VIBRATION THEORY, VOL. 1
Linear Vibration Theory

Søren R. K. Nielsen

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PREFACE

The present textbook has been written for a basic course on linear vibration theory that is being given on the 8th semester at Aalborg University for M.Sc. students in structural engineering. Basically, the present 3rd edition of the book is a translation into English of the preceding 1st and 2nd editions in Danish published in 1991 and 1993. Chapters 1, 2 and 3 have in broad terms been carried over unchanged, except for a revised outline of section 3.9 on damping models. Further, most example problems have been modified or further worked up. In contrast, chapter 4 has been completely revised. Further, a new chapter 5 on dynamic modelling of continuous systems has been added to establish the connection to computational structural dynamics as carried out in finite element codes. On the other hand, a method for analytical determination of eigenfrequencies of plane frames has been omitted. Such methods were considered obsolete in comparison to modern numerical methods already at the time of appearance of the 1st edition, but were included for historical reasons at that time.

Part of the manuscript has been typed by Mrs. Solveig Hesselvang. Mrs. Norma Hornung prepared the drawings and Mrs. Kirsten Aakjær has proofread and corrected my modest English. The help from all of them is gratefully acknowledged.

Aalborg University, July 1998
Søren R. K. Nielsen

The present 3rd edition of my textbook on linear structural dynamics is unchanged in comparison to the 2nd edition. Only discovered typing errors have been corrected.

Aalborg University, June 2004
Søren R. K. Nielsen
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1. INTRODUCTION

The purpose of vibration analysis is to determine the motion, i.e. the time-dependence of the displacements or some other response quantities, of a mass system due to time-dependent external dynamic loads, time-dependent boundary conditions or due to non-zero initial conditions.

The motion is assumed to be completely described by a set of time-dependent coordinates \( \mathbf{x}^T(t) = [x_1(t), \ldots, x_n(t)] \) in an \( n \)-dimensional Euclidian space of motion. \( T \) as upper index denotes transpose. The vector function \( \mathbf{x}(t) \) specifies the state of the structure at any instant of time. Based on this vector it is in principle possible to determine approximately the displacements, stresses, strains etc. anywhere in the structure. The coordinates \([x_1(t), \ldots, x_n(t)]\) will be referred to as the degrees of freedom of the structure. In general it can be stated that, as more and more degrees-of-freedom are associated with the structure, the more accurate will be the description of its motion. However, as will be shown later, not all degrees of freedom carry the same information about the motion. Since the dimension \( n \) of the problem in numerical analyses needs to be finite, the selection of degrees of freedom that are most important for the accuracy of the prediction of the system response then becomes an important problem, which is considered in some detail in section 3.8 on system reduction and in chapter 5 on dynamic modelling of continuous systems.

Dependent on the number \( n \) of degrees-of-freedom, the structure is categorized as a single degree-of-freedom (SDOF) system, if \( n = 1 \), as a multi degree-of-freedom (MDOF) system, if \( 1 < n < \infty \), and as a continuous system, if \( n = \infty \). Collectively, SDOF and MDOF systems are denoted discrete systems.

Further, a differentiation is made between linear systems and non-linear systems, for which the differential equations of motion are linear and non-linear, respectively. In what follows, exclusively linear systems are considered. For linear systems the superposition principle is applicable.

A motion is denoted harmonic, if it can be written

\[
\mathbf{x}(t) = \mathbf{A} \cos(\omega t - \Psi)
\]  

(1-1)

where \( \mathbf{A} \in \mathbb{R}^n \) is a time-dependent amplitude vector. (1-1) implies, that all components \( x_1(t), x_2(t), \ldots \) are moving with the same circular frequency \( \omega \) and the same phase \( \Psi \). The proportion between any two components is time-independent. As an example one has

\[
\frac{x_1(t)}{x_2(t)} = \frac{A_1}{A_2} = \text{constant}
\]

(1-2)

Especially, all components are zero at the same instant of time. Occasionally, a harmonic motion is alternatively defined by the following complex notation

\[
\mathbf{x}(t) = \text{Re}(\mathbf{B}e^{i\omega t}) = \text{Re}(\mathbf{B})\cos(\omega t) - \text{Im}(\mathbf{B})\sin(\omega t), \quad \mathbf{B} \in \mathbb{C}^n
\]  

(1-3)
Re (·) and Im (·) denote the real and imaginary part of a complex number. C is the set of complex numbers. (1-3) represents a more general definition of a harmonic motion, for which (1-2) is not necessarily fulfilled, and for which the components are not zero at the same time. This implies, that the individual components \( x_i(t) \) are related with different phases. On component form (1-3) can be written

\[
x_i(t) = A_i \cos(\omega t - \Psi_i) \quad , \quad i = 1, \ldots, n
\]

(1-4)

where

\[
\begin{align*}
A_i \cos \Psi_i &= \text{Re}(B_i) \\
A_i \sin \Psi_i &= -\text{Im}(B_i)
\end{align*}
\]

\[
\Rightarrow \quad A_i = \left( (\text{Re}(B_i))^2 + (\text{Im}(B_i))^2 \right)^{\frac{1}{2}} = |B_i| \quad , \quad i = 1, \ldots, n
\]

(1-5)

\[
\tan \Psi_i = -\frac{\text{Im}(B_i)}{\text{Re}(B_i)} \quad , \quad i = 1, \ldots, n
\]

(1-6)

All components of the motion (1-3) are performing harmonic motions with the same circular frequency \( \omega \), but with different phases \( \Psi_i \). Consequently, the complex notation (1-3) implicitly defines both an amplitude \( A_i \) and a phase \( \Psi_i \) of all components.

A motion is denoted periodic, if a positive real number \( T \) exists, so that

\[
x(t + T) = x(t)
\]

(1-7)

The smallest number \( T \) for which (1-7) is fulfilled is called the period of the motion. The period signifies the shortest time-interval until the motion starts repeating itself. Upon replacing \( t \) with \( t + T \) in (1-1) or (1-3) it is seen, that a harmonic motion is periodic with the period \( T = \frac{2\pi}{\omega} \). Periodic motions can be expanded into a uniformly convergent Fourier series of harmonic component motions, see appendix A.

The number of repetitions per unit of time is denoted the frequency of vibration, defined as

\[
f = \frac{1}{T} \quad , \quad [\text{s}^{-1} = \text{Hz}]
\]

(1-8)

The circular frequency of vibration can be written

\[
\omega = \frac{2\pi}{T} \quad , \quad [\text{rad/s}]
\]

(1-9)
2. VIBRATIONS OF SDOF SYSTEMS

2.1 Eigenvibrations of Undamped Systems

![Diagram showing the eigenvibrations of a linear undamped SDOF system]

Fig. 2-1: Eigenvibrations of linear undamped SDOF system. a) Undeformed state. b) Static equilibrium state. c) Initial state. d) Dynamic deformed state. e) Free mass loaded with external and internal forces.

Fig. 2-1 shows an oscillator made up of a point mass $m$ suspended at the free end of a massless linear elastic spring with the spring constant $k$. The system is only allowed to move in the vertical direction. The position of the mass is then described by a single coordinate, and the system has a single degree of freedom. Elongation of any real spring is related with dissipation of mechanical energy into heat, which is referred to as damping. In the present case it is assumed that the spring is ideally elastic, corresponding to damping forces being ignored.

Fig. 2-1a shows the spring in its undeformed state before the mass is attached to the spring. The gravity force $mg$ on the mass causes an elongation of the spring of magnitude $x_s$, which defines the static equilibrium state, see fig. 2-1b. The degree of freedom, $x(t)$, is considered positive for motions downward from the equilibrium state. Hence, the mass is in the static equilibrium state, whenever $x = 0$.

The velocity $\dot{x}(t)$ and the acceleration $\ddot{x}(t)$ are considered positive in the same directions as $x(t)$. At the time $t = 0$ a set of initial conditions is applied to the system. Both the initial displacement and the velocity need to be specified for a SDOF oscillator, see fig. 2-1c

$$[x(0), \dot{x}(0)] = [x_0, \dot{x}_0] \quad (2 - 1)$$

Hereafter, the system is left to itself. Vibrations take place with the static gravity force as the only external load, fig. 2-1d. Since no external dynamic loads are present, these vibrations are called eigenvibrations. Now, the problem is to determine the motion $x(t)$, so the initial conditions (2-1) are fulfilled.

The general method for formulating the equations of motion of a dynamic system can be summarized in the following three steps
1. All masses are cut free.

2. All external and internal forces are applied as external forces on the free masses with positive sign when acting in the direction of the corresponding degrees of freedom.

3. Newton's 2nd law of motion is applied to the each of the free masses.

In the present case only one mass with a single degree of freedom is present. The external force is the gravity force \( mg \), and the internal force is the spring force \( k(x + x_s) \). These are applied to the free mass with a definition of sign as shown in fig. 2-1e. Newton's 2nd law of motion then gives

\[
m\ddot{x} = mg - k(x + x_s)
\]  

(2-2)

The right-hand side of (2-2) indicates the sum of all forces on the free mass considered positive in the direction of the degree of freedom \( x \).

The deformation \( x_s \) of the spring in the static equilibrium state is given by

\[
mg = kx_s
\]  

(2-3)

From (2-2) and (2-3) it follows that

\[
m\ddot{x} + kx = 0
\]  

(2-4)

From (2-4) it is seen that the gravity force disappears from the equation of motion. Generally, the gravity forces (and any other static forces) causing the static equilibrium state will disappear from the dynamic equations of motion, if the following conditions are fulfilled:

1. The motions are measured from the static equilibrium state.

2. The system is linear.

On the other hand, whenever these conditions are fulfilled gravity forces and other static forces can formally be ignored in the formulation of the equations of motion. In what follows it is always assumed that the static equilibrium state is known from an initial static analysis, and the motions are measured from this state. Then only the additional dynamic external loads will enter the dynamic equations of motion.

**Example 2-1: Motion Measured from the Undeformed State**

If the displacement \( x(t) \) is measured from the undeformed state in fig. 2-1a the equation of motion becomes

\[
m\ddot{x} + kx = mg
\]  

(2-5)

In this case the static load does not disappear from the equation.
(2-4) can be written
\[ \ddot{x} + \omega_0^2 x = 0 \] (2-6)

where
\[ \omega_0 = \sqrt{\frac{k}{m}} \] (2-7)

The solution of (2-6), which also fulfils the initial conditions (2-1), reads
\[ x(t) = x_0 \cos(\omega_0 t) + \frac{x_0}{\omega_0} \sin(\omega_0 t), \quad t \geq 0 \] (2-8)

Alternatively, (2-8) can be written in the following form
\[ x(t) = A \cos(\omega_0 t - \Psi) \] (2-9)

where
\[
\begin{align*}
A \cos \Psi &= x_0 \\
A \sin \Psi &= \frac{x_0}{\omega_0}
\end{align*} \Rightarrow
\]
\[ A = \left( x_0^2 + \left( \frac{x_0}{\omega_0} \right)^2 \right)^{\frac{1}{2}} \] (2-10)
\[ \tan \Psi = \frac{\dot{x}_0}{x_0 \omega_0} \] (2-11)

Comparison of (2-9) and (1-1) shows that (2-8) and (2-9) describe a harmonic motion with the circular eigenfrequency \( \omega_0 \) determined by (2-7) and the amplitude \( A \) given by (2-10).

The eigenvibration period becomes
\[ T_0 = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{m}{k}} \] (2-12)

The eigenfrequency becomes
\[ f_0 = \frac{1}{T_0} = \frac{1}{2\pi} \omega_0 = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \] (2-13)
For most civil engineering structures the fundamental (lowest) eigenfrequency $f_0$ is placed in the interval $[0.1 \text{Hz}, 2.0 \text{Hz}]$, where the lower bound of the interval is valid for relatively flexible structures such as high-rise buildings and suspension bridges.

**Example 2-2: Torsional Vibrations of Flywheel**

![Diagram of Torsional Vibrations of Flywheel](image)

A circular cylindrical bar with the diameter $d$ and the length $l$ is fixed at one end. At the other end it is supporting a flywheel with the mass moment of inertia $J$. The bar is assumed to be massless and linear elastic with the shear modulus $G$. The problem is to determine the eigenfrequency of the system.

The flywheel is assumed to be inflexible. Then the system has but a single degree of freedom, which is selected as the torsional angle $\theta$. The flywheel is cut free from the bar, and the torsional moment $M$ is applied as an external load with the sign defined in fig. 2-2. For the free flywheel the equation of moment of momentum applies

$$J\ddot{\theta} = -M \left( = \sum \text{all external moments in the direction of } \theta \right) \quad (2 - 14)$$

For the circular cylindrical bar the following constitutive equation is valid (St. Venant torsion)

$$M = \frac{Gl_t}{l} \theta \quad (2 - 15)$$

where $I_t = \frac{\pi}{32} d^4$ is the torsional moment of inertia of the bar. From (2-14) and (2-15) it follows that

$$\ddot{\theta} + \frac{\omega_0^2}{J l} \theta = 0 \quad (2 - 16)$$

where

$$\omega_0^2 = \frac{Gl_t}{J l} = \frac{\pi}{32} \frac{Gd^4}{J l} \quad (2 - 17)$$
Multiplication of (2-4) with $\dot{x}$ provides

$$(m\ddot{x} + kx)\dot{x} = 0 \Rightarrow$$

$$\frac{d}{dt} \left( \frac{1}{2} mx^2 + \frac{1}{2} kx^2 \right) = 0 \Rightarrow$$

$$\ddot{T} + \ddot{U} = 0$$

(2 - 18)

where

$$T = \frac{1}{2} mx^2$$

(2 - 19)

$$U = \frac{1}{2} kx^2$$

(2 - 20)

$T$ is the kinetic energy and $U$ is the potential energy. (2-18) shows that the mechanical energy of the system is constant with time. This is a consequence of the system being free of damping. In some cases (2-18) can be used to formulate the equation of motion of a system of 1 degree of freedom as demonstrated in the following example 2-3.

Example 2-3: Rotational Vibrations of Pulley

Fig. 2-3 shows a pulley made up of a circular cylinder with the radius $r_1$, which is fixed to another cylinder with the radius $r_2$. A massless inflexible string is attached to the cylinder with the radius $r_1$. The string is supporting a point mass $m$ at the other end, which can only move in the direction of the string. Another inflexible string is attached to the cylinder with the radius $r_2$, which at the other end is connected to a linear elastic spring with the spring constant $k$. The body is infinitely stiff, and is only allowed to perform rotational vibrations around the axis of symmetry. The mass moment of inertia around this axis is $J$. The problem is to formulate the equations of motion of the system, and to determine the circular eigenfrequency.
The problem will be solved by means of (2-18). Let $\theta$ signify rotation from the static equilibrium state. Then

$$T = \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} m (r_1 \dot{\theta})^2$$

$$U = \frac{1}{2} k (r_2 \theta)^2$$

Hence

$$\frac{d}{dt} \left( \frac{1}{2} J \dot{\theta}^2 + \frac{1}{2} m (r_1 \dot{\theta})^2 + \frac{1}{2} k (r_2 \theta)^2 \right) = 0 \Rightarrow$$

$$\left( (J + m r_1^2) \ddot{\theta} + k r_2 \theta \right) \dot{\theta} = 0 \Rightarrow$$

$$(J + m r_1^2) \ddot{\theta} + k r_2 \theta = 0$$

From (2-23) it follows directly that

$$\omega_0^2 = \frac{k r_2}{J + m r_1^2} \Rightarrow \omega_0 = \sqrt{\frac{k r_2}{J + m r_1^2}}$$

2.2 Basic Equation for Forced Vibrations of Linear Viscous Damped Systems

Fig. 2-4: Forced vibrations of damped SDOF system.

Damping, i.e. transformation of mechanical energy into heat, always takes place due to molecular dissipation in the material, friction in structural joints, plastic deformations, etc. In order to make the model more realistic a damping element is included, which is connected to the mass $m$ in parallel to the linear elastic spring with the spring constant $k$, see fig. 2-4. The spring is assumed free of damping, so all energy dissipation in the system takes place in the damping element. At the same time an external dynamic force $f(t)$ is applied to the model.
The external forces on the mass $m$ are made up of the force of gravity $mg$ and the time varying force $f(t)$. $f(t)$ is considered positive in the same direction as the degree of freedom $x(t)$, which is selected as the displacement from the static state of equilibrium, see fig. 2-4. As mentioned in section 2.1, one may then formally ignore the gravity force or other static loads at the formulation of the equation of motion of the system.

The mass is cut free from the spring and the damping element, and the internal and external dynamic force are applied as external loads on the mass. The damping element introduces the damping force $f_d$ on the free mass, which is considered positive in the same direction as the spring force $kx$, see fig. 2-4.

Next, the equation of motion follows upon applying Newton’s 2nd law of motion to the free mass

$$m\ddot{x} = f(t) - kx - f_d \Rightarrow$$

$$m\ddot{x} + kx = f(t) - f_d \tag{2-25}$$

Multiplication of (2-25) with $\dot{x}$ provides

$$(m\ddot{x} + kx)\dot{x} = f(t)\dot{x} - f_d \dot{x} \Rightarrow$$

$$d\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = f(t)\dot{x}dt - f_d \dot{x}dt \Rightarrow$$

$$d(T + U) = f(t)\dot{x}dt - f_d \dot{x}dt \tag{2-26}$$

where the kinetic energy $T$ is given by (2-19), and the potential energy $U$ is given by (2-20). Temporarily, assume that $f(t) \equiv 0$. Then, (2-26) can be written

$$f_d \dot{x} = -\frac{d}{dt}(T + U) \tag{2-27}$$

The right-hand side of (2-27) indicates the loss of mechanical energy per unit time. Consequently, $f_d \dot{x}$ is equal to the energy, which is dissipated as heat per unit time. The damping force $f_d(t)$ is designated as dissipative, if for any velocity $\dot{x} \neq 0$ of the system

$$f_d \dot{x} > 0 \tag{2-28}$$

If (2-28) is fulfilled, a steady loss of mechanical energy takes place according to (2-27). (2-27) reduces to (2-18) for $f_d(t) \equiv 0$.

Solution of (2-25) requires a constitutive relation $f_d = f_d(x, \dot{x})$, specifying the dependence of the damping force on $x$ and $\dot{x}$. Examples of dissipative damping models are the following
The damping models (2-30) and (2-31) are non-linear. For these a doubling of the velocity \( \dot{x} \) does not imply a doubling of the damping force. (2-30) is a useful damping model in case of vibrations of the mass in a still fluid. (2-31) is called Coulomb's damping model. This may be used as a model for the friction force at horizontal motions of the mass on a dry surface.

In contrast, the model (2-29) is linear and is designated the linear viscous damping model. The importance of this model is that analytically tractable equations of motion are obtained. In what follows linear viscous damping is always assumed. The constant \( c \) in (2-29) is termed the damping constant.

Insertion of (2-29) into (2-26) provides the basic equation of motion for forced vibrations of a linear viscous damped SDOF oscillator

\[
m\ddot{x} + c\dot{x} + kx = f(t), \quad t > 0
\]  

which must be solved with the initial conditions

\[
x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0
\]  

Assume that \( x(t) \) is periodic with the period \( T \). Integration of (2-32) over one period provides

\[
\int_0^T f(t)\dot{x}(t)dt - \int_0^T f_d(t)\dot{x}(t)dt = \left[ \frac{1}{2}m\dot{x}^2(t) + \frac{1}{2}kx^2(t) \right]_0^T
\]  

Since \( x(0) = x(T) \) and \( \dot{x}(0) = \dot{x}(T) \), the right-hand side of (2-34) is equal to 0. Consequently,

\[
E_e = E_d \tag{2-35}
\]

where

\[
E_e = \int_0^T f(t)\dot{x}(t)dt \tag{2-36}
\]
\[ E_d = \int_0^T f_d(t) x(t) \, dt \quad (2-37) \]

\( E_d \) signifies the work performed by the external force on the system during one period. \( E_d \) is the energy dissipated by the system during the same period. (2-35) is a necessary but not sufficient condition for the motion being periodic.

![Hysteretic loops for damping force in harmonic motion](image)

Fig. 2-5: Hysteretic loops for the damping force in harmonic motion. a) Linear viscous damping. b) Fluid damping. c) Coulomb damping.

If corresponding values of \( f_d(t) \) and \( x(t) \) are plotted in case of a periodic motion, a so-called hysteretic loop is obtained. Fig. 2-5 shows the result in case of a harmonic motion with the amplitude \( A \) and the circular frequency \( \omega \) for each of the damping models (2-29), (2-30) and (2-31).

### 2.3 Eigenvibrations of Linear Viscous Damped System

The equation of motion follows from (2-32), (2-33) with \( f(t) = 0 \). After division with \( m \) the following initial-value problem is obtained

\[
\begin{align*}
\ddot{x} + 2\zeta \omega_0 \dot{x} + \omega_0^2 x &= 0 , \quad t > 0 \\
x(0) &= x_0 , \quad \dot{x}(0) = \dot{x}_0
\end{align*}
\quad (2-38)
\]

\( \omega_0 \) is the circular eigenfrequency for undamped eigenvibrations as given by (2-7). The parameter \( \zeta \) is defined as

\[
2\zeta \omega_0 = \frac{c}{m} \Rightarrow \quad \zeta = \frac{c}{2\omega_0 m} = \frac{c}{2\sqrt{km}}
\quad (2-39)
\]

The characteristics of the solution to (2-38) depends on the magnitude of \( \zeta \). Immediately, it is clear that \( \zeta \in [0, \infty[ \). For each of the intervals of \( \zeta \) with qualitatively identical motion the following solutions are obtained:
1. $\zeta = 0 : \text{Undamped System}$

The motion is given by (2-8).

2. $\zeta \in ]0,1[ : \text{Undercritical Damped System}$

\[
x(t) = e^{-\zeta \omega_0 t} \left( x_0 \cos(\sqrt{1-\zeta^2} \omega_0 t) + \frac{\dot{x}_0 + \zeta \omega_0 x_0}{\omega_0 \sqrt{1-\zeta^2}} \sin(\sqrt{1-\zeta^2} \omega_0 t) \right), \quad t \geq 0
\]

(2-40)

3. $\zeta = 1 : \text{Critical Damped System}$

\[
x(t) = e^{-\omega_0 t} \left( x_0 + (\dot{x}_0 + \omega_0 x_0) t \right), \quad t \geq 0
\]

(2-41)

4. $\zeta \in ]1,\infty[ : \text{Overcritical Damped System}$

\[
x(t) = e^{-\zeta \omega_0 t} \left( x_0 \cosh(\sqrt{\zeta^2 - 1} \omega_0 t) + \frac{\dot{x}_0 + \zeta \omega_0 x_0}{\omega_0 \sqrt{\zeta^2 - 1}} \sinh(\sqrt{\zeta^2 - 1} \omega_0 t) \right), \quad t \geq 0
\]

(2-42)

(2-40), (2-41), (2-42) are proved upon insertion into (2-38), verifying that both the differential equation and the initial values are fulfilled.

The critical damping constant $c_{cr}$ signifies the value of $c$ for which (2-39) provides the value $\zeta = 1$. Hence,

\[
c_{cr} = 2 \sqrt{km}
\]

(2-43)

From (2-39) and (2-43) it then follows that

\[
\zeta = \frac{c}{c_{cr}}
\]

(2-44)

Hence, $\zeta$ is interpreted as the actual damping constant in proportion to the critical value of this quantity. For this reason $\zeta$ is designated the damping ratio.

![Fig. 2-6: Displacement motion of undercritically damped system. $\zeta = 0.05$, $\omega_0 = 1$, $x_0 = 0.7$, $\dot{x}_0 = 0.7$.](image)
Fig. 2-6 shows the motion of an undercritically damped system. The time has been normalized with respect to the undamped eigenvibration period $T_0$. The motion can be be written, see (2-40)

$$x(t) = Ae^{-\zeta\omega_0 t} \cos \left( \sqrt{1 - \zeta^2} \omega_0 t - \Psi \right) \tag{2-45}$$

where

$$A \cos \Psi = x_0$$

$$A \sin \Psi = \frac{\dot{x}_0 + \zeta \omega_0 x_0}{\omega_0 \sqrt{1 - \zeta^2}} \Rightarrow$$

$$A = \left( x_0^2 + \left( \frac{\dot{x}_0 + \zeta \omega_0 x_0}{\omega_0 \sqrt{1 - \zeta^2}} \right)^2 \right)^{\frac{1}{2}} \tag{2-46}$$

$$\tan \Psi = \frac{\dot{x}_0 + \zeta \omega_0 x_0}{x_0 \omega_0 \sqrt{1 - \zeta^2}} \tag{2-47}$$

The motion (2-45) is non-periodic due to the factor $e^{-\zeta\omega_0 t}$, which specifies the decrease of the vibration amplitude with the time. The damped eigenvibration period $T_d$ is defined as the period of the harmonically varying factor in (2-45), which means that

$$T_d = \frac{2\pi}{\omega_0 \sqrt{1 - \zeta^2}} \tag{2-48}$$

Evidently $T_d$ is the time-interval between two subsequent upcrossings of the time axis by $x(t)$. The damped circular eigenfrequency is defined as

$$\omega_d = \frac{2\pi}{T_d} = \omega_0 \sqrt{1 - \zeta^2} \tag{2-49}$$

At the instant of time $t$ the motion is given by (2-45). At the time $t + nT_d$ after the elapse of $n$ damped vibration period the displacement becomes

$$x(t + nT_d) = Ae^{-\zeta\omega_0 (t + nT_d)} \cos \left( \omega_d t + \omega_d nT_d - \Psi \right) = e^{-\zeta\omega_0 nT_d} x(t) \Rightarrow$$

$$\frac{x(t + nT_d)}{x(t)} = e^{-\zeta\omega_0 nT_d} = \exp \left( -2\pi n \frac{\zeta}{\sqrt{1 - \zeta^2}} \right) \tag{2-50}$$

where the last statement of (2-50) follows from (2-48). Further, $\omega_0 nT_d = 2\pi n$ has been used. (2-50) shows, that if the displacement at a given instant of time $t$ is $x(t)$,
the displacement at the time \( t + nT_d \) has decreased with the factor \( \exp(-2\pi n \frac{\zeta}{\sqrt{1-\zeta^2}}) \), independently of the selected time \( t \). The logarithmic decrement \( \delta \) is defined by

\[
\delta = \ln \left( \frac{x(t)}{x(t+T_d)} \right) = 2\pi \frac{\zeta}{\sqrt{1-\zeta^2}}
\]  

(2-51)

where (2-50) has been used. From (2-51) it follows that \( \zeta \) and \( \delta \) are equivalent measures for the damping in an undercritically damped system.

Assume that the motion of the structure has been measured by an eigenvibration test, and the result is plotted as shown in fig. 2-6. On the graph the time-interval \( T_d \) between two subsequent upcrossings of the time axis is measured. Next, the displacements \( A_0 \) and \( A_n \), placed with the time interval \( nT_d \), are registrated. The position on the time axis for measuring these displacements is arbitrary. The logarithmic decrement then follows from (2-50), (2-51)

\[
\delta = \frac{1}{n} \ln \left( \frac{A_0}{A_n} \right)
\]  

(2-52)

Next, the damping ratio is calculated from (2-51)

\[
\zeta = \frac{\delta}{2\pi} \frac{1}{\sqrt{1 + (\frac{\delta}{2\pi})^2}}
\]  

(2-53)

Finally, the undamped circular eigenfrequency is determined from (2-49)

\[
\omega_0 = \frac{2\pi}{T_d \sqrt{1-\zeta^2}}
\]  

(2-54)

(2-52), (2-53) and (2-54) demonstrate how the parameters of the differential equation in (2-38) may be identified by an eigenvibration test in the undercritically damped case.

**Fig. 2-7:** Motion of critically damped system as a function of \( \dot{x}_0 \). \( \omega_0 = 1, \ x_0 = 0.7 \).

**Fig. 2-8:** Motion of overcritically damped system as a function of \( \dot{x}_0 \). \( \zeta = 1.5, \ \omega_0 = 1, \ x_0 = 0.7 \).
Fig. 2-7 shows the motion of a critically damped system with the initial displacement \(x_0 = 0.7\) as a function of the initial velocity. Again, time has been normalized with respect to \(T_0\). As seen the motion is no longer oscillatory. If \(\dot{x}_0 \geq -\omega_0 x_0\) the motion is monotonously decreasing to the static equilibrium state, without crossing the time axis. If \(\dot{x}_0 < -\omega_0 x_0\) a crossing of the time-axis takes place at the time \(t = -\frac{x_0}{\dot{x}_0 + \omega_0 x_0}\), and next a local minimum is passed followed by an monotonously increasing convergence to the static equilibrium state.

Figur 2-8 shows the corresponding motion for an overcritically damped system. The qualitative behaviour of the motions is quite similar to those of the critically damped system. However, in this case a crossing of the time axis only takes place, if \(\dot{x}_0 < -\omega_0 x_0(\zeta + \sqrt{\zeta^2 - 1})\). Unless \(\dot{x}_0 = -\omega_0 x_0(\zeta + \sqrt{\zeta^2 - 1})\) the motion is damped more slowly than in the critical damped case, even though the damping ratio is larger. The explanation of this apparent paradox is left to the reader as an exercise.

All civil engineering structures are undercritical damped. Typically, \(\zeta \in [0.005, 0.05]\). Structures with \(\zeta \leq 0.02\) are classified as lightly damped.

2.4 Forced Harmonic Vibrations

Fig. 2-9: Forced harmonic vibrations of linear viscous damped SDOF system.

The external dynamic force \(f(t)\) is assumed to be harmonic varying with time, i.e.

\[
f(t) = \text{Re}(Fe^{j\omega t})
\]  

\((2-55)\)

\(F\) is a complex amplitude defined as

\[
F = |F|e^{-j\alpha}
\]  

\((2-56)\)

\((2-55)\) can then be written

\[
f(t) = |F|\cos(\omega t - \alpha)
\]  

\((2-57)\)
Hence, \(|F|\) is the amplitude and \(\alpha\) is the phase of the harmonic load. After division by \(m\) (2-32), (2-33) take the form
\[
\begin{align*}
\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2x &= \text{Re}\left(\frac{F}{m}e^{i\omega t}\right), \quad t > 0 \\
x(0) &= x_0, \quad \dot{x}(0) = \dot{x}_0
\end{align*}
\] (2-58)

If a system as shown in fig. 2-9 is loaded with a harmonic external force, it is observed experimentally that the motion eventually becomes harmonic with the same circular frequency as the loading, as the motion due to the initial conditions is damped out. For this reason a particular solution to the inhomogenous differential equation (2-58) is searched for on the form
\[
x(t) = \text{Re}(Xe^{i\omega t})
\] (2-59)

where \(X\) is a complex amplitude. Inserting (2-59) into (2-58) provides
\[
\text{Re}(-\omega^2Xe^{i\omega t}) + \text{Re}(2\zeta\omega_0i\omega Xe^{i\omega t}) + \text{Re}(\omega_0^2Xe^{i\omega t}) = \text{Re}\left(\frac{F}{m}e^{i\omega t}\right) \Rightarrow \\
\text{Re}\left([\omega_0^2 - \omega^2 + 2\zeta\omega_0\omega i)X - \frac{F}{m}]e^{i\omega t}\right) = 0
\] (2-60)

(2-59) is a possible motion, if and only if (2-60) is fulfilled at all times. This is only possible, if the term within the sharp-edged brackets is equal to zero, leading to
\[
X = H(\omega)F
\] (2-61)
\[
H(\omega) = \frac{1}{m(\omega_0^2 - \omega^2 + 2\zeta\omega_0\omega i)}
\] (2-62)

\(H(\omega)\) is designated the frequency response function. This merely depends on the system parameters \(m, c, k\), whereas the load only enters through the circular frequency \(\omega\). It follows from (2-61) that \(H(\omega)\) represents the complex amplitude \(X\) of the displacement response, when \(F = 1\). \(X\) is written in the following polar complex form
\[
X = |X|e^{-i\psi} = Ae^{-i\psi}
\] (2-63)

Insertion of (2-63) into (2-59) provides
\[
x(t) = A\cos(\omega t - \Psi)
\] (2-64)

(2-64) signifies a harmonic motion with the amplitude \(A = |X|\) and the phase \(\Psi\). Next, these quantities will be determined. The frequency response function (2-62) is equally written in complex polar form
\[
H(\omega) = \frac{\omega_0^2 - \omega^2 - 2\zeta\omega_0\omega i}{m((\omega_0^2 - \omega^2)^2 + 4\zeta^2\omega_0^2\omega^2)} = |H(\omega)|e^{-i\psi_0}
\] (2-65)
\[
|H(\omega)| = \frac{1}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\zeta^2\omega_0^2\omega^2}} \quad (2-66)
\]
\[
|H(\omega)| \cos \Psi_0 = \frac{\omega_0^2 - \omega^2}{m((\omega_0^2 - \omega^2)^2 + 4\zeta^2\omega_0^2\omega^2)} \quad (2-67)
\]
\[
|H(\omega)| \sin \Psi_0 = \frac{2\zeta\omega_0\omega}{m((\omega_0^2 - \omega^2)^2 + 4\zeta^2\omega_0^2\omega^2)} \quad (2-68)
\]

From (2-67) and (2-68) it follows that
\[
\tan \Psi_0 = \frac{2\zeta\omega_0\omega}{\omega_0^2 - \omega^2} = \frac{2\zeta \frac{\omega}{\omega_0}}{1 - \left(\frac{\omega}{\omega_0}\right)^2} \quad (2-69)
\]

From (2-61), (2-63), (2-65) and (2-66) it follows that
\[
A e^{-i\Psi} = |H(\omega)| e^{-i\Psi_0} |F| e^{-ia} \Rightarrow
\]
\[
A = |H(\omega)||F| = \frac{|F|}{m\sqrt{(\omega_0^2 - \omega^2)^2 + 4\zeta^2\omega_0^2\omega^2}} \quad (2-70)
\]
\[
\Psi = \Psi_0 + \alpha \quad (2-71)
\]

Hence, \(\Psi_0\) may be interpreted as the phase delay of the motion relative to the loading. By the use of (2-7), (2-70) can be written on the following form
\[
A = |X| = D\left(\zeta, \frac{\omega}{\omega_0}\right) \frac{|F|}{k} \quad (2-72)
\]
\[
D\left(\zeta, \frac{\omega}{\omega_0}\right) = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + 4\zeta^2\left(\frac{\omega}{\omega_0}\right)^2}} \quad (2-73)
\]

\(|F|/k\) denotes the amplitude of the displacement in case of quasi-static excitation, i.e. for a harmonic external load with the amplitude \(|F|\) and an infinitely small circular frequency. In this case the inertial term \(m\ddot{x}\) and the damping term \(c\dot{x}\) in (2-32) are both ignorable. The factor \(D(\zeta,\omega/\omega_0)\), which is denoted the dynamic amplification factor, determines the relative increase of the amplitude \(|X|\), when these forces have significant influence on the displacement response. By the use of conventional function analysis it can be shown, that a local maximum of \(D(\zeta,\omega/\omega_0)\) for \(\zeta < \sqrt{2}/2\) exists, as shown in fig. 2-10a. The maximum is attained at \(\frac{\omega}{\omega_0} = \sqrt{1 - 2\zeta^2}\), and the maximum value becomes \(D(\zeta, \sqrt{1 - 2\zeta^2}) = 1/2\zeta\sqrt{1 - \zeta^2}\). Harmonic excitations with \(\omega = \omega_0\) are denoted resonance, in which case the dynamic amplification factor becomes \(D(\zeta, 1) = \).
1/2ζ. For lightly damped systems this value is only slightly smaller than the maximum value, and the resonance circular frequency \( \omega = \omega_0 \) is only slightly larger than the circular frequency at maximum response \( \omega = \omega_0 \sqrt{1 - 2\zeta^2} \). If \( \zeta \geq \sqrt{2}/2 \) the maximum is attained at \( \omega = 0 \). In this case the dynamic amplitude response is smaller than the quasi-static response at all circular frequencies. For \( \zeta = \sqrt{2}/2 \) the graph of \( D(\zeta, \omega/\omega_0) \) has a horizontal tangent at \( \omega = 0 \).

