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Directed Strongly Regular Graphs with rank 5

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Abstract

From the parameters (n, k, t, λ, μ) of a directed strongly regular graph (dsrg) A. Duval (1988) showed how to compute the eigenvalues and multiplicities of the adjacency matrix, and thus the rank of the adjacency matrix. For every rational number q, where $\frac{1}{5} \leq q \leq \frac{7}{10}$, there is feasible (i.e., satisfying Duval's conditions) parameter set for a dsrg with rank 5 and with $\frac{k}{n} = q$.

In this paper we show that there exist a dsrg with such a feasible parameter set only if $\frac{k}{n}$ is $\frac{1}{5}$, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{1}{2}$, $\frac{3}{5}$, $\frac{2}{3}$. Every dsrg with rank 5 therefore has parameters of a known graph. The proof is based on an enumeration of 5×5 matrices with entries in $\{0, 1\}$.

1 Introduction

A directed strongly regular graph with parameters (n, k, t, λ, μ) is a k-regular directed graph on n vertices such that every vertex is on t 2-cycles (which may be thought of as undirected edges), and the number of paths of length 2 from a vertex x to a vertex y is λ if there is an edge directed from x to y and it is μ otherwise. Thus the adjacency matrix A satisfies

$$A^2 = tI + \lambda A + \mu(J - I - A) \quad \text{and} \quad AJ = JA = kJ.$$

It is usually assumed that $0 < t < k$. These graphs were introduced by A. Duval [4], who also shoved that the spectrum of A may be computed from the parameters.

In some cases 0 is an eigenvalue of large multiplicity and then the rank of A is small. In [8], we proved that there exists a dsrg with parameters (n, k, t, λ, μ) and with adjacency matrix of rank 3 if and only if the parameters are either $(6m, 2m, m, 0, m)$ or $(8m, 4m, 3m, m, 3m)$, for some integer m, and there exists one with rank 4 if and only if (n, k, t, λ, μ) is either $(6m, 3m, 2m, m, 2m)$ and $(12m, 3m, m, 0, m)$, for an integer m. For rank 3, this was proved independently in [6].

The main theorem in this paper is a characterization of parameters with rank 5.

Theorem 1 *There exists a directed strongly regular graph with parameters (n, k, t, λ, μ) and with adjacency matrix of rank 5 if and only if the parameter set is one of the following: $(20m, 4m, m, 0, m)$, $(36m, 12m, 5m, 2m, 5m)$, $(10m, 4m, 2m, m, 2m)$, $(16m, 8m, 5m, 3m, 5m)$, $(20m, 12m, 9m, 6m, 9m)$, or $(18m, 12m, 10m, 7m, 10m)$, for some positive integer m.*

Note that in these results we assume that $t < k$. A dsrg with $t = k$, eigenvalue 0 and with rank r is an undirected complete r-partite graph, that exists for every $r \geq 2$. In the following table we list the possible values of $\frac{k}{n}$ for which there exists a dsrg with rank $r \leq 5$, including $\frac{r-1}{r}$ that we get when $t = k$. We see that then the list is symmetric around $\frac{1}{2}$ for each $r \leq 5$. It would be interesting to know if this is also true for $r \geq 6$.

Rank	Values of k/n
2	$\frac{1}{2}$
3	$\frac{1}{3}, \frac{1}{2}, \frac{2}{3}$
4	$\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$
5	$\frac{1}{5}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{4}{5}$

For the proof of the *if* part of Theorem 1 we refer to known constructions. Duval [4] proved that if for one of the six families of parameters sets there exists a dsrg for $m = 1$ then there exists a dsrg for every parameter set in that

family. This construction replaces each vertex by a set of m independent vertices, and it works when $t = \mu$. Graphs with parameters $(20, 4, 1, 0, 1)$ and $(10, 4, 2, 1, 2)$ were also constructed by Duval. A dsrg with parameters $(20, 7, 4, 3, 2)$ was constucted in [10]. Its complement has parameters $(20, 12, 9, 6, 9)$. A dsrg with parameters $(18, 5, 3, 2, 1)$ and the complementary dsrg with parameters $(18, 12, 10, 7, 10)$ was constructed in [5]. A dsrg with parameters $(36, 12, 5, 2, 5)$ was constructed in [3]. In [7] we constructed dsrgs with parameters $(4r-4, 2r-3, r-1, r-2, r-2)$, for every $r \geq 3$. The complement has parameters $(4r-4, 2r-2, r, r-2, r)$ and it has rank r. This proves that a $(16, 8, 5, 3, 5)$ exists and that for every $r \geq 3$ there is a dsrg with rank r and $\frac{k}{n} = \frac{1}{2}$. In several of these cases, there exist (many) non-isomorphic graphs. In fact some constructions of directed strongly regular graphs with low rank involve some degree of randomness, see e.g. [1].

