSION

In this paper, a new method for fundamental frequency estimation has been presented. The method, which is a subspace method, avoids the commonly used asymptotic approximations of other methods, including also the classical MUSIC algorithm. Instead, the method is based on an exact measure of the angles between subspaces. In simulations, the method was demonstrated to outperform its approximate counterpart for low fundamental frequencies, a situation where the aforementioned asymptotic approximations become inaccurate.

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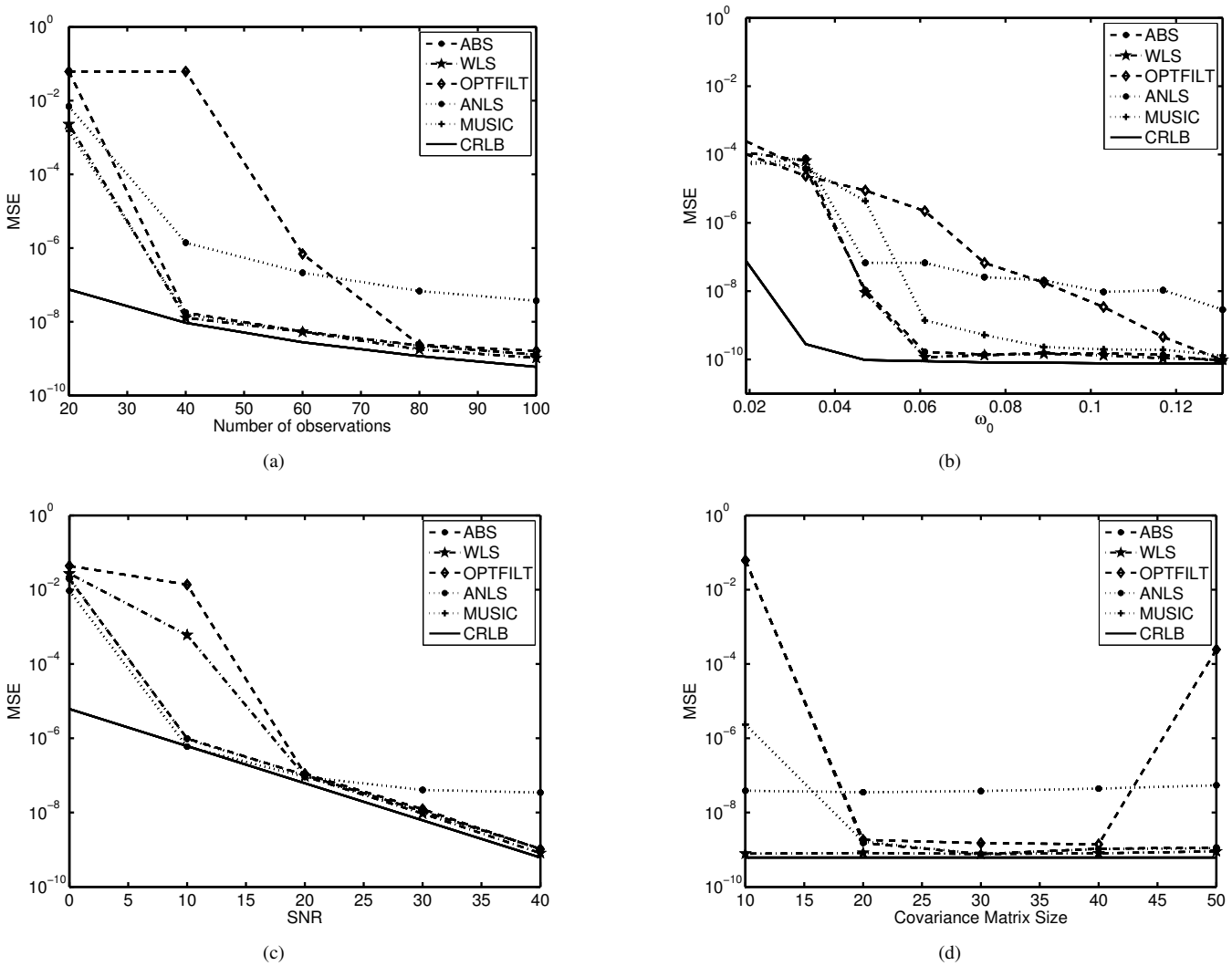

Fig. 1. Performance measured in terms of the mean square estimation error (MSE) as a function of (a) the number of samples, N, and (b) the fundamental frequency, ω_0, (c) the SNR, and (d) the covariance matrix size M.

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