The variation of the phase angle \( \Psi_0 \) with the frequency ratio \( \omega/\omega_0 \) is shown in fig. 2-10b for the same values of the damping ratio \( \zeta \) as considered in fig. 2-10a. As seen \( \Psi_0 \simeq 0 \) for \( \omega/\omega_0 \ll 1 \) for all values of \( \zeta \), implying that the motion is in phase with the excitation. Physically, this means that the elastic restoring force \( kx \) is dominating the left-hand side of (2-58). At resonance, \( \omega/\omega_0 = 1 \), \( \Psi_0 = 90^\circ \), independent of the damping ratio. For \( \omega/\omega_0 \gg 1 \) the phase angle \( \Psi_0 \) approaches \( 180^\circ \), which means that the response is in counter phase to the excitation. Under these circumstances the term \( mx \) is dominating the left-hand side of (2-58).

The complete solution of (2-58) is made up by a linear combination of (2-59) and the solution to the homogenous differential equation. The use of (2-45) for the complementary solution provides

\[
x(t) = A \cos(\omega t - \Psi) + A_1 e^{-\zeta \omega_0 t} \cos(\omega_0 t - \Psi_1)
\]  

(2 - 74)

In dynamics the eigenvibration part is termed the transient motion, and the forced vibration part is termed the stationary motion, equivalent to the mathematical designations complementary and particular solutions. \( A_1 \) and \( \Psi_1 \) are integration parameters,
which are determined by the following equations, resulting from the initial conditions (2-33)

\[
\begin{align*}
    x_0 &= A \cos \Psi + A_1 \cos \Psi_1 \\
    \dot{x}_0 &= A \omega \sin \Psi + A_1 \omega_d \left( \sin \Psi_1 - \frac{\zeta}{\sqrt{1 - \zeta^2}} \cos \Psi_1 \right)
\end{align*}
\]

(2 - 75)

From (2-29) and (2-64) it follows that

\[
f_d(t) = c \dot{x} = -A \omega \sin(\omega t - \Psi)
\]

(2 - 76)

(2-64) and (2-76) then provide

\[
\left( \frac{x}{A} \right)^2 + \left( \frac{f_d}{c \omega A} \right)^2 = 1
\]

(2 - 77)

(2-77) shows that the hysteresis loop at harmonic motion of a linear SDOF system with linear viscous damping is an ellipse with centre at \((x, f_d) = (0, 0)\) and semi-axes \(A\) and \(c \omega A\), see fig. 2-5a. The dissipated energy per period is equal to the area of the ellipse, i.e.

\[
E_d = \pi c A^2 \omega
\]

(2 - 78)

The external energy supplied per period becomes, cf. (2-36), (2-57), (2-64) and (2-71)

\[
E_e = \int_0^T |F| \cos(\omega t - \alpha)( - A \omega \sin(\omega t - \Psi_0 - \alpha)) dt =
\]

\[
A \omega |F| \int_0^T \cos(\omega t - \alpha)( - \sin(\omega t - \alpha) \cos \Psi_0 + \cos(\omega t - \alpha) \sin \Psi_0) dt =
\]

\[
\pi A |F| \sin \Psi_0
\]

(2 - 79)

From (2-39), (2-66) and (2-68) follow

\[
\sin \Psi_0 = |H(\omega)|2 \zeta \omega_0 \omega m = |H(\omega)|\omega
\]

(2 - 80)

Finally, (2-70), (2-79) and (2-80) provide

\[
E_c = \pi A |F||H(\omega)|\omega = \pi c A^2 \omega = E_d
\]

(2 - 81)

(2-81) represents a formal proof of (2-35) for harmonic motions of a linearly viscous damped SDOF system.
Fig. 2-11: Hysteretic loop for the total internal force in harmonic motion.

The total internal force becomes $f_d(t) + kx(t)$. The hysteretic loop for this quantity is also an ellipse, which is brought forward upon rotating the hysteretic loop for $f_d$, see fig. 2-11. Since $\int_0^T kx \dot{x} \, dt = 0$, the area of the hysteretic loops in fig. 2-5a and fig. 2-11 is of equal magnitude.

Example 2-4: Determination of the Half-Band Width

From (2-61), (2-72) follow

$$|H(\omega)|^2 = \frac{1}{k^2} D^2 \left( \zeta, \frac{\omega}{\omega_0} \right)$$

$D^2(\zeta, \frac{\omega}{\omega_0})$ has been shown as a function of the frequency ratio $\frac{\omega}{\omega_0}$ in fig. 2-12. As follows from (2-73)
\[ D^2(\zeta, 1) = \frac{1}{8\zeta^2} \text{ at resonance } \frac{\omega_1}{\omega_0} = 1. \]  
Now, two circular frequencies, \( \omega_1 < \omega_0 < \omega_2 \), are sought, for which \( D^2(\zeta, \frac{\omega_1}{\omega_0}) = D^2(\zeta, \frac{\omega_2}{\omega_0}) = \frac{1}{2} D^2(\zeta, 1) = \frac{1}{8\zeta^2} \). Both of these points exist, only if \( \frac{1}{8\zeta^2} > 1 \Rightarrow \zeta < \frac{\sqrt{2}}{4} \). \( \omega_1, \omega_2 \) are denoted the half-band points. At these points the value of \( D^2 \) is exactly half of the value at the resonance point. The half-band width is defined as the interval length
\[ \Delta \omega = \omega_2 - \omega_1 \]  
(2-83)

\( \frac{\omega_1}{\omega_1} \) and \( \frac{\omega_1}{\omega_2} \) are determined from
\[
\frac{1}{8\zeta^2} = \frac{1}{(1 - (\frac{\omega_1}{\omega_0})^2)^2 + 4\zeta^2(\frac{\omega_1}{\omega_0})^2} \Rightarrow \\
\left\{ \begin{array}{l}
(\frac{\omega_2}{\omega_0})^2 = (1 - 2\zeta^2) \pm 2\zeta \sqrt{1 + \zeta^2} = 1 \pm 2\zeta + O(\zeta^3) \Rightarrow \\
(\frac{\omega_1}{\omega_0})^2 = 1 \pm \zeta + O(\zeta^3)
\end{array} \right.
\]  
(2-84)

The order symbol \( O(x) \) is defined by
\[
\lim_{x \to 0} \frac{O(x)}{x} = A
\]  
(2-85)

where \( A \) is a constant. Then, the half-band width becomes
\[
\Delta \omega = \omega_0 \left( \frac{\omega_2}{\omega_0} - \frac{\omega_1}{\omega_0} \right) = (2\zeta + O(\zeta^3))\omega_0
\]  
(2-86)

For lightly damped structures, where \( \zeta \ll 1 \), one can ignore the remainder in (2-86). Further, the maximum value of the dynamic amplification factor is almost equal to the resonance value, as mentioned subsequent to (2-73). This observation is the basis of an alternative approach to the identification of parameters in a lightly damped system. The frequency response function \( H(\omega) \) is determined as a function of the circular frequency in a vibration test. On the graph of \( |H(\omega)|^2 \) the half-band width is measured together with the circular frequency \( \omega_{\text{max}} \), at which the maximum dynamic amplification is attained. Then the circular eigenfrequency and the damping ratio are identified from
\[
\omega_0 \simeq \omega_{\text{max}}
\]  
(2-87)

\[
\zeta = \frac{\Delta \omega}{2\omega_{\text{max}}}
\]  
(2-88)

Alternatively the half-band width may be measured on the graph of \( |H(\omega)| \). In this case the half-band width is displayed by the frequency width at the height \( \frac{\sqrt{2}}{2} \) times the peak height.

Up to now the external force \( f(t) \) has been assumed to be harmonic, cf. (2-55). Next, it is only assumed that \( f(t) \) is periodic with the period \( T \). Then, \( f(t) \) can be expanded into a Fourier series, see (A-1), see
\[
f(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} a_m \cos(\omega_m t) + b_m \sin(\omega_m t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \text{Re}(F_m e^{i\omega_m t})
\]  
(2-89)
where
\[ \omega_m = \frac{2\pi m}{T}, \quad m = 1, 2, \ldots \]  
\[ a_m = \frac{2}{T} \int_0^T f(t) \cos(\omega_m t) dt, \quad m = 0, 1, 2, \ldots \]  
\[ b_m = \frac{2}{T} \int_0^T f(t) \sin(\omega_m t) dt, \quad m = 1, 2, \ldots \]  
\[ \left\{ \begin{array}{l} a_m = \frac{2}{T} \int_0^T f(t) \cos(\omega_m t) dt, \quad m = 0, 1, 2, \ldots \\ b_m = \frac{2}{T} \int_0^T f(t) \sin(\omega_m t) dt, \quad m = 1, 2, \ldots \end{array} \right. \]  
(2-90)
(2-91)

The complex force amplitude is written in the following polar form
\[ F_m = |F_m|e^{-i\alpha_m}, \quad m = 1, 2, \ldots \]  
(2-92)

Insertion of (2-92) into (2-89) provides
\[ |F_m| \cos \alpha_m = a_m, \quad |F_m| \sin \alpha_m = b_m \]  
\[ |F_m| = \left( a_m^2 + b_m^2 \right)^{\frac{1}{2}}, \quad m = 1, 2, \ldots \]  
(2-93)
\[ \tan \alpha_m = \frac{b_m}{a_m}, \quad m = 1, 2, \ldots \]  
(2-94)

If (2-89) is inserted on the right-hand side of (2-32) the stationary motion can be determined from superposition of the stationary motion from each of the harmonic components. Use of (2-61) provides
\[ x(t) = \frac{a_0}{2k} + \sum_{m=1}^{\infty} \text{Re} \left( X_m e^{i\omega_m t} \right) = \frac{a_0}{2k} + \sum_{m=1}^{\infty} |X_m| \cos(\omega_m t - \Psi_m) \]  
(2-95)
\[ X_m = H(\omega_m)F_m \]  
(2-96)

where \( H(\omega) \) is given by (2-62), and \( X_m = |X_m|e^{-i\Psi_m}, \quad m = 1, 2, \ldots \) has been used, cf. (2-63). The amplitude \( |X_m| \) and the phase \( \Psi_m \) are determined from (2-69), (2-70) og (2-71)
\[ |X_m| = |H(\omega_m)| |F_m| = \frac{|F_m|}{m\sqrt{(\omega_0^2 - \omega_m^2)^2 + 4\zeta^2\omega_0^2\omega_m^2}} \]  
(2-97)
\[ \Psi_m = \Psi_0(\omega_m) + \alpha_m = \arctan \left( \frac{2\zeta\omega_0\omega_m}{\omega_0^2 - \omega_m^2} \right) + \alpha_m \]  
(2-98)
Example 2-5: Rectangular Periodic Pulse Load

![Rectangular pulse load](image)

Fig. 2-13: Rectangular pulse load.

Fig. 2-13 shows a periodic load with the period $T$, consisting of rectangular pulses of the height $F_0$ and the width $\kappa \cdot \frac{T}{2}$, where $\kappa \in [0, 2]$. The Fourier series converge to $\frac{F_0}{2}$ at the discontinuity points. Application of (2-91) provides

$$
\begin{align*}
  a_0 &= \kappa F_0 \\
  a_m &= \frac{F_0}{m\pi} \sin(m\kappa \pi) \\
  b_m &= \frac{F_0}{m\pi} (1 - \cos(m\kappa \pi))
\end{align*}
$$

Finally, the stationary part of the motion is determined upon insertion of (2-99) into (2-95)-(2-98).

2.5 Forced Vibrations due to Arbitrary Excitation

In this section the motion is determined by a linearly viscous damped SDOF system excited by an external dynamic load $f(t)$, which is neither harmonic nor periodic. The force is assumed to be applied during the time interval $[0, \infty]$, i.e.

$$
f(t) = 0 \quad , \quad t < 0
$$

(2-100)

The *impulse* is defined by the time integral

$$
I(t) = \int_{0^-}^{t} f(\tau)d\tau
$$

(2-101)
Fig. 2-14: Impulsive load.

Quite often forces of large intensity are acting during a short time interval, so the impulse becomes limited. Such forces are denoted impulsive. Fig. 2-14 shows an impulsive force of magnitude \( f(\tau) = \frac{I}{\varepsilon} \), which is applied at the time \( t \), and is constant during an interval of the length \( \varepsilon \). Obviously, the impulse of the load is \( I \). Next, let \( \varepsilon \to 0 \), so the impulse is constantly equal to \( I \) during the limit passing. Then \( \lim_{\varepsilon \to 0} f(\tau) = \lim_{\varepsilon \to 0} \frac{I}{\varepsilon} = \infty \). Hence, in the limit the load can formally be described by the Dirac’s delta function

\[
f(\tau) = I\delta(t - \tau)
\]  \hspace{1cm} (2-102)

Especially, \( f(\tau) \) is denoted a unit impulse if \( I = 1 \). The impulsive load (2-102) is applied to the mass. From the conservation of momentum it follows that

\[
\int_{t-}^{t+} f(\tau) d\tau = I = m\Delta \dot{x} \Rightarrow \\
\Delta \dot{x} = \frac{I}{m}
\]  \hspace{1cm} (2-103)

Consequently, an impulsive load causes a discontinuous change of the velocity. Since the load is acting during an infinitely short time interval, the displacement will not change due to the inertia, i.e.

\[
\Delta x = 0
\]  \hspace{1cm} (2-104)

Consider a linear viscous damped SDOF system at rest at the time \( t = 0^- \). At the time \( t = 0 \) a unit impulse is applied. The problem is then to determine the motion \( h(t) \) for \( t > 0 \). The equation of motion becomes, cf. (2-32) and (2-102)

\[
m(\ddot{h} + 2\zeta \omega_0 \dot{h} + \omega_0^2 h) = \delta(t)
\]  \hspace{1cm} (2-105)

The system is at rest before the excitation is applied, corresponding to

\[
h(0^-) = \dot{h}(0^-) = 0
\]  \hspace{1cm} (2-106)
From (2-104) it follows that
\[ h(0^+) = 0 \quad (2-107) \]

(2-105) is integrated from \( t = 0^- \) to \( t = 0^+ \). Use of (2-106) and (2-107) provides
\[
m \left( \int_{0^-}^{0^+} \ddot{h}(t) \, dt + \int_{0^-}^{0^+} 2\zeta \omega_0 \dot{h}(t) \, dt + \omega_0^2 \int_{0^-}^{0^+} h(t) \, dt \right) = \int_{0^-}^{0^+} \delta(t) \, dt \Rightarrow \\
m \left( \dot{h}(0^+) - \dot{h}(0^-) + 2\zeta \omega_0 (h(0^+) - h(0^-)) \right) = 1 \Rightarrow \\
\dot{h}(0^+) = \frac{1}{m} \quad (2-108)
\]

Alternatively, (2-108) follows directly from (2-103). \( h(t) \) is designated the impulse response function. From (2-105), (2-107), (2-108) it follows that this quantity may be determined as the solution to the initial value problem
\[
\begin{align*}
\dddot{h} + 2\zeta \omega_0 \ddot{h} + \omega_0^2 h &= 0 \quad , \quad t > 0 \\
h(0^+) &= 0 \text{,} \quad \dot{h}(0^+) = \frac{1}{m}
\end{align*} \quad (2-109)
\]

As seen \( h(t) \) for \( t > 0 \) is an eigenvibration and hence a solution to (2-38). Then the solution to (2-109) follows upon specialization of (2-40)
\[
h(t) = \begin{cases} 
0 & \text{,} \quad t < 0 \\
\frac{1}{m \omega_d} e^{-\zeta \omega dt} \sin(\omega_d t) & \text{,} \quad t \geq 0
\end{cases} \quad (2-110)
\]
where \( \omega_d \) is given by (2-49). From the differential equation (2-109) at the time \( t = 0^+ \) it follows that
\[
\ddot{h}(0^+) = -2\zeta \omega_0 \frac{1}{m} \quad (2-111)
\]

Differentiation of (2-109) implies, that \( \dot{h}(t) \) for \( t > 0 \) also becomes a solution to (2-38). The initial conditions are given by (2-108) and (2-111). This leads to the initial value problem
\[
\begin{align*}
\frac{d^2}{dt^2} \ddot{h} + 2\zeta \omega_0 \frac{d}{dt} \dot{h} + \omega_0^2 \ddot{h} &= 0 \quad , \quad t > 0 \\
\dot{h}(0^+) &= \frac{1}{m} \text{,} \quad \frac{d}{dt} \dot{h}(0^+) = -2\zeta \omega_0 \frac{1}{m}
\end{align*} \quad (2-112)
\]
Consider the quantity
\[ x^{(1)}(t) = \int_{0}^{t} h(t - \tau) f(\tau) d\tau, \quad t > 0 \] (2 - 113)

Differentiation of (2-113) and use of (2-107), (2-108) provides
\[ \ddot{x}^{(1)}(t) = \dot{h}(t - t^-) f(t) + \int_{0}^{t} \dot{h}(t - \tau) f(\tau) d\tau, \quad t \geq 0 \] (2 - 114)
\[ \dddot{x}^{(1)}(t) = h(t - t^-) f(t) + \int_{0}^{t} \ddot{h}(t - \tau) f(\tau) d\tau = \] \[ \frac{1}{m} f(t) + \int_{0}^{t} \ddot{h}(t - \tau) f(\tau) d\tau, \quad t > 0 \] (2 - 115)

From (2-113), (2-114), (2-115) the following result is obtained
\[ \dddot{x}^{(1)} + 2\zeta \omega_0 \dot{x}^{(1)} + \omega_0^2 x^{(1)} = \frac{1}{m} f(t) + \] \[ \int_{0}^{t} \left[ \dddot{h}(t - \tau) + 2\zeta \omega_0 \ddot{h}(t - \tau) + \omega_0^2 \dot{h}(t - \tau) \right] f(\tau) d\tau, \quad t > 0 \] (2 - 116)

It follows from (2-109) that the term within sharp-edged bracket is equal to 0. Consequently one has
\[ \dddot{x}^{(1)} + 2\zeta \omega_0 \dot{x}^{(1)} + \omega_0^2 x^{(1)} = \frac{1}{m} f(t), \quad t > 0 \] (2 - 117)

(2-117) shows that \( x^{(1)}(t) \) defined by (2-113) is a particular (stationary) solution to (2-32). The precise specification of the upper integration limit as \( t^- \) in (2-113) is immaterial and will be omitted in the following. From (2-113) and (2-114) it follows that the obtained particular integral fulfils the initial conditions
\[ x^{(1)}(0) = \dot{x}^{(1)}(0) = 0 \] (2 - 118)

The complete solution of (2-32) can be written
\[ x(t) = x^{(0)}(t) + x^{(1)}(t), \quad t > 0 \] (2 - 119)
\( x^{(o)}(t) \) is the complementary (transient) solution to the homogeneous differential equation (2-38). As seen from (2-33), (2-118), (2-119) the eigenvibration search for must fulfill the initial conditions

\[
x^{(o)}(0) = x_0 \quad , \quad \dot{x}^{(o)}(0) = \dot{x}_0
given by (2 - 120).
\]

(2-120) implies that the transient part of the motion is given by (2-40). Hence the motion searched for is given by

\[
x(t) = e^{-\zeta \omega_0 t} \left( x_0 \cos \left( \sqrt{1-\zeta^2} \omega_0 t \right) + \frac{\dot{x}_0 + \zeta \omega_0 x_0}{\omega_0 \sqrt{1-\zeta^2}} \sin \left( \sqrt{1-\zeta^2} \omega_0 t \right) \right) + \int_0^t \dot{h}(t-\tau) f(\tau) d\tau \quad , \quad t > 0
\]

(2 - 121)

As seen from (2-109) and (2-112) \( h(t) \) and \( \dot{h}(t) \) represent 2 linearly independent solutions to the homogeneous differential equation of the second order. Then \( h(t) \) and \( \dot{h}(t) \) form a fundamental set of solutions, and \( x^{(o)}(t) \) can be written as the following linear combination

\[
x^{(o)}(t) = a h(t) + b \dot{h}(t) \quad \Rightarrow
\]

\[
\dot{x}^{(o)}(t) = a \dot{h}(t) + b \ddot{h}(t)
\]

The expansion coefficients \( a \) and \( b \) are determined from the initial conditions (2-107), (2-108), (2-111)

\[
\begin{aligned}
x_0 &= a_0 + b \frac{1}{m} \\
\dot{x}_0 &= a \frac{1}{m} + b \left( -2\zeta \omega_0 \frac{1}{m} \right) \\
a &= m x_0 + m 2 \zeta \omega_0 x_0 \\
b &= m x_0
\end{aligned}
\]

(2 - 124)

Upon eliminating \( \zeta \) in favour of \( c \) by use of (2-39) the following result is obtained for the transient part of the motion

\[
x^{(o)}(t) = (m \ddot{h}(t) + c h(t)) x_0 + m h(t) \dot{x}_0
\]

(2 - 135)

Then (2-121) may be written on the form

\[
x(t) = (m \ddot{h}(t) + c h(t)) x_0 + m h(t) \dot{x}_0 + \int_0^t h(t-\tau) f(\tau) d\tau \quad , \quad t > 0
\]

(2 - 126)
(2-126) shows that the motion caused by arbitrary initial conditions \( x_0, \dot{x}_0 \) and an arbitrary dynamic excitation \( f(t) \) is determined solely by the impulse response function \( h(t) \) as given by (2-110). The stationary part of the motion as defined by (2-113) is denoted Duhamel's integral.

\[
(2-127) \text{ shows that the motion caused by arbitrary initial conditions } x_0, \dot{x}_0 \text{ and an arbitrary dynamic excitation } f(t) \text{ is determined solely by the impulse response function } h(t) \text{ as given by (2-110). The stationary part of the motion as defined by (2-113) is denoted Duhamel's integral.}
\]

Fig. 2-15: Physical interpretation of Duhamel's integral.

\( f(\tau)d\tau \) in (2-113) represents a differential impulse, which is applied at the time \( t = \tau \). The magnitude of the impulse has been marked by the hatched area in fig. 2-15. The motion at the subsequent time \( t \) from the impulse \( f(\tau)d\tau \) is equal to \( h(t - \tau)f(\tau)d\tau \). Then (2-113) specifies the superposed motion at the time \( t \) from all previous differential impulses of this kind. These partial solutions from each differential impulse assume the system to be at rest with zero displacement and velocity prior to the time they are applied, cf. (2-106). The oscillation described by (2-113) must then be at rest with zero displacement and velocity prior to the application of the first impulse at the time \( t = 0 \). Consequently the initial conditions (2-118) also follow from physical reasons.

In section 2.4 the frequency response function \( H(\omega) \) was defined as the complex amplitude of the displacement response as a unit amplitude harmonic excitation \( f(t) = \exp(i\omega t) \) had been acting on the system for infinitely long. Similarly, \( h(t) \) describes the response from a unit impulse \( \delta(t) \). Hence both functions describe the behaviour of the system due to well-defined excitations, and may then be called system describing functions. Actually, if either \( H(\omega) \) or \( h(t) \) are known, the system parameters \( m, c \) and \( k \) may be identified from the analytical expressions (2-62) or (2-110). Since the functions describe one and the same system a relation between them may be anticipated. In order to determine this relation the unit amplitude harmonic excitation \( f(\tau) = \exp(i\omega \tau) \) is applied to the system. From (2-59) and (2-113) the following displacement response is obtained after infinitely long time of excitation

\[
Xe^{i\omega t} = H(\omega)e^{i\omega t} = \int_{-\infty}^{t} h(t - \tau)e^{i\omega \tau}d\tau \tag{2-127}
\]

In (2-127) the real operator has been skipped for ease on both sides of the equation. Introduction of the integral substitution \( u = t - \tau \) provides

\[
H(\omega)e^{i\omega t} = -\int_{\infty}^{0} h(u)e^{i\omega(t-u)}du = e^{i\omega t}\int_{0}^{\infty} h(u)e^{-i\omega u}du \Rightarrow
\]
At the derivation of (2-128) it has been used that the factor \( \exp(i\omega t) \) is independent of the integration variable \( u \), and hence may placed outside the integral. Since \( h(u) = 0 \) for \( u < 0 \) the lower limit of the integral in (2-128) may be changed to \(-\infty\). Renaming the integration variable from \( u \) to \( t \) one finally has

\[
H(\omega) = \int_{-\infty}^{\infty} h(t)e^{-i\omega t} dt \tag{2-129}
\]

Consequently \( H(\omega) \) is the Fourier transform of \( h(t) \), cf. (A-15). \( h(t) \) may then be determined from the inverse transform, cf. (A-15)

\[
h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)e^{i\omega t} d\omega , \quad t > 0 \tag{2-130}
\]

(2-129), (2-130) is not restricted to the SDOF system, but is valid for any linear MDOF or continuous system for which the response (e.g. a displacement or stress component) at a certain point is searched for due to a force component exciting the structure at another point. \( H(\omega) \) then represents the complex amplitude of the harmonic varying response quantity, due to a unit amplitude harmonic varying force \( \exp(i\omega t) \) in the indicated position and direction of the force. Similarly \( h(t) \) represents the time-variation of the response quantity due to unit impulse applied in the direction and the position of the force. Of course the analytical form of \( H(\omega) \) and \( h(t) \) are different than (2-62) and (2-110) in these cases. Even extension to non-structural systems is possible. Actually, the essence of the problem is that the system is linear and dissipative, so a harmonic excitation eventually results in a harmonic response.

2.6 Vibrations due to Movable Support

Fig. 2-16: Movable support.

Vibrations caused by the motion of the support occur in various cases. Important examples are earthquakes and traffic induced vibrations of nearby buildings. Consider
the system shown in fig. 2-16, where \( x(t) \) signifies the displacement of the mass from the static equilibrium state, and \( y(t) \) is the displacement of the support considered positive in the same direction as \( x(t) \). Then the displacement of the mass relative to the support is given as

\[
z(t) = x(t) - y(t)
\]  
(2-131)

The elongation of the spring is \( z = x - y \), and the rate of elongation of the dashpot is \( \dot{z} = \dot{x} - \dot{y} \). Consequently the free mass is excited by the forces shown in fig. 2-16. Newton’s 2nd law provides

\[
m\ddot{x} = -k(x - y) - c(\dot{x} - \dot{y}) \quad \Rightarrow
\]
\[
m\ddot{x} + c\dot{x} + kx = cy(t) + ky(t) , \quad t > 0
\]  
(2-132)

(2-133)

Introduction of (2-131) into (2-132) provides the following equation of motion formulated in the relative displacements

\[
m\ddot{z} + c\dot{z} + kz = -my(t) , \quad t > 0
\]  
(2-134)

In (2-133) and (2-134) it has been assumed that the motion of the support \( y(t) \), and hence the velocity \( \dot{y}(t) \) and acceleration \( \ddot{y}(t) \), is a known function of the time. Both (2-133) and (2-134) are on the standard form (2-32), and represent the equation of motion for the determination of the total displacement \( x(t) \) and the relative displacement \( z(t) \), respectively. The formulation (2-134) is usually preferred in earthquake resistant design, because the acceleration time-series \( \ddot{y}(t) \) is measured directly at seismological stations, and because the deformation of structural components, in casu the spring and the damper element, depends on the relative displacement, which is directly determined. Comfort criteria within accommodations and other applicability limit states are usually described in terms of the total acceleration \( \ddot{x}(t) \). In these cases the formulation in the total displacement (2-133) may be preferred.

The force on the support \( f_s(t) \) considered positive in the same direction as \( x(t) \) and \( y(t) \) becomes, cf. fig. 2-16

\[
f_s(t) = kz(t) + c\dot{z}(t)
\]  
(2-135)

Example 2-6: Harmonic Varying Ground Surface Motion

The stationary motion is to be determined, when the motion of the support is harmonic, i.e.

\[
y(t) = \text{Re}(Ye^{i\omega t}) = |Y|\cos(\omega t - \alpha)
\]  
(2-136)

\[
Y = |Y|e^{-i\alpha}
\]  
(2-137)
The force term of the differential equation (2-133) then becomes

\[ f(t) = \text{Re}(Fe^{i\omega t}) \]  
(2 – 138)

\[ F = (k + i\omega c)Y \]  
(2 – 139)

The stationary motion to (2-133) is given by (2-59). The complex amplitude \( X \) follows from (2-61), (2-62), (2-139)

\[ X = H(\omega)(k + i\omega c)Y = H_{xy}(\omega)Y \]  
(2 – 140)

\[ H_{xy}(\omega) = \frac{\omega^2 + 2\zeta_0\omega_i}{\omega_0^2 - \omega^2 + 2\zeta_0\omega_i} \]  
(2 – 141)

(2-7) and (2-39) have been used at the derivation of (2-141). \( H_{xy}(\omega) \) signifies the frequency response function relating harmonic varying ground surface motion to the harmonic varying total displacement.

The force term of the differential equation (2-134) is still determined by (2-138) with the complex amplitude now given as

\[ F = \omega^2 mY \]  
(2 – 142)

The complex amplitude amplitude \( Z \) of the harmonic varying relative displacement then becomes

\[ Z = H(\omega)\omega^2 mY = H_{xy}(\omega)Y \]  
(2 – 143)

\[ H_{xy}(\omega) = \frac{\omega^2}{\omega_0^2 - \omega^2 + 2\zeta_0\omega_i} \]  
(2 – 144)

\( H_{xy}(\omega) \) denotes the frequency response function relating harmonic varying ground surface motion to the harmonic varying relative displacement.

The force on the support follows from (2-135), (2-140), (2-143)

\[ f_s(t) = \text{Re}(F_s e^{i\omega t}) \]  
(2 – 145)

\[ F_s = (k + i\omega c)Z = (k + i\omega c)H(\omega)\omega^2 mY = H_{fxy}(\omega)Y \]  
(2 – 146)

\[ H_{fxy}(\omega) = m\omega^2 H_{xy}(\omega) = m\omega^2 \frac{\omega^2 + 2\zeta_0\omega_i}{\omega_0^2 - \omega^2 + 2\zeta_0\omega_i} \]  
(2 – 147)

If (2-141) and (2-147) are written in polar form it is seen that the argument of these frequency response functions are identical. From this follows that the total displacement \( x(t) \) and the force on the support \( f_s(t) \) are in phase in case of excitation of harmonic varying ground motions.

Example 2-7: Earthquake Excitation of a Single Storey Framed Structure

![Example 2-7: Earthquake Excitation of a Single Storey Framed Structure](image)

Fig. 2-17: Earthquake excitation of SDOF system.
Fig. 2-17 shows a single storey frame, which is excited by a horizontal ground motion $y(t)$. The beam has the total mass $m$, and is assumed to be infinitely stiff against both axial and bending deformations. The columns are massless, linear elastic and are fixed to the support. The shear stiffness of both columns is $k$ and the dissipation taken in the columns during deformations is modelled by a linear viscous damping element with the damping constant $c$.

The system has a single degree of freedom, which is selected as the horizontal displacement of the beam $x(t)$. The beam is cut free from the columns and the damping element, and the elastic restoring force, $kz(t)$, and the damping force, $ci(t)$, are applied as external forces with the sign shown in fig. 2-17. The total shear force transmitted through the columns to the supports then becomes $f_s(t) = kz(t) + ci(t)$. As seen this force depends on the relative displacement $z(t) = x(t) - y(t)$ rather than the total displacement $x(t)$. Application of Newton's 2nd law of motion provides

$$m\ddot{z} = -kz - ci \quad \Rightarrow$$

$$m\ddot{z} + ci + kx = -m\ddot{y}(t) \quad \Rightarrow$$

$$\ddot{z} + 2\zeta\omega_0\dot{z} + \omega_0^2z = -\ddot{y}(t), \quad t > 0$$

where (2-7) and (2-39) have been applied in the last statement.

It is remarkable that the mass of the beam has disappeared from (2-148). Hence $z(t)$ merely depends on the damping ratio $\zeta$, the circular eigenfrequency $\omega_0$ and the acceleration time-series $\ddot{y}(t)$. For a given acceleration time-series one is primarily interested in the maximum of the relative displacement, since this quantity determines the stresses in the columns. The response spectrum $R(\zeta, \omega_0)$ is defined as

$$R(\zeta, \omega_0) = \max_{t \in [0, \infty]} |z(t)|$$

(2 - 149)

Response spectra are widely used in earthquake resistant structural design. For a structure with given circular eigenfrequency $\omega_0$ and damping ratio $\zeta$ the maximum numerical relative response can directly be seen on the graph of $R(\zeta, \omega_0)$. A response spectrum is a characterization of a given earthquake and indicates the value hability of a structure defined by the parameters $\zeta$ and $\omega_0$ when exposed to the earthquake. At the time where $z(t)$ is at a maximum one has $\dot{z}(t) = 0$, which means that...
the numerical value of the shear force is given as \( f_s(t) = kR(\zeta, \omega_0) \). Corresponding to (2-149) the maximum numerical shear force during the excitation is determined from
\[
\max_{t \in [0, \infty]} |f_s(t)| = \max_{t \in [0, \infty]} |kz(t) + cz(t)| \tag{2 - 150}
\]

Since the structural damping is small (2-150) will only be insignificantly larger than \( kR(\zeta, \omega_0) \). In fig. 2-18a is shown the ground surface acceleration time-series of the North-South component of the 1940 El Centro Earthquake, which is famous, since it was the first time a strong motion earthquake was recorded on a seismograph. The peak ground acceleration was \( y_{\text{max}} = 0.32 \) g, which was recorded after approximate 2 s. The corresponding response spectrum is shown in fig. 2-18b. The spectrum has been shown in the non-dimensional form \( \omega_0 R(\omega_0, \zeta)/y_{\text{max}} \) where \( y_{\text{max}} = 0.348 \) is the maximum recorded ground surface velocity. Since \( \omega_0 R(\omega_0, \zeta) \) is the amplitude of \( \dot{z}(t) \) in case of harmonic response the fraction \( \omega_0 R(\omega_0, \zeta)/y_{\text{max}} \) may be interpreted as a kind of dynamic amplification factor. As seen relatively large amplifications may occur for civil engineering structures with eigenfrequency \( f_0 = \frac{\omega_0}{2\pi} \in [0.1 \text{Hz}, 2.0 \text{Hz}] \) and damping ratio \( \zeta = 0.01 \). Very flexible structures such as long suspension bridges with \( f_0 < 0.1 \text{Hz} \) or very stiff structures with \( f_0 > 5.0 \text{Hz} \) are generally not sensible to earthquakes. Because of the variable frequency contents in different recorded ground motion accelerations, and because of the rather erratic appearance of most response spectra, a so-called design response spectrum is used in earthquake resistant design, which is merely a smoothed upper envelope curve through the response spectra of a number of recorded strong motion earthquakes of equal magnitude and different frequency content.

\section*{2.7 D'Alembert's Principle}

Fig. 2-19: Illustration of d'Alembert's principle.

Newton's 2nd law of motion for the free mass reads
\[
m\ddot{x} = -kx - c\dot{x} + f(t) \tag{2 - 151}
\]

The equation may be written in the form
\[
0 = -m\ddot{x} - kx - c\dot{x} + f(t) \tag{2 - 152}
\]

Formally (2-151) may be interpreted as a statical equation of equilibrium, where the inertial force \( f_I(t) = -m\ddot{x}(t) \) is applied as an external load on the free mass. \( f_I(t) \) is
considered positive in the direction of the degree of freedom \( x(t) \). (2-151) may be interpreted as a statical equilibrium equation, where the inertial force \( f_I(t) \) is an equilibrium with the elastic force \( -kx(t) \), the damping force \( \dot{x}(t) \) and the external dynamic force \( f(t) \). This expresses d'Alembert's principle: \( f_I = -m\ddot{x} \) is applied as an external force, and all calculations are next performed as formally statical calculations. D'Alembert's principle may be applied to multi degree-of-freedom systems and to continuous systems as well. It should be stressed that d'Alembert's principle is a formal operation without any physical substance. The only motivation for the use of the principle is that it facilitate the formulation of the equations of motion. Actually, the fictitious inertial force does not exist in reality. Instead (2-151) is an observable fact. When a force of magnitude as specified on the right-hand side of the equation is applied to the free mass an acceleration of the indicated magnitude is measured.