2 Preliminaries

Duval [4] proved the following conditions for the existence of a dsrg.

Theorem 2 *Suppose that there exists a directed strongly regular graph with parameters (n, k, μ, λ, t).*

Then the parameters satisfy

$$k(k + (\mu - \lambda)) = t + (n-1)\mu \tag{1}$$

and

$$0 \leq \lambda < t, \quad 0 < \mu \leq t, \quad -2(k-t-1) \leq \mu - \lambda \leq 2(k-t). \tag{2}$$

The eigenvalues of the adjacency matrix are

$$k > \rho = \frac{1}{2}(-(\mu - \lambda) + d) > \sigma = \frac{1}{2}(-(\mu - \lambda) - d),$$

for some positive integer d, where $d^2 = (\mu - \lambda)^2 + 4(t - \mu)$. The multiplicities are

$$1, \ -\frac{k + \sigma(n-1)}{\rho - \sigma}, \ \frac{k + \rho(n-1)}{\rho - \sigma}, \tag{3}$$

respectively.

We say that (n, k, μ, λ, t) is a feasible parameter set if the conditions (1) and (2) are satisfied and the multiplicities in (3) are positive integers, and we say that (n, k, μ, λ, t) is a realizable parameter set if there exists a directed strongly regular graph with these parameters.

From Theorem 2 it follows that we have an eigenvalue $\rho = 0$ if and only if $d = \mu - \lambda$, i.e., $t = \mu$. If 0 is an eigenvalue of the adjacency matrix then the rank of the adjacency matrix is the sum of multiplicities of non-zero eigenvalues, i.e.,

$$\text{rank } = 1 + \frac{k + \rho(n-1)}{\rho - \sigma} = 1 + \frac{k}{\mu - \lambda} = 1 + \frac{k}{d}.$$

Thus we define the rank of a feasible parameter set with $t = \mu$ to be $1 + \frac{k}{\mu - \lambda}$ (even if no directed strongly regular graph exists with these parameters).

If the rank of a dsrg and the value of $\frac{k}{n}$ is known then we can find the parameters.

Proposition 3 *If (n, k, t, λ, μ) are the parameters of a dsrg with rank 5 and with $\frac{k}{n} = \frac{a}{b}$ where a and b are relatively prime integers then*

$$(n, k, t, \lambda, \mu) = (\frac{(r-1)b^2}{c}m, \frac{(r-1)ab}{c}m, \frac{ra^2}{c}m, \frac{(ar-b)a}{c}m, \frac{ra^2}{c}m),$$

for some positive integer m, where c is the greatest common divisor of $((r-1)b^2, (r-1)ab, ra^2, (ar-b)a, ra^2)$.

Proof Using that $1 + \frac{k}{d} = r$, $d = \mu - \lambda$, $\mu = t$, $\frac{k}{n} = \frac{a}{b}$ and equation 1, we get $(n, k, t, \lambda, \mu) = (\frac{b}{a}(r-1)d, (r-1)d, \frac{a}{b}rd, (\frac{a}{b}r - 1)d, \frac{a}{b}rd)$, where d is an integer satisfying that all parameters are integers. Replacing d by $ab\frac{m}{c}$ we get the required result. $\square$

In [9], we proved from Theorem 2 that for a positive integer r there exists a feasible parameter set (n, k, μ, λ, t) with rank r and with $\frac{k}{n} = q$ if and only if q is a rational number in the interval $[\frac{1}{r}, \frac{2r-3}{2r}]$.

However, we also proved in [9] that there are only finitely many values of $\frac{k}{n}$ for which there exists a k-regular directed graph on n vertices.

We say that a $\{0, 1\}$ matrix is k-regular if it has exactly k ones in each row and in each column.

Theorem 4 ([9]) *For any positive integer r the set of values of $\frac{k}{n}$, for which there exists a k-regular $n \times n$ matrix of rank r is finite.*

In particular, if there is a k-regular $n \times n$ matrix of rank 5 then

$$\frac{k}{n} \in \{\frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}\}.$$

The result for rank 5 was based on computer enumeration of all 5×5 $\{0,1\}$-matrices with rank 5.