**Example 2-8: Equation of Motion of Infinitely Stiff Beam by D'Alembert's Principle**

\[ E I = \infty, \mu = \frac{m}{l}, \]

\[ -\gamma l \theta \frac{m}{l} \]

\[ k \gamma l \theta \]

\[ (-\frac{m}{l} \Delta x)x \theta \]

Fig. 2-20: Infinitely stiff homogeneous beam of a single degree of freedom. a) Definitions. b) Spring force and inertial loading on free beam.

Fig. 2-20a shows a plane infinitely stiff homogenous beam \( ABC \) with the total length \( l \) and the mass \( m \). Then the mass per unit length is constantly equal to \( \mu = \frac{m}{l} \). \( AB \) has the length \( \gamma l \) and \( BC \) has the length \( (1-\gamma)l \), where \( \gamma \in [0,1] \). The beam is simply supported in \( B \) and supported in \( A \) by a linear elastic spring with the spring stiffness \( k \). In the figure the beam is shown in the statical equilibrium state. Since the beam is infinite stiff, the system has but a single degree of freedom, which is selected as the rotation \( \theta \) around the support at \( B \) with positive direction defined in fig. 2-20a.

A coordinate \( x \) along the beam is introduced, which is measured from the point \( B \) in the direction of the free end \( C \), see fig. 2-20b. A differential beam element of the length \( dx \) placed at the coordinate \( x \) is considered. The element has the mass \( \frac{m}{l} \Delta x \), and the displacement \( x \theta \). Then the acceleration becomes \( x \theta \), providing the inertial load on the element \( -\frac{m}{l} \Delta x x \theta \), acting in the direction of the displacement. The linearly varying inertial load per unit length has been shown in fig. 2-20b. The beam is cut free.
from the spring at the point \(A\), and the spring force \(k \cdot \gamma l \dot{\theta}\) is applied as an external load. According to d'Alemberts principle the system can now be analysed based on the formal statical equations of equilibrium.

Equilibrium of forces in the vertical direction determines the unknown reaction on the beam at the support in point \(B\). If the moment equilibrium is formulated around point \(B\) the reaction force is eliminated from the moment equation, which leads to the equation of motion

\[
k \cdot \gamma l \dot{\theta} - m \frac{d^2}{dx^2}z = \int_{-\gamma l}^{(1-\gamma)l} \left( - \frac{m}{l} dx \dot{\theta} \right) \cdot z \Rightarrow
\]

\[
\frac{1}{3}((1-\gamma)^3 + \gamma^3) ml^2 \ddot{\theta} + k \gamma^2 \dot{\theta}^2 = 0 \Rightarrow
\]

\[
\ddot{\theta} + \omega_0^2 \theta = 0
\]

\[
\omega_0^2 = \frac{3 \gamma^2}{1 - 3\gamma + 3\gamma^2} \frac{k}{m}
\]

(2 - 153)

(2 - 154)

Obviously, the present application of d'Alemberts principle is tantamount to the equation of moment of momentum formulated in the point \(B\). Since the beam is infinitely stiff, the inertial load is statical equivalent to a single force \(f_I\) and a moment \(M_I\) referred to an arbitrary point. If \(f_I\) is referred to the point \(B\) one finds

\[
f_I^{(B)} = \int_{-\gamma l}^{(1-\gamma)l} \left( - \frac{m}{l} dx \dot{\theta} \right) = -m \left( \frac{1}{2} - \gamma \right) l \ddot{\theta}
\]

(2 - 155)

\[
M_I^{(B)} = \int_{-\gamma l}^{(1-\gamma)l} \left( - \frac{m}{l} dx \dot{\theta} \right) \dot{z} = -\frac{1}{3}(1 - 3\gamma + 3\gamma^2) ml^2 \ddot{\theta}
\]

(2 - 156)

The coordinate of the mass center of gravity \(G\) is \(z_G = (\frac{1}{2} - \gamma)l\). If the moment is referred to mass center of gravity the statical equivalent loads then become

\[
f_I^{(G)} = f_I^{(B)} = -m \ddot{y}_G
\]

(2 - 157)

\[
M_I^{(G)} = M_I^{(B)} - f_I^{(G)} z_G = -J_G \ddot{\theta}_G
\]

(2 - 158)

\[
J_G = \frac{1}{12} ml^2
\]

(2 - 159)

where \(y_G = z_G \dot{\theta}\) is the displacement and \(\theta_G = \dot{\theta}\) is the rotation of the mass center of gravity, and \(J_G\) is the mass moment of inertia around the mass center of gravity.

(2 - 157) and (2 - 158) show that the inertial load per unit length is statical equivalent to a discrete inertial force \(f_I^{(G)} = -m \ddot{y}_G\) and a moment \(M_I^{(G)} = -J_G \ddot{\theta}_G\) referred to the mass center of gravity. Although this result has been derived for a very special system it has general validity: To express the inertial loading on an infinitely stiff body the displacement and rotation of the mass center of gravity is at first expressed in terms of the degrees of freedom of the body. Next, the statical equivalent load and moment are calculated from (2 - 157) and (2 - 158), where \(m\) is the total mass of the body and \(J_G\) is the mass moment of inertia related to the corresponding degrees of freedom of the mass center of gravity.
2.8 Vibrations due to Indirectly Acting Dynamic Loads

![Diagram of a linearly elastic massless beam with masses 1 and 2 and a force function \( f(t) \).]

Indirect excitation of the system means that the external load \( f(t) \) is not acting directly on the mass, but is transmitted through the elastic structure to the mass. Fig. 2-21 shows a linear elastic, simply supported beam with a concentrated mass and an indirectly acting external force \( f(t) \). The displacement at the mass (point 1) is denoted \( x_1 \), and the displacement at the force (point 2) is denoted \( x_2 \). A unit force at the point 2 causes the displacement \( \delta_{12} \) at the point 1. \( \delta_{ij} \) is the flexibility coefficient. A unit force at the point 1 causes the displacement \( \delta_{11} \) at the point 1. At the point 1 the inertial force \( f_I = -m \ddot{x}_1 \) is acting, and the external force \( f(t) \) is acting at the point 2. \( x_1(t) \) is composed of contributions from both \( f(t) \) and \( f_I(t) \). Hence

\[
x_1(t) = \delta_{11} f_I(t) + \delta_{12} f(t) \Rightarrow
\]

\[
m \ddot{x}_1 + \frac{1}{\delta_{11}} x_1 = \frac{\delta_{12}}{\delta_{11}} f(t)
\]

(2-160)

Generally, if the system is excited by the directly acting force \( f_I(t) \) at the mass in point 1 and the indirectly acting forces \( f_2(t), \ldots, f_n(t) \) at the points 2, \ldots, \( n \) one has

\[
x_1(t) = \delta_{11} (-m \ddot{x}_1 + f_I(t)) + \sum_{j=2}^{n} \delta_{1j} f_j(t) \Rightarrow
\]

\[
m \ddot{x}_1 + \frac{1}{\delta_{11}} x_1 = f_I(t) + \frac{1}{\delta_{11}} \sum_{j=2}^{n} \delta_{1j} f_j(t)
\]

(2-161)

(2-160) and (2-161) have then both been reduced to the standard form (2-32).

For a number of statiscal determinate Bernouilli-Euler beam structures with constant bending stiffness the flexibility coefficient \( \delta_{ij} \) have been indicated in appendix B.
3. VIBRATIONS OF MDOF SYSTEMS

The motion of a system with \( n \) degrees of freedom is specified by \( n \) coordinates. The system may be interpreted as an assembly of concentrated masses or distributed infinitely stiff bodies connected by a massless linear elastic medium. The description of the motion of each of the concentrated masses requires 1, 2 or 3 coordinates depending on the number of geometrical constraints. For distributed masses 1, 2 or 3 coordinates are required for the determination of the position of the mass centre of gravity and additionally 1, 2 or 3 coordinates are required for the determination of rotations around this centre. At this point it should be noticed that the described system is only one of several realizations of a MDOF system, which has merely been adopted because it allows a simple generalization of the SDOF case of chapter 2. In section 5.2 a general interpretation of a MDOF system is given.

3.1 Basic Equation of Motion for Forced Vibrations of Linear Viscous Damped Systems

Fig. 3-1: Forced damped vibrations of MDOF system.

Translational- and rotational degrees of freedom are both designated as \( x_j \), and the corresponding mass or mass moment of inertia is denoted \( m_j \). All degrees of freedom are measured from the static equilibrium state, so possible static forces disappear from the dynamic equations of motion, see fig. 3-1. For distributed masses the translational degree of freedom are referred to the mass centre of gravity, and \( m_j \) then signifies the mass moment of inertia around this centre.

\( f_j(t) \) denotes the external dynamic load acting on \( m_j \). Dependent on whether \( x_j \) signifies a translational or a rotational degree of freedom, \( f_j(t) \) represents a time-varying force or moment. \( f_j(t) \) is considered positive when acting in the direction of \( x_j \). Any discrete damping elements are cut free from the system, and the damping force in the \( j \)th degree of freedom, \( f_{dj}(t) \), is applied as an external load on \( m_j \). Again, depending on whether \( x_j \) is a translational or a rotational degree of freedom, \( f_{dj}(t) \) should be interpreted as a force or a moment. \( f_{dj}(t) \) is considered positive in the opposite direction of the velocity \( \dot{x}_j \) as seen in fig. 3-1, i.e. the damping force is positive whenever it is acting against the present direction of motion. In the formulation of the equations of motion d’Alembert’s principle is applied. The inertial load related to \( x_j \) becomes \( f_{ij}(t) = -m_j \ddot{x}_j(t) \), which is considered positive in the same direction as \( x_j \). \( f_{ij}(t) \) will be referred to as the inertial...
force if \( x_j \) is a translational degree of freedom, whereas it is denoted \( \text{d' Alembert moment} \) in case \( x_j \) is a rotation. The latter designation is coined in favour of the straightforward inertial moment in order to prevent misapprehensions with the concepts of mass moment of inertia and moment of inertia in bending, which are used in the text elsewhere. Assuming linearity, the deformation \( x_i \) in the \( i \)th degree of freedom is composed of a sum of deformation contributions from the external dynamic loads, \( f_j(t) \), the damping forces, \( f_{dj}(t) \), and the inertial loads \(-m_j\ddot{x}_j\) from all degrees of freedom \( x_j, j = 1, \ldots, n \)

\[
x_i = \sum_{j=1}^{n} \delta_{ij}(f_j(t) - f_{dj}(t) - m_j\ddot{x}_j)
\]

\((3-1)\) is formulated for all degrees of freedom \( x_i, i = 1, \ldots, n \). \( \delta_{ij} \) signifies the flexibility coefficient for the translational or rotational degree of freedom \( x_i \) due to a static unit force or moment \( f_j = 1 \) acting in the direction of the degree of freedom \( x_j \). From the \textit{Maxwell reciprocal theorem} it follows that

\[
\delta_{ij} = \delta_{ji}
\]

\((3-2)\)

\((3-1)\) may be written in the following matrix form

\[
x = D(f(t) - f_d(t) - M\ddot{x})
\]

\((3-3)\)

\[
x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad f(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}, \quad f_d(t) = \begin{bmatrix} f_{d1}(t) \\ \vdots \\ f_{dn}(t) \end{bmatrix}
\]

\((3-4)\)

\[
M = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}
\]

\((3-5)\)

\[
D = \begin{bmatrix} \delta_{11} & \delta_{12} & \cdots & \delta_{1n} \\ \delta_{21} & \delta_{22} & \cdots & \delta_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{n1} & \delta_{n2} & \cdots & \delta_{nn} \end{bmatrix}
\]

\((3-6)\)

As seen the \textit{mass matrix} \( M \) is diagonal. This is a consequence of referring the translational degrees of freedom to the mass centre of gravity of the stiff bodies. If these degrees of freedom are referred to some other point within the stiff body, the mass matrix will generally contain off-diagonal terms.
\( \mathbf{D} \) as given by (3-6) is denoted the \textit{flexibility matrix}. As a consequence of (3-2) it is a symmetric matrix, which means

\[
\mathbf{D} = \mathbf{D}^T \tag{3-7}
\]

The upper index \( T \) in (3-7) signifies transpose. (3-3) is written in the following form

\[
\mathbf{D}\dot{\mathbf{x}} + \mathbf{x} = \mathbf{D}(\mathbf{f}(t) - \mathbf{f}_d(t)) \tag{3-8}
\]

The potential energy \( U \) from an arbitrary static load \( \mathbf{f} \in \mathbb{R}^n \) becomes

\[
U = \frac{1}{2} \mathbf{f}^T \mathbf{D}\mathbf{f} \tag{3-9}
\]

Since \( U \) is positive for all \( \mathbf{f} \neq 0 \), (3-9) implies that \( \mathbf{D} \) is positive definite. In turn this means that \( \mathbf{D} \) has an inverse, \( \mathbf{K} \), which is denoted the \text{stiffness matrix}

\[
\mathbf{K} = \mathbf{D}^{-1} \tag{3-10}
\]

(3-7) implies that \( \mathbf{K} \) is a symmetric matrix as well. Actually, \( \mathbf{K}\mathbf{D} = \mathbf{I} \Rightarrow \mathbf{D}^T\mathbf{K}^T = \mathbf{I}^T \Rightarrow \mathbf{D}\mathbf{K} = \mathbf{I} \Rightarrow \mathbf{K}^T = \mathbf{D}^{-1} = \mathbf{K} \), where \( \mathbf{I} \) signifies the identity matrix. Premultiplication of (3-8) by \( \mathbf{D}^{-1} = \mathbf{K} \) gives the following formulation of the equations of motion

\[
\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}(t) - \mathbf{f}_d(t) \tag{3-11}
\]

Introduction of \( \mathbf{f} = \mathbf{K}\mathbf{x} \) in (3-9) provides

\[
\forall \mathbf{x} \in \mathbb{R}^n : |\mathbf{x}| > 0 \Rightarrow U = \frac{1}{2} \mathbf{x}^T \mathbf{K}^T \mathbf{D}\mathbf{x} = \frac{1}{2} \mathbf{x}^T \mathbf{K}\mathbf{x} > 0 \tag{3-12}
\]

Hence, \( \mathbf{K} \) is also a positive definite matrix. The previous statements on the definite properties of the matrices \( \mathbf{D} \) and \( \mathbf{K} \) presume that the structure is well supported against stiffbody motions. For certain structures, e.g. ships and airplanes, this is not the case. In such cases (3-11) is still valid. However, \( \mathbf{K} \) is now singular with a rank \( n - n_s \), where \( n_s \) is the number of independent stiffbody modes, and \( \mathbf{K} \) is merely \textit{positive semi-definite}. In contrast \( \mathbf{D} \), and hence the formulation (3-8), does not exists in this case.

From (3-11) it follows that the \( j \)th column in \( \mathbf{K} \) may be determined as the static load vector \( \mathbf{f} \) on the masses, which causes the states of deformation

\[
x_i = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} \tag{3-13}
\]
Example 3-1: Flexibility Matrix for Three-Storey Frame

The storey beams of the three-storey building shown in fig. 3-2 are assumed to be infinitely stiff. The columns are Bernoulli-Euler beams, all with constant bending stiffness $EI$. The system has 3 degrees of freedom, which are selected as the horizontal displacements of the storey beams from the static equilibrium state.

The 1st column of the flexibility matrix $D$ is determined as the deformation along the degrees of freedom $x_1, x_2, x_3$, if a force $f_1 = 1$ is applied at $x_1$. In this case the shear force is 1 between the 3rd and 2nd storey, between the 2nd and 1st storey, and between the 1st storey and the support. Then, the 3rd storey is distorted $\frac{a^3}{24EI}$ relative to the 2nd storey, the 2nd storey is distorted $\frac{a^3}{6EI}$ relative to the 1st storey, and the 1st storey is distorted $\frac{a^3}{6EI}$ relative to the support. Hence

\[
\begin{align*}
\delta_{11} &= \frac{a^3}{24EI} + \frac{a^3}{24EI} + \frac{a^3}{6EI} = \frac{6a^3}{24EI} \\
\delta_{21} &= \frac{a^3}{24EI} + \frac{a^3}{6EI} = \frac{5a^3}{24EI} \\
\delta_{31} &= \frac{a^3}{6EI} = \frac{4a^3}{24EI}
\end{align*}
\]
The 2nd column of the flexibility matrix is determined as the deformation along the degrees of freedom $x_1, x_2, x_3$, if a unit force $f_2 = 1$ is applied at $x_2$. In this case the shear force is 0 between the 3rd and 2nd storey, and 1 between the 2nd and 1st storey, and between the 1st storey and the support. Then, the 3rd storey is not distorted relative to the 2nd storey, the 2nd storey is distorted $\frac{a^3}{6EI}$ relative to the 1st storey, and the 1st storey is distorted $\frac{a^3}{6EI}$ relative to the support. Hence

$$
\delta_{12} = \delta_{22} = \frac{a^3}{24EI} = \frac{a^3}{6EI} = \frac{5}{24 EI}
$$

$$
\delta_{32} = \frac{a^3}{6EI} = \frac{4}{24 EI}
$$

In the same way it is seen that

$$
\delta_{13} = \delta_{23} = \delta_{33} = \frac{4}{24 EI}
$$

Then

$$
D = \frac{a^3}{24EI} \begin{bmatrix}
6 & 5 & 4 \\
5 & 5 & 4 \\
4 & 4 & 4
\end{bmatrix} \tag{3 - 14}
$$

Example 3-2: Stiffness Matrix for Three-Storey Frame

The stiffness matrix of the plane frame described in example 3-1 may be determined analysing the system shown in fig. 3-3. The unknown external forces $K_{ij}$ are applied to the free masses together with the elastic restoring forces in order to produce the relevant deformation states and to insure static
equilibrium, which leads to
\[
\begin{align*}
K_{11} &= 24 \frac{EI}{a^3}, & K_{12} &= -24 \frac{EI}{a^3}, & K_{13} &= 0 \\
K_{21} &= -24 \frac{EI}{a^3}, & K_{22} &= 48 \frac{EI}{a^3}, & K_{23} &= -24 \frac{EI}{a^3} \\
K_{31} &= 0, & K_{32} &= -24 \frac{EI}{a^3}, & K_{33} &= 30 \frac{EI}{a^3}
\end{align*}
\]

\[K = \frac{EI}{a^3} \begin{bmatrix}
24 & -24 & 0 \\
-24 & 48 & -24 \\
0 & -24 & 30
\end{bmatrix}\]  \hspace{1cm} (3-15)

Scalar multiplication of (3-11) with \( \dot{x} \) provides
\[
\dot{x}^T (M\ddot{x} + Kx) = \dot{x}^T f(t) - \dot{x}^T f_d(t) \quad \Rightarrow
\]
\[
\frac{d}{dt} (T + U) = \dot{x}^T f(t) - \dot{x}^T f_d(t)
\]  \hspace{1cm} (3-16)

\[
T = \frac{1}{2} \dot{x}^T M \dot{x}
\]  \hspace{1cm} (3-17)

\( f_i(t)\dot{x}_i dt = f_i(t)dx_i \) signifies the work performed by the external load component \( f_i \) on the mass or mass moment of inertia \( m_i \) during the differential time interval \( dt \). Consequently, \( \dot{x}^T f = \sum_{i=1}^n \dot{x}_i f_i \) represents the total work from all external dynamic loads on the system per unit time. In the same way \( -\dot{x}^T f_d = -\sum_{i=1}^n \dot{x}_i f_{di} \) indicates the total work from the damping forces on the system per unit time. The negative sign is brought forward because the damping forces are considered positive in the opposite direction of the velocities.

The right-hand side of (3-16) represents the power supply or the network per unit time performed on the system. This must balance the increase per unit time of the mechanical energy, \( T + U \), of the system as indicated on the left-hand side of (3-16), where the kinetic energy \( T \) is given by (3-17), and the potential energy \( U \) is given by (3-12). At the derivation of (3-16) the mass matrix \( M \) has been assumed to be symmetric, i.e.
\[
M = M^T
\]  \hspace{1cm} (3-18)

On the other hand the energy equation as formulated by (3-16) only makes sense, when (3-18) is fulfilled. Since \( T \) is positive for all \( \dot{x} \neq 0 \), it follows that \( M \) is positive definite.

Temporarily, assume that \( f(t) = 0 \). Then, (3-16) can be written
\[
\dot{x}^T f_d(t) = -\frac{d}{dt} (T + U)
\]  \hspace{1cm} (3-19)

The right-hand side of (3-19) represents the loss of mechanic energy per unit time. Consequently, \( \dot{x}^T f_d(t) \) is equal to the energy, which is dissipated as heat per unit time.
For any real system performing eigenvibration corresponding to $f(t) \equiv 0$ the mechanical energy will be continuously decreasing (i.e. $\frac{d}{dt}(T + U) < 0$), for which reason the following inequality is valid at all times

$$\forall \dot{x} \in \mathbb{R}^n : |\dot{x}| > 0 \Rightarrow \dot{x}^T f_d(t) > 0$$

(3 - 20)

The system of damping forces $f_d(t)$ is denoted dissipative, if (3-20) is fulfilled. In order to solve the equation of motion (3-11) a constitutive condition has to be formulated for the damping forces $f_d$. In its most general form $f_d$ may depend explicitly on the displacement $x$, the velocity $\dot{x}$ and the time $t$

$$f_d = f_d(x, \dot{x}, t)$$

(3 - 21)

The simplest possible form of the relationship (3-21) is provided by the linear viscous damping

$$f_d = C\dot{x}$$

(3 - 22)

$C$ is denoted the damping matrix. This may depend on the time corresponding to the explicit time dependency indicated in the general model (3-21). The damping load component $f_{di}(t)$ acting on $m_i$ generally depends not only on the velocity $\dot{x}_i$ of this mass, but also on the velocity of other masses, which means that $C$ is not a diagonal matrix. Further, in contrast to the mass matrix $M$ and the stiffness matrix $K$, the damping matrix $C$ is generally neither symmetric nor positive definite.

From (3-20) and (3-22) it follows that a linearly viscous damping model is dissipative, if and only if the damping matrix is positive definite

$$\forall \dot{x} \in \mathbb{R}^n : |\dot{x}| > 0 \Rightarrow \dot{x}^T C\dot{x} > 0$$

(3 - 23)

Example 3-3: Damping Model for Cantilever Beam

The damping forces for the system shown in fig. 3-1 become

$$f_{d1} = c_0(\dot{x}_1 - \dot{x}_2)
\begin{align*}
f_{d2} = c_0(\dot{x}_2 - \dot{x}_1) \\
f_{d3} = c_3\dot{x}_3 \\
f_{d4} = 0
\end{align*}
\Rightarrow
f_d = C\dot{x}$$

(3 - 24)

$$C = \begin{bmatrix}
c_0 & -c_0 & 0 & 0 \\
-c_0 & c_0 & 0 & 0 \\
0 & 0 & c_3 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}$$

(3 - 25)
As seen the damping model is linearly viscous. The following velocity vectors are considered

\[
\begin{align*}
\dot{x} &= \dot{x}_1 = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ 0 \\ 0 \end{bmatrix} \Rightarrow f_d = 0 \\
\dot{x} &= \dot{x}_4 = \begin{bmatrix} 0 \\ 0 \\ \dot{x}_3 \\ 0 \end{bmatrix} \Rightarrow f_d = 0 \\
\dot{x} &= a\dot{x}_1 + b\dot{x}_4 \Rightarrow f_d = 0
\end{align*}
\]

(3-26)

where \( \dot{x}_1 \) and \( \dot{x}_4 \) are non-zero quantities. \( a \) and \( b \) are constants, so the last velocity vector represents an arbitrary linear combination of the first two. (3-26) shows that non-zero velocity vectors may exist, which will not cause dissipation in the system. This defect is caused by the present damping matrix being only positive semi-definite, i.e.

\[
\forall x \in \mathbb{R}^n: |x| > 0 \Rightarrow x^T C x \geq 0
\]

(3-27)

\( C \) can be written as a sum of symmetric matrix \( C_s \) and an anti-symmetric matrix \( C_a \)

\[
C = C_s + C_a
\]

(3-28)

\[
C_s = \frac{1}{2} (C + C^T)
\]

(3-29)

\[
C_a = \frac{1}{2} (C - C^T)
\]

(3-30)

Since \( \dot{x}^T C_a \dot{x} = \frac{1}{2} \dot{x}^T C \dot{x} - \frac{1}{2} \dot{x}^T C^T \dot{x} = 0 \), it then follows that

\[
\forall \dot{x} \in \mathbb{R}^n: \dot{x}^T C \dot{x} = \dot{x}^T C_s \dot{x}
\]

(3-31)

Hence, \( C \) is a dissipative damping matrix, if and only if \( C_a \) is positive definite. Theorems from linear algebra state that, since \( C_s \) is symmetric, it has real eigenvalues, and it will be positive definite, if all these eigenvalues are positive. \( C \) is then a dissipative damping matrix, if all eigenvalues of \( C_s \) are positive. From (3-22) and (3-28) it follows that the damping force can be written

\[
f_d = f_{ds} + f_{da}
\]

(3-32)

\[
f_{ds} = C_s \dot{x}
\]

(3-33)

\[
f_{da} = C_a \dot{x}
\]

(3-34)

Since \( \dot{x}^T f_{da} = \dot{x}^T C_a \dot{x} = \frac{1}{2} (\dot{x}^T C \dot{x} - \dot{x}^T C^T \dot{x}) = 0 \), it follows that \( \dot{x} \) and \( f_{da} \) are orthogonal vectors, so \( f_{da} \) does not perform any work on the system. Hence, the dissipative
work is solely related to the component $f_{ds}$. Non-zero damping forces orthogonal to the velocity vector such as $f_{ds}(t)$ are denoted gyroscopic forces.

The equation of motion is obtained upon insertion of (3-22) into (3-11)

$$M\ddot{x} + C\dot{x} + Kx = f(t), \quad t > 0$$  \hspace{1cm} (3-35)

(3-35) is solved with respect to the initial conditions

$$x(0) = x_0, \quad \dot{x}(0) = \dot{x}_0$$  \hspace{1cm} (3-36)

(3-35), (3-36) are the basic equations of motion for forced vibrations of a linear viscous damped MDOF system.

**Example 3-4: Basic Equations for 2 DOF System**

![Diagram of a 2 DOF system](image)

Fig. 3-4: Forced vibrations of a linear viscous damped 2 DOF system.

The system has two degrees of freedom, which are selected as the displacements $x_1, x_2$ of the masses from the static equilibrium state. The masses are cut free from the springs and damping elements, and the spring forces and damping forces in the dynamically deformed state are applied as external forces on the masses with the sign shown in fig. 3-4. Newton’s 2nd law of motion is formulated for each of the masses, which leads to the following equation of motion

$$\begin{align*}
    m_1\ddot{x}_1 &= -k_1x_1 + k_2(x_2 - x_1) - c_1\dot{x}_1 + c_2(\dot{x}_2 - \dot{x}_1) + f_1(t) \\
    m_2\ddot{x}_2 &= -k_3x_2 - k_2(x_2 - x_1) - c_3\dot{x}_2 - c_2(\dot{x}_2 - \dot{x}_1) + f_2(t)
\end{align*}$$  \hspace{1cm} (3-37)

(3-37) may be written in the following matrix form

$$M\ddot{x} + C\dot{x} + Kx = f(t)$$  \hspace{1cm} (3-38)

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}$$

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix}, \quad K = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}$$  \hspace{1cm} (3-39)
As seen the system is linearly viscous damped. The damping matrix is symmetric and has positive eigenvalues, so the system is dissipative. Notice, that although the analysed system may appear rather specialised, the obtained equations (3-39), (3-40) indicate the equations of motion of an arbitrary 2 DOF system upon specific identification of the parameters $m_1, m_2, k_1, k_2, k_3, c_1, c_2, c_3$. An example of such an identification has been given in example 3-6 below.

### 3.2 Eigenvibrations of Undamped Systems

The differential equation for undamped eigenvibrations of a MDOF system follows from (3-35), (3-36) for $C = 0$, $f(t) \equiv 0$

$$
\begin{align*}
M \ddot{x} + Kx &= 0 , \quad t > 0 \\
x(0) &= x_0 , \quad \dot{x}(0) = \dot{x}_0
\end{align*}
$$

(3-40)

Solution to (3-40) is sought in the form

$$
x(t) = \text{Re}(\Phi e^{i\omega t})
$$

(3-41)

(3-41) signifies a harmonic motion if $\omega$ is real, cf. (1-3). This will be shown later. For the time being, $\omega$ is merely considered an unknown complex parameter, and $\Phi$ an unknown complex amplitude vector.

Insertion of (3-41) into (3-40) and use of the same arguments as applied in relation to (2-60) provide

$$
\forall t \in R : \text{Re}\left(( - \omega^2 M + K)\Phi e^{i\omega t}\right) = 0 \Rightarrow
$$

$$(K - \omega^2 M)\Phi = 0
$$

(3-42)

(3-42) represents a homogeneous system of $n$ linear equations for the determination of $\Phi$. A necessary condition for non-trivial solutions $\Phi \neq 0$ is that the determinant of the coefficient matrix is equal to 0, i.e.

$$
\text{det} (K - \omega^2 M) = 0
$$

(3-43)

(3-43) represents a polynomial equation of $n$th degree in $\omega^2$, which is called the characteristic equation or the frequency condition. The left-hand side is called the characteristic polynomial. The roots of this equation, which forms the eigenvalues of (3-43), are denoted $\omega_1^2, \omega_2^2, \ldots, \omega_n^2$. For each of the roots $\omega_1^2, \omega_2^2, \ldots, \omega_n^2$ a non-trivial solution $\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(n)}$ exists to (3-42). These solutions are denoted the undamped eigenmodes of the system defined by the mass matrix $M$ and the stiffness matrix $K$.

**Theorem 3-1:** If $M$ and $K$ are both symmetric matrices, and if either $M$ or $K$ is positive definite, then the roots of the characteristic equation and the associated eigenmodes are all real.
Proof: Assume that e.g. $M$ is positive definite. (3-42) is formulated for the $k$th eigenmode, i.e.

$$K\Phi^{(k)} = \omega^2_k M \Phi^{(k)}$$

(3 - 44)

Complex conjugation of (3-44) then provides

$$K\Phi^{(k)*} = (\omega^2_k)^* M \Phi^{(k)*}$$

(3 - 45)

where the symbol $*$ applied as an upper index denotes complex conjugation. Further, the rule $(ab)^* = a^*b^*$ for complex conjugation of the product of two complex numbers $a$ and $b$ has been applied. Notice that $M$ and $K$ are real matrices, so e.g. $M = M^*$. From (3-45) it follows that if $(\omega^2_k, \Phi^{(k)})$ is a solution to (3-42), then $((\omega^2_k)^*, (\Phi^{(k)*}))$ is another solution, which will be different if the solutions are complex. Hence, the solutions to (3-43) are either real or complex conjugates in pair.

Scalar multiplication of (3-44) with $(\Phi^{(k)*})^T$, and scalar multiplication of (3-45) with $\Phi^{(k)T}$ provides the following identities

$$((\Phi^{(k)*})^T)K\Phi^{(k)} = \omega^2_k((\Phi^{(k)*})^T)M\Phi^{(k)}$$

(3 - 46)

$$\Phi^{(k)T}K\Phi^{(k)*} = (\omega^2_k)^*\Phi^{(k)T}M\Phi^{(k)*}$$

(3 - 47)

(3-47) is transposed. From $K^T = K$ and $M^T = M$ it follows that

$$((\Phi^{(k)*})^T)K\Phi^{(k)} = (\omega^2_k)^*(\Phi^{(k)*})^TM^T\Phi^{(k)}$$

$$\Phi^{(k)*}K\Phi^{(k)} = (\omega^2_k)^*(\Phi^{(k)*})^TM\Phi^{(k)}$$

(3 - 48)

Upon subtraction of (3-48) from (3-46) the following identity is provided

$$((\omega^2_k)^* - (\omega^2_k))((\Phi^{(k)*})^T)M\Phi^{(k)} = 0$$

(3 - 49)

(3-49) can only be fulfilled if at least one of the following identities is fulfilled

$$\omega^2_k = (\omega^2_k)^*$$

(3 - 50)

$$((\Phi^{(k)*})^T)M\Phi^{(k)} = 0$$

(3 - 51)

First it is assumed that (3-50) is not fulfilled, i.e. $\omega^2_k \neq (\omega^2_k)^*$, which means that $\omega^2_k$ is complex. It is then necessary that (3-51) is fulfilled. Since it has been assumed that $\omega^2_k \neq (\omega^2_k)^*$ the eigenmode $\Phi^{(k)}$ is complex too, i.e. $\Phi^{(k)} = a + ib$, where $a$ og $b$ are real vectors. Then

$$(\Phi^{(k)*})^T M \Phi^{(k)} = (a - ib)^T M (a + ib) =$$

$$a^TMa + b^TMb + i(a^TMb - b^TMa) =$$

$$a^TMa + b^TMb > 0$$

(3 - 52)

The imaginary part in the second last statement cancels, since $b^TMa = (b^TMa)^T = a^TM^Tb = a^TMb$. The last statement of (3-52) follows because of the premise that $M$ is positive definite. Since (3-51) can never be fulfilled it is then necessary that (3-50) is valid, i.e. $\omega^2_k$ is real. Then the coefficient matrix in (3-42) is real for which reason the solution $\Phi^{(k)}$ must be real as well.
Alternatively, if $K$ is positive definite whereas $M$ does not fulfil any definite properties, (3-44) is reformulated as $\lambda_k K \Phi^{(k)} = M \Phi^{(k)}$, $\lambda_k = 1/\omega_k^2$. It is then proven in the same way that $(\lambda_k, \Phi^{(k)})$ and hence $(\omega_k^2, \Phi^{(k)})$ are real.

Especially, if $M = I$ the general eigenvalue problem (3-42) reduces to the conventional eigenvalue problem for the matrix $K$. Since the unit matrix $I$ is symmetric and positive definite the theorem in this case states that a symmetric real matrix (with arbitrary definite properties) has real eigenvalues, a well-known result from matrix algebra.

**Theorem 3-2:** If $M$ and $K$ are both symmetric and positive definite matrices, then all eigenvalues are positive.

**Proof:** From (3-44) it follows that

$$\omega_k^2 = \frac{\Phi^{(k)T} K \Phi^{(k)}}{\Phi^{(k)T} M \Phi^{(k)}}, \quad k = 1, 2, \ldots, n \quad (3-53)$$

Both the numerator and the denominator of the fraction on the right-hand side of (3-52) are positive, since $K$ and $M$ are positive definite. It then follows that $\omega_k^2 > 0$.

Since $\omega_k^2 > 0$ it follows that both $\omega_k$ and $\Phi^{(k)}$ are real. Then, (3-41) can be written

$$x(t) = \Phi^{(k)} \cos(\omega_k t), \quad k = 1, \ldots, n \quad (3-54)$$

(3-54) specifies a harmonic motion with all components in phase, cf. (1-1), which is denoted the eigenvibration in the $k$th mode, and $\omega_k$ is the $k$th undamped circular eigenfrequency. In what follows the eigenmodes are assumed to be ordered according to ascending circular eigenfrequencies. i.e. $\omega_1 \leq \omega_2 \leq \cdots \leq \omega_n$, unless otherwise stated.

The eigenvalue problem can alternatively be formulated in terms of the flexibility matrix $D = K^{-1}$. (3-42) is premultiplied by $-\omega_k^{-2} K^{-1}$ which leads to conventional eigenvalue problem for the matrix $DM$

$$(DM - \omega_k^{-2} I) \Phi^{(k)} = 0 \quad (3-55)$$

(3-55) determines the same eigenmodes $\Phi^{(k)}$ as (3-42) and its eigenvalues are $\omega_k^{-2}$. If (3-55) is multiplied by $DM$, the following eigenvalue problem for $(DM)^2 = (DM)(DM)$ is obtained using (3-55)

$$

\left( (DM)^2 - \omega_k^{-2} DM \right) \Phi^{(k)} = \left( (DM)^2 - \omega_k^{-4} I \right) \Phi^{(k)} = 0 \quad (3-56)

$$

Premultiplication by $DM$ and use of the derived equation for the case $m - 1$ provide the following sequence of eigenvalue problems for the matrix products $(DM)^m = (DM) \cdots (DM)$

$$

\left( (DM)^m - \omega_k^{-2m} I \right) \Phi^{(k)} = 0, \quad m = 1, 2, \ldots \quad (3-57)

$$
If (3-55) is premultiplied by $M$ and next multiple times premultiplied by $MK^{-1}$, the following sequence of generalized eigenvalue problem is obtained for the matrices $(MK^{-1})^m M = (MK^{-1}) \cdots (MK^{-1}) M$ and $M$

$$(MK^{-1} M - \omega_k^{-2} M) \Phi^{(k)} = 0 \quad \Rightarrow$$

$$(MK^{-1})^2 M - \omega_k^{-2} MK^{-1} M) \Phi^{(k)} = \left( (MK^{-1})^2 M - \omega_k^{-4} M \right) \Phi^{(k)} = 0 \quad \Rightarrow$$

$$(MK^{-1})^m M - \omega_k^{-2 m} M) \Phi^{(k)} = 0 \quad , \quad m = 1, 2, \ldots \quad (3-58)$$

(3-58) is proved by induction using the same argument as leading to (3-57).

In (3-42) the identity $K = KM^{-1} M$ is used. Multiple premultiplications by $KM^{-1}$ then lead to the following sequence of generalized eigenvalue problems for the matrices $(KM^{-1})^m M = (KM^{-1}) \cdots (KM^{-1}) M$ and $M$

$$(KM^{-1} M - \omega_k^2 M) \Phi^{(k)} = 0 \quad \Rightarrow$$

$$(KM^{-1})^2 M - \omega_k^2 KM^{-1} M) \Phi^{(k)} = \left( (KM^{-1})^2 M - \omega_k^4 M \right) \Phi^{(k)} = 0 \quad \Rightarrow$$

$$(KM^{-1})^m M - \omega_k^{2 m} M) \Phi^{(k)} = 0 \quad , \quad m = 1, 2, \ldots \quad (3-59)$$

For $m = -1, -2, \ldots$ the following definition is applied $(KM^{-1})^m = (MK^{-1})^{-1} \cdots (MK^{-1})^{-1}$. Since $KM^{-1} MK^{-1} = I \Rightarrow (KM^{-1})^{-1} = MK^{-1}$ the eigenvalue problems (3-58) and (3-59) can then be combined in the following equation

$$\left( (KM^{-1})^m M - \omega_k^{2 m} M \right) \Phi^{(k)} = 0 \quad , \quad m = 0, \pm 1, \pm 2, \ldots \quad (3-60)$$

Negative values of $m$ in (3-60) indicate the eigenvalue problem (3-58). In (3-60) the definition $(KM^{-1})^0 = I$ has been applied for the case $m = 0$. The eigenvalue problems (3-60) all determine the same eigenvectors $\Phi^{(k)}$ as (3-42), whereas the eigenvalues are $\omega_k^{2 m}$.

Notice, that the matrix $K_m = (KM^{-1})^m M$ is symmetric for all values of $m$. As an example, for $m = -2$ one has that $K_{-2}^T = ((KM^{-1})^{-2} M)^T = (MK^{-1} MK^{-1} M)^T = M^T (K^{-1})^T M^T (K^{-1})^T M^T = (MK^{-1}) (MK^{-1}) M = (KM^{-1})^{-1} (KM^{-1})^{-1} M = K_{-2}$. (3-60) is used in section 3.9 on damping models.

If $\Phi^{(k)}$ is a solution to (3-41), then $c \Phi^{(k)}$ is another solution, where $c$ is an arbitrary constant. Hence, the eigenmodes can only be determined within an undetermined common factor to all components. This means that an arbitrary component in $\Phi^{(k)}$ can be selected as 1. Then the remaining $n - 1$ components are determined by choosing $n - 1$ of the $n$ equation (3-42) which will determine the remaining $n - 1$ components of the
eigenmode. There are \( n \) different choices for these \( n - 1 \) equations, which will all determine the same solution, as a consequence of the eigenvalues fulfilling the characteristic equation. This point is illustrated in the following example 3-5.

**Example 3-5: Eigenvibrations of 2 DOF System**

Circular eigenfrequencies and eigenmodes of the system defined in example 3-4 follow from (3-39) and (3-42)

\[
\begin{bmatrix}
  k_1 + k_2 - \omega^2 m_1 & -k_2 \\
  -k_2 & k_2 + k_3 - \omega^2 m_2
\end{bmatrix}
\begin{bmatrix}
  \Phi_1 \\
  \Phi_2
\end{bmatrix}
= \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\]  

(3 - 61)

The characteristic equation (3-43) becomes

\[
m_1 m_2 \omega^4 - \left( m_1 (k_2 + k_3) + m_2 (k_1 + k_2) \right) \omega^2 + k_1 k_2 + k_1 k_3 + k_2 k_3 = 0 \quad \Rightarrow
\]

\[
\omega_1^2, \omega_2^2 \quad \frac{1}{2} \left( \frac{k_1 + k_2}{m_1} + \frac{k_2 + k_3}{m_2} \pm \sqrt{\left( \frac{k_1 + k_2}{m_1} - \frac{k_2 + k_3}{m_2} \right)^2 + 4 \frac{k_2^2}{m_1 m_2}} \right)
\]  

(3 - 62)

If \( \omega^2 = \omega_1^2 \) or \( \omega^2 = \omega_2^2 \) is inserted into (3-61), the associated eigenmodes \( \Phi^{(1)} \) and \( \Phi^{(2)} \) are determined. Setting the component \( \Phi^{(2)}_1 = 1 \), (3-61) provides the following equations for the determination of the component \( \Phi^{(i)}_1 \)

\[
(k_1 + k_2 - \omega^2 m_1) \Phi^{(i)}_1 - k_2 1 = 0 \quad \Rightarrow \quad \Phi^{(i)}_1 = \frac{k_2}{k_1 + k_2 - \omega^2 m_1}
\]

\[
-k_2 \Phi^{(i)}_1 + (k_2 + k_3 - \omega^2 m_2) 1 = 0 \quad \Rightarrow \quad \Phi^{(i)}_1 = \frac{k_2 + k_3 - \omega^2 m_2}{k_2}
\]

(3 - 63)

The two solutions indicated in (3-63), resulting from choosing the first and the second equation respectively, will be identical if

\[
\frac{k_2}{k_1 + k_2 - \omega_1^2 m_1} = \frac{k_2 + k_3 - \omega_1^2 m_2}{k_2} \quad \Rightarrow
\]

\[
(k_1 + k_2 - \omega_1^2 m_1) (k_2 + k_3 - \omega_1^2 m_2) - k_2^2 = 0
\]  

(3 - 64)

The left-hand side of (3-64) is the characteristic polynomial in the first statement of (3-62). (3-64) is then fulfilled since \( \omega_1^2 \) is an eigenvalue. Hence, the eigenmodes become

\[
\Phi^{(i)} = \begin{bmatrix}
  \frac{k_2}{k_1 + k_2 - \omega_1^2 m_1} \\
  1
\end{bmatrix}, \quad i = 1, 2
\]  

(3 - 65)
From (3-65) it follows that
\[
\begin{align*}
\Phi^{(1)} &= \begin{bmatrix} \sqrt{3} - 1 \\ 1 \end{bmatrix} \\
\Phi^{(2)} &= \begin{bmatrix} -\sqrt{3} - 1 \\ 1 \end{bmatrix}
\end{align*}
\] (3-66)

The eigenmodes have been sketched in fig. 3-5.

**Example 3-6: Eigenvibrations of Mathematical Double Pendulum**

![Diagram of Mathematical Double Pendulum]

Fig. 3-6: Mathematical double pendulum.
Two mathematical pendulums are coupled through a weak horizontal spring $k$, which is undeformed when the pendulums are in the vertical position. The position of the pendulums is determined by the rotational angles $\theta_1$ and $\theta_2$, which are considered positive in the counter-clockwise direction.

It is assumed that $|\theta| \ll 1 \Rightarrow \sin \theta \simeq \theta, \cos \theta \simeq 1$, which will result in linear equations of motion. The pendulums are cut free from the spring, and the internal spring force $ka(\theta_2 - \theta_1)$ along with the inertial forces $-ml\dot{\theta}_1$ and $-ml\dot{\theta}_2$ are applied as external forces on the pendulums with the signs shown in fig. 3-6.

According to d’Alembert’s principle the equations of motion are obtained by expressing the static moment equilibrium around the support points, which will eliminate the unknown reaction forces from the equations of motion. These become

$$\begin{align*}
-ml^2\ddot{\theta}_1 - mgl\theta_1 + ka^2(\theta_2 - \theta_1) &= 0 \\
-ml^2\ddot{\theta}_2 - mgl\theta_2 - ka^2(\theta_2 - \theta_1) &= 0
\end{align*}$$

(3-68) can be written in the following matrix form

$$\begin{bmatrix}
ml^2 & 0 \\
0 & ml^2
\end{bmatrix} \begin{bmatrix}
\ddot{\theta}_1 \\
\ddot{\theta}_2
\end{bmatrix} + \begin{bmatrix}
mgl + ka^2 & -ka^2 \\
-ka^2 & mgl + ka^2
\end{bmatrix} \begin{bmatrix}
\theta_1 \\
\theta_2
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \end{bmatrix}$$

(3-69)

The mass- and stiffness matrices of (3-69) are of the type (3-39). The results of example 3-5 can directly be transferred using the parameter identification

$$\begin{align*}
k_1 &= mgl, \ k_2 = ka^2, \ k_3 = mgl \\
m_1 &= ml^2, \ m_2 = ml^2
\end{align*}$$

(3-70)

Circular eigenfrequencies then follow from (3-62)

$$\begin{align*}
\omega_1^2 &= \frac{\xi}{\xi + 2\frac{k}{m}a^2} \\
\omega_2^2 &= \frac{\xi}{\xi + 2\frac{k}{m}a^2}
\end{align*}$$

(3-71)

The eigenmodes follows from (3-65)

$$\begin{align*}
\Phi^{(1)} &= \begin{bmatrix}
\frac{ka^2}{mgl + ka^2 - \frac{4l^2}{m}} \\
\frac{-4l^2}{m}
\end{bmatrix} = \begin{bmatrix} 1 \\
1 \end{bmatrix} \\
\Phi^{(2)} &= \begin{bmatrix}
\frac{ka^2}{mgl + ka^2 - \left(\frac{4l^2}{m} + \frac{k}{m}a^2\right)} \\
\frac{-4l^2}{m}
\end{bmatrix} = \begin{bmatrix} -1 \\
1 \end{bmatrix}
\end{align*}$$

(3-72)

$\Phi^{(1)}$ and $\Phi^{(2)}$ correspond to the masses moving in phase and counterphase, respectively. Of course the indicated eigenmodes could have been obtained from the symmetry of the system alone without any calculation. During vibrations in the first mode the spring remains undeformed. The circular eigenfrequency is then identical to that of a single mathematical pendulum. During vibrations in the second eigenmode the spring is deformed with a node in the middle corresponding to 2 springs of magnitude $2k$ in series, fixed at the midpoint. These eigenvibrations can then be analysed by a single mathematical pendulum with a spring of magnitude $2k$ attached at the distance $a$ from the support, which will have the circular eigenfrequency $\omega_2$ given in (3-71). Use of symmetry conditions for the reduction of the number of degrees of freedom is further explained in section 3.8 on system reduction.
Example 3-7: Eigenvibrations of 3 DOF System

![Beam structure diagram]

Fig. 3-7: Beam structure with 3 degrees-of-freedom.

The Bernoulli-Euler beams AB, BC and BD of the structure shown in fig. 3-7 are placed in a horizontal plane with the beams BC and BD orthogonal to the beam AB. All beams are assumed to be massless and infinitely stiff against axial deformations. The beam AB has the constant bending stiffness $EI$ against vertical bending deformations and the St. Venant torsional stiffness $GI_t = \frac{1}{2}EI$ against torsional deformations, cf. (2-15). The beams BC and BD both have the constant bending stiffness $\frac{1}{2}EI$ against vertical deformations. At the free ends C and D and at the joint B point masses of magnitude $m$, $m$ and $2m$, respectively, are attached. Although it is not shown in fig. 3-7 the masses at C and D are supported in such a way that they can only move in the vertical direction. Only small vertical vibrations of the system are considered, and no warping deformations of the cross-sections of the beam AB are assumed. The system has three degrees of freedom, which are selected as the vertical displacement $x_1$, $x_2$ and $x_3$ of the points C, D and B, respectively. The flexibility coefficients become

\[
\begin{align*}
\delta_{11} &= \frac{1}{3} \frac{a^3}{EI} + \frac{a^3}{3EI/2} = \frac{3a^3}{EI} = \delta_{22} \\
\delta_{12} &= \frac{1}{3} \frac{a^3}{EI} - \frac{a^3}{GI_t} = \delta_{21} \\
\delta_{13} &= \frac{1}{3} \frac{a^3}{EI} = \delta_{31} = \delta_{23} = \delta_{32} = \delta_{33}
\end{align*}
\]

(3-73)

Only the evaluation of $\delta_{11}$ will be explained. A unit force of magnitude 1 applied at point C causes a vertical displacement of magnitude $\frac{1}{3} \frac{a^3}{EI}$ at that point, if point B was fixed. Now point B can move in the vertical direction, and is free to rotate around the axis of the beam AB. The vertical displacement of point B from the unit force becomes $\frac{1}{3} \frac{a^3}{EI}$, which must be added to the displacement of point C. The beam AB is exposed to a torsional moment of magnitude $a$ acting at B. The rotation at point B then becomes $\frac{\partial \delta_{11}}{\partial r_t} \cdot a = \frac{a^2}{EI}$. This rotation results in a vertical displacement at C of magnitude $\frac{\partial^2 \delta_{11}}{\partial r_t^2} \cdot a = \frac{a^3}{3EI}$, so the total displacement $\delta_{11}$ becomes as indicated in (3-73). The mass matrix $M$, the flexibility matrix $D$ and the product $DM$ then become

\[
M = \begin{bmatrix}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & 2m
\end{bmatrix}, \quad D = \frac{a^3}{3EI} \begin{bmatrix}
9 & -5 & 1 \\
-5 & 9 & 1 \\
1 & 1 & 1
\end{bmatrix} \Rightarrow
\]

\[
DM = ma^3 \begin{bmatrix}
9 & -5 & 2 \\
-5 & 9 & 2 \\
1 & 1 & 2
\end{bmatrix}
\]

(3-74)
The eigenvalue problem (3.55) may be written
\[
\begin{bmatrix}
9 - \mu_k & -5 & 2 \\
-5 & 9 - \mu_k & 2 \\
1 & 1 & 2 - \mu_k
\end{bmatrix}
\begin{bmatrix}
\Phi_1^{(k)} \\
\Phi_2^{(k)} \\
\Phi_3^{(k)}
\end{bmatrix}
= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},
\mu_k = \frac{3EI}{ma^3\omega_k^2}
\] (3.75)

The characteristic equation becomes
\[-\mu_k^3 + 20\mu_k^2 - 88\mu_k + 56 = 0 \implies
\mu_k = \begin{cases}
14, & k = 1 \\
3 + \sqrt{5}, & k = 2 \\
3 - \sqrt{5}, & k = 3
\end{cases}
\]
\[\omega_k = \begin{cases}
\frac{3}{14}\sqrt{\frac{EI}{ma^3}}, & k = 1 \\
\frac{3 + \sqrt{5}}{3}\sqrt{\frac{EI}{ma^3}}, & k = 2 \\
\frac{3 - \sqrt{5}}{3}\sqrt{\frac{EI}{ma^3}}, & k = 3
\end{cases}
\] (3.76)

The eigenmodes are normalized as follows
\[\Phi^{(k)} = \begin{bmatrix} 1 \\ \phi_2^{(k)} \\ \phi_3^{(k)} \end{bmatrix}
\] (3.77)

The first and third equation of (3.75) are used for the determination of the components \(\Phi_2^{(k)}\) and \(\Phi_3^{(k)}\), leading to the following system of equations
\[
\begin{align*}
(9 - \mu_k - 5\Phi_2^{(k)} + 2\Phi_3^{(k)} &= 0 \\
1 - 1 - \Phi_2^{(k)} + (2 - \mu_k)\Phi_3^{(k)} &= 0
\end{align*}
\] \(\implies
\Phi^{(k)} = \begin{bmatrix} \frac{1}{14} - \mu_k + 16 \\
\frac{5\mu_k - 12}{5\mu_k - 12} \\
\frac{5\mu_k - 12}{5\mu_k - 12}
\end{bmatrix}
\] \(\implies
\Phi^{(1)} = \begin{bmatrix} 1 \\ -1 \\ 0
\end{bmatrix}, \quad \Phi^{(2)} = \begin{bmatrix} 1 \\ \frac{1}{2}(\sqrt{5} - 1)
\end{bmatrix}, \quad \Phi^{(3)} = \begin{bmatrix} 1 \\ \frac{1}{2}(\sqrt{5} + 1)
\end{bmatrix}
\] (3.78)

The first eigenmode is a purely torsional mode, with opposite displacements of the points C and D, leaving point B at rest (\(\Phi_3^{(1)} = 0\)). The second and third eigenmodes take place without torsion in beam AB (\(\Phi_1^{(k)} = \Phi_2^{(k)} = 1\)). In the second mode the masses are all moving in phase. In the third mode the mass at point B is moving in counterphase to the masses at the points C and D.

(3.54) determines \(n\) linearly independent solutions \(x(t) = \Phi^{(k)}(t)\cos(\omega_k t), \ k = 1, \ldots, n,\) to the homogeneous differential equations (3.40). It is easily proven by insertion that \(x(t) = \Phi^{(k)}(t)\sin(\omega_k t), \ k = 1, \ldots, n\) additionally represent \(n\) linearly independent solutions. Hence \(2n\) linearly independent solutions have been indicated, which may be used as a fundamental set of solutions. Then any solution to the homogeneous differential
equations (3-40), including the one which fulfils the initial conditions of the problem, can be written as a linear combination of these fundamental solutions

\[
x(t) = a_1 \Phi^{(1)}(t) \cos(\omega_1 t) + \cdots + a_n \Phi^{(n)}(t) + \\
b_1 \Phi^{(1)}(t) \sin(\omega_1 t) + \cdots + b_n \Phi^{(n)}(t) \sin(\omega_n t)
\]

The solution to the initial value problem (3-40) is given by (3-79), if the coefficients 
\[a^T = [a_1, \ldots, a_n]\] and 
\[b^T = [b_1, \ldots, b_n]\] are selected so that the initial conditions are fulfilled. For \(t = 0\) one has

\[
x_0 = a_1 \Phi^{(1)} + \cdots + a_n \Phi^{(n)} \\
\dot{x}_0 = b_1 \omega_1 \Phi^{(1)} + \cdots + b_n \omega_n \Phi^{(n)}
\]  

\[
a = P^{-1} x_0
\]

\[
b = \omega^{-1} P^{-1} \dot{x}_0
\]

where

\[
P = \begin{bmatrix} \Phi^{(1)} \ldots \Phi^{(n)} \end{bmatrix}
\]

\[
\omega = \begin{bmatrix} \omega_1 & 0 & \cdots & 0 \\
0 & \omega_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \omega_n
\end{bmatrix}
\]

The matrix \(P\) is denoted the modal matrix. \(P^{-1}\) exists because the eigenmodes, which form the columns of \(P\), are linearly independent.

**Example 3-8: Eigenvibrations of 2 DOF System**

Eigenvibrations of the system shown in fig. 3-5 is determined with the initial conditions

\[
x_0 = \begin{bmatrix} A \\
0
\end{bmatrix}, \quad \dot{x}_0 = \begin{bmatrix} 0 \\
0
\end{bmatrix}
\]

The modal matrix \(P\) follows from (3-67) and (3-83)

\[
P = \begin{bmatrix} \sqrt{3} - 1 & -(\sqrt{3} + 1) \\
1 & 1
\end{bmatrix} \quad \Rightarrow \quad P^{-1} = \begin{bmatrix} \frac{\sqrt{3}}{6} & \frac{1}{2} + \frac{\sqrt{3}}{6} \\
-\frac{\sqrt{3}}{6} & \frac{1}{2} - \frac{\sqrt{3}}{6}
\end{bmatrix}
\]

From (3-81), (3-82), (3-85) it then follows that

\[
\begin{bmatrix} a_1 \\
a_2
\end{bmatrix} = \begin{bmatrix} 1 \\
-1
\end{bmatrix} \frac{\sqrt{3}}{6} A, \quad \begin{bmatrix} \omega_1 b_1 \\
\omega_2 b_2
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix} \Rightarrow \begin{bmatrix} b_1 \\
b_2
\end{bmatrix} = \begin{bmatrix} 0 \\
0
\end{bmatrix}
\]
The motion then follows upon insertion of (3-87) into (3-79)

\[
x(t) = \frac{A\sqrt{3}}{6} \left( \begin{bmatrix} \sqrt{3} - 1 \\ 1 \end{bmatrix} \cos(\omega_1 t) + \begin{bmatrix} \sqrt{3} + 1 \\ -1 \end{bmatrix} \cos(\omega_2 t) \right), \quad t \geq 0
\]

\[
\omega_1 = \sqrt{\frac{3 - \sqrt{3}}{2}} \sqrt{\frac{k}{m}}, \quad \omega_2 = \sqrt{\frac{3 + \sqrt{3}}{2}} \sqrt{\frac{k}{m}}
\]  

(3 - 88)

**Example 3-9: Eigenvibrations of Mathematical Double Pendulum**

Eigenvibrations of the system defined in example 3-6 are determined with the initial conditions

\[
\begin{bmatrix} \theta_1(0) \\ \theta_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \dot{\theta}_1(0) \\ \dot{\theta}_2(0) \end{bmatrix} = \begin{bmatrix} \dot{\theta}_0 \\ 0 \end{bmatrix}
\]

(3 - 89)

The modal matrix \( P \) follows from (3-72)

\[
P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}
\]

(3 - 90)

Then, from (3-81), (3-82), (3-89) it follows that

\[
\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \omega_1 b_1 \\ \omega_2 b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega_1} \\ -\frac{1}{\omega_2} \end{bmatrix} \frac{\dot{\theta}_0}{2}
\]

(3 - 91)

where the circular eigenfrequencies are given by (3-71). Insertion of (3-91) into (3-89) then provides the motion of the system

\[
\begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix} = \frac{\dot{\theta}_0}{2} \begin{bmatrix} \frac{1}{\omega_1} \\ 1 \end{bmatrix} \sin(\omega_1 t) - \frac{1}{\omega_2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \sin(\omega_2 t), \quad t \geq 0
\]

\[
\omega_1 = \sqrt{\frac{g}{l}}, \quad \omega_2 = \sqrt{\frac{g}{l} + \frac{2k a^2}{ml^2}}
\]

(3 - 92)

Introduce the circular frequency deviation \( \Delta \omega = \omega_2 - \omega_1 \). (3-92) may then be written

\[
\begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix} = \frac{\dot{\theta}_0}{2\omega_1 \omega_2} \begin{bmatrix} \omega_2 \sin(\omega_1 t) + \omega_1 \sin(\omega_2 t) \end{bmatrix} = \\
\frac{\dot{\theta}_0}{2\omega_1 \omega_2} \begin{bmatrix} \omega_2 \pm \omega_1 \cos(\Delta \omega t) \end{bmatrix} \sin(\omega_1 t) \pm \omega_1 \sin(\Delta \omega t) \cos(\omega_1 t)
\]

\[
\begin{bmatrix} A_1(t) \cos(\omega_1 t - \Psi_1(t)) \\ A_2(t) \cos(\omega_1 t - \Psi_2(t)) \end{bmatrix}
\]

(3 - 93)

where + refers to \( \theta_1(t) \) and - refers to \( \theta_2(t) \). The time-dependent amplitudes \( A_j(t) \) and phases \( \Psi_j(t) \) are given as

\[
A_j(t) \cos \Psi_j(t) = \frac{\dot{\theta}_0}{2\omega_1 \omega_2} \begin{bmatrix} \pm \omega_1 \sin(\Delta \omega t) \\ \omega_2 \pm \omega_1 \cos(\Delta \omega t) \end{bmatrix} \Rightarrow \\
A_j(t) \sin \Psi_j(t) = \frac{\dot{\theta}_0}{2\omega_1 \omega_2} \begin{bmatrix} \pm \omega_1 \sin(\Delta \omega t) \\ \omega_2 \pm \omega_1 \cos(\Delta \omega t) \end{bmatrix}
\]
It is assumed the spring between the pendulums is sufficiently soft, so that

\[ \frac{2 \frac{k a^2}{m}}{l^2} \ll \frac{g}{l} \Rightarrow \Delta \omega = \sqrt{\frac{g}{l} + \frac{k a^2}{m l^2}} - \sqrt{\frac{g}{l} \approx \frac{k a^2}{m l^2} \sqrt{\frac{l}{g}}} \ll \omega_1 \]

(3-95)

At the derivation of (3-95) the Taylor expansion \( \sqrt{1+2x} = 1+x+O(x^2) \) has been used. The implication of (3-95) is that \( A_j(t) \) and \( \Psi_j(t) \) will both be slowly varying functions with the time.

The graph of the motions (3-93) is shown in fig. 3-8 for the circular frequency deviation \( \Delta \omega = 0.2 \omega_1 \). The time is normalized with respect to the eigenperiod \( T_1 = \frac{2\pi}{\omega_1} \) and the rotations are normalized with respect to \( \frac{\theta_0}{\omega_1} \). The amplitude functions \( \pm A_j(t) \) given by (3-94) are shown as a dashed signature. As follows from (3-94) the amplitudes vary between the minimum value \( A_{\text{min}} = \frac{\theta_0}{2\omega_1} \frac{\Delta \omega}{\omega_1+\Delta \omega} \) attained for \( \pm \cos(\Delta \omega t) = -1 \), and the maximum value \( A_{\text{max}} = \frac{\theta_0}{2\omega_1} \frac{2\omega_1+\Delta \omega}{\omega_1+\Delta \omega} = 2 \frac{\theta_0}{2\omega_1} - A_{\text{min}} \) attained for \( \pm \cos(\Delta \omega t) = 1 \). The said phenomenon is denoted amplitude modulation or beating, and is often observed in measured response signals of civil engineering structures. The phenomenon is always due to the interference of two or more harmonic components in the response signal with closely separated circular eigenfrequencies. These harmonic components are either caused by two close circular eigenfrequencies as in the present system, or they are caused by forced harmonic excitation of the structure, where two or more harmonic components with closely separated circular frequencies are present in the excitation. The period of the amplitude modulation is \( \frac{2\pi}{\Delta \omega} \). The fraction \( \frac{2\pi}{\Delta \omega} \) indicates the number of vibration periods per period of the amplitude modulation, which is 5 in fig. 3-8. The
squared amplitude $A_j^2(t)$ is a measure of the mechanical energy of the $j$th pendulum. As seen from fig. 3-8 the energy is then shifting between the pendulums with time. The mechanical energy of both pendulums is of course constant, since no damping mechanism is present in the system, so when the amplitude is small in one pendulum it must be large in the other. Actually, from (3-94) it follows that $A_1^2(t) + A_2^2(t) = \left( \frac{\beta_0}{\omega_1 \omega_2} \right)^2 (\omega_1^2 + \omega_2^2)$ is indeed time-invariant.

### 3.3 Forced Harmonic Vibrations

In this section the external dynamic force vector $f(t)$ is assumed to be harmonic varying with time, i.e.

$$f(t) = \text{Re}(F e^{i\omega t}) \quad (3-96)$$

In component form (3-96) reads

$$
\begin{bmatrix}
  f_1(t) \\
  \vdots \\
  f_n(t)
\end{bmatrix} = 
\begin{bmatrix}
  \text{Re}(F_1 e^{i\omega t}) \\
  \vdots \\
  \text{Re}(F_n e^{i\omega t})
\end{bmatrix} = 
\begin{bmatrix}
  \text{Re}(|F_1| e^{-i\alpha_1} e^{i\omega t}) \\
  \vdots \\
  \text{Re}(|F_n| e^{-i\alpha_n} e^{i\omega t})
\end{bmatrix} = 
\begin{bmatrix}
  |F_1| \cos(\omega t - \alpha_1) \\
  \vdots \\
  |F_n| \cos(\omega t - \alpha_n)
\end{bmatrix} \quad (3-97)
$$

The phases $\alpha_j$ are given as

$$
\begin{align*}
\text{Re}(F_j) &= \text{Re}(|F_j| e^{-i\alpha_j}) = |F_j| \cos \alpha_j \\
\text{Im}(F_j) &= \text{Im}(|F_j| e^{-i\alpha_j}) = -|F_j| \sin \alpha_j
\end{align*} \quad \Rightarrow
$$

$$\alpha_j = -\arctan \left( \frac{\text{Im}(F_j)}{\text{Re}(F_j)} \right), \quad j = 1, \ldots, n \quad (3-98)$$

(3-35), (3-36) can then be written

$$
\begin{align*}
M \ddot{x} + C \dot{x} + K x &= \text{Re}(F e^{i\omega t}) \quad , \quad t > 0 \\
x(0) &= x_0 , \quad \dot{x}(0) = \dot{x}_0
\end{align*} \quad (3-99)
$$

The stationary solution to (3-99) is searched for in the form

$$x(t) = \text{Re}(X e^{i\omega t}) , \quad X \in C^n \quad (3-100)$$

Insertion of (3-100) into (3-98) and use of the argumentation applied in relation to (2-60) provide

$$\forall t \in \mathbb{R} : \text{Re} \left[ \left( -\omega^2 M + i\omega C + K \right) X - F \right] e^{i\omega t} = 0 \quad \Rightarrow$$
\[
X = H(\omega)F \\
H(\omega) = (-\omega^2 M + i\omega C + K)^{-1} 
\]

(3-100) is a solution to the inhomogenous differential equation (3-99), if and only if the complex amplitude vector \(X\) is given as (3-101), which forms the MDOF generalization to (2-61). \(H(\omega)\) is denoted the frequency response matrix. Obviously, this depends on the structural system as specified by the matrices \(M\), \(C\), \(K\). The load only enters through the circular frequency \(\omega\).

\[
\begin{align*}
    x_1(t) &= \text{Re}(H_{11}e^{i\omega t}) \\
    x_2(t) &= \text{Re}(H_{22}e^{i\omega t}) \\
    f_1(t) &= \text{Re}(e^{i\omega t}) \\
    f_2(t) &= 0
\end{align*}
\]

Fig. 3-9: Harmonic excitation with a unit amplitude in the first degree of freedom of MDOF system.

Assume that
\[
F = e_j \tag{3-103}
\]
\[
e_j^T = [0 \ 0 \ \cdots \ 0 \ 1 \ 0 \ \cdots \ 0] \tag{3-104}
\]

where the \(j\)th component of the vector \(e_j\) is 1, and the other components are all 0. As seen from (3-101) the amplitude response \(X = H_j\) from the load with the amplitude (3-103) forms the \(j\)th column in the frequency response matrix \(H(\omega)\). The loading condition is illustrated in fig. 3-9 for the case \(j = 1\).

Next, it is assumed that the load vector \(f(t)\) is periodic in all its components with the period \(T\), i.e.
\[
f(t) = f(t + T) \tag{3-105}
\]

Then a Fourier series of the type (2-89) is valid for each component \(f_j(t)\). Consequently, \(f(t)\) can be written
\[
f(t) = \frac{1}{2} a_0 + \sum_{m=1}^{\infty} \text{Re}(F_m e^{i\omega_m t}) \tag{3-106}
\]
\[ \omega_m = \frac{2\pi}{T}, \quad m = 1, 2, \ldots \]  \hfill (3 - 107)

\(a_0\) are vectors of the Fourier coefficients \(a_0\) and \(F_m\) in (2-89) for each of the load components \(f_i(t)\). The stationary motion from the loading (3-106) is determined by superposition of the stationary motion from each of the harmonic components. From (3-35) and (3-101) it follows that

\[
x(t) = \frac{1}{2}K^{-1}a_0 + \sum_{m=1}^{\infty} \text{Re}(X_m e^{i\omega_m t})
\hfill (3 - 108)
\]

\[X_m = H(\omega_m)F_m\]  \hfill (3 - 109)

(3-108), (3-109) represent the MDOF generalization to (2-95), (2-96).

**Example 3-10: Forced Harmonic Vibrations of 2 DOF System**

The system defined in example 3-4 is excited by the harmonically varying external load vector

\[ f(t) = \text{Re}(Fe^{i\omega t}) = \begin{bmatrix} \text{Re}(F_1e^{i\omega t}) \\ \text{Re}(F_2e^{i\omega t}) \end{bmatrix} \]  \hfill (3 - 110)

The stationary harmonic motion becomes

\[ x(t) = \text{Re}(Xe^{i\omega t}) = \begin{bmatrix} \text{Re}(X_1e^{i\omega t}) \\ \text{Re}(X_2e^{i\omega t}) \end{bmatrix} \]  \hfill (3 - 111)

where \(X\) is related to \(F\) as specified by (3-101). The frequency response matrix \(H(\omega)\) follows from (3-39), (3-102)

\[
H(\omega) = \begin{bmatrix} k_1 + k_2 - \omega^2 m_1 + i\omega(c_1 + c_2) & -k_2 - i\omega c_2 \\ -k_2 - i\omega c_2 & k_2 + k_3 - \omega^2 m_2 + i\omega(c_2 + c_3) \end{bmatrix}^{-1} = \frac{1}{D} \begin{bmatrix} k_2 + k_3 - \omega^2 m_2 + i\omega(c_2 + c_3) & k_2 + i\omega c_2 \\ k_2 + i\omega c_2 & k_1 + k_2 - \omega^2 m_1 + i\omega(c_1 + c_2) \end{bmatrix}
\]  \hfill (3 - 112)

\[
D = \det(-\omega^2 M + i\omega C + K) = \left( k_1 + k_2 - \omega^2 m_1 + i\omega(c_1 + c_2) \right) \left( k_2 + k_3 - \omega^2 m_2 + i\omega(c_2 + c_3) \right) - (k_2 + i\omega c_2)^2
\]  \hfill (3 - 113)
Example 3-11: Passive Vibration Control of Harmonically Excited 2 DOF System

a) \( f(t) = f_0 \sin(\omega t) \)

b) \( f(t) = f_0 \sin(\omega t) \)

![Diagram](image)

Fig. 3-10: Harmonic excitation of primary mass. a) Definition of system. b) Forces on free masses.

The system shown in fig. 3-10a can be analysed by specializing the equations given in example 3-10 upon setting \( k_3 = 0, c_1 = c_2 = c_3 = 0, f_1(t) = f(t) = f_0 \sin(\omega t) \) and \( f_2(t) \equiv 0 \). For the given circular frequency \( \omega \) of the excitation one may ask, whether it is possible to select the mass \( m_2 \) and the spring \( k_2 \) in a way that the mass \( m_1 \) remains at rest, i.e. \( x_1(t) \equiv 0 \). Since a non-zero force is acting on \( m_1 \), this seems impossible at first sight. Nevertheless, it is indeed possible. The resolution of the paradox will be explained after the solution to the problem has been given. Since, \( f(t) = f_0 \sin(\omega t) = f_0 \cos(\omega t - \frac{\pi}{2}) = \text{Re}(f_0 e^{i(\omega t - \frac{\pi}{2})}) \) and \( \exp(-i\frac{\pi}{2}) = -i \) the loading vector (3-110) can be written

\[
f(t) = \begin{bmatrix} \text{Re}(-if_0 e^{i\omega t}) \\ 0 \end{bmatrix}
\]

(3 - 114)

The complex amplitudes \( X_1 \) and \( X_2 \) then follow from (3-112), (3-113)

\[
\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{D} \begin{bmatrix} k_2 - \omega^2 m_2 & k_2 \\ k_2 & k_1 + k_2 - \omega^2 m_1 \end{bmatrix} \begin{bmatrix} -i f_0 \\ 0 \end{bmatrix} = -i f_0 \begin{bmatrix} k_2 - \omega^2 m_2 \\ k_2 \end{bmatrix}
\]

\[D = (k_1 + k_2 - \omega^2 m_1)(k_2 - \omega^2 m_2) - k_2^2 \]

(3 - 115)

(3 - 116)

It follows from (3-115) that \( X_1 = 0 \) if \( m_2 \) and \( k_2 \) are selected so that

\[
k_2 - \omega^2 m_2 = 0 \quad \Rightarrow \quad \omega_0 = \sqrt{\frac{k_2}{m_2}} = \omega
\]

(3 - 117)

\( \omega_0 \) is identified as the circular eigenfrequency of the secondary system made up of the mass \( m_2 \) and spring \( k_2 \), when the primary system consisting of the mass \( m_1 \) and the spring \( k_1 \) is at rest. (3-117) is known as the tuning condition. Assume that \( \omega \neq \omega_0 \). From (3-115) and (3-117) it follows that \( X_1 \) and \( X_2 \) are related as

\[
X_2 = \frac{\omega_0^2}{\omega_0^2 - \omega^2} X_1
\]

(3 - 118)
(3-118) shows that if \( \omega_0 \neq \omega \) and \( X_1 = 0 \) then one always has \( X_2 = 0 \). However, as \( \omega_0 \to \omega \) the numerator and denominator of (3-118) both pass to zero in a way that the finite limit given by the 2nd equation of (3-115) is achieved. Under harmonic excitations with \( \omega = \omega_0 \) it follows from (3-116) and (3-117) that \( D = -k_2^2 \). The finite amplitude for \( X_2 \) then becomes

\[
X_2 = -i\frac{f_0 k_2}{k_2} = i\frac{f_0}{k_2} \quad (3-119)
\]

(3-119) implies that the harmonic motion of the mass \( m_2 \) is given as \( x_2(t) = -\frac{k_2}{k_2} \sin(\omega t) \), cf. the derivations leading to (3-114). With \( x_1(t) = 0 \) the spring \( k_2 \) is acting on \( m_1 \) with a force \( k_2 x_2(t) = -f_0 \sin(\omega t) = -f_1(t) \), see fig. 3-11b. Hence, under tuned conditions a force is developed in the spring \( k_2 \), which at every instant of time is of equal magnitude and in opposite direction of the external force on the primary mass. The primary mass must then be at rest, which explains the apparent paradox stated above.