The idea of the proof of Theorem 1 is that a large k-regular matrix with low rank is forced to have many identical rows (Section 3), but the adjacency matrix of a dsrg can not have too many identical rows (Section 4).

3 Regular rank 5 matrices

Lemma 5 *A k-regular $n \times n$ matrix with rank 5 and $\frac{k}{n} = \frac{4}{7}$ or $\frac{k}{n} = \frac{3}{7}$ has a set of $\frac{2}{7}n$ identical rows or a set of $\frac{2}{7}n$ identical columns.*

Proof Let A be a k-regular $n \times n$ matrix with rank 5. If $\frac{k}{n} = \frac{4}{7}$ then $J - A$ is a $(n-k)$-regular matrix with rank 5 and with $\frac{n-k}{n} = \frac{3}{7}$. So let us assume that $\frac{k}{n} = \frac{3}{7}$.

Let M be $5 \times n$ submatrix of A of rank 5 and let $C = [\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3, \mathbf{c}_4, \mathbf{c}_5]$ be a 5×5 submatrix of M of rank 5 and with columns $\mathbf{c}_1, \ldots, \mathbf{c}_5$. There exists unique real numbers $\alpha_1, \ldots, \alpha_5$ so that $\sum_{i=1}^{5} \alpha_i \mathbf{c}_i = \mathbf{j}_5$, the all 1 vector, as $k\mathbf{j}_5$ is the sum of all columns of M. If $\mathbf{a}_1, \ldots, \mathbf{a}_5$ are the corresponding columns of A then $\sum_{i=1}^{5} \alpha_i \mathbf{a}_i = \mathbf{j}_n$. Taking dot product of this equation with $\mathbf{j}_n$ shows that $k \sum_{i=1}^{5} \alpha_i = n$, i.e, $\alpha_1 + \ldots + \alpha_5 = \frac{7}{3}$.

We say that two matrices are equivalent if one of them can be obtained from the other by permuting rows and permuting columns. Using computer we find that there are four equivalence classes of 5×5 $\{0,1\}$-matrices of rank 5 satisfying $\sum_{i=1}^{5} \alpha_i \mathbf{c}_i = \mathbf{j}_n$ with $\alpha_1 + \ldots + \alpha_5 = \frac{7}{3}$. These equivalence classes are represented by

$$C_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, C_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and C_2^T.

Each matrix C (other than C_2^T) representing an equivalence class is chosen so that the binary number $C_{11} \ldots C_{15} C_{21} \ldots C_{25} \ldots C_{51} \ldots C_{55}$ is largest. (We assume that the rows of M are permuted similarly.)

Any additional column of M is a linear combination

$$\sum_{i=1}^{5} \beta_i \mathbf{c}_i, \quad \text{where} \quad \beta_1 + \ldots + \beta_5 = 1. \tag{4}$$

The last equation is to ensure that the sum of entries in the new column of A is k. For the matrix $C = C_3$ this condition is not satisfied by any $\{0,1\}$ vector other that $\mathbf{c}_1, \ldots, \mathbf{c}_5$. Thus every column of M is identical to one of $\mathbf{c}_1, \ldots, \mathbf{c}_5$. As $\frac{1}{3}(\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 + \mathbf{c}_4 + 3\mathbf{c}_5) = \mathbf{j}_5$, column $\mathbf{c}_5$ appears $\frac{3}{7}n$ times.

Also if $C = C_2$ then every column must be identical to a column of C. As $\frac{1}{3}(\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 + 2\mathbf{c}_4 + 2\mathbf{c}_5) = \mathbf{j}_5$, columns $\mathbf{c}_4$ and $\mathbf{c}_5$ each appear $\frac{2}{7}n$ times. If $C = C_2^T$ then C_2 is submatrix A^T, therefore A has $\frac{2}{7}n$ identical rows.