![Diagram of vibration protection system](image)

Fig. 3-11: Passive vibration protection of vortex induced steel chimney. a) Principle of system. b) Practical layout of system.

The described principle finds practical application in so-called passive vibration control of structural systems. It should be clear from the previous outline that a passive vibration control system can only be expected to be effective, if a single dominating circular frequency \( \omega \) is present in the excitation of the system. This is the case in vortex induced vibrations of steel chimneys. Under certain conditions so-called frequency lock-in may occur. Then the vibrations of the cylinder controls the vortex shedding process so the circular shedding frequency \( \omega \) of the vortices is equal to the lowest circular eigenfrequency of the chimney, leading to resonance excitation. Since the damping in such structures is very low, large amplitude vibrations take place. The problem can be cured by adding a secondary system to the chimney as sketched in fig. 3-11a, which is tuned so \( \omega_0 = \omega_1 \). Typically, \( m_2 = 0.01m_1 \), where \( m_1 \) is of the same magnitude as the total mass of the chimney. A practical layout of the system is shown in fig. 3-11b. From the outside the system appears as a local thickening of the chimney close to the top. The principle has also been used to damp torsional and bending motions in long bridges from vortex excitation, and to control roll motions in passenger ships. The last application is less obvious, since the roll motion is dominated by amplitude modulations as explained in example 3-9, indicating that several harmonic components are present in the signal.
3.4 Forced Vibrations due to Arbitrary Excitation

Consider a linear viscous damped system of \( n \) degrees of freedom, which is assumed at rest at the time \( t = 0^- \). At the time \( t = 0^+ \) a unit impulse is applied in the \( j \)th degree of freedom, whereas the remaining degrees of freedom are unloaded. The loading condition has been illustrated in fig. 3-12 for the case \( j = 1 \). The resulting motion is denoted \( \mathbf{h}_j(t) \). According to (3-35), \( \mathbf{h}_j(t) \) is then determined as the solution of the vector differential equation

\[
M\ddot{\mathbf{h}}_j + C\dot{\mathbf{h}}_j + K\mathbf{h}_j = \mathbf{e}_j \delta(t)
\]  

where \( \mathbf{e}_j \) is defined in (3-104). Unit impulses are applied to all \( n \) degrees of freedom in turn, resulting in \( n \) solution vectors \( \mathbf{h}_1(t), \ldots, \mathbf{h}_n(t) \). These vectors are organized as columns in a matrix \( \mathbf{h}(t) \), which is denoted the *impulse response matrix*, i.e.

\[
\mathbf{h}(t) = [\mathbf{h}_1(t) \mathbf{h}_2(t) \cdots \mathbf{h}_n(t)]
\]  

The set of differential equations (3-120) can then be collected into the following matrix differential equation

\[
M\ddot{\mathbf{h}} + C\dot{\mathbf{h}} + K\mathbf{h} = \mathbf{I}\delta(t)
\]  

where \( \mathbf{I} \) denotes the identity matrix. Because the system is at rest before the excitation is applied, it follows that

\[
\mathbf{h}(0^-) = \dot{\mathbf{h}}(0^-) = 0
\]

As was the case for SDOF system, impulse loads cause continuous displacements, whereas the velocities are discontinuous. Hence

\[
\mathbf{h}(0^+) = 0
\]
(3-116) is integrated from \( t = 0^- \) to \( t = 0^+ \). From (3-123) and (3-124) it follows that

\[
M \int_{0^-}^{0^+} \dot{h}(t) dt + C \int_{0^-}^{0^+} \dot{h}(t) dt + K \int_{0^-}^{0^+} h(t) dt = I \int_{0^-}^{0^+} \delta(t) dt \Rightarrow
\]

\[
M(\dot{h}(0^+) - \dot{h}(0^-)) + C(h(0^+) - h(0^-)) = I \Rightarrow
\]

\[
\dot{h}(0^+) = M^{-1}
\] (3 - 125)

From (3-122), (3-124), (3-125) it then follows that the impulse response matrix can be determined as the solution to the initial value problem

\[
\begin{align*}
M \ddot{h} + C \dot{h} + K h &= 0 \quad , \quad t > 0 \\
h(0^+) &= 0 \quad , \quad \dot{h}(0^+) = M^{-1}
\end{align*}
\] (3 - 126)

From (3-126) it follows that the column vectors \( h_j(t) \), \( j = 1, \ldots, n \) of \( h(t) \) are all solutions to the homogeneous differential equations (3-35), and then represent \( n \) linearly independent solutions to the damped eigenvalue problem. Further,

\[
\ddot{h}(0^+) = -M^{-1}CM^{-1}
\] (3 - 127)

(3-125), (3-127) and differentiation of (3-126) provide the following initial value problem for the determination of \( \dot{h}(t) \)

\[
\begin{align*}
M \frac{d^2}{dt^2} \dot{h} + C \frac{d}{dt} \dot{h} + Kh &= 0 \quad , \quad t > 0 \\
\dot{h}(0^+) &= M^{-1} \quad , \quad \frac{d}{dt} \dot{h}(0^+) = -M^{-1}CM^{-1}
\end{align*}
\] (3 - 128)

From (3-128) it follows that the column vectors \( \dot{h}_j(t) \), \( j = 1, \ldots, n \) of \( \dot{h}(t) \) represent additional \( n \) linearly independent solution vectors to the damped eigenvalue problem. Consider the integral

\[
x^{(1)}(t) = \int_{0^-}^{t^-} \dot{h}(t - \tau)f(\tau)d\tau
\] (3 - 129)

Upon differentiation of (3-129), and use of (3-124), (3-125) it follows that

\[
x^{(1)}(t) = h(t - t^-)f(t) + \int_{0^-}^{t^-} \dot{h}(t - \tau)f(\tau)d\tau = \int_{0^-}^{t^-} \ddot{h}(t - \tau)f(\tau)d\tau
\] (3 - 130)

\[
x^{(1)}(t) = \dot{h}(t - t^-)f(t) + \int_{0^-}^{t^-} \ddot{h}(t - \tau)f(\tau)d\tau =
\]
\[ M^{-1} f(t) + \int_0^t \dot{h}(t - \tau) f(\tau) d\tau \]  
(3-131)

(3-129), (3-130), (3-131) provide
\[
M\ddot{x}^{(1)} + C\dot{x}^{(1)} + Kx^{(1)} = \\
f(t) + \int_0^t \left[ M\ddot{h}(t - \tau) + Ch(t - \tau) + Kh(t - \tau) \right] f(\tau) d\tau \text{, } t > 0 
(3 - 132)
\]

It follows from (3-128) that the term within the square bracket in the integrand on the right-hand side is equal to 0. Consequently one has
\[
M\ddot{x}^{(1)} + C\dot{x}^{(1)} + Kx^{(1)} = f(t) \text{, } t > 0 
(3 - 133)
\]

(3-133) shows that \( x^{(1)}(t) \) is a particular integral to (3-35). The precise specification of the upper integration limit as \( t^- \) is immaterial and will be omitted in what follows. From (3-129) and (3-130) it follows that the particular integral fulfills the initial conditions
\[
x^{(1)}(0) = \dot{x}^{(1)}(0) = 0 
(3 - 134)
\]

The complete solution of (3-35), (3-36) can be written
\[
x(t) = x^{(0)}(t) + x^{(1)}(t) \text{, } t > 0 
(3 - 135)
\]

\( x^{(0)}(t) \) indicates the damped eigenvibrations, which must fulfill the same initial conditions as \( x(t) \) due to (3 - 134). From (3-36) it then follows that
\[
x^{(0)}(0) = x_0 \text{, } \dot{x}^{(0)}(0) = \dot{x}_0 
(3 - 136)
\]

Above, \( 2n \) linearly independent solutions \( h_j(t), \dot{h}_j(t), j = 1, \ldots, n \) to the homogeneous differential equation have been indicated. These may be used as a fundamental set of solutions to the homogenous version of the differential equations (3-35). Any damped eigenvibration, including \( x^{(0)}(t) \), can then be written as a linear combination of these fundamental solutions, i.e.
\[
x^{(0)}(t) = \sum_{j=1}^{n} h_j(t)a_j + \sum_{j=1}^{n} \dot{h}_j(t)b_j 
(3 - 137)
\]

On matrix form (3-137) reads
\[
x^{(0)}(t) = h(t)a + \dot{h}(t)b 
(3 - 138)
\[
\mathbf{a}^T = [a_1, \ldots, a_n], \quad \mathbf{b}^T = [b_1, \ldots, b_n]
\]  
\hspace{1cm} (3 - 139)

Differentiation of (3-138) provides
\[
\dot{x}^{(0)}(t) = \dot{h}(t)a + \ddot{h}(t)b
\]  
\hspace{1cm} (3 - 140)

The expansion coefficients \(a, b\) are determined upon insertion of the initial conditions (3-124), (3-125), (3-127), (3-136) into (3-138), (3-140)
\[
\begin{align*}
\begin{cases}
x_0 = 0a + M^{-1}b \\
\dot{x}_0 = M^{-1}a - M^{-1}CM^{-1}b
\end{cases}
\Rightarrow
\end{align*}
\]
\hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} \hspace{1cm} (3 - 141)

Insertion of (3-141) into (3-138) gives
\[
x^{(0)}(t) = (\dot{h}(t)M + h(t)C)x_0 + h(t)M\dot{x}_0
\]  
\hspace{1cm} (3 - 142)

Finally, insertion of (3-129) and (3-142) in (3-135) provides the following representation of the solution to (3-35), (3-36)
\[
x(t) = (\dot{h}(t)M + h(t)C)x_0 + h(t)M\dot{x}_0 + \int_0^t h(t - \tau)f(\tau)d\tau
\]  
\hspace{1cm} (3 - 143)

The damped eigenvibration, which fulfils the initial conditions (3-36), follows from \(f(\tau) \equiv 0\) in (3-143). This eigenvibration is equal to \(x^{(0)}(t)\) as given by (3-142)
\[
x(t) = (\dot{h}(t)M + h(t)C)x_0 + h(t)M\dot{x}_0
\]  
\hspace{1cm} (3 - 144)

Hence, both eigenvibrations and forced vibrations of a linearly viscous damped system is determined if only the impulse response matrix \(h(t)\) is known. In principle, the linear vibration theory for systems of \(n\) degrees of freedom deals with the determination of this quantity. (3-129) is the MDOF generalization of Duhamel's integral.
Fig. 3-13: Physical interpretation of Duhamel’s integral for MDOF system. Differential impulse loading in the $j$th degree of freedom and resulting response in the $i$th degree of freedom.

Fig. 3-13 shows the time series for the load $f_j(\tau)$ acting in the $j$th degree of freedom. During the time interval $[\tau, \tau + d\tau]$ a differential impulse of the magnitude $f_j(\tau)d\tau$ is applied to the system. The system response at the succeeding time $t > \tau$ from this impulse is $h_j(t-\tau)f_j(\tau)d\tau$. The component of this motion in the $i$th degree of freedom, $h_{ij}(t-\tau)f_j(\tau)d\tau$, has been illustrated in the figure. Differential impulses are acting on all $n$ masses during the said interval $[\tau, \tau + d\tau]$. The motion from these becomes $\sum_{j=1}^{n} h_j(t-\tau)f_j(\tau)d\tau$. Finally, the motion at the time $t$ is obtained upon summing the contribution from each of these differential time-intervals during the period $[0, t]$, i.e.

$$x^{(1)}(t) = \int_0^t \sum_{j=0}^{n} h_j(t-\tau)f_j(\tau)d\tau = \int_0^t h(t-\tau)f(\tau)d\tau \quad (3-145)$$

$x^{(1)}(t)$ may be defined as a superposition of solutions to (3-35), where the dynamic load has been decomposed into impulses. Hence, (3-140) is itself a particular solution to (3-35). The partial solutions from these differential impulses presumes the system to be at rest with a zero displacement and velocity vector prior to the time they are applied, cf. (3-123). The particular solution (3-129) must then be at rest with zero displacement and velocity vector prior to the applicance of the first impulse at the time $t = 0$. Hence, the initial conditions (3-134) must prevail for Duhamel’s integral (3-140).

In section 3.3 the $j$th column of the frequency response matrix $H_j(\omega)$ was defined as the complex amplitude of the harmonic response vector due to a harmonic excitation, $f(t) = e_j \exp(i\omega t)$, acting with a unit amplitude in the $j$th degree of freedom alone. Similarly, $h_j(t)$ signifies the transient response vector from an impulse loading $f(t) = e_j \delta(t)$, acting with a unit impulse in the $j$th degree of freedom alone. Since both $H_j(\omega)$ and $h_j(t)$ describe the response of the same system to well-defined loads a relation is likely to exist between these quantities. In order to unveil this relation the excitation $f(t) = e_j \exp(i\omega t)$ is applied to the system. After infinitely long time, so the response from the initial conditions has passed away, the following displacement response follows
from (3-100) and (3-143)

\[ X e^{i\omega t} = H_j(\omega) e^{i\omega t} = \int_{-\infty}^{t} h(t-\tau)e^{i\omega \tau} d\tau = \int_{-\infty}^{t} h_j(t-\tau)e^{i\omega \tau} d\tau \quad (3-146) \]

Using the same operations as in relation to (2-128) it then follows that

\[ H_j(\omega) e^{i\omega t} = -\int_{0}^{\infty} h_j(u)e^{i\omega(t-u)} du + e^{i\omega t} \int_{0}^{\infty} h_j(u)e^{-i\omega u} du \Rightarrow \]

\[ H_j(\omega) = \int_{0}^{\infty} h_j(u)e^{-i\omega u} du \quad (3-147) \]

(3-147) determines each component of \( H_j(\omega) \) as the Fourier transform of the corresponding components of \( h_j(t) \). (3-147) is valid for any corresponding columns in \( H(\omega) \) and \( h(t) \). The relations from all columns can be assembled in the following matrix equation

\[ H(\omega) = \int_{-\infty}^{\infty} h(u)e^{-i\omega u} du \quad (3-148) \]

where it has been utilized that \( h(u) \equiv 0 \) for \( u < 0 \), so the lower integration limit can be changed to \(-\infty\). The inverse relation follows from (A-15)

\[ h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega)e^{i\omega t} d\omega \quad , \quad t > 0 \quad (3-149) \]

In principle the problem of calculating the impulse response matrix is solved using (3-149) in combination with the result (3-102) for the frequency response matrix \( H(\omega) \). However, such a procedure is numerically intractable even for small-size problems. Instead, various analytical solutions for the impulse response matrix \( h(t) \) will be indicated in sections 3.6 and 3.7, valid for a large class of linear structures.

### 3.5 Orthogonality Properties of Undamped Eigenmodes

**Theorem 3-3:** If the eigenmodes \( \Phi^{(i)} \) and \( \Phi^{(j)} \) are associated with different eigenvalues \( \omega_i^2 \) and \( \omega_j^2 \) (i.e. \( \omega_i^2 \neq \omega_j^2 \)) they fulfill the orthogonality conditions

\[ \Phi^{(i)^T} M \Phi^{(j)} = \begin{cases} 0 & , \quad i \neq j \\ M_i & , \quad i = j \end{cases} \quad (3-150) \]

\[ \Phi^{(i)^T} K \Phi^{(j)} = \begin{cases} 0 & , \quad i \neq j \\ \omega_i^2 M_i & , \quad i = j \end{cases} \quad (3-151) \]
The parameter $M_i$, which is denoted the *undamped modal mass*, is defined as
\[ M_i = \Phi^{(i)T} M \Phi^{(i)} , \quad i = 1, \ldots, n \tag{3-152} \]

**Proof:** The eigenvibration condition (3-42) is formulated for both the $i$th and the $j$th mode
\[ K \Phi^{(i)} = \omega_i^2 M \Phi^{(i)} \tag{3-153} \]
\[ K \Phi^{(j)} = \omega_j^2 M \Phi^{(j)} \tag{3-154} \]
(3-153) is premultiplied by $\Phi^{(j)T}$ and (3-154) is premultiplied by $\Phi^{(i)T}$. Then
\[ \Phi^{(j)T} K \Phi^{(i)} = \omega_i^2 \Phi^{(j)T} M \Phi^{(i)} \tag{3-155} \]
\[ \Phi^{(i)T} K \Phi^{(j)} = \omega_j^2 \Phi^{(i)T} M \Phi^{(j)} \tag{3-156} \]
(3-155) is transposed. Use of the symmetry properties $K^T = K$, $M^T = M$ then provides
\[ \Phi^{(i)T} K \Phi^{(j)} = \omega_i^2 \Phi^{(i)T} M \Phi^{(j)} \Rightarrow \]
\[ \Phi^{(i)T} K \Phi^{(j)} = \omega_j^2 \Phi^{(i)T} M \Phi^{(j)} \tag{3-157} \]
Subtraction of (3-156) from (3-157) provides
\[ (\omega_i^2 - \omega_j^2) \Phi^{(i)T} M \Phi^{(j)} = 0 \tag{3-158} \]
Since $\omega_i^2 \neq \omega_j^2$ as stated in the premises of the theorem, (3-150) then follows. However, if (3-150) is valid, then it follows from (3-157) that (3-151) must also be valid. This concludes the proof of the theorem.

Along with the assumption $\omega_i^2 \neq \omega_j^2$ the proof of the theorem relies on the symmetry properties of $K$ and $M$. The symmetry of $K$ is a consequence of Maxwell's theorem. As is the case for theorem 3-1, theorem 3-3 may then be considered merely as a lemma to Maxwell's theorem. The modal mass $M_i$ is a pure mathematical symbol without physical substance, since it depends on the arbitrary normalization of the eigenmode $\Phi^{(i)}$. Hence the designation of this quantity as a mass may appear somewhat misleading. However, the rationale for the use of this name will be clear in the following section 3.6.

If the circular eigenfrequencies for all $n$ modes are different as assumed in the theorem the eigenvalues $\omega_i^2$ are characterized as *simple*. Next, assume that $\omega_i^2$ is a *multiple* root of multiplicity $k$ to the characteristic equation (3-43), where $i + k - 1 \leq n$. Then the corresponding linear independent eigenmodes $\Phi^{(i)}, \Phi^{(i+1)}, \ldots, \Phi^{(i+k-1)}$ all fulfill (3-42) with the same eigenvalue $\omega_i^2$, whereas (3-150) and (3-151) are not necessarily fulfilled among these eigenmodes, i.e. for $j = i, \ldots, i + k - 1$. However, it can be shown that it is always possible to specify linear combinations of $\Phi^{(i)}, \Phi^{(i+1)}, \ldots, \Phi^{(i+k-1)}$, which mutually fulfill (3-150) and (3-151). It is easy to show that any such linear combination fulfills (3-42) with the same eigenvalue $\omega_i^2$, and is then an eigenmode itself associated to this eigenvalue. Without restrictions it can then be assumed in what follows that a set of
eigenmodes \( \Phi^{(1)}, \Phi^{(i+1)}, \ldots, \Phi^{(n)} \) has been determined, which are linear independent and additionally fulfill the orthogonality property (3-150), (3-151), no matter if the eigenvalues are simple or multiple.

Comparing the eigenvalue problems (3-42) and (3-60) it is seen that the following orthogonality property must hold for \( K_m = (KM^{-1})^m M \) for \( m = 0, \pm 1, \pm 2, \ldots \):

\[
\Phi^{(i)T} K_m \Phi^{(j)} = \Phi^{(i)T} (KM^{-1})^m M \Phi^{(j)} = \begin{cases} 0 & , \ i \neq j \\ \omega_i^{2m} M_i & , \ i = j \end{cases} \tag{3-159}
\]

The proof for (3-159) follows upon inserting \( K_m = (KM^{-1})^m M \) for \( K \) into (3-42) and shifting the eigenvalue \( \omega_i^2 \) with \( \omega_i^{2m} \). The essential point of the proof of (3-151) is the symmetry property of \( K \), which needs to be fulfilled by \( K_m \) as well. However, as shown subsequently to (3-60), \( K_m \) is indeed a symmetric matrix for all values of \( m \).

The relations (3-150) can be assembled into the following matrix equations

\[
\begin{bmatrix} \Phi^{(1)T} \\ \vdots \\ \Phi^{(n)T} \end{bmatrix} M \begin{bmatrix} \Phi^{(1)} \ldots \Phi^{(n)} \end{bmatrix} = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_n \end{bmatrix} \Rightarrow \tag{3-160}
\]

\[
P^T M P = m \quad \Rightarrow \quad M = (P^T)^{-1} m P^{-1} = (P^{-1})^T m P^{-1} \tag{3-161}
\]

where \( P \) is the modal matrix (3-83) and \( m \) is a diagonal matrix with the modal masses entering the main diagonal

\[
m = \begin{bmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_n \end{bmatrix} \tag{3-162}
\]

The identity \( (P^T)^{-1} = (P^{-1})^T \) used in the last statement of (3-161) follows from \( I = (PP^{-1})^T = (P^{-1})^T P^T \). The orthogonality properties (3-151) and (3-159) can be assembled into matrix relations similar to (3-160). The representations similar to (3-161) read

\[
K = (P^{-1})^T k P^{-1} \tag{3-163}
\]

\[
K_m = (KM^{-1})^m M = (P^{-1})^T k_m P^{-1} \tag{3-164}
\]
where

\[ k = k_1 = \begin{bmatrix}
\omega_1^2 M_1 & 0 & \cdots & 0 \\
0 & \omega_2^2 M_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \omega_n^2 M_n \\
\end{bmatrix} \]

\[ k_m = \begin{bmatrix}
\omega_1^{2m} M_1 & 0 & \cdots & 0 \\
0 & \omega_2^{2m} M_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \omega_n^{2m} M_n \\
\end{bmatrix}, \quad m = 0, \pm 1, \pm 2, \ldots \] (3 - 165)

Example 3-12: Eigenvibrations of 2 DOF System with Multiple Eigenvalues

The plane structure shown in fig. 3-14a consists of the horizontal homogenous Bernoulli-Euler beam ABC of the length 2a and with the constant bending stiffness EI. At the mid-point C the vertical infinitely stiff beam CD of the length \( \sqrt{3}a \) and the mass per unit length \( \mu \) is attached. Both beams are infinitely stiff against axial deformations. The structure then has 2 degrees of freedom, which are selected as the vertical displacement \( x_1 \) and the rotation \( x_2 \) of the point C.

The total mass of CD is \( m_1 = \sqrt{3} \mu a \). The mass moment of inertia of CD around point C is \( m_2 = \frac{1}{3} \mu (\sqrt{3}a)^3 = \sqrt{3} \mu a^3 \). The flexibility coefficients become \( \delta_{11} = \frac{1}{6} \frac{g}{EI}, \delta_{22} = \frac{1}{6} \frac{g}{EI}, \delta_{12} = \delta_{21} = 0 \). The mass and stiffness matrices then become

\[ M = \begin{bmatrix}
\sqrt{3} \mu a & 0 \\
0 & \sqrt{3} \mu a^3 \\
\end{bmatrix}, \quad D = \begin{bmatrix}
\frac{1}{6} \frac{g}{EI} & 0 \\
0 & \frac{1}{6} \frac{g}{EI} \\
\end{bmatrix} \] (3 - 167)

Since both the mass matrix and the flexibility matrices are diagonal it is straightforward to calculate the circular eigenfrequencies and eigenmodes. These become

\[ \omega_1 = \omega_2 = \sqrt{\frac{6}{\sqrt{3}}} \sqrt{\frac{EI}{\mu a^3}} = 2 \sqrt{3} \sqrt{\frac{EI}{\mu a^3}} \] (3 - 168)

\[ \Phi^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \Phi^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \] (3 - 169)
The eigenmodes $\Phi^{(1)}$ and $\Phi^{(2)}$ have been illustrated in fig. 3-14b. These fulfil the orthogonality condition (3-150) with the mass matrix (3-167), and illustrate the statement subsequent to the proof of theorem 3-3 that such eigenmodes do exist even if the eigenvalues are multiple.

3.6 Expansion in Undamped Eigenmodes

Fig. 3-15: Expansion of displacement vector in Cartesian and eigenmode based coordinate systems.

Hitherto, no distinction has been made between a vector $x(t)$, i.e. a geometric quantity of length and orientation in the $n$-dimensional space, and the column matrix of its coordinates $[x_1(t), \ldots, x_n(t)]^T$, since it has been implicitly assumed that the vector was referred to a fixed Cartesian coordinate system with the basis $i_1, i_2, \ldots, i_n$. Actually, (3-35) is a matrix equation, and correctly $x(t)$ should here be interpreted as a column matrix of Cartesian coordinates. Since the eigenmodes $\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(n)}$ are linearly independent these vectors may alternatively be used as a basis in another coordinate system. Hence one has the following expansions of the displacement vector $x(t)$

$$x(t) = x_1(t)i_1 + x_2(t)i_2 + \cdots + x_n(t)i_n =$$

$$q_1(t)\Phi^{(1)} + q_2(t)\Phi^{(2)} + \cdots + q_n(t)\Phi^{(n)} \quad (3-170)$$

The coordinates $[q_1(t), q_2(t), \ldots, q_n(t)]^T$ in the new coordinate system are termed the undamped modal coordinates. These change with time as do the corresponding Cartesian coordinates, because the length and orientation of $x(t)$ are changing with the time. The expansion of $x(t)$ in the two coordinate systems has been illustrated in fig. 3-15. $q_j(t)\Phi^{(j)}$ signifies the projection vector of $x(t)$ on the direction specified by $\Phi^{(j)}$, which is a physical quantity. In contrast, both $\Phi^{(j)}$ and $q_j(t)$ are mathematical quantities which depend on the chosen normalization of the eigenmode. However, if the components of $\Phi^{(j)}$ are proportionally increased, $q_j(t)$ will reduce comparably making the product $q_j(t)\Phi^{(j)}$ invariant.

The modal coordinates are assembled in the column matrix $q^T(t) = [q_1(t), \ldots, q_n(t)]$. From (3-170) it is seen that the following relation must be valid between Cartesian and modal coordinates

$$x(t) = Pq(t) \quad (3-171)$$
\( P \) is the modal matrix (3-83) with the components of the eigenmodes \( \Phi^{(j)} \) specified in the Cartesian coordinate system. This matrix constitutes the transformation matrix from modal coordinates back to Cartesian coordinates. Next, the equations of motion are formulated in the modal coordinates. The expansion (3-170) is inserted into (3-35) and the resulting equation is premultiplied by \( \Phi^{(i)T} \). It then follows that

\[
\sum_{j=1}^{n} (M \Phi^{(j)} \ddot{q}_j + C \Phi^{(j)} \dot{q}_j + K \Phi^{(j)} q_j) = f(t) \quad \Rightarrow
\]

\[
\sum_{j=1}^{n} (\Phi^{(i)T} M \Phi^{(j)} \ddot{q}_j + \Phi^{(i)T} C \Phi^{(j)} \dot{q}_j + \Phi^{(i)T} K \Phi^{(j)} q_j) = \Phi^{(i)T} f(t) \quad \Rightarrow
\]

\[
\ddot{q}_i + \frac{1}{M_i} \sum_{j=1}^{n} (\Phi^{(i)T} C \Phi^{(j)} \dot{q}_j) + \omega_i^2 q_i = \frac{1}{M_i} F_i(t) \quad , \quad i = 1, \ldots, n \quad , \quad t > 0(3-172)
\]

\[
F_i(t) = \Phi^{(i)T} f(t)
\]

At the derivation of the final statement of (3-172) the orthogonality properties (3-150) and (3-151) have been applied. A non-zero contribution to the sums only appears for \( j = i \). \( F_i(t) \), which is denoted the modal load, indicates the scalar product of the loading vector \( f(t) \) and the \( j \)th eigenmode \( \Phi^{(j)} \). In order to solve (3-172) initial conditions \( q_i(0), \dot{q}_i(0) \) need to be known. In order to formulate these initial conditions (3-170) is inserted into (3-36) and next the equations are premultiplied by \( \Phi^{(i)T} M \). Then

\[
x_0 = \sum_{j=1}^{n} \Phi^{(j)} q_j(0) \quad \Rightarrow
\]

\[
\dot{x}_0 = \sum_{j=1}^{n} \Phi^{(j)} \dot{q}_j(0)
\]

\[
\Phi^{(i)T} M x_0 = \sum_{j=1}^{n} \Phi^{(i)T} M \Phi^{(j)} q_j(0)
\]

\[
\Phi^{(i)T} M \dot{x}_0 = \sum_{j=1}^{n} \Phi^{(i)T} M \Phi^{(j)} \dot{q}_j(0)
\]

From the orthogonality properties (3-150) it then follows that

\[
q_{i0} = q_i(0) = \frac{1}{M_i} \Phi^{(i)T} M x_0 \quad , \quad i = 1, \ldots, n
\]

\[
\dot{q}_{i0} = \dot{q}_i(0) = \frac{1}{M_i} \Phi^{(i)T} M \dot{x}_0
\]
From (3-171) it follows that the initial conditions for the modal coordinates alternatively are given as

\[
\begin{align*}
q(0) &= P^{-1}x_0 \\
\dot{q}(0) &= P^{-1}\dot{x}_0
\end{align*}
\]

Comparison of (3-175) and (3-176) provides the following expression for the inverse modal matrix

\[
P^{-1} = \begin{bmatrix}
\frac{1}{M_1} & 0 & \cdots & 0 \\
0 & \frac{1}{M_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{1}{M_n}
\end{bmatrix}
\]

where \( m \) is defined by (3-162). If \( M = I \) and the eigenmodes are normalized so that \( m = I \), (3-177) specifies the well-known result from linear algebra that the inverse of an orthonormal transformation matrix is equal to its transpose.

For linear undamped eigenvibrations one has \( C = 0, f(t) \equiv 0 \). Then (3-172) reduces to

\[
\begin{align*}
\ddot{q}_i + \omega_i^2 q_i &= 0, \quad i = 1, \ldots, n, \quad t > 0 \\
q_i(0) &= q_{i0}, \quad \dot{q}_i(0) = \dot{q}_{i0}
\end{align*}
\]

(3-178) has the solution, cf. (2-8)

\[
q_i(t) = q_i(0) \cos(\omega_i t) + \frac{\dot{q}_i(0)}{\omega_i} \sin(\omega_i t), \quad i = 1, \ldots, n, \quad t \geq 0
\]

Insertion of (3-179) into (3-170) provides the following solution to the undamped eigen vibration problem

\[
x(t) = \sum_{j=1}^{n} \Phi^{(j)} \left( q_j(0) \cos(\omega_j t) + \frac{\dot{q}_j(0)}{\omega_j} \sin(\omega_j t) \right)
\]

If (3-176) is inserted into (3-180) the solution (3-79), (3-81), (3-82) is again obtained. (3-172) is written on the form

\[
\ddot{q}_i + 2\omega_i \left( \zeta_i \dot{q}_i + \sum_{j=1}^{n} \frac{\omega_j M_j}{\omega_i M_i} \zeta_{ij} \dot{q}_j \right) + \omega_i^2 q_i = \frac{1}{M_i} F_i(t), \quad i = 1, \ldots, n, \quad t > 0
\]
where

\[ \zeta_i = \frac{\Phi^{(i)^T} C \Phi^{(i)}}{2\omega_i M_i} , \quad i = 1, \ldots, n \]  

\[ \zeta_{ij} = \frac{\Phi^{(i)^T} C \Phi^{(j)}}{2\sqrt{\omega_i \omega_j} M_i M_j} , \quad i, j = 1, \ldots, n , \quad j \neq i \]

(3 - 182)  

(3 - 183)

Notice the quantities \( \zeta_i \) and \( \zeta_{ij} \) are independent of the normalization of the eigenmodes \( \Phi^{(i)} \) and \( \Phi^{(j)} \). \( \zeta_{ij} \) will be referred to as the modal coupling coefficients. In contrast to the most modal parameters these are physical quantities, which can be measured physically. (3-181) is in no way analytically more tractable than are the original system of differential equations (3-35), unless the coefficients \( \zeta_{ij} = 0 \). Then the system of differential equations (3-181) decouples into simple differential equations of 2nd order with constant coefficients, each of which determines a modal coordinate. Apparently the decoupling condition is

\[ \Phi^{(i)^T} C \Phi^{(j)} = \begin{cases} 0 & , \quad i \neq j \\ 2\zeta_i \omega_i M_i & , \quad i = j \end{cases} \]

(3 - 184)

In case of decoupling of the modal differential equations the eigenmodes are orthogonally weighted to all 3 system matrices \( M, C, K \) as indicated by (3-150), (3-151), (3-184). In the same way as the representations (3-161) and (3-163) for the mass- and stiffness matrices have been derived based on the orthogonality conditions (3-150), (3-151) the following representation of the damping matrix can be derived based on (3-184)

\[ C = (P^{-1})^T C P^{-1} \]  

(3 - 185)

\[ c = \begin{bmatrix} 2\zeta_1 \omega_1 M_1 & 0 & \cdots & 0 \\ 0 & 2\zeta_2 \omega_2 M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2\zeta_n \omega_n M_n \end{bmatrix} \]

(3 - 186)

If (3-184) is valid, or if the coupling terms are otherwise ignored, one has

\[ \ddot{q}_i + 2\zeta_i \omega_i \dot{q}_i + \omega_i^2 q_i = \frac{1}{M_i} F_i(t) , \quad i = 1, \ldots, n , \quad t > 0 \]

(3 - 187)

(3-187) is identical to the equation of motion of a SDOF system (2-7), (2-32), (2-39), with \( M_i, \omega_i, \zeta_i \) and \( F_i(t) \) replacing the mass \( m \), the circular eigenfrequency \( \omega_0 \), the damping ratio \( \zeta \) and the external loading \( f(t) \). This equivalence is the reason for naming \( M_i \) the modal mass despite the previously mentioned problems with that naming. In the same
way the parameter $\zeta_i$ is called the *modal damping ratio*. The solution of (3-187) follows immediately from (2-121) upon substituting the described parameters

$$q_i(t) = e^{-\zeta_i\omega_i t} \left( q_i(0) \cos(\omega_{d,i} t) + \frac{\dot{q}_i(0) + \zeta_i \omega_i q_i(0)}{\omega_{d,i}} \sin(\omega_{d,i} t) \right) +$$

$$\int_0^t h_i(t - \tau) F_i(\tau) d\tau , \quad i = 1, 2, \ldots, n$$

$$\omega_{d,i} = \omega_i \sqrt{1 - \zeta_i^2}$$

$$h_i(t) = \begin{cases} 0 & , \quad t \leq 0 \\ \frac{1}{M_i \omega_{d,i}} e^{-\zeta_i \omega_i t} \sin(\omega_{d,i} t) & , \quad t > 0 \end{cases}$$

(3-188) \hspace{1cm} (3-189) \hspace{1cm} (3-190)

$\omega_{d,i}$ and $h_i(\cdot)$ are denoted the *modal damped circular eigenfrequency* and the *modal impulse response function*, respectively. It should be realized that these concepts only make sense for systems for which the decoupling condition (3-184) is fulfilled.

In practical dynamic calculations there will be a strong tendency to model the damping matrix so the decoupling condition (3-184) is fulfilled. Actually in most commercial computer programs this is implicitly assumed. In these programs the user is usually asked to specify the damping properties in terms of the modal damping ratios, from which the underlying damping matrix may be synthesized by (3-185), (3-186). From a practical point of view it is then of importance to have guidelines for cases where such an approach works well. Based on extensive analytical and numerical investigations it has been concluded that the coupling between the modal differential equations via modal velocities is of minor importance to the structural response if the following condition is fulfilled for any two sequential circular eigenfrequencies

$$\omega_i(1 + a\zeta_i) < \omega_{i+1}(1 - a\zeta_{i+1}) , \quad i = 1, \ldots, n - 1$$

(3-191)

where $a \simeq 2 - 3$. The condition (3-191) can be stated that modal decoupling can be anticipated, whenever the circular eigenfrequencies are well separated and the system is lightly damped. Notice, $2\zeta_i \omega_i$ indicates the half-band width (2-86) related to the $i$th mode. Hence, (3-191) states that the circular eigenfrequencies must be separated at least two half-band widths if modal decoupling is to be ignored. Land based civil engineering structures are generally lightly damped, unless they are exposed to plastic deformations. However, for offshore structures the damping ratios may be relatively large due to the dissipation caused by the drag component of the wave loading, and then (3-191) is not equally well fulfilled. The importance of the condition (3-191) has been illustrated in the following example 3-13.

If (3-188) is inserted into (3-170) the solution to (3-35), (3-36) is obtained. Further, upon insertion of (3-173), (3-175) the solution is obtained in terms of the physical load vector $f(t)$ and the initial conditions $x_0$ and $\dot{x}_0$. The result becomes

$$x(t) = A(t)x_0 + B(t)\dot{x}_0 + \int_0^t h(t - \tau)f(\tau)d\tau$$

(3-192)
\[
A(t) = \left[ \sum_{j=1}^{n} (\dot{h}_j(t) + 2\zeta_j\omega_j h_j(t)) \Phi^{(j)}(\Phi^{(j)})^T \right] M
\] (3 - 193)

\[
B(t) = \left[ \sum_{j=1}^{n} h_j(t) \Phi^{(j)}(\Phi^{(j)})^T \right] M
\] (3 - 194)

\[
h(t) = \sum_{j=1}^{n} h_j(t) \Phi^{(j)}(\Phi^{(j)})^T
\] (3 - 195)

(3-195) denotes the analytical solution for the impulse response matrix, which is valid whenever the decoupling condition (3-184) is valid.

The frequency response matrix is obtained from (2-62), (2-129), (3-102), (3-148), (3-195)

\[
H(\omega) = (-\omega^2 M + i\omega C + K)^{-1} = \sum_{j=1}^{n} H_j(\omega) \Phi^{(j)}(\Phi^{(j)})^T
\] (3 - 196)

\[
H_j(\omega) = \int_{-\infty}^{\infty} h_j(t)e^{-i\omega t} dt = \frac{1}{M_j(\omega_j^2 - \omega^2 + 2\zeta_j\omega_j\omega t)}
\] (3 - 197)

\[H_j(\omega)\] is denoted the modal frequency response function. Still, this quantity only exists for systems for which the decoupling conditions (3-184) are fulfilled. Setting \(\omega = 0\) in (3-196) and (3-197) the following interesting series expansion for \(K^{-1}\) is obtained, which is denoted Mercer's theorem

\[
K^{-1} = D = \sum_{j=1}^{n} \frac{1}{\omega_j^2 M_j} \Phi^{(j)}(\Phi^{(j)})^T
\] (3 - 198)

In what follows some of the properties of the coupling constants \(\zeta_{ij}\) will be revealed. Assume that \(C\) is a symmetrical matrix, i.e. \(C = C^T\). From (3-183) then it follows that \(\zeta_{ij}\) fulfils the symmetry property

\[
\zeta_{ij} = \zeta_{ji}
\] (3 - 199)

Consider the matrix

\[
c = \begin{bmatrix}
2\omega_1 M_1 \zeta_1 & 2\sqrt{\omega_1 \omega_2} M_1 M_2 \zeta_{12} & \cdots & 2\sqrt{\omega_1 \omega_n} M_1 M_n \zeta_{1n} \\
2\sqrt{\omega_2 \omega_1} M_2 M_1 \zeta_{21} & 2\omega_2 M_2 \zeta_2 & \cdots & 2\sqrt{\omega_2 \omega_n} M_2 M_n \zeta_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
2\sqrt{\omega_n \omega_1} M_n M_1 \zeta_{n1} & 2\sqrt{\omega_n \omega_2} M_n M_2 \zeta_{n2} & \cdots & 2\omega_n M_n \zeta_n
\end{bmatrix}
\]
The damping matrix $C$ is positive definite since the structural system is assumed to be dissipative, cf. (3-23). The modal matrix $P$ is non-singular, because the column vectors (the eigenmodes) are linearly independent. It then follows that $\varpi$ is also a positive definite matrix. Let \(x^T = [0 \cdots 0 \ x_i \ 0 \cdots 0 \ x_j \ 0 \cdots 0]\), where only the $i$th and the $j$th components are different from 0. Hence

\[
x^T \varpi x = \begin{bmatrix} x_i \\ x_j \end{bmatrix}^T \begin{bmatrix} 2\omega_i M_i \zeta_i & 2\sqrt{\omega_j \omega_i M_i M_j \zeta_{ij}} \\ 2\sqrt{\omega_j \omega_i M_i M_j \zeta_{ij}} & 2\omega_j M_j \zeta_j \end{bmatrix} \begin{bmatrix} x_i \\ x_j \end{bmatrix} > 0	ag{3-201}
\]

The coefficient matrix of the quadratic form on the right-hand side of (3-201) is also positive definite. Since $\zeta_i > 0 \land \zeta_j > 0$, this is the case if and only if the determinant is positive. This leads to the following upper bound for the coupling coefficients

\[
\zeta_{ij} \zeta_{ji} < \zeta_i \zeta_j
\tag{3-202}
\]

Further, if $C = C^T$ the following inequality, valid for dissipative symmetric damping matrices, follows from (3-199) and (3-202)

\[
\left| \zeta_{ij} \right| < \sqrt{\zeta_i \zeta_j}
\tag{3-203}
\]

In case (3-184) is not fulfilled the relative magnitude of the left and right-hand sides of (3-203) can be used as a measure of the strength of the coupling between the $i$th and the $j$th modal coordinate, i.e. the closer the fraction $|\zeta_{ij}|/\sqrt{\zeta_i \zeta_j}$ is to 1 the stronger the modal coupling.

3.7 Expansion in Damped Eigenmodes

Consider the identity

\[
M \dot{x} - \dot{M} x = 0
\tag{3-204}
\]

If (3-204) is combined with (3-35) the following system of differential equations is obtained

\[
\begin{bmatrix} C & M \\ M & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} + \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} f(t) \\ 0 \end{bmatrix}
\tag{3-205}
\]

which may be written in the following matrix form

\[
A \dot{z} + Bz = F(t) \quad , \quad t > 0
\]

\[
z(0) = z_0
\tag{3-206}
\]
\[ z(t) = \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix}, \quad z_0 = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} \]  

(3-207)

\[ F(t) = \begin{bmatrix} f(t) \\ 0 \end{bmatrix} \]  

(3-208)

\[ A = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}, \quad B = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix} \]  

(3-209)

(3-206) is called a state vector formulation of the equations of motion and \( z(t) \) is denoted the state vector. In what follows \( C \) is assumed to be symmetric and positive definite. Since both \( M \) and \( K \) are symmetric matrices, the enlarged matrices \( A \) and \( B \) of dimension \( 2n \times 2n \) are symmetric, too. Even though \( M \), \( C \) and \( K \) are all positive definite, this is not the case for neither \( A \) nor \( B \). Actually, for \( z^T = [x^T, \dot{x}^T] \), one has

\[
\begin{align*}
  z^T Az &= x^T C x + 2x^T M \dot{x} \\
  z^T Bz &= x^T K x - x^T M \dot{x}
\end{align*}
\]  

(3-210)

Dependent on the magnitude and the sign of the components of the vectors \( x \) and \( \dot{x} \), the right-hand sides of (3-210) may be both positive and negative.

The eigenvibrations of the system (3-206) are given as

\[
\begin{align*}
  A \dot{z} + B z &= 0, \quad t > 0 \\
  z(0) &= z_0
\end{align*}
\]  

(3-211)

Solutions to (3-211) are searched for in the form

\[ z(t) = \text{Re}(\Psi \lambda^t) \]  

(3-212)

where \( \Psi \) is a constant complex vector of the dimension \( 2n \), and \( \lambda \) is a complex constant. Upon insertion, (3-212) is seen to be a solution to (3-211) if and only if \( \Psi \) is a solution to the following linear eigenvalue problem of dimension \( 2n \)

\[ (\lambda A + B) \Psi = 0 \]  

(3-213)

The necessary and sufficient condition for non-trivial solutions to the homogeneous system of equations is that the determinant to the coefficient matrix is equal to 0. This leads to the characteristic equation

\[ \det(\lambda A + B) = 0 \]  

(3-214)
(3-214) represents a polynomial equation of the 2nth degree in \( \lambda \). The roots of this equation determine the eigenvalues \( \lambda_j, j = 1, \ldots, 2n \). For each eigenvalue a non-trivial solution \( \Psi^{(j)} \) to (3-213) exists, which is determined within an arbitrary factor. The eigenvalues \( \lambda_j \) and the corresponding eigenvector \( \Psi^{(j)} \) are either both real or both complex.

(3-213) is of the same form as (3-42). \( B \) corresponds to \( K \), \( A \) corresponds to \( M \) and \( \lambda \) corresponds to \( -\omega^2 \). Similar to \( M \) and \( K \) the matrices \( A \) and \( B \) are both symmetric. This property and the assumption of simple eigenvalues were necessary in order to prove the orthogonality properties (3-150), (3-151). Hence, the following orthogonality conditions must also be valid in case the eigenvalues \( \lambda_i \) are all simple

\[
\Psi^{(i)^T} A \Psi^{(j)} = \begin{cases} 0, & i \neq j \\ m_j, & i = j \end{cases} \tag{3 - 215}
\]

\[
\Psi^{(i)^T} B \Psi^{(j)} = \begin{cases} 0, & i \neq j \\ -\lambda_j m_j, & i = j \end{cases} \tag{3 - 216}
\]

where

\[
m_j = \Psi^{(j)^T} A \Psi^{(j)} \tag{3 - 217}
\]

\( m_j \) is denoted the damped modal mass, which is a complex quantity whenever \( \Psi^{(j)} \) is complex. As was the case for the modal mass (3-152) at expansion in undamped eigenmodes, \( m_j \) is a pure mathematical symbol without physical substance. It follows from complex conjugation of (3-213) that \( (\lambda_j^*, \Psi^*_j) \) is a solution to the eigenvalue problem, whenever \( (\lambda_j, \Psi_j) \) is a solution, cf. (3-45). Hence, the solutions to (3-213) are either real or complex conjugates in pairs. The total number of complex eigenvalues is then an even number. Since the total number of eigenvalues is 2n, the number of real eigenvalues is also even.

The decisive difference between the eigenvalue problems (3-42) and (3-213) is that neither \( A \) nor \( B \) are positive definite in contrast to \( M \) and \( K \). In the proof for the theorem 3-1 the property that \( M \) is positive definite was used to exclude the possibility (3-51), so (3-50) must be valid with necessity in order to fulfil (3-49). If \( \lambda_j \) is complex then \( \Psi^{(j)*} \neq \Psi^{(j)} \). According to (3-215) og (3-216) it then follows that

\[
(\Psi^{(j)*})^T A \Psi^{(j)} = (\Psi^{(j)*})^T B \Psi^{(j)} = 0. \tag{3 - 219}
\]

Since the condition analogue to (3-50) is no longer fulfilled, because complex eigenvalues mean that \( \lambda_j \neq \lambda_j^* \), then the condition analogue to (3-51) must necessarily be fulfilled in order to fulfil the condition analogue to (3-49).

The eigenvalues are written as

\[
\lambda_j = -\mu_j + i\nu_j \tag{3 - 218}
\]

where \( \mu_j \) and \( \nu_j \) are real. Then the \( k \)th component of the solutions (3-212) can be written

\[
z_k(t) = \text{Re}(\Psi_k^{(j)} e^{(-\mu_j + i\nu_j)t}) = \text{Re}\left(\left|\Psi_k^{(j)}\right| e^{-i\theta_k^{(j)}} e^{(-\mu_j + i\nu_j)t}\right) =
\]
\[
\left| \Psi_k^{(j)} \right| e^{-\mu_j t} \cos(\nu_j t - \alpha_k^{(j)}) , \quad k = 1, \ldots, 2n \quad (3-219)
\]

\[
\tan(\alpha_k^{(j)}) = -\frac{\text{Im}(\Psi_k^{(j)})}{\text{Re}(\Psi_k^{(j)})} , \quad k = 1, \ldots, 2n \quad (3-220)
\]

The eigenvibrations (3-219) must all die away for \( t \rightarrow \infty \), since the system has been assumed to be dissipative (\( C \) is positive definite). Then the real part \( \mu_j \) must be positive for all eigenvalues. If \( \lambda_j \) is complex then (3-219) describes an undercritically damped motion with the damped circular eigenfrequency \( \nu_j \), cf. (2-45) and fig. 2-6. Correspondingly, if \( \lambda_j \) is real (\( \nu_j = 0 \)), then (3-218) represents an overcritically damped motion, cf. (2-42) and fig. 2-8.

From (3-207) and (3-212) it follows that the eigenvectors must have the form

\[
\Psi = \begin{bmatrix} \Phi \\ \lambda \Phi \end{bmatrix} \quad (3-221)
\]

Upon insertion of (3-209) and (3-221) into (3-213) it is seen that \( \Phi \) is the solution to the eigenvalue problem

\[
(\lambda^2 M + \lambda C + K) \Phi = 0 \quad (3-222)
\]

(3-222) is another homogeneous system of linear equations. The characteristic equation becomes

\[
\det(\lambda^2 M + \lambda C + K) = 0 \quad (3-223)
\]

(3-222) is a non-linear (quadratic) eigenvalue problem of the order \( n \). The roots \( \lambda_j \) to the characteristic equations (3-214) and (3-223) are identical, and the eigenvectors \( \Phi^{(j)} \) to (3-222) form half of the eigenvectors \( \Psi^{(j)} \) to (3-213). Despite the doubled size one usually prefer to solve the eigenvalue problem (3-213), because effective numerical algorithms are available for linear eigenvalue problems. A straightforward solution of (3-222) via the solution of the characteristic equation (3-223) is only possible for small size problems. The eigenvectors \( \Phi^{(j)} \) of the dimension \( n \) to the quadratic eigenvalue problem (3-222) are denoted the damped eigenmodes. The naming is motivated by the fact that (3-222) determines the same undamped eigenmodes as (3-42) if \( C = 0 \). From (3-209), (3-217), (3-221) it follows that the damped modal mass can be written

\[
m_j = \Phi^{(j)T} C \Phi^{(j)} + 2 \lambda_j \Phi^{(j)T} M \Phi^{(j)} \quad (3-224)
\]

In what follows it is assumed that all eigenvibrations are undercritically damped. Additionally, for the sake of simplicity the eigenvalues are assumed to be simple. The
eigenmodes are assembled in the following complex modal matrix of the dimension $n \times 2n$

$$
Q = [\Phi^{(1)} \Phi^{(1)*} \Phi^{(2)} \Phi^{(2)*} \ldots \Phi^{(n)} \Phi^{(n)*}] \quad (3-225)
$$

$\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(n)}$ are the eigenmodes for which the corresponding eigenvalues have positive imaginary part, i.e. $\nu_j > 0$. Then the eigenmodes $\Phi^{(1)*}, \Phi^{(2)*}, \ldots, \Phi^{(n)*}$ are all associated with eigenvalues with negative imaginary part. Further, it is assumed that the eigenmodes $\Phi^{(1)}, \Phi^{(2)}, \ldots, \Phi^{(n)}$ have been ordered according to ascending values of $\nu_j$, i.e. $0 < \nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$. The solution to (3-206) is a real $2n$-dimensional vector. Consider the following tentative expansion for $z(t)$

$$
z(t) = \sum_{j=1}^{n} \Psi^{(j)} q_j(t) + \Psi^{(j)*} q_j^*(t) \quad (3-226)
$$

Notice that the right-hand side of (3-222) is always a real quantity. (3-226) can be written in the following equivalent forms

$$
z(t) = 2 \sum_{j=1}^{n} \text{Re} \left( \Psi^{(j)} q_j(t) \right) \quad (3-227)
$$

$$
z(t) = 2 \sum_{j=1}^{n} \left( \text{Re}(\Psi^{(j)}) \text{Re}(q_j(t)) - \text{Im}(\Psi^{(j)}) \text{Im}(q_j(t)) \right) \quad (3-228)
$$

(3-228) may be interpreted as an expansion of the state vector $z(t)$ in the basis made up of the $2n$ linearly independent base vectors $\text{Re}(\Psi^{(j)}), \text{Im}(\Psi^{(j)}), j = 1, \ldots, n$, with the coordinates $2\text{Re}(q_j(t))$ and $-2\text{Im}(q_j(t))$. For this reason the quantities $q_j(t)$ will be denoted the damped modal coordinates. Hence, for suitable choice of the real- and imaginary part of the coefficients $q_j(t)$, (3-226) will be a valid representation of $z(t)$. Notice that $q_j(t)$ is complex for modes with undercritically damped eigenvibrations, and real for modes with critically or overcritically damped eigenvibrations.

(3-226) is inserted into the differential equation and initial values of (3-206)

$$
\begin{align*}
\sum_{j=1}^{n} A(\Psi^{(j)} \dot{q}_j + \Psi^{(j)*} \dot{q}_j^*) &+ \sum_{j=1}^{n} B(\Psi^{(j)} q_j + \Psi^{(j)*} q_j^*) = F(t) \quad , \quad t > 0 \\
\sum_{j=1}^{n} (\Psi^{(j)} q_j(0) + \Psi^{(j)*} q_j^*(0)) &= z_0
\end{align*}
\quad (3-229)
$$

The differential equation of (3-229) is premultiplied by $\Psi^{(i)}$ and the initial value equation is premultiplied by $\Psi^{(i)*} A$. The vector $\Psi^{(i)}, i = 1, \ldots, n$ is different from any of the vectors $\Psi^{(j)*}, j = 1, \ldots, n$ as well as the vectors $\Psi^{(j)}, j \neq i$. Hence, $\Psi^{(i)}$ is orthogonal
to any of these vectors weighted with the matrices $A$ or $B$. From (3-215), (3-216) it then follows that

$$
\begin{align*}
\dot{q}_i - \lambda_i q_i &= \frac{1}{m_i} \Psi^{(i)T} F(t) , \quad t > 0 \\
q_i(0) &= \frac{1}{m_i} \Psi^{(i)T} A z_0
\end{align*}
$$

(3-230)

is a complex differential equation of the 1st order, which is equivalent to two real coupled 1st order differential equations for the real and imaginary parts of $q_i(t)$, respectively. This should be compared with the case of undamped modal coordinates determined by the real differential equation of the 2nd order (3-187), which is also equivalent to 2 differential equations of the 1st order. The solution of (3-230) reads

$$
q_i(t) = e^{\lambda_i t} \left( \int_0^t e^{-\lambda_i \tau} \frac{1}{m_i} \Psi^{(i)T} F(\tau) d\tau + q_i(0) \right) , \quad i = 1, \ldots, n
$$

(3-231)

From (3-208), (3-221) it follows that

$$
\Psi^{(i)T} F(\tau) = \Phi^{(i)T} f(\tau)
$$

(3-232)

(3-207), (3-209), (3-221) provide the following solution for the initial values $q_i(0)$ as given by (3-230)

$$
q_i(0) = \frac{1}{m_i} \left[ \Phi^{(i)} \right]^T \left[ \begin{array}{cc} C & M \\ M & 0 \end{array} \right] \left[ \begin{array}{c} x_0 \\ \dot{x}_0 \end{array} \right] =
$$

$$
\frac{1}{m_i} \Phi^{(i)T} ((C + \lambda_i M)x_0 + M\dot{x}_0) , \quad i = 1, \ldots, n
$$

(3-233)

The displacement $x(t)$ follows from (3-207), (3-221), (3-227)

$$
x(t) = 2 \sum_{j=1}^n \text{Re}(\Phi^{(j)T} q_j(t))
$$

(3-234)

Finally, insertion of (3-231), (3-232), (3-233) into (3-234) provides the solution

$$
x(t) = 2 \sum_{j=1}^n \text{Re}\left( \frac{1}{m_j} \Phi^{(j)T} \Phi^{(j)T} \left[ \int_0^t e^{\lambda_j (t-\tau)} f(\tau) d\tau + e^{\lambda_j t} (C + \lambda_j M)x_0 + e^{\lambda_j t} M\dot{x}_0 \right] \right) =
$$

$$
(h(t)M + h(t)C)x_0 + h(t)M\dot{x}_0 + \int_0^t h(t - \tau) f(\tau) d\tau
$$

(3-235)
\[
\begin{align*}
\mathbf{h}(t) &= 2 \sum_{j=1}^{n} \text{Re} \left( \frac{e^{\lambda_j t}}{m_j} \Phi(j) \Phi(j)^T \right) = \\
&= 2 \sum_{j=1}^{n} e^{-\nu_j t} \left( \text{Re} \left( \frac{1}{m_j} \Phi(j) \Phi(j)^T \right) \cos(\nu_j t) - \text{Im} \left( \frac{1}{m_j} \Phi(j) \Phi(j)^T \right) \sin(\nu_j t) \right) \\
\end{align*}
\]

(3-235) is identical to (3-143) which was derived without the assumption of symmetric damping matrices. (3-236) represents the analytical solution for the impulse response matrix based on the expansion into damped eigenmodes. This solution is more general than the corresponding solution (3-195) based on expansion into undamped eigenmodes, because (3-236) merely presumes that \( C \) is symmetric. The property \( C = C^T \) is necessary in order to prove the orthogonality properties (3-215), (3-216).

The \textit{quasi-static response} is obtained upon ignoring \( \dot{q}_i \) in (3-230) leading to

\[
\begin{align*}
q_i(t) &= \frac{1}{-\lambda_i m_i} \Psi(i)^T \mathbf{f}(t) = \frac{1}{-\lambda_i m_i} \Phi(i)^T \mathbf{f}(t) \\
\end{align*}
\]

(3-237)

Insertion of (3-237) into (3-234) provides

\[
\begin{align*}
\mathbf{x}(t) &= 2 \sum_{j=1}^{n} \text{Re} \left( \Phi(j) \frac{1}{-\lambda_j m_j} \Phi(j)^T \mathbf{f}(t) \right) = \left( 2 \sum_{j=1}^{n} \text{Re} \left( \frac{1}{-\lambda_j m_j} \Phi(j) \Phi(j)^T \right) \right) \mathbf{f}(t) \\
\end{align*}
\]

(2-238)

Since the quasi-static response is given as \( \mathbf{x}(t) = \mathbf{K}^{-1} \mathbf{f}(t) \) it follows from (2-238) that

\[
\begin{align*}
\mathbf{K}^{-1} &= 2 \sum_{j=1}^{n} \text{Re} \left( \frac{1}{-\lambda_j m_j} \Phi(j) \Phi(j)^T \right) \\
\end{align*}
\]

(3-239) represents the analogy to Mercer's theorem (3-198) based on expansion into undamped eigenmodes.

Assume that the decoupling condition (3-184) is fulfilled upon expansion into undamped eigenmodes. Then the representations (3-161), (3-163) and (3-185) are valid for the matrices \( \mathbf{M}, \mathbf{K} \) and \( \mathbf{C} \). Upon insertion of these representations, into the characteristic equation (3-223) one has

\[
\begin{align*}
\det \left( \lambda^2 (\mathbf{P}^{-1})^T \mathbf{m} \mathbf{p}^{-1} + \lambda (\mathbf{P}^{-1})^T \mathbf{c} \mathbf{p}^{-1} + (\mathbf{P}^{-1})^T \mathbf{k} \mathbf{p}^{-1} \right) &= \\
\det \left( (\mathbf{P}^{-1})^T \right) \det \left( \lambda^2 \mathbf{m} + \lambda \mathbf{c} + \mathbf{k} \right) \det \left( \mathbf{P}^{-1} \right) &= 0 \\
\Rightarrow
\end{align*}
\]
\[ \text{det} \left( \lambda^2 \mathbf{m} + \lambda \mathbf{c} + \mathbf{k} \right) = \text{det}(\mathbf{d}) = 0 \quad (3-240) \]

\[ \mathbf{d} = \lambda^2 \mathbf{m} + \lambda \mathbf{c} + \mathbf{k} = \]
\[
\begin{bmatrix}
(\lambda^2 + 2\zeta_1 \omega_1 \lambda + \omega_1^2)M_1 & 0 & \cdots & 0 \\
0 & (\lambda^2 + 2\zeta_2 \omega_2 \lambda + \omega_2^2)M_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & (\lambda^2 + 2\zeta_n \omega_n \lambda + \omega_n^2)M_n
\end{bmatrix}
\]

(3-241)

where the well-known result from matrix algebra \( \text{det}(\mathbf{A}\mathbf{B}) = \text{det}(\mathbf{A})\text{det}(\mathbf{B}) \) has been used at the derivation of (3-240). (3-241) follows upon insertion of the diagonal matrices \( \mathbf{m}, \mathbf{k} \) and \( \mathbf{c} \) as given by (3-162), (3-165) and (3-186). (3-240), (3-241) provide the characteristic equation

\[ \prod_{j=1}^{n} \left( \lambda^2 + 2\zeta_j \omega_j \lambda + \omega_j^2 \right)M_j = 0 \quad \Rightarrow \]
\[ \begin{aligned}
\lambda_j \\
\lambda_j^* \end{aligned} = -\zeta_j \omega_j \pm i\omega_j \sqrt{1 - \zeta_j^2} , \quad j = 1, \ldots, n \quad (3-242) \]

Hence, in case the decoupling condition (3-184) is fulfilled the parameters \( \mu_j \) and \( \nu_j \) defined in (3-218) become

\[ \mu_j = \zeta_j \omega_j , \quad \nu_j = \omega_j \sqrt{1 - \zeta_j^2} \quad (3-243) \]

where \( \omega_j \) are the undamped circular eigenfrequencies and \( \zeta_j \) are the modal damping ratios as given by (3-182). In this case \( \nu_j \) can be interpreted as the modal damped circular eigenfrequency \( \omega_{d,j} \) given by (3-189).
Example 3-13: Forced Vibrations of 2 DOF System

Fig. 3-16: 2 degrees of freedom system with close circular eigenfrequencies. a) Definitions of geometrical and physical parameters. b) Forces on free beam.

Fig. 3-16 shows a continuous plane horizontal massless linear elastic Bernoulli-Euler beam $ABCD$, free of damping and with the constant bending stiffness $EI$. Beams $AB$ and $CD$ have the length $a$, whereas beam $BC$ has the length $c$. At points $A$ and $D$ point masses $m$ and linear viscous dampers with the damping coefficients $c_1 = c$ and $c_2 = 2c$ are attached. Further, points $A$ and $D$ are connected with a linear viscous damping element with the damping constants $c$, active against relative displacements of $A$ and $D$. All damping elements are active only against vertical velocities of their points of attachment. The mass at the point $A$ is subjected to a vertical ramp force $f(t)$ with the maximum value $f_0$ and the duration $\Delta t$. The sign of the force is defined in fig. 3-16a. The system is starting from the rest in the static equilibrium state at the time $t = 0$, and only small vertical displacements are considered.

Since the beam $ABCD$ is assumed massless and free of damping, the system has 2 degrees of freedom, which are selected as the vertical displacements $x_1(t)$ and $x_2(t)$ of the points $A$ and $B$ from their static equilibrium state with signs as shown in fig. 3-16b. Next, the beam is cut free from the damping elements, and the damping forces $c_1 \ddot{x}_1 + c(\dot{x}_1 - \dot{x}_2)$ and $c_2 \ddot{x}_2 + c(\dot{x}_2 - \dot{x}_1)$ are applied as external forces with positive sign in opposite direction of the selected degrees of freedom. Further, the inertial forces $-m\ddot{x}_1$ and $-m\ddot{x}_2$ are applied as external loads according to d’Alembert’s principle. The equations of motion read, cf. (3-1)

$$
x_1(t) = \delta_{11}(f(t) - m\ddot{x}_1 - c\dot{x}_1 - c(\dot{x}_1 - \dot{x}_2)) + \delta_{12}(-m\ddot{x}_2 - c_2\dot{x}_2 - c(\dot{x}_2 - \dot{x}_1))
$$

$$
x_2(t) = \delta_{21}(f(t) - m\ddot{x}_1 - c\dot{x}_1 - c(\dot{x}_1 - \dot{x}_2)) + \delta_{22}(-m\ddot{x}_2 - c_2\dot{x}_2 - c(\dot{x}_2 - \dot{x}_1))
$$

(3-244)

The flexibility coefficients are given as, see (B-3), (B-4)

$$
D = \begin{bmatrix}
\delta_{11} & \delta_{12} \\
\delta_{21} & \delta_{22}
\end{bmatrix} = \frac{a^3}{6EI} \begin{bmatrix}
2(1+\varepsilon) & \varepsilon \\
\varepsilon & 2(1+\varepsilon)
\end{bmatrix}
$$

(3-245)
The equations of motion for the displacement of points $A$ and $D$ can then be written in the standard form (3-35) with

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad \mathbf{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad \mathbf{C} = c \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \quad \mathbf{K} = \mathbf{D}^{-1} = \frac{EI}{a^3} \begin{bmatrix} 6 & 2(1+\varepsilon) & -\varepsilon \\ -\varepsilon & 2(1+\varepsilon) \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} f(t) \right\}$$

Undamped circular eigenfrequencies $\omega_j$ and eigenmodes $\Phi^{(i)} = \begin{bmatrix} \phi_1^{(i)} \\ \phi_2^{(i)} \end{bmatrix}$ are obtained from, cf. (3-42)

$$\begin{bmatrix} 2(1+\varepsilon) - \lambda_j \\ -\varepsilon \\ 2(1+\varepsilon) - \lambda_j \end{bmatrix} \begin{bmatrix} \phi_1^{(i)} \\ \phi_2^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\lambda_j = \frac{(4 + 8\varepsilon + 3\varepsilon^2) ma^2}{6 EI \omega_j^2}$$

The frequency condition then becomes

$$\left(2(1+\varepsilon) - \lambda_j\right)^2 - \varepsilon^2 = 0 \Rightarrow \lambda_j = \begin{cases} 2 + \varepsilon, & j = 1 \\ 2 + 3\varepsilon, & j = 2 \end{cases} \Rightarrow \omega_j = \Omega_j(\varepsilon)\omega_0, \ j = 1, 2$$

where

$$\omega_0 = \sqrt{\frac{3}{\sqrt{\frac{EI}{ma^3}}} \left(\frac{4 + 2\varepsilon}{4 + 8\varepsilon + 3\varepsilon^2}, \ j = 1 \right)}$$

$$\Omega_j(\varepsilon) = \begin{cases} \sqrt{\frac{4 + 2\varepsilon}{4 + 8\varepsilon + 3\varepsilon^2}}, & j = 1 \\ \sqrt{\frac{4 + 6\varepsilon}{4 + 8\varepsilon + 3\varepsilon^2}}, & j = 2 \end{cases}$$

$\omega_0$ is identified as the undamped circular eigenfrequency of beam $AB$ fixed at point $B$. As seen the circular eigenfrequencies approach $\omega_0$ from below, as $\varepsilon \to 0$, i.e. $\Omega_j(\varepsilon) < 1$. The undamped eigenmodes are normalized as follows

$$\Phi^{(i)} = \begin{bmatrix} \phi_1^{(i)} \\ 1 \end{bmatrix}$$

The component $\Phi_1^{(j)}$ is determined from the 2nd equation of (3-247)

$$-\varepsilon \phi_1^{(j)} + (2(1+\varepsilon) - \lambda_j) \cdot 1 = 0 \Rightarrow \phi_1^{(j)} = \frac{2(1+\varepsilon) - \lambda_j}{\varepsilon} = \begin{cases} 1, & j = 1 \\ -1, & j = 2 \end{cases}$$
Of course, (3-253) follows immediately from symmetry conditions. The modal masses, modal loads and modal damping ratios follow from (3-152), (3-173), (3-182), (3-183)

\[ M_j = \Phi^{(j)T} M \Phi^{(j)} = 2m, \quad j = 1, 2 \quad (3-254) \]

\[ F_j(t) = \Phi^{(j)T} f(t) = \Phi^{(j)} \cdot f(t), \quad j = 1, 2 \quad (3-255) \]

\[ \zeta_j = \frac{\Phi^{(j)T} C \Phi^{(j)}}{2\omega_j \cdot M_j} = \zeta_0 \cdot \left\{ \begin{array}{ll}
\frac{3}{2\Omega_1(e)} & , \quad j = 1 \\
\frac{7}{2\Omega_2(e)} & , \quad j = 2
\end{array} \right. \quad (3-256) \]

\[ \zeta_{12} = \frac{\Phi^{(1)T} C \Phi^{(2)}}{2\sqrt{\omega_1 \omega_2 M_1 M_2}} = \zeta_0 \frac{1}{2\sqrt{\Omega_1(e) \Omega_2(e)}} \quad (3-257) \]

where

\[ \zeta_0 = \frac{c}{2\omega_0 m} \quad (3-258) \]

\( \zeta_0 \) is identified as the damping ratio of beam AB fixed at point B, if only the damping element with damping constant \( c_1 = c \) is acting on the mass at point A. As seen \( \zeta_{12} = \sqrt{\frac{\zeta_1 \zeta_2}{\sqrt{\omega_1 \omega_2}}}. \) Since \( \zeta_{12} \) in the extreme case is given by the physical bound \( \zeta_{12} = \sqrt{\zeta_1 \zeta_2} \) as indicated by (3-203) the modal coupling of the present example should be classified as small to moderate. Despite \( \zeta_{12} \neq 0 \), decomposition into undamped modal coordinates will be applied at first. Since the system starts with the initial conditions \( x_0 = \dot{x}_0 = 0 \) the response becomes, cf. (3-170), (3-188)

\[ x(t) = \sum_{j=1}^{2} \Phi^{(j)} q_j(t) = \begin{bmatrix} q_1(t) - q_2(t) \\ q_1(t) + q_2(t) \end{bmatrix} \quad (3-259) \]

\[ q_j(t) = \int_0^t h_j(t - \tau) F_j(\tau) d\tau = \Phi^{(j)} \int_0^t h_j(t - \tau) f(\tau) d\tau = \Phi^{(j)} \int_{\min(t, \Delta t)}^t \frac{e^{-\zeta_j \omega_j (t - \tau)}}{2m \omega \omega_j} \sin(\omega \omega_j (t - \tau)) f_0 \frac{\tau}{\Delta t} d\tau = \Phi^{(j)} \frac{f_0}{2m \omega \omega_j \Delta t} g_1(t, \Delta t, \zeta_j \omega_j, \omega \omega_j) \quad (3-260) \]

where the modal impulse response function \( h_j(t) \) is given by (3-190) and the damped circular eigen-frequency \( \omega \omega_j \) by (3-189). The function \( g_1(t, \Delta t, a, b) \) is defined by

\[
g_1(t, \Delta t, a, b) = \int_{\min(t, \Delta t)}^t e^{-a(t-\tau)} \sin(b(t-\tau)) \tau d\tau = \int_{t_0}^t e^{-a u \sin(\tau)} (t - u) du =
\]

\[
\left[ -e^{-au} \frac{a \sin(bu) + b \cos(bu)}{a^2 + b^2} (t - u) + e^{-au} \frac{(a^2 - b^2) \sin(bu) + 2ab \cos(bu)}{(a^2 + b^2)^2} \right]_{t_0}^t
\]

\[ t_0 = t - \min(t, \Delta t) \quad (3-261) \]

\[ \int_{t_0}^t e^{-au} \sin(\tau) (t - u) du =
\]

\[ \int_{t_0}^t e^{-au} \sin(\tau) (t - u) du =
\]

\[ \int_{t_0}^t e^{-au} \sin(\tau) (t - u) du =
\]
Since \( C = G^T \) the exact solution may alternatively be obtained by decomposition into damped modal coordinates. Initially, the eigenvalues \( \lambda_j \) and damped eigenmodes \( \Phi^{(j)} \) are obtained from the quadratic eigenvalue problem (3-322). Introducing (3-250) and (3-258) into (3-222) leads to

\[
(\mu_j^2 I + \mu_j C_0 + K_0)\Phi^{(j)} = 0
\]  
(3-263)

\[
\mu_j = \frac{\lambda_j}{\omega_0}
\]

\[
C_0 = \frac{1}{\omega_0^2} C = 2\zeta_0 \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}
\]

\[
K_0 = \frac{1}{\omega_0^2} K = \frac{2}{4 + 8\varepsilon + 3\varepsilon^2} \begin{bmatrix} 2(1 + \varepsilon) & -\varepsilon \\ -\varepsilon & 2(1 + \varepsilon) \end{bmatrix}
\]  
(3-264)

(3-263) is solved by solving the equivalent linear eigenvalue problem (3-213) of dimension 4. Analytical solutions for \( \Phi^{(j)} \) and \( \mu_j = \frac{\lambda_j}{\omega_0} \) as functions of \( \varepsilon \) and \( \zeta_0 \) are hardly possible. Instead a numerical solution by means of MATLAB is applied. The normalization \( \Phi_2^{(j)} = 1 \), \( j = 1, 2 \), is used again as in (3-252). However, the component \( \Phi_1^{(j)} \) now becomes complex. For various combinations of \( (\varepsilon, \zeta_0) \) the results for \( \Phi_1^{(j)} \) and the modal masses \( m_j = \Phi_1^{(j)T} C \Phi_1^{(j)} + 2\lambda_j \Phi_1^{(j)T} M \Phi_1^{(j)} = m_{\omega_0} (\Phi_1^{(j)T} C_0 \Phi_1^{(j)} + 2\lambda_j \Phi_1^{(j)T} M \Phi_1^{(j)}) \) have been indicated in table 3-1.