Thus we may assume that every 5×5 submatrix of A of rank 5 is equivalent to C_1. For $C = C_1$ the possible $\{0,1\}$ vectors satisfying equation 4 are $\mathbf{c}_1, \ldots, \mathbf{c}_5, \mathbf{c}_6 = -\mathbf{c}_2 + \mathbf{c}_3 + \mathbf{c}_4 = (0,1,0,0,1)^T$ and $\mathbf{c}_7 = \mathbf{c}_2 - \mathbf{c}_3 + \mathbf{c}_5 = (0,0,1,1,0)^T$. We know that the columns $\mathbf{c}_1, \ldots, \mathbf{c}_5$ appear in M. If $\mathbf{c}_6$ also appears in M then $[\mathbf{c}_1\ \mathbf{c}_3\ \mathbf{c}_4\ \mathbf{c}_5\ \mathbf{c}_6]$ is equivalent to C_2^T. If $\mathbf{c}_7$ appears in M then $[\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3\ \mathbf{c}_4\ \mathbf{c}_7]$ is equivalent to C_2^T. Thus every column of M is identical to one of $\mathbf{c}_1, \ldots, \mathbf{c}_5$. As $\frac{1}{3}(\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 + 2\mathbf{c}_4 + 2\mathbf{c}_5) = \mathbf{j}_5$, columns $\mathbf{c}_4$ and $\mathbf{c}_5$ each appear $\frac{2}{7}n$ times in M. $\square$

In fact a stronger result follows from the above proof.

Corollary 6 *Let A be $\frac{3}{7}n$-regular $n \times n$ matrix of rank 5. Then either A or A^T satisfies one of the following properties.*

- *There is a set of $\frac{3}{7}n$ identical rows, or*
- *There are two disjoint sets of $\frac{2}{7}n$ identical rows.*

Lemma 7 *A k-regular $n \times n$ matrix with rank 5 and $\frac{k}{n} = \frac{3}{8}$ or $\frac{k}{n} = \frac{5}{8}$ has a set of $\frac{3}{8}n$ identical rows and a set of $\frac{3}{8}n$ identical columns.*

Proof There is only one equivalence class of 5×5 $\{0,1\}$-matrices $C = [\mathbf{c}_1 \ldots \mathbf{c}_5]$ of rank 5 satisfying $\sum_{i=1}^{5} \alpha_i \mathbf{c}_i = \mathbf{j}_5$ with $\alpha_1 + \ldots + \alpha_5 = \frac{8}{3}$. This class is represented by the matrix $C = C_{\frac{3}{8}}$ shown below. The columns $\mathbf{c}_1 \ldots \mathbf{c}_5$

are the only $\{0,1\}$-vectors of the form $\sum_{i=1}^{5} \beta_i \mathbf{c}_i$ with $\beta_1 + \ldots + \beta_5 = 1$. As $\frac{1}{3}(\mathbf{c}_1 + \mathbf{c}_2 + \mathbf{c}_3 + 2\mathbf{c}_4 + 3\mathbf{c}_5)$, any $\frac{3}{8}n$-regular $n \times n$ matrix with rank 5 has a set of $\frac{3}{8}n$ identical columns. As C is equivalent to its transpose, the same holds for rows. $\square$

$$C_{\frac{3}{8}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, C_{\frac{2}{7}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, C_{\frac{4}{9}} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

The cases $\frac{2}{7}$ and $\frac{4}{9}$ have similar proofs.

Lemma 8 *A k-regular $n \times n$ matrix with rank 5 and $\frac{k}{n} = \frac{2}{7}$ or $\frac{k}{n} = \frac{5}{7}$ has a set of $\frac{2}{7}n$ identical rows and a set of $\frac{2}{7}n$ identical columns.*

Lemma 9 *A k-regular $n \times n$ matrix with rank 5 and $\frac{k}{n} = \frac{4}{9}$ or $\frac{k}{n} = \frac{5}{9}$ has a set of $\frac{1}{3}n$ identical rows and a set of $\frac{1}{3}n$ identical columns.*

Lemma 10 *A k-regular $n \times n$ matrix with rank 5 and $\frac{k}{n} = \frac{1}{4}$ or $\frac{k}{n} = \frac{3}{4}$ has a set of $\frac{1}{4}n$ identical rows or a set of $\frac{1}{4}n$ identical columns.*

Proof Let $C = [\mathbf{c}_1 \ldots \mathbf{c}_5]$ be a full rank 5×5 submatrix of a $\frac{n}{4}$-regular $n \times n$ matrix A. Then C is equivalent to one of the following 9 matrices.

$$C_1 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, C_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$C_4 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, C_5 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, C_6 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

C_3^T, C_4^T and C_6^T. If $C = C_4$ then no further columns satisfy equation 4, but then $\mathbf{c}_2 + \mathbf{c}_3 + \mathbf{c}_4 + \mathbf{c}_5 = \mathbf{j}_5$, and column $\mathbf{c}_1$ does not appear, a contradiction. Thus C_4 and C_4^T can not be submatrices of A. Similarly, C_6 and C_6^T can not be submatrices of A, as the coefficient of $\mathbf{c}_1$ is negative. Thus every full rank 5×5 submatrix has at most three ones in each row/column.