<table>
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<tr>
<th>((\varepsilon, \zeta_0))</th>
<th>(\Phi_1^{(1)})</th>
<th>(\lambda_1 / \omega_0)</th>
<th>(m_1 / m_{\omega_0})</th>
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<tr>
<td>((0.01,0.01))</td>
<td>1.573773+0.167356i</td>
<td>-0.013873+0.992378i</td>
<td>-1.046557+6.847186i</td>
</tr>
<tr>
<td>((0.01,0.10))</td>
<td>-0.615654+0.006265i</td>
<td>-0.361798+0.923021i</td>
<td>0.014685+2.562985i</td>
</tr>
<tr>
<td>((0.10,0.01))</td>
<td>1.067862+0.205396i</td>
<td>-0.014796+0.932866i</td>
<td>-0.819166+3.910936i</td>
</tr>
<tr>
<td>((0.10,0.10))</td>
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<td>-0.361354+0.903821i</td>
<td>0.132605+2.458690i</td>
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<tr>
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<td>-0.149944+0.632396i</td>
<td>-0.120565+2.531504i</td>
</tr>
<tr>
<td>((1.00,0.10))</td>
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<td>-1.220793+2.855643i</td>
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</table>

<table>
<thead>
<tr>
<th>((\varepsilon, \zeta_0))</th>
<th>(\Phi_1^{(2)})</th>
<th>(\lambda_2 / \omega_0)</th>
<th>(m_2 / m_{\omega_0})</th>
</tr>
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<td>-0.626794+0.066985i</td>
<td>-0.036127+0.996650i</td>
<td>0.167662+2.767361i</td>
</tr>
<tr>
<td>((0.01,0.10))</td>
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<td>-0.132802+0.983100i</td>
<td>-0.113563+7.126805i</td>
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<tr>
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<td>-0.035204+0.974671i</td>
<td>0.634901+4.660034i</td>
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<tr>
<td>((0.10,0.10))</td>
<td>1.601856+0.160906i</td>
<td>-0.138460+0.945262i</td>
<td>0.958191+6.543822i</td>
</tr>
<tr>
<td>((1.00,0.01))</td>
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<td>-0.035006+0.815531i</td>
<td>0.196597+3.285806i</td>
</tr>
<tr>
<td>((1.00,0.10))</td>
<td>1.655790+0.257495i</td>
<td>-0.354480+0.724255i</td>
<td>0.530168+1.976761i</td>
</tr>
</tbody>
</table>

Table 3-1: Components \( \Phi_1^{(j)} \), eigenvalues \( \lambda_j \) and modal masses \( m_j \) as a function of \( (\varepsilon, \zeta_0) \).

The response follows from (3-231) and (3-234) as

\[
x(t) = 2 \sum_{j=1}^{2} \text{Re}(\Phi^{(j)}_1 q_1(t)) = 2 \left[ \frac{\text{Re}(\Phi_1^{(1)} q_1(t) + \Phi_1^{(2)} q_2(t))}{\text{Re}(q_1(t) + q_2(t))} \right]
\]  
(3-265)

\[
q_j(t) = \frac{1}{m_j} \int_0^t e^{\lambda_j(t-\tau)} \Phi^{(j)}_1 T(t) d\tau = \frac{\Phi^{(j)}_1}{m_j} \int_0^t e^{\lambda_j(t-\tau)} f(\tau) d\tau
\]
\[
\frac{\Phi^{(j)}}{m_j} \min(t,\Delta t) \int_0^\min(t,\Delta t) e^{\lambda_j (t-\tau)} \frac{f_0}{\Delta t} d\tau = \frac{\Phi^{(j)}}{m_j} \int_0^\Delta t g_2(t,\Delta t, \lambda_j) \quad (3-266)
\]

\[
g_2(t,\Delta t, \lambda) = \int_0^{\min(t,\Delta t)} e^{\lambda(t-\tau)} \tau d\tau = \int_0^t e^{\lambda u} (t-u) du = \left[ e^{\lambda u} \left( \frac{1}{\lambda^2} + \frac{t-u}{\lambda} \right) \right]_0^t \quad (3-267)
\]

where \(t_0\) is given by (3-262)

The solutions for \(x_1(t)\) and \(x_2(t)\), obtained by (3-259) and (3-265), respectively, have been compared in fig. 3-17 for the combinations of \(\varepsilon\) and \(\zeta_0\) indicated in table 3-1. The approximate solutions (3-259) are shown as a dashed line, whereas the exact solutions (3-265) are shown as an unbroken line. The duration of the ramp function has been chosen as \(\Delta t = \frac{1}{10} T_0\), \(T_0 = \frac{2\pi}{\omega_0}\). Both \(x_1(t)\) and \(x_2(t)\) are normalized with respect to the quantity \(\frac{f_0 \Delta t}{m \omega_0}\), and the time is normalized with respect to \(T_0 = \frac{2\pi}{\omega_0}\). As seen in figures 3-17a, 3-17b, 3-17c, which all display the response of lightly damped systems, the agreement for the solutions for \(x_1(t)\) eventually become better as \(\varepsilon\) is increased, and the undamped circular eigenfrequencies become more separated. However, as seen the responses for \(x_1(t)\) of the strongly damped systems shown in figures 3-17d, 3-17e and 3-17f are deviating for all considered values of \(\varepsilon\).

In all cases the solutions for \(x_2(t)\) are in better agreement than are the solutions for \(x_1(t)\). All the indicated observations can be explained with reference to (3-191) that expansion in undamped modal coordinated requires well-separated circular eigenfrequencies and a lightly damped system. The first of these requirements is not fulfilled for the cases shown in figures 3-17a and 3-17b and the second is not fulfilled in figures 3-17d, 3-17e and 3-17f.
Fig. 3-17: Normalized displacements $\frac{x_1(t)\omega_0}{f_0\Delta t}$ and $\frac{x_2(t)\omega_0}{f_0\Delta t}$ as functions of $(\varepsilon, \zeta_0)$. —— : Expansion in damped eigenmodes. —- : Expansion in undamped eigenmodes. a) $(\varepsilon, \zeta_0) = (0.01, 0.01)$. b) $(\varepsilon, \zeta_0) = (0.1, 0.01)$. c) $(\varepsilon, \zeta_0) = (1.0, 0.01)$. 
Fig. 3-17 (continued): d) $(\varepsilon, \zeta_0) = (0.01, 0.1)$. e) $(\varepsilon, \zeta_0) = (0.1, 0.1)$. f) $(\varepsilon, \zeta_0) = (1.0, 0.1)$. 
3.8 System Reduction

Generally, the calculation time for dynamic structural problems is much longer than for comparable static problems, because the time dependence increases the number of independent variable by one. In the previous sections 3.6 and 3.7 analytical methods have been indicated for the determination of the response of a system of \( n \) degrees of freedom. Both methods were based on a change of description from Cartesian coordinates to expansions of the response vector in various modal bases. The method in section 3.6 decouples the mass- and stiffness matrices, whereas modal couplings in general still persist via the damping matrix. The method in section 3.7 decouples all 3 system matrices, if only the damping matrix is symmetric. Both analytical methods presume that all \( n \) damped or undamped eigenmodes are known. Generally, the calculation time for obtaining these modes increases with the number of degrees of freedom as \( n^3 \). For this reason one may be interested in techniques, which can reduce the number of dynamic degrees of freedom from \( n \) to \( n_1 \leq n \), reducing the calculation time by the factor \((\frac{n_1}{n})^3\). A procedure which reduces the number of degrees of freedom is referred to as a system reduction scheme. In most cases system reduction is related with a loss of accuracy. The minimization of the inherent inaccuracy can only be achieved if the remaining dynamic degrees of freedom are selected from physical understanding of the dynamic behaviour of the structure. First, symmetric structures are considered, which may be reduced without any loss of accuracy.

![Fig. 3-18: Plane symmetric structure. a) Partitioning of dynamic loads into anti-symmetric and symmetric components. b) Equivalent systems for analysis of anti-symmetric vibrations. c) Equivalent system for analysis of symmetric vibrations.](image)
Fig. 3-18 shows a plane structure of 10 degrees of freedom for which the geometry as well as the elasticity-, damping- and mass distributions are symmetric around a certain plane, whereas the external dynamic loads need not fulfil any symmetry properties. However, as shown in fig. 3-18a, the load may be partitioned into an anti-symmetric and a symmetric load distribution. The anti-symmetric load distribution causes zero vertical displacement and a point of inflection at the material points originally placed in the symmetry plane. Hence, the anti-symmetric vibrations can be analysed by the system shown in fig. 3-18b with 5 degrees of freedom. Notice that the mass \( \frac{1}{2}m_3 \) is related to the horizontal degree of freedom \( x_5 \), and that the vertical load \( F_2(t) \) placed in the symmetry plane can be totally ignored. The symmetric load distribution causes zero horizontal displacements and zero tangential slopes at the material points in the symmetry plane. Symmetric vibrations can then be analysed by the system shown in fig. 3-18c with 5 degrees of freedom. The mass \( \frac{1}{2}m_3 \) is now related to the vertical degree of freedom \( x_6 \), and the horizontal load \( F_3(t) \) acting in the symmetry plane is ignored. It follows from the example that if a system has a structural symmetry plane, the system of \( n \) degrees of freedom can be reduced to two equivalent systems of \( \frac{n}{2} \) degrees of freedom. In doing this the calculation effort is reduced by a factor \( 2 \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{4} \). Notice that additional to the symmetry with respect to geometry and elasticity as in statics, symmetry requirements in dynamics also include symmetric mass and damping distributions.

(3-181) indicates the differential equations for the modal coordinates in the basis made up of the undamped eigenmodes. Assume that the excitation time is given by the interval \([0, T]\). Formally, the load vector may then be considered periodic with the interval length \( T \) as period, and a harmonic expansion as given by (3-106) can be determined. In general, harmonic components with large magnitude amplitude will influence the response more than small ones. It is thus assumed that harmonic amplitudes \( F_m \) of significant magnitude have been identified from an initial harmonic analysis. The corresponding circular frequencies \( \omega_m \) are confined to a certain closed interval. Within this interval a relatively small number \( n_1 \) of undamped circular eigenfrequencies of the structure is assumed to be positioned. Especially these \( n_1 \) modes are likely to contribute significantly to the response, since dynamic amplification due to resonance acts together with relatively large harmonic amplitudes. The corresponding \( n_1 \) modal coordinates will be denoted the dynamic modal coordinates. For the remaining \( n - n_1 \) modal coordinates the influence of inertial and damping forces on the dynamic response is ignorable, and the modal load \( F_i(t) \) in (3-181) is primarily balanced by the modal elastic restoring force \( \omega_i^2 M_i q_i(t) \). For this reason these degrees of freedom are denoted the quasi-static modal coordinates. The eigenmodes are numbered, so the dynamic modal coordinates form the first \( n_1 \) degrees of freedom. Ignoring \( \ddot{q}_i \) and \( \dddot{q}_i \) for \( i = n_1 + 1, \ldots, n \) implies that (3-181) can be written

\[
\ddot{q}_i + 2 \omega_i \left( c_i \dot{q}_i + \sum_{j=1,\neq i}^{n_1} \sqrt{\frac{\omega_j M_j}{\omega_i M_i}} \zeta_{ij} \dot{q}_j \right) + \omega_i^2 q_i = \frac{1}{M_i} F_i(t) \quad , \quad i = 1, \ldots, n_1 (3 - 268a)
\]
\[ q_i(t) = \frac{1}{\omega_i^2 M_i} F_i(t) = \frac{1}{\omega_i^2 M_i} \Phi^{(i)T} f(t), \quad i = n_1 + 1, \ldots, n \]  

(3-268b)

where (3-173) has been used. Insertion of (3-268b) into (3-170) and use of (3-198) provide:

\[
\begin{align*}
\mathbf{x}(t) &= \sum_{i=1}^{n_1} q_i(t) \Phi^{(i)} + \sum_{i=n_1+1}^{n} q_i(t) \Phi^{(i)} \\
&= \sum_{i=1}^{n_1} q_i(t) \Phi^{(i)} + \left( \sum_{i=n_1+1}^{n} \frac{1}{\omega_i^2 M_i} \Phi^{(i)T} \right) f(t) \\
&= \sum_{i=1}^{n_1} q_i(t) \Phi^{(i)} + \left( \sum_{i=1}^{n_1} \frac{1}{\omega_i^2 M_i} \Phi^{(i)T} - \sum_{i=n_1+1}^{n} \frac{1}{\omega_i^2 M_i} \Phi^{(i)T} \right) f(t) \\
&= \sum_{i=1}^{n_1} q_i(t) \Phi^{(i)} + \left( \mathbf{K}^{-1} - \sum_{i=1}^{n_1} \frac{1}{\omega_i^2 M_i} \Phi^{(i)T} \right) f(t)
\end{align*}
\]

(3-269)

The transformation (3-269) merely requires knowledge of \((\omega_i, \Phi^{(i)}), i = 1, \ldots, n_1,\) along with the flexkisity matrix \(\mathbf{K}^{-1}\). The dynamic modal coordinates \(q_i(t), i = 1, \ldots, n_1,\) are determined from the coupled, but reduced system of differential equations (3-268a). The last term on the right-hand side of (3-269) represents the quasi-static part of the response. As seen from (3-198), this term becomes increasingly less important as \(n_1\) approaches \(n\). Occasionally, this part of the response is ignored, resulting in the truncated series:

\[
\mathbf{x}(t) \simeq \sum_{i=1}^{n_1} q_i(t) \Phi^{(i)}
\]

(3-270)

In practice, the validity of (3-270) is verified by testing the sensitivity of the estimate on \(\mathbf{x}(t)\), when \(n_1\) is increased by 1.

In the same way the modal coordinates in case of expansion in damped eigenmodes may be separated into \(n_1\) dynamic and \(n - n_1\) quasi-static modal coordinates. The quasi-static modal coordinates are given by (3-237). A derivation similar to (3-269) is performed for the series (3-234). Use of (3-237) and (3-239) then provides:

\[
\mathbf{x}(t) = 2 \sum_{i=1}^{n_1} \text{Re} \left( q_i(t) \Phi^{(i)} \right) + \left( \mathbf{K}^{-1} - 2 \sum_{i=1}^{n_1} \text{Re} \left( \frac{1}{\lambda_i m_i} \Phi^{(i)T} \right) \right) f(t)
\]

(3-271)

The retained dynamic modal coordinates are unchanged given by (3-231).
Both of the indicated system reduction schemes are based on a separation of the modal coordinates into dynamic and quasi-static degrees of freedom, where the dynamic degrees of freedom are assumed to be affected by both inertial, damping and elastic forces, whereas only elastic restoring forces affect the quasi-static degrees of freedom. Guyan\(^1\) has indicated an alternative reduction scheme, based on an analogue separation of the Cartesian coordinates \(x(t)\) into a sub-vector of degrees of freedom \(x_1(t)\) influenced by both inertial-, damping- and elastic forces, and a sub-vector \(x_2(t)\) for which the external loads are totally balanced by the elastic restoring forces. The dimension of \(x_1(t)\) is \(n_1\) and the dimension of \(x_2(t)\) is \(n_2 = n - n_1\). On this assumption (3-35) can approximately be written in the form

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} + \begin{bmatrix}
C_{11} & 0 \\
0 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
K_{11} & K_{12} \\
K_{12}^T & K_{22}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} = \begin{bmatrix}
f_1(t) \\
f_2(t)
\end{bmatrix}, \quad t > 0
\]  

(3-272)

\(-M_{11}\ddot{x}_1\) and \(-C_{11}\dot{x}_1\) denote the inertial and damping loads acting on the dynamic degrees of freedom \(x_1(t)\). \(f_1(t)\) og \(f_2(t)\) are sub-vectors of \(f(t)\), indicating the external loads acting in the degrees of freedom \(x_1(t)\) and \(x_2(t)\), respectively. \(K_{11}, K_{12}\) and \(K_{22}\) signify the sub-matrices in the partitioned stiffness matrix \(K\). From the last matrix equation in (3-272) it follows that

\[
K_{12}^T x_1 + K_{22} x_2 = f_2(t) \quad \Rightarrow
\]

\[
x_2(t) = K_{22}^{-1} f_2(t) - K_{22}^{-1} K_{12}^T x_1(t)
\]

(3-273)

is inserted into the first matrix equation of (3-272).

\[
M_{11}\ddot{x}_1 + C_{11}\dot{x}_1 + \left( K_{11} - K_{12} K_{22}^{-1} K_{12}^T \right) x_1 = f_1(t) - K_{12} K_{22}^{-1} f_2(t)
\]

(3-274)

is a matrix differential equation on the standard form (3-35). However, the dimension has been reduced to \(n_1 < n\). The reduced stiffness matrix is given as \( \tilde{K}_{11} = K_{11} - K_{12} K_{22}^{-1} K_{12}^T \), and the reduced load vector is given as \( \tilde{f}_1(t) = f_1(t) - K_{12} K_{22}^{-1} f_2(t) \).

At first \(x_1(t)\) is determined from (3-274). The remaining quasi-static degrees of freedom \(x_2(t)\) are next determined from the linear transformation (3-273).

Alternatively, Guyan reduction is denoted static condensation. In this case \(x_1(t)\) and \(x_2(t)\) are denoted master and slave degrees of freedom, respectively.

---

Example 3-15: Guyan Reduction for Two-Storey Frame

Fig. 3-19: Two-storey plane frame with 6 degrees of freedom.

The plane frame in fig. 3-19 consists of Bernoulli-Euler beams. All beams have constant bending stiffness $EI$ and constant mass per unit length $\mu$. The beams are assumed to be infinitely stiff against axial deformations. The frame then has 6 degrees of freedom, which are selected as shown in the figure.

The corresponding mass- and stiffness matrices can be shown to be

\[
M = \frac{\mu a}{420} \begin{bmatrix}
1044 & 108 & 0 & -13a & -13a & 0 \\
108 & 732 & 13a & -22a & -22a & 13a \\
0 & 13a & 12a^2 & -3a^2 & 0 & -3a^2 \\
-13a & -22a & -3a^2 & 8a^2 & -3a^2 & 0 \\
-13a & -22a & 0 & -3a^2 & 8a^2 & -3a^2 \\
0 & 13a & -3a^2 & 0 & -3a^2 & 12a^2 \\
\end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix} \quad (3-275)
\]

\[
K = \frac{EI}{a^3} \begin{bmatrix}
48 & -24 & 0 & 6a & 6a & 0 \\
-24 & 24 & -6a & -6a & -6a & -6a \\
0 & -6a & 12a^2 & 2a^2 & 0 & 2a^2 \\
6a & -6a & 2a^2 & 8a^2 & 2a^2 & 0 \\
6a & -6a & 0 & 2a^2 & 8a^2 & 2a^2 \\
0 & -6a & 2a^2 & 0 & 2a^2 & 12a^2 \\
\end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{bmatrix} \quad (3-276)
\]

The mass matrix (3-275) is the so-called consistent mass matrix, which is considered to provide the most accurate distribution of inertial loads on the structure for a given number of degrees of freedom. The formulation of consistent mass matrices is explained in section 5.2. In the present example, the two lowest circular eigenfrequencies of the system will be calculated by a Guyan reduction of the order $n_1 = 2$, and the result is compared to the exact solution of the system defined by the system matrices (3-275), (3-276). In the lowest modes significant inertial loads are acting along the storey beams, whereas the inertial loads induced by the rotational degrees of freedom are relatively small. For this reason the storey displacements $x_1(t)$ and $x_2(t)$ are selected as master degrees of freedom, whereas the slave degrees of freedom are made up of the nodal point rotations $x_1(t), x_2(t), x_3(t)$ and $x_4(t)$, i.e.

\[
x_1(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad x_2(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} \quad (3-277)
\]
The corresponding partitioning of the system matrices has been indicated in (3-275) and (3-276). At first the inverse matrix $K_{22}^{-1}$ is calculated

$$K_{22}^{-1} = \alpha \frac{1}{476EI} \begin{bmatrix} 43 & -12 & 5 & -8 \\ -12 & 67 & -18 & 5 \\ 5 & -18 & 67 & -12 \\ -8 & 5 & -12 & 43 \end{bmatrix}$$ (3 - 278)

The reduced stiffness matrix then becomes

$$\tilde{K}_{11} = K_{11} - K_{12} K_{22}^{-1} K_{12}^T = \frac{42}{119} \frac{EI}{a^3} \begin{bmatrix} 115 & -50 \\ -50 & 38 \end{bmatrix}$$ (3 - 279)

The characteristic equation (3-42) for the reduced system becomes

$$\det(\tilde{K}_{11} - \omega^2 M_{11}) = 0 \ \Rightarrow \ \det\left(\begin{bmatrix} 115 - 522\lambda & -50 - 54\lambda \\ -50 - 54\lambda & 38 - 366\lambda \end{bmatrix}\right) = 0 , \ \lambda = \omega^2 \frac{119}{42 \cdot 210} \frac{\mu a^4}{EI} \Rightarrow$$

$$188136\lambda^2 - 67326\lambda + 1870 = 0 \ \Rightarrow$$

$$\lambda = \begin{cases} 0.327509 , & j = 1 \\ 0.030349 , & j = 2 \end{cases} \ \Rightarrow$$

$$\omega_j = \begin{cases} 1.4998 \sqrt{\frac{EI}{\mu a^4}} , & j = 1 \\ 4.9269 \sqrt{\frac{EI}{\mu a^4}} , & j = 2 \end{cases}$$ (3 - 280)

The exact eigenvalues of the system (3-275) and (3-276) can be shown to be

$$\omega_j = \begin{cases} 1.4994 \sqrt{\frac{EI}{\mu a^4}} , & j = 1 \\ 4.9361 \sqrt{\frac{EI}{\mu a^4}} , & j = 2 \\ 12.2788 \sqrt{\frac{EI}{\mu a^4}} , & j = 3 \\ 20.4939 \sqrt{\frac{EI}{\mu a^4}} , & j = 4 \\ 22.6599 \sqrt{\frac{EI}{\mu a^4}} , & j = 5 \\ 43.4139 \sqrt{\frac{EI}{\mu a^4}} , & j = 6 \end{cases}$$ (3 - 281)

As seen the first two circular eigenfrequencies are calculated with ignorable errors by the reduced system.
Fig. 3-20: Two-storey plane frame. Lumped mass model.

The fact that the inertial loads in the two lower modes are primarily acting in the translational degrees of freedom is the basis of the so-called lumped mass method. The mass of the storeys and the adjacent columns is lumped in discrete masses of magnitude \( m_1 = 3 \mu a \) and \( m_2 = 2 \mu a \) in the translational degrees of freedom as shown in figure 3-20. In this case the mass matrix becomes

\[
\begin{bmatrix}
3 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^{T} & M_{22} \end{bmatrix}
\]

(3-282)

Hence, \( M_{12} = M_{22} = 0 \), so both the lumped mass method and the Guyan reduction scheme are based on the assumption that the inertial loads caused by \( M_{12} \) and \( M_{22} \) are ignorable. The circular eigenfrequencies for the systems (3-279), (3-282) become

\[
\omega_j = \begin{cases}
1.4650 \sqrt{\frac{EI}{\mu a^4}}, & j = 1 \\
4.2531 \sqrt{\frac{EI}{\mu a^4}}, & j = 2
\end{cases}
\]

(3-283)

As seen, larger errors are introduced by the indicated rather crude modelling of the inertial loads in the master degrees of freedom. However, the lumped mass method may still work fine for the lower modes of multi-storey buildings, using the storey displacements as master degrees of freedom, i.e. for cases where \( n_1 \gg 2 \).

### 3.9 Damping Models

In most commercial computer programs dynamic analyses are based on the system reduction schemes (3-269) or (3-270) for the undamped modal coordinates. Since modal decoupling is implicitly assumed, the damping properties of the system are defined completely from a limited number \( n_1 \) of modal damping ratios \( \zeta_1, \zeta_2, \ldots, \zeta_{n_1}, n_1 \leq n \). The
modal damping ratios have either been measured on an existing structure for which further dynamic analysis is going to be performed, or have been estimated for a design proposal of a new building based on measurements from comparable structures. Typically, the measurement of these damping ratios is performed by means of the half-band width method described in example 2.4, where the half-band width is measured on each modal frequency response function (3-197). Although not needed in the described semi-analytical solution methods, a synthesized damping matrix may still be requested in some numerical algorithms for the solution of (3-35). In this connection it is demonstrated in this section how a class of damping matrices may be constructed, for which the decoupling condition (3-184) is fulfilled. For ease, the eigenvalues $\omega_i^2$ of (3-42) are all assumed to be simple.

First, the case is considered where all $n$ damping ratios are available. Then, the damping matrix is uniquely synthesized by (3-185).

Next, the other extreme is considered, where only two modal damping ratios $\zeta_1$ and $\zeta_2$ are available from measurements. It may then be attempted to write the damping matrix as a linear combination of the mass- and the stiffness matrix, i.e.

$$C = a_0 M + a_1 K$$  \hspace{1cm} (3 - 284)

The model (3-284) is denoted Rayleigh's damping model or proportional damping. From (3-150), (3-151) it follows that

$$\Phi^{(i)T} C \Phi^{(j)} = a_0 \Phi^{(i)T} M \Phi^{(j)} + a_1 \Phi^{(i)T} K \Phi^{(j)} =$$

$$\begin{cases} 
0, & i \neq j \\
 a_0 M_i + a_1 \omega_i^2 M_i, & i = j 
\end{cases}$$  \hspace{1cm} (3 - 285)

Consequently, the damping model (3-284) permits modal decoupling. Comparison with (3-184) shows that the modal damping ratios are related to the coefficients $a_0$ and $a_1$ by the relation

$$\zeta_i = \frac{1}{2\omega_i} \left( a_0 + a_1 \omega_i^2 \right) = \frac{a_0}{2\omega_i} + \frac{a_1}{2} \omega_i, \quad i = 1, \ldots, n$$  \hspace{1cm} (3 - 286)

Assuming $\zeta_1$ og $\zeta_2$ to be known, $a_0$ and $a_1$ can then be calibrated from (3-286) for $i = 1, 2$

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2\omega_1} & \frac{1}{2} \omega_1 \\ \frac{1}{2\omega_2} & \frac{1}{2} \omega_2 \end{bmatrix} \begin{bmatrix} \omega_2 & -\omega_1 \\ -\frac{1}{\omega_2} & \frac{1}{\omega_1} \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix}$$  \hspace{1cm} (3 - 287)

With $a_0$ and $a_1$ determined by (3-287), the model (3-284) provides the exact modal damping ratios for the first two modes. The remaining modal damping ratios of the
model are determined by (3-286) for \( i = 3, \ldots, n \). However, these formal damping ratios cannot be given any physical significance, i.e. they will in general be different from the damping ratios of the real structure, if these have been measured.

The orthogonality property (3-159) for the symmetric matrix \( K_m = (K M^{-1})^m M \) is a generalization of the orthogonality properties (3-150) and (3-151) for \( M \) and \( K \). For the case where a number \( k \) of modal damping ratios, \( 2 < k < n \), one may then generalize (3-285) by writing the damping matrix as a linear combination of \( k \) arbitrary, but different matrices of the type \( K_m \)

\[
C = \sum_{p=1}^{k} a_{m_p} K_{m_p} \tag{3-288}
\]

where the \( k \) different indices \( m_1, m_2, \ldots, m_k \) can be selected arbitrarily. The generalization (3-288) is denoted Caughey’s damping model\(^1\). Rayleigh’s damping model is obtained as the special case of \( k = 2 \) with \( m_1 = 0 \) and \( m_2 = 1 \). From (3-159) and (3-288) it follows that

\[
\Phi^{(i)}^T C \Phi^{(j)} = \sum_{p=1}^{k} a_{m_p} \Phi^{(i)}^T K_{m_p} \Phi^{(j)} = \begin{cases} 0, & i \neq j \\ \left( \sum_{p=1}^{k} a_{m_p} \omega_i^{2m_p} \right) M_i, & i = j \end{cases} \tag{3-289}
\]

Comparison with (3-184) provides the following relation for the modal damping ratios

\[
\zeta_i = \frac{1}{2\omega_i} \sum_{p=1}^{k} a_{m_p} \omega_i^{2m_p} = \frac{1}{2} \sum_{p=1}^{k} a_{m_p} \omega_i^{2m_p-1}, \quad i = 1, \ldots, n \tag{3-290}
\]

Assume, that \( k = 3 \) so that \( \zeta_1, \zeta_2, \zeta_3 \) are known. If the indices \( m_p \) are chosen as \( m_1 = 0, m_2 = 1 \) og \( m_3 = 2 \), the following damping model is obtained

\[
C = a_0 M + a_1 K + a_2 K M^{-1} K \tag{3-291}
\]

\( a_0, a_1, a_2 \) are determined from the linear equations

\[
\begin{bmatrix}
\frac{1}{\omega_1} & \omega_1^3 \\
\frac{1}{\omega_2} & \omega_2^3 \\
\frac{1}{\omega_3} & \omega_3^3
\end{bmatrix}
\begin{bmatrix}
a_0 \\ a_1 \\ a_2
\end{bmatrix}
= 
\begin{bmatrix}
\zeta_1 \\ \zeta_2 \\ \zeta_3
\end{bmatrix} \tag{3-292}
\]

Alternatively, one may choose \( m_1 = -1, m_2 = 0 \) og \( m_3 = 1 \), which leads to the damping model

\[
C = a_{-1} M K^{-1} M + a_0 M + a_1 K \tag{3-293}
\]

where it has been used that $K_{-1} = (KM^{-1})^{-1}M = MK^{-1}M$, cf. the comment subsequent to (3.59). $a_{-1}, a_0, a_1$ are determined from the linear equations

$$\frac{1}{2} \begin{bmatrix} \frac{1}{\xi_1^3} & \frac{1}{\xi_2^3} & \omega_1 \\ \frac{1}{\xi_1^2} & \frac{1}{\xi_2^2} & \omega_2 \\ \frac{1}{\xi_1} & \frac{1}{\xi_2} & \omega_3 \end{bmatrix} \begin{bmatrix} a_{-1} \\ a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}$$

(3-294)

If $n > 3$ then both (3.291) and (3.293) will provide the first three modal damping ratios exactly, whereas the damping ratios $\zeta_i, i = 4, \ldots, n$ will be mutually different and different from the real damping ratios of the structure. However, if $n = 3$ one may ask whether the representations (3.291) and (3.293) will be identical. The answer to this question is affirmative. In the following example 3-16 it is shown that any n-term Caughey series will give the same damping matrix, no matter which indices $m_1, \ldots, m_n$ are selected, if only the corresponding expansion coefficients $a_{m_1}, \ldots, a_{m_n}$ are calibrated by (3.290).