Suppose $C = C_5$. Then the only solutions to equation 4 other the columns of C_5 are $(1,0,1,1,0)^T$ and $(0,0,0,1,0)^T$. Thus in order to have k ones in row 5, $\mathbf{c}_3$ must appear k times and $\mathbf{c}_1$ and $\mathbf{c}_2$ will not appear, a contradiction.

Suppose $C = C_3$. The additional solutions to equation 4 are $\mathbf{c}_6 = \mathbf{c}_1 - \mathbf{c}_4 + \mathbf{c}_5 = (1,0,0,0,1)^T$, $\mathbf{c}_7 = (0,0,1,0,0)^T$ and $\mathbf{c}_8 = (0,0,0,1,0)^T$. Since $[\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_3\ \mathbf{c}_4\ \mathbf{c}_6]$ is a rank 5 matrix with four ones in row 1, $\mathbf{c}_6$ does not appear. Then column $\mathbf{c}_5$ must appear $k = \frac{n}{4}$ times.

We may now assume that every full rank 5×5 submatrix is equivalent to C_1 or C_2. If $C = C_2$ then the additional solutions to equation 4 are $\mathbf{c}_6 = (1,0,0,1,0)^T$, $\mathbf{c}_7 = (1,0,0,0,1)^T$ and $\mathbf{c}_8 = (0,0,1,0,0)^T$. Since $[\mathbf{c}_1\ \mathbf{c}_2\ \mathbf{c}_6\ \mathbf{c}_3\ \mathbf{c}_5] = C_3$, column $\mathbf{c}_6$ does not appear and column $\mathbf{c}_4$ then appears k times.

Suppose now that $C = C_1$. Additional solutions to equation 4 are $\mathbf{c}_6 = \mathbf{c}_1 - \mathbf{c}_2 + \mathbf{c}_3 = (0,1,1,0,0)^T$, $\mathbf{c}_7 = (0,1,0,1,0)^T$ and $\mathbf{c}_8 = (0,1,0,0,1)^T$. As $[\mathbf{c}_1\ \mathbf{c}_6\ \mathbf{c}_3\ \mathbf{c}_4\ \mathbf{c}_5]$ is equivalent to C_2, we may assume that $\mathbf{c}_6$ does not appear. Then there must be k copies of column $\mathbf{c}_3$. $\square$

4 Proof of Theorem 1

In this section we will show that there are no directed strongly regular graphs with rank 5, $t < k$ and with $\frac{k}{n} \in \{\frac{1}{4}, \frac{2}{7}, \frac{3}{8}, \frac{3}{7}, \frac{4}{9}, \frac{5}{9}, \frac{4}{7}, \frac{5}{8}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}\}$. Then the proof of Theorem 1 follows from Theorem 4 and Proposition 3.

Note first that for a dsrg with rank $r = 5$, the above mentioned result that $\frac{k}{n} \leq \frac{2r-3}{2r} = \frac{7}{10}$, excludes $\frac{k}{n} = \frac{5}{7}, \frac{3}{4}$, and $\frac{4}{5}$.

Lemma 11 *If a dsrg with parameters (n, k, t, λ, μ) has a set S of vertices, all of which have the same set N of out-neighbours (or they all have the same set in-neighbours), then*

$$|S| \leq k - \lambda, \quad \textit{and} \quad |S| \leq n - 2k + t.$$

Proof We have that $|N| = k$ and $S \cap N = \emptyset$. Let $v \in S$ and $w \in N$. Then there are λ paths of length 2 from v to w, i.e., w has λ in-neighbours in N. Also every vertex in S is an in-neighbour of w. Thus $|S| + \lambda \leq k$.

The number of in-neighbours of v in $S \cup N$ is t. Thus v has $k-t$ in-neighbours outside $S \cup N$. Thus $|S| + k + (k-t) \leq n$. $\square$

Proposition 12 *A dsrg with rank 5 and $\frac{k}{n} = \frac{4}{7}$ does not exist.*

Proof Suppose that (n, k, t, λ, μ) are the parameters of a dsrg with rank 5 and with $\frac{k}{n} = \frac{4}{7}$. By Proposition 3, we have $(n, k, t, \lambda, \mu) = (49m, 28m, 20m, 13m, 20m)$. By Lemma 5, this graph has a set of $\frac{2}{7} \cdot 49m = 14m$ vertices that either all have the same out-neighbours or all have the same in-neighbours. But this contradicts Lemma 11, as $n - 2k + t = 13m$. $\square$

Proposition 13 *A dsrg with rank 5 and $\frac{k}{n} = \frac{1}{4}, \frac{2}{7}, \frac{3}{8}, \frac{4}{9}, \frac{5}{9}$, or $\frac{5}{8}$, does not exist.*

Proof In each case we get the parameters from Proposition 3 and apply the first part of Lemma 11.

$\frac{k}{n} = \frac{1}{4}$: A dsrg with parameters $(64m, 16m, 5m, m, 5m)$ has $\frac{1}{4}n = 16m > k - \lambda = 15m$, a contracdiction to Lemma 10.

$\frac{k}{n} = \frac{2}{7}$: A dsrg with parameters $(98m, 28m, 10m, 3m, 10m)$ has $\frac{2}{7}n = 28m > k - \lambda = 25m$, a contracdiction to Lemma 8.

$\frac{k}{n} = \frac{3}{8}$: A dsrg with parameters $(256m, 96m, 45m, 21m, 45m)$ has $\frac{3}{8}n = 96m > k - \lambda = 75m$, a contracdiction to Lemma 7.

$\frac{k}{n} = \frac{4}{9}$: A dsrg with parameters $(81m, 36m, 20m, 19m, 20m)$ has $\frac{1}{3}n = 27m > k - \lambda = 17m$, a contracdiction to Lemma 9.

$\frac{k}{n} = \frac{5}{9}$: A dsrg with parameters $(324m, 180m, 125m, 90m, 125m)$ has $\frac{1}{3}n = 108m > k - \lambda = 90m$, a contracdiction to Lemma 9.

$\frac{k}{n} = \frac{5}{8}$: A dsrg with parameters $(256m, 160m, 125m, 85m, 125m)$ has $\frac{3}{8}n = 96m > k - \lambda = 75m$, a contracdiction to Lemma 7.

$\square$

Proposition 14 *A dsrg with rank 5 and $\frac{k}{n} = \frac{3}{7}$ does not exist.*

Proof Suppose that (n, k, t, λ, μ) are the parameters of a dsrg with rank 5 and with $\frac{k}{n} = \frac{3}{7}$. By Proposition 3, we have $(n, k, t, \lambda, \mu) = (196m, 84m, 45m, 24m, 45m)$. Let A be the adjacency matrix of this dsrg. Then A^T is the adjacency matrix of a dsrg with the same parameters. By Lemma 11, a set S of vertices with identical out- (or in-) neighbour sets has $|S| \leq k - \lambda = 60m < \frac{3}{7}n$. Therefore by Corollary 6, using A^T if necessary, we have two disjoint sets S_1 and S_2 of vertices with $|S_1| = |S_2| = \frac{2}{7}n = 56m$ and sets N_1 and N_2 with $|N_1| = |N_2| = 84m$ so that every vertex in S_i has out-neighbour set N_i. Then $N_1 \cap N_2 = \emptyset$, as a vertex in $N_1 \cap N_2$ would have in-degree at least $|S_1 \cup S_2| = 112m$. We also have $S_i \cap N_i = \emptyset$, but since there are only $28m$ vertices outside $N_1 \cup N_2$, the sets $S_1 \cap N_2$ and $S_2 \cap N_1$ are non-empty. Let $v \in S_1 \cap N_2$. A subset of N_1 of exactly $t = 45m$ vertices are in-neighbours of v. $S_2 \cap N_1$ is contained in this set. Thus $|S_2 \cap N_1| \leq 45m$, and so there exists a vertex $x \in S_2 \setminus N_1$. Similarly there exists $y \in S_1 \setminus N_2$. Since x and y are non-adjacent there are $45m$ paths from x to y of length 2. The internal vertex of such a path belongs to $N_2 \setminus S_1$, as S_1 is independent. Thus $|N_2 \setminus S_1| \geq 45m$ and so $|S_1 \setminus N_2| \geq 17m$. Similarly, $|S_2 \setminus N_1| \geq 17m$. But these sets are disjoint subsets of a set of $28m$ vertices, a contradiction. $\square$

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