**Example 3-16: Cayley-Hamilton’s Theorem and the Uniqueness of n-term Caughey Series**

The characteristic polynomial (3.43) reads

$$p(\omega^2) = \det(K - \omega^2 M) = (-1)^n I_0(\omega^2)^n + (1)^{n-1} I_1(\omega^2)^{n-1} + \cdots + (-1)I_{n-1} \omega^2 + I_n$$

(3-295)

where the coefficients $I_m$ are the invariants of the eigenvalue problem. Especially, $I_0 = \det(M)$ and $I_n = \det(K)$. Assume that the terms $(\omega^2)^m$ in the characteristic equation are replaced by the matrix $K_m = (KM^{-1})^m M = (P^{-1})^T k_m P^{-1}$, cf. (3.164). $P$ is the modal matrix of the undamped eigenmodes (3.83) and the diagonal matrix $k_m$ is given by (3.166). Notice, that $k_0$ and $k_1$ are equal to the diagonal matrices $m$ and $k$ as given by (3.162) and (3.165), respectively. It then follows that

$$(-1)^n I_0 K_n + (-1)^{n-1} I_1 K_{n-1} + \cdots + (-1)I_{n-1} K_1 + I_n K_0 =$$

$$(P^{-1})^T \left((-1)^n I_0 k_n + (-1)^{n-1} I_1 k_{n-1} + \cdots + (-1)I_{n-1} k_1 + I_n k_0\right) P^{-1} =$$

(3-296)

where the diagonal matrix $p$ is given as

$$p = \begin{bmatrix} p(\omega_1^2)M_1 & 0 & \cdots & 0 \\ 0 & p(\omega_2^2)M_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p(\omega_n^2)M_n \end{bmatrix}$$

(3-297)

Since $\omega_i^2$ is an eigenvalue of (3.42) it follows from (3.295) that $p(\omega_i^2) = 0, i = 1, \ldots, n$, so $p = 0$. Hence, the following matrix identity is obtained, which is known as *Cayley-Hamilton’s theorem*

$$(-1)^n I_0 k_n + (-1)^{n-1} I_1 k_{n-1} + \cdots + (-1)I_{n-1} k_1 + I_n k_0 = 0$$

(3-298)
(3-298) is pre-multiplied by \( \text{KM}^{-1} \). Since \( \text{K}_{m+1} = \text{KM}^{-1} \text{K}_m \), the following expansion is obtained

\[
\text{K}_{n+1} = \frac{I_1}{I_0} \text{K}_n + (-1) \frac{I_2}{I_0} \text{K}_{n-1} + \cdots + (-1)^{n-1} \frac{I_n}{I_0} \text{K}_1 = \\
\sum_{p=1}^{n} b_p^{(n+1)} \text{K}_p , \quad b_p^{(n+1)} = (-1)^{n-p} \frac{I_{n-p+1}}{I_0}
\]  \hspace{1cm} (3 - 299)

Next, (3-299) is pre-multiplied by \( \text{KM}^{-1} \), which leads to

\[
\text{K}_{n+2} = b_n^{(n+1)} \text{K}_{n+1} + \sum_{p=1}^{n-1} b_p^{(n+1)} \text{K}_{p+1} + b_n^{(n+1)} \text{K}_n = \\
\sum_{p=1}^{n} b_p^{(n+2)} \text{K}_p , \quad b_p^{(n+2)} = b_n^{(n+1)} b_p^{(n+1)} + b_p^{(n+1)} , \quad b_0^{(n+1)} = 0
\]  \hspace{1cm} (3 - 300)

where \( \text{K}_{n+1} \) has been eliminated by the use of (3-299). (3-299) and (3-300) show that \( \text{K}_{n+1} \) and \( \text{K}_{n+2} \) can both be expressed as a linear expansion of the \( n \) matrices \( \text{K}_p, p = 1, \ldots, n \). Proceeding with multiple premultiplication it can sequentially be proved that any matrix \( \text{K}_m, m = n + 1, n + 2, \ldots \) can be expressed by a linear expansion of the same matrices. From (3-298) follows that

\[
\text{K}_0 = \text{M} = (-1)^{n-1} \frac{I_{n-1}}{I_n} \text{K}_1 + (-1)^{n-2} \frac{I_{n-2}}{I_n} \text{K}_2 + \cdots + (-1)^{n-1} \frac{I_1}{I_n} \text{K}_{n-1} + (-1)^{n-1} \frac{I_0}{I_n} \text{K}_n = \\
\sum_{p=1}^{n} b_p^{(0)} \text{K}_p , \quad b_p^{(0)} = (-1)^{p-1} \frac{I_{n-p}}{I_n}
\]  \hspace{1cm} (3 - 301)

(3-301) is pre-multiplied by \( (\text{KM}^{-1})^{-1} = \text{MK}^{-1} \). Since \( \text{K}_{m-1} = (\text{KM}^{-1})^{-1} \text{K}_m \), the following expansion is obtained for \( \text{K}_{-1} \)

\[
\text{K}_{-1} = b_1^{(0)} \text{K}_0 + \sum_{p=2}^{n} b_p^{(0)} \text{K}_{p-1} = b_1^{(0)} \sum_{p=1}^{n} b_p^{(0)} \text{K}_p + \sum_{p=1}^{n-1} b_p^{(0)} \text{K}_{p+1} = \\
\sum_{p=1}^{n} b_p^{(-1)} \text{K}_p , \quad b_p^{(-1)} = b_1^{(0)} b_p^{(0)} + b_p^{(0)} , \quad b_p^{(0)} = 0
\]  \hspace{1cm} (3 - 302)

where \( \text{K}_0 \) has been eliminated by the use of (3-301). Proceeding with multiple pre-multiplications by \( (\text{KM}^{-1})^{-1} \) it can sequentially be shown that any matrix \( \text{K}_m, m = 0, -1, -2, \ldots \) can be expressed as a linear combination of the \( n \) matrices \( \text{K}_p, p = 1, \ldots, n \). Next, consider an \( n \)-term Caughey series

\[
\text{C} = \sum_{p=1}^{n} a_{mp} \text{K}_{mp}
\]  \hspace{1cm} (3 - 303)

Some of the indices \( m_p \) may be outside the range 1, \ldots, \( n \). In these cases the corresponding matrices \( \text{K}_{mp} \) in (3-303) are replaced by an equivalent linear expansion in the matrices \( \text{K}_p, p = 1, \ldots, n \) as described above. It then follows that any \( n \)-term Caughey series can always be reduced to the form

\[
\text{C} = \sum_{p=1}^{n} b_p \text{K}_p
\]  \hspace{1cm} (3 - 304)
The coefficients $b_p$ are calibrated by the use of (3-290)

$$\zeta_i = \frac{1}{2} \sum_{p=1}^{n} b_p \omega_i^{2p-1}, \quad i = 1, \ldots, n \quad (3-305)$$

(3-305) represents $n$ linear equations from which the $n$ unknown coefficients $b_p$ can be uniquely determined (since the eigenvalues have been assumed to be simple, and $\omega_i$ thus is different). All $n$-term Caughey series of the type (3-288) are then reducible to one and the same unique series (3-304). Any $n$-term Caughey series, for which the expansion coefficients $a_{m_p}$ are determined from (3-290), will then represent one and the same damping matrix.

### 3.10 Rayleigh’s Fraction

Let $x$ signify the displacement, which is caused by the static load $f$, i.e.

$$Kx = f \quad (3-306)$$

According to (3-170) the displacement vector $x$ can be expanded into the the basis made up of the linear independent undamped eigenmodes in the following way

$$x = q_1 \Phi^{(1)} + q_2 \Phi^{(2)} + \cdots + q_n \Phi^{(n)} \quad (3-307)$$

Next, the following fraction may be formulated

$$\omega^2_R = \frac{x^T f}{x^T M x} = \frac{x^T K x}{x^T M x} \quad (3-308)$$

If (3-307) is inserted into (3-308) it follows from (3-150), (3-151) that

$$\omega^2_R = \frac{\sum_{i=1}^{n} \sum_{j=1}^{n} q_i q_j \Phi^{(i)^T} K \Phi^{(j)}}{\sum_{i=1}^{n} \sum_{j=1}^{n} q_i q_j \Phi^{(i)^T} M \Phi^{(j)}} = \frac{q_1^2 M_1 \omega_1^2 + q_2^2 M_2 \omega_2^2 + \cdots + q_n^2 M_n \omega_n^2}{q_1^2 M_1 + q_2^2 M_2 + \cdots + q_n^2 M_n} \quad (3-309)$$

The previously described ordering of the circular eigenfrequencies $0 < \omega_1 \leq \omega_2 \leq \cdots \leq \omega_n$ is applied. Consequently, the following bounds prevail

$$\omega^2_R \geq \frac{q_1^2 M_1 \omega_1^2 + q_2^2 M_2 \omega_1^2 + \cdots + q_n^2 M_n \omega_1^2}{q_1^2 M_1 + q_2^2 M_2 + \cdots + q_n^2 M_n} = \omega_1^2 \quad (3-310)$$

$$\omega^2_R \leq \frac{q_1^2 M_1 \omega_n^2 + q_2^2 M_2 \omega_n^2 + \cdots + q_n^2 M_n \omega_n^2}{q_1^2 M_1 + q_2^2 M_2 + \cdots + q_n^2 M_n} = \omega_n^2 \quad (3-311)$$
corresponding to
\[
\omega_1 \leq \omega_R \leq \omega_n^2
\] (3-312)

\( \omega_R^2 \) is denoted the Rayleigh fraction and the bounding (3-312) specifies Rayleigh's principle. The numerator of this fraction is equal to the maximum potential energy, and the denominator times \( \omega_R^2 \) signifies the maximum kinetic energy, when the system is performing harmonic motions with the amplitude vector \( \mathbf{x} \) and the circular frequency \( \omega_R \).

Based on the eigenvalue (3-60) the following fraction analog to (3-308) may be calculated
\[
\omega_R^{2m} = \frac{\mathbf{x}^T \mathbf{K}_m \mathbf{x}}{\mathbf{x}^T \mathbf{M} \mathbf{x}}
\] (3-313)

where \( \mathbf{K}_m = (\mathbf{K} \mathbf{M}_m)^{-1} \mathbf{M} \). \( \omega_R^{2m} \) should be interpreted as a new symbol, rather than \( \omega_R \) as given by (3-308) raised to the power \( m \). The rationale for this symbol becomes obvious from the relation (3-315) and (3-316) below. Insertion of (3-307) into (3-313) and use of the orthogonality properties (3-150) and (3-159) provide the following relation analogue to (3-309)
\[
\omega_R^{2m} = \sum_{i=1}^{n} \sum_{j=1}^{n} q_i q_j \Phi^{(i)^T} \mathbf{K}_m \Phi^{(j)} = \frac{q_1^2 M_1 \omega_1^{2m} + q_2^2 M_2 \omega_2^{2m} + \cdots + q_n^2 M_n \omega_n^{2m}}{q_1^2 M_1 + q_2^2 M_2 + \cdots + q_n^2 M_n}, \quad m = \pm 1, \pm 2, \ldots
\] (3-314)

Depending on whether \( m > 0 \) or \( m < 0 \), one has the orderings \( \omega_1^{2m} \leq \omega_2^{2m} \leq \cdots \leq \omega_n^{2m} \) and \( \omega_1^{2m} \geq \omega_2^{2m} \geq \cdots \geq \omega_n^{2m} \), respectively. Corresponding to (3-310) the following bounds then prevail for the fundamental circular eigenfrequency
\[
\omega_R^{2m} \geq \frac{q_1^2 M_1 \omega_1^{2m} + q_2^2 M_2 \omega_1^{2m} + \cdots + q_n^2 M_n \omega_1^{2m}}{q_1^2 M_1 + q_2^2 M_2 + \cdots + q_n^2 M_n} = \omega_1^{2m}, \quad m = 1, 2, \ldots
\] (3-315)
\[
\omega_R^{2m} \leq \frac{q_1^2 M_1 \omega_1^{2m} + q_2^2 M_2 \omega_1^{2m} + \cdots + q_n^2 M_n \omega_1^{2m}}{q_1^2 M_1 + q_2^2 M_2 + \cdots + q_n^2 M_n} = \omega_1^{2m}, \quad m = -1, -2, \ldots
\] (3-316)

From (3-315) and (3-316) it follows that the following upper bound is obtained for the fundamental circular eigenfrequency
\[
\omega_1 \leq \left( \frac{\mathbf{x}^T \mathbf{K}_m \mathbf{x}}{\mathbf{x}^T \mathbf{M} \mathbf{x}} \right)^{\frac{1}{2m}}, \quad m = \pm 1, \pm 2, \ldots
\] (3-317)
(3-317) is easily obtained from (3-315) when \( m > 0 \). For \( m < 0 \) (3-317) is obtained from (3-316) and (3-313) in the following few steps

\[
\frac{x^T K_m x}{x^T M x} \leq \left( \frac{1}{\omega_1} \right)^{2|m|} \quad \Rightarrow \quad \omega_1 \leq \left( \frac{x^T M x}{x^T K_m x} \right)^{\frac{1}{2|m|}} = \left( \frac{x^T M x}{x^T K_m x} \right)^{\frac{1}{2}} \quad \Rightarrow \quad \omega_1 = \left( \frac{x^T K_m x}{x^T M x} \right)^{\frac{1}{2|m|}}
\]

(3-318)

It can be proved that the right-hand side of (3-317) decreases monotonously towards \( \omega_1 \) as \( m \to -\infty \), and increases monotonously towards \( \omega_n \) as \( m \to \infty \). Especially, the case \( m = -1 \) is of interest. In this case (3-317) becomes

\[
\omega_1 \leq \left( \frac{x^T (KM^{-1}) M x}{x^T M x} \right)^{-\frac{1}{2}} = \sqrt{\frac{x^T M x}{x^T MDM x}}
\]

(3-319)

where the identity \((KM^{-1})^{-1} M = MK^{-1} M = MDM\) has been used, cf. the comments subsequent to (3-59). For the same displacement vector \( x \), (3-319) provides a sharper upper bound than the case (3-308) corresponding to \( m = 1 \), because of the mentioned monotonous decrease of the right-hand side of (3-317) with \( m \).

Rayleigh’s fraction is used to calculate an approximate value of the 1st circular eigenfrequency. First, the displacement \( x \) is determined. If the stiffness matrix \( K \) is given, where the flexibility matrix \( D \) is unknown unless it is calculated as \( K^{-1} \), it is use to estimate \( x \), so it has qualitative resemblance to the 1st undamped eigenmode \( \Phi^{(1)} \). If \( D \) is known, a load vector \( f \) is estimated with qualitative resemblance to the inertial load vector in the 1st mode \( \omega_1^2 M \Phi^{(1)} \), and \( x \) is next calculated from (3-306). \( \omega_R^2 \) is then calculated from (3-308) or (3-319) depending on whether \( K \) or \( D \) is given.

Consider the conventional eigenvalue problem (3-57) with matrix \( A = (DM)^m \) and the eigenvalues \( \lambda_k = \omega_k^{-2m}, m = 1,2, \ldots \) From linear algebra it is well-known that the trace of the matrix \( A \) is equal to the sum of its eigenvalues, i.e.

\[
\text{tr}(A) = A_{11} + A_{22} + \cdots + A_{nn} = \lambda_1 + \lambda_2 + \cdots + \lambda_n \quad \Rightarrow
\]

\[
\text{tr}((DM)^m) = \omega_1^{-2m} + \omega_2^{-2m} + \cdots + \omega_n^{-2m} \geq \omega_1^{-2m}, \quad m = 1,2, \ldots
\]

(3-320)

The last statement of (3-320) follows since all terms in the sum are positive. From (3-320) the following lower bounds are next obtained for \( \omega_1 \)

\[
\omega_1 > \left( \frac{1}{\text{tr}((DM)^m)} \right)^{\frac{1}{2m}}, \quad m = 1,2, \ldots
\]

(3-321)

The lower bounds (3-321) can be shown to increase monotonously towards \( \omega_1 \) as \( m \to \infty \).
Example 3-17: Bounds for the Fundamental Circular Eigenfrequency of Two-Storey Frame

The system described in example 3-15 with the mass- and stiffness matrices given by (3-275) and (3-276) is considered again. First the flexibility matrix corresponding to (3-276) is evaluated

\[
D = K^{-1} = \frac{1}{9240} \begin{bmatrix}
532a^2 & 700a^2 & 294a & 42a & 42a & 294a \\
700a^2 & 1610a^2 & 630a & 420a & 420a & 630a \\
294a & 630a & 1083 & -81 & 249 & 93 \\
42a & 420a & -81 & 1497 & -153 & 249 \\
42a & 420a & 249 & -153 & 1497 & -81 \\
294a & 630a & 93 & 249 & -81 & 1083
\end{bmatrix}
\]  

(3 - 322)

Notice that the upper left 2 x 2 sub-matrix \(D_{11}\) is equal to \(\tilde{K}_{11}^{-1}\), where \(\tilde{K}_{11}\) is the reduced stiffness matrix given by (3-279). A unit horizontal force applied at the top storey (i.e. in the degree of freedom \(x_2(t)\)). Then \(x = Df\) becomes equal to the second column of \(D\), i.e.

\[
f = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad x = \begin{bmatrix} 10a \\ 23a \\ 9 \\ 6 \\ 6 \\ 9 \end{bmatrix} \quad \begin{array}{c}
\text{(3 - 323)}
\end{array}
\]

Notice that (3-323) merely indicates the correct shape of the static deformation rather than the correct magnitude and dimensions from the load \(f\). Actually, multiplication of the vector \(x\) by an arbitrary factor does not affect the value of the Rayleigh fraction (3-308) and (3-313). The 2nd column of \(D\) is obtained upon multiplying the indicated vector \(x\) by the factor \(\frac{70}{9240} \frac{\alpha^2}{EI}\). The following results from (3-317) are obtained for a number of values of \(m\)

\[
\begin{align*}
\text{m} = -3 & \quad \Rightarrow \quad \omega_1 < 1.500761 \sqrt{\frac{EI}{\mu a^5}} \\
\text{m} = -2 & \quad \Rightarrow \quad \omega_1 < 1.501418 \sqrt{\frac{EI}{\mu a^5}} \\
\text{m} = -1 & \quad \Rightarrow \quad \omega_1 < 1.503083 \sqrt{\frac{EI}{\mu a^5}} \\
\text{m} = 1 & \quad \Rightarrow \quad \omega_1 < 1.539548 \sqrt{\frac{EI}{\mu a^5}} \\
\text{m} = 2 & \quad \Rightarrow \quad \omega_1 < 1.804913 \sqrt{\frac{EI}{\mu a^5}} \\
\text{m} = 3 & \quad \Rightarrow \quad \omega_1 < 3.705289 \sqrt{\frac{EI}{\mu a^5}}
\end{align*}
\]  

(3 - 324)

As seen the upper bounds decrease monotonously towards the exact value \(\omega_1 = 1.499416 \sqrt{\frac{EI}{\mu a^5}}\), cf. (3-281), as \(m \to -\infty\), and the drift towards \(\omega_6\) as \(m \to \infty\) is also noticed. Further, the improvement for the case \(m = -1\) corresponding to (3-319) in comparison to the case \(m = 1\) corresponding to (3-308) is noticed. Next, the lower bounds (3-321) are considered. At first the matrix \(DM\) is calculated

\[
DM = \frac{1}{4620 \cdot 420} \frac{\mu a^4}{EI} \begin{bmatrix}
314958 & 287826 & 5810a & -11494a & -11494a & 5810a \\
446830 & 626010 & 12670a & -22155a & -22155a & 12670a \\
186396 & 252252 & 10575 & -11163 & -7863 & 2655 \\
35868 & 142296 & -375 & 1446 & -8124 & 4575 \\
35868 & 142296 & 4575 & -8124 & 1446 & -375 \\
186396 & 252252 & 2655 & -7863 & -11163 & 10575
\end{bmatrix}
\]  

(3 - 325)
From (3-321) the following results are then obtained for various values of $m$

\[
\text{tr}(DM) = \frac{966102}{4620-4390} \frac{\mu a^4}{EI} = 0.497325 \frac{\mu a^4}{EI} \Rightarrow \omega_1 > 1.418011 \sqrt{\frac{EI}{\mu a^4}}
\]

\[
\text{tr}((DM)^2) = 0.199577 \left( \frac{\mu a^4}{EI} \right)^2 \Rightarrow \omega_1 > 1.496140 \sqrt{\frac{EI}{\mu a^4}}
\]

\[
\text{tr}((DM)^3) = 0.088066 \left( \frac{\mu a^4}{EI} \right)^3 \Rightarrow \omega_1 > 1.499219 \sqrt{\frac{EI}{\mu a^4}}
\]

(3 - 326)

The monotonous increase of the lower bounds as $m \to \infty$ is clearly noticed from the results (3-326).

### 3.11 Vibrations due to Movable Support

Vibrations of buildings caused by the dynamic motions of the foundations have significant practical importance. Smaller damages may occur due to traffic induced ground motions. Earthquakes may cause severe damage or total collapse in inappropriately designed seismic buildings.

The displacement and rotation components from all the movable foundation points of the structure are assembled in the vector $y(t)$ of dimension $m$. For beam-column structures each support point may have up to 3 translational components and 3 rotational components. $y(t)$ (or the acceleration time history $\ddot{y}(t)$) is assumed to be known. This deformation vector forms the external disturbance, which induces the dynamic motion of the structure.

At first the quasi-static response $x^{(0)}(t)$ are determined. These are the motion of the degrees of freedom from the deformations $y(t)$ of the movable support points, when inertial- and damping forces are ignored. Due to the linearity of the system it follows directly that

\[
x^{(0)}(t) = Uy(t)
\]

(3 - 327)

The $j$th column of the influence matrix $U$ of the dimension $n \times m$ is determined as the displacement in the degrees of freedom of the structure from the support motion $y_j(t) = 1$, $y_i(t) = 0$, $i \neq j$. For statically determinate structures and for some statically indeterminate structures, the corresponding motion corresponds to a stiff-body motion, which can be geometrically determined. In other cases the motion can be determined from usual statical methods. Examples of the calculation of the influence matrix of a three-storey building have been given in the following example 3-18.
Example 3-18: Determination of Influence Matrix for Three-Storey Building

The 3-storey frame described in example 3-1 is exposed to various deformations of the support points. In example a) both support points are subjected to the same horizontal motion \( y \). In example b) the foundation of the structure is rotating as a stiff body, and in example c) the support points are subjected to different horizontal displacements \( y_1 \) and \( y_2 \). The corresponding influence matrices have been indicated below the figures.

\( x(t) \) signifies the total displacements in the selected degrees of freedom. \( x(t) \) is made up of \( x^{(0)}(t) \) together with the displacements caused by the inertial forces \( f_{Ij}(t) \) and the damping forces \( f_{dij}(t) \). Hence

\[
x_i(t) = x_i^{(0)}(t) + \sum_{j=1}^{n} \delta_{ij} \left( f_{Ij} - f_{dij} \right), \quad f_{Ij} = -\sum_{j=1}^{n} M_{ij} \ddot{x}_j
\]

(3-328)

On matrix form the relation reads

\[
x = x^{(0)}(t) - D(M\ddot{x} + f_d(t)) \Rightarrow
M\ddot{x} + Kx = Kx^{(0)}(t) - f_d(t) = KUy(t) - f_d(t)
\]

(3-329)

where (3-327) has been used. The displacements of the mass points relative to the quasi-static deformations \( x^{(0)}(t) \) are defined by

\[
z(t) = x(t) - x^{(0)}(t)
\]

(3-330)

A linear viscous damping model is introduced in which the damping forces are assumed to depend on the relative velocities \( \dot{z}(t) \) rather than the total velocities \( \dot{x}(t) \), cf. (3-24)

\[
f_d(t) = C \left( \dot{x}(t) - \dot{x}^{(0)}(t) \right)
\]

(3-331)
In cases where \( \dot{x}^{(0)}(t) \) signifies a stiff-body motion, (3-331) insures that such motions will not introduce damping forces. Actually, the relevance of the damping model (3-331) can merely be argued for quasi-static motions without shape distortions such as those shown in the figs. 3-20a and 3-20b, whereas the model may be questioned for the case shown in fig. 3-20c. Using (3-330), (3-331) the following formulations are then obtained from (3-329) for the equations of motion formulated in total and relative displacements, respectively,

\[
\begin{align*}
M\ddot{x} + C\dot{x} + Kx &= C\dot{x}^{(0)}(t) + Kx^{(0)}(t) = CU\dot{y}(t) + KU\dot{y}(t) \quad (3 - 332) \\
M\ddot{\mathbf{z}} + C\dot{\mathbf{z}} + K\mathbf{z} &= -M\ddot{x}^{(0)}(t) = -MU\dot{y}(t) \quad (3 - 333)
\end{align*}
\]

The equations of motion (3-332) og (3-333) for the total and the relative displacements are both on the standard form (3-35) with the formal external dynamic load given as \( f(t) = CU\dot{y}(t) + KU\dot{y}(t) \) and \( f(t) = -MU\dot{y}(t) \), respectively. The solution methods presented in the previous sections can then be applied to both of these formulations.

Earthquake waves have wave-lengths typically in the interval 1000-5000 m. This means that the motion of all support points for ordinary building structures will be identical (in phase). Then, for horizontal ground surface accelerations, merely a single scalar quantity \( y(t) \) need to be given as shown in fig. 3-20a. For long-span suspension bridges and other buildings with large horizontal dimensions the support displacements \( y_1(t), y_2(t), \ldots \) may be out of phase and need to be specified individually corresponding to the case shown in fig. 3.20c. In what follows the first case is presumed, i.e. the ground surface acceleration of all support points can be specified by a single scalar acceleration time series \( \ddot{y}(t) \) corresponding to \( m = 1 \). At first, the relative displacement vector \( \mathbf{z}(t) \) is expanded into the base of the undamped eigenmodes, cf. (3-170)

\[
\mathbf{z}(t) = \sum_{j=1}^{n} \Phi^{(j)}q_j(t) \quad (3 - 334)
\]

(3-334) is inserted into (3-333), and the orthogonality conditions (3-150) and (3-151) are applied. Further, the modal decoupling condition (3-184) is assumed. The modal coordinates are then determined by (3-187), which is written in the following form

\[
\ddot{q}_j + 2\zeta_j\omega_j\dot{q}_j + \omega_j^2 q_j = -\beta_j\ddot{y}(t) \quad , \quad j = 1, \ldots, n \quad , \quad t > 0 \quad (3 - 335)
\]

\[
\beta_j = \frac{\Phi^{(j)T}MU}{\Phi^{(j)T}M\Phi^{(j)}} \quad j = 1, \ldots, n \quad (3 - 336)
\]

The modal differential equations (3-336) have the same form as the differential equation (2-148) for the relative displacement of an SDOF system besides the factor \( \beta_j \), which is denoted the mode participation factor. In earthquake analysis of multi-storey buildings it is common practice to normalize the modes, so the horizontal component at the
top-storey in the direction of the ground surface acceleration becomes equal to 1. Then, the modal coordinate \( q_j \) signifies the contribution from the \( j \)th mode to the relative displacement of the top-storey, \( z_{\text{top}}(t) \). With such a normalization the modal participation factors become non-dimensional, and \( \beta_j \) will in most cases be of magnitude 1.0-1.5. For the given ground surface acceleration \( \ddot{y}(t) \) the response spectrum \( R(\zeta, \omega_0) \) may be calculated as explained in example 2-7. The maximum of the modal coordinate \( q_j(t) \) then becomes

\[
\max_{t \in [0, \infty]} |q_j(t)| = \beta_j R(\zeta_j, \omega_j)
\] (3 - 337)

With the described normalization of the eigenmodes the relative top-storey displacement becomes \( z_{\text{top}}(t) = q_1(t) + \cdots + q_n(t) \). The maximum of \( z_{\text{top}}(t) \) rather than the maximum of the modal coordinates is of concern. Because the modal coordinates will not attain their maximum at the same time a simple addition of the modal maxima (3-337) will overestimate the maximum of \( z_{\text{top}}(t) \) unacceptably. Instead, the following combination rule is commonly used, which is merely empirically motivated

\[
\max_{t \in [0, \infty]} |z_{\text{top}}(t)| \approx \sqrt{\sum_{j=1}^{n} \left( \max_{t \in [0, \infty]} |q_j(t)| \right)^2} = \sqrt{\sum_{j=1}^{n} \beta_j^2 R(\zeta_j, \omega_j)^2}
\] (3 - 338)

A certain theoretical support for (3-338) is given in stochastic vibration theory, where the modal coordinates can be proved to be uncorrelated on the assumption of (3-191) for well-separated circular eigenfrequencies. Then the variance of the top-storey displacement becomes equal to the sum of the variance of the modal coordinates similar to the rule (3-338). (3-338) demonstrates how a seismic analysis of the top-storey displacement of an MDOF building may be performed based on the response spectrum for the earthquake under consideration. In the following example 3-19 it is shown how the response spectrum method may be used for the analysis of other response quantities of interest such as the bending moment in critical sections.

---

Example 3-19: Earthquake Analysis of Two-Storey Frame

Fig. 3-21: Horizontal earthquake excitation two-storey plane frame.

The two-storey frame in example 3-15 is assumed to be exposed to a horizontal ground surface acceleration $\ddot{y}(t)$ as shown in fig. 3-21. A horizontal stiff-body displacement $y(t) = 1$ causes the quasi-static deformations $x_1 = x_2 = 1$ and $x_3 = x_4 = x_5 = x_6 = 0$. With the mass matrix $M$ given by (3-275) the influence vector $U$ and the vector $MU$ become

$$U = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad MU = \begin{bmatrix} 1152 \\ 840 \\ 13a \\ -35a \\ -35a \\ 13a \end{bmatrix}$$

(3 - 339)

The structural data are selected so that $\sqrt{\frac{EI}{\mu_a}} = \frac{2}{3}\pi$ s$^{-1}$, and the eigenmodes are normalized to unit displacement at the top storey (degree of freedom $x_2 = 1$). Then, the following undamped eigenmodes $\Phi_j$, eigenfrequencies $\omega_j^2$, damping ratios $\zeta_j$ and modal participation factors $\beta_j$ are obtained, cf. (3-281), (3-336)

<table>
<thead>
<tr>
<th>Mode $j$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Phi_j$</td>
<td>0.5259</td>
<td>-1.1214</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.2236</td>
<td>0.7266</td>
</tr>
<tr>
<td></td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
<tr>
<td></td>
<td>0.4080</td>
<td>0.3080</td>
<td>1.0000</td>
<td>1.0000</td>
<td>-11.2977</td>
<td>15.2222</td>
</tr>
<tr>
<td></td>
<td>0.1873</td>
<td>1.1975</td>
<td>-1.5000</td>
<td>1.0000</td>
<td>9.8371</td>
<td>30.1384</td>
</tr>
<tr>
<td></td>
<td>0.1873</td>
<td>1.1975</td>
<td>1.5000</td>
<td>-1.0000</td>
<td>9.8371</td>
<td>30.1384</td>
</tr>
<tr>
<td></td>
<td>0.4080</td>
<td>0.3080</td>
<td>-1.0000</td>
<td>-1.0000</td>
<td>-11.2977</td>
<td>15.2222</td>
</tr>
</tbody>
</table>

| $\omega_j^2$ [Hz] | 0.9996 | 13.6626 | 15.1066 | 15.1066 | 15.1066 |
| $\zeta_j$         | 1.05 | 0.05 | 0.05 | 0.05 | 0.05 | 0.05 |
| $\beta_j$         | 1.2701 | -0.2940 | 0.0000 | 0.0000 | 0.0298 | -0.0059 |
| $R(\zeta_j, \omega_j)$ [m] | 0.111 | 0.0202 | - | - | 0.00044 | 0.00011 |

Table 3-2: Undamped eigenmodes, eigenfrequencies, modal damping ratios, mode participation factors and modal response spectrum values for two-storey frame.
As seen, high modal damping ratios of $\zeta_j = 0.05$ have been assumed. This is in accordance with common practice in earthquake engineering in order to compensate for the additional dissipation taking place during moderate plastic deformations, which are ignored in the present linear analysis. Further, it is seen that the 3rd and 4th eigenmodes are symmetric ($x_1 = x_2 = 0$ and $x_3 = -x_6$, $x_4 = -x_5$, see fig. 3-21), and that the mode participation factors for these modes are zero. This is because the considered antisymmetric horizontal earthquake loading will only excite the antisymmetric modes. The El Centro earthquake shown in fig. 2-18a is considered. With the indicated damping ratios and circular eigenfrequencies the response spectrum values shown in table 3-2 are next read in fig. 2-18b with $g_{\text{max}} = 0.348 \frac{m}{s^2}$. From (3-338) it follows that $\max |z_2(t)| = 0.14111$ m. If only the first mode is taken into consideration the corresponding result becomes $\max |z_2(t)| = 0.14098$ m. Obviously, most of the response is carried by the 1st mode. Generally, this is the case for normal high-rise building structures, which constitute the rationale for most code provisions for the earthquake resistant design of such buildings, where an SDOF model is explicitly or implicitly applied.

Next, the bending moment $M_0(t)$ shown in fig. 3-21 is requested. Assume that the frame is deformed statically as the jth eigenmode, i.e. $z = \Phi_j$. Then a corresponding bending moment of magnitude $M_0^{(j)}$ occurs at the support point. $M_0^{(j)}$, which is denoted the modal bending moment, may be interpreted as the bending moment, when $q_j(t) = 1$ and $q_i(t) = 0$, $i \neq j$. From the linearity it follows that the bending moment can be written

$$M_0(t) = \sum_{j=1}^{n} M_0^{(j)} q_j(t)$$  \hspace{1cm} (3 - 340)

Using the same combination rule as (3-338), the maximum of the bending moment then becomes

$$\max_{t \in [0, \infty]} |M_0(t)| \simeq \left( \sum_{j=1}^{n} \left( \max_{t \in [0, \infty]} |M_0^{(j)} q_j(t)| \right)^2 \right)^{1/2} = \sum_{j=1}^{n} \left( M_0^{(j)} \beta_j R(\zeta_j, \omega_j) \right)^2$$  \hspace{1cm} (3 - 341)

(3-341) indicates the calculation of the maximum bending moment in a critical section based on the response spectrum method. Compared to (3-341) one also need to calculate the modal bending moments in order to perform the analysis. These can be derived from the jth modal eigenmode $\Phi_j$ and the local stiffness- and mass matrices as shown in section 5.2.

### 3.12 Vibrations due to Indirectly Acting Dynamic Loads

![Fig. 3-22: Indirect excitation of MDOF system.](image)

Up to now it has been assumed that the external loads are acting at the masses of the structure. In this section the case is analysed, where a structural system is excited by an external force $f(t)$ acting indirectly on the mass system, see fig. 3-22, and the motion as given by the degrees of freedom $x_1, \ldots, x_n$ and the displacement of the point of action $x_0$ of the external force are determined.
The deformations $x_i$, $i = 0, 1, \ldots, n$ are produced as a sum of contributions from the external forces $f(t)$ acting along $x_0$, from the inertial forces $f_{Ij}(t)$ and from the damping forces $f_{dj}$ acting along the degree of freedom $x_j$. With the positive sign of the damping forces defined in the opposite direction of $x_j$, cf. fig. 3-1, the following identities may be formulated

$$x_0(t) = \delta_{00}f(t) + \delta_{01}(f_{I1} - f_{d1}) + \cdots + \delta_{0n}(f_{In} - f_{dn})$$  \hspace{1cm} (3-342)

$$x_1(t) = \delta_{10}f(t) + \delta_{11}(f_{I1} - f_{d1}) + \cdots + \delta_{1n}(f_{In} - f_{dn})$$  

$$\vdots$$

$$x_n(t) = \delta_{n0}f(t) + \delta_{n1}(f_{I1} - f_{d1}) + \cdots + \delta_{nn}(f_{In} - f_{dn})$$  \hspace{1cm} (3-343)

where, cf. (3-22)

$$f_{di} = \sum_{j=1}^{n} C_{ij} \ddot{x}_j \quad , \quad f_{li} = -\sum_{j=1}^{n} M_{ij} \ddot{x}_j$$  \hspace{1cm} (3-344)

(3-342), (3-343), (3-344) may be written in the following matrix form

$$x_0(t) = \delta_{00}f(t) - d_0^T\left(M\ddot{x} + C\dot{x}\right)$$  \hspace{1cm} (3-345)

$$x(t) = d_0f(t) - D\left(M\ddot{x} + C\dot{x}\right)$$  \hspace{1cm} (3-346)

where

$$d_0 = \begin{bmatrix} \delta_{10} \\ \vdots \\ \delta_{n0} \end{bmatrix}$$  \hspace{1cm} (3-347)

$$x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$  \hspace{1cm} (3-348)

Upon pre-multiplication with $K = D^{-1}$ (3-346) attains the form

$$M\ddot{x} + C\dot{x} + Kx = Kd_0f(t)$$  \hspace{1cm} (3-349)

(3-349) is on the standard form (3-35) with the formal external load given as $f(t) = Kd_0f(t)$, and can hence be analysed by the methods given in the previous sections. At
first (3-349) is solved. Then \( x_0(t) \) is determined from (3-345). As an example, assume that \( f(t) \) is harmonically varying on the form

\[
f(t) = F \cos(\omega t) = \text{Re} \left( F e^{i\omega t} \right)
\]

Then the system is described by the equations of motion (3-99) with the amplitude of the load vector given as

\[
F = Kd_0 F
\]  

(3 - 351)

The stationary response is then given by (3-100), (3-101) as

\[
x(t) = \text{Re} (Xe^{i\omega t})
\]

(3 - 352)

\[
X = H(\omega)Kd_0 F = F \sum_{j=1}^{n} \frac{\Phi^{(j)T}Kd_0}{M_j(\omega_j^2 - \omega^2 + 2\zeta_j\omega_j\omega)} \Phi^{(j)}
\]

(3 - 353)

where the modal decoupling condition (3-184) has been assumed, leading to the frequency response matrix \( H(\omega) \) as given by (3-196), (3-197). The stationary harmonic motion of the point of attack of the external force is obtained upon insertion of (3-350), (3-352), (3-353) into (3-345)

\[
x_0(t) = \text{Re} \left( X_0 e^{i\omega t} \right)
\]

(3 - 354)

\[
X_0(\omega) = F \left( \delta_{00} + \sum_{j=1}^{n} \frac{d_0^T(\omega^2 M - i\omega C)\Phi^{(j)T}Kd_0}{M_j(\omega_j^2 - \omega^2 + 2\zeta_j\omega_j\omega)} \Phi^{(j)} \right)
\]

(3 - 355)
4. VIBRATION OF BEAM ELEMENTS

In dynamics, continuous structures refer to continuous distribution of masses. At each material point of the structure a differential mass is attached. Since an infinite number of such differential masses can be identified, continuous structures are defined by an infinite number of dynamic degrees of freedom. In this section only the bending vibrations of plane Bernoulli-Euler beam elements will be considered as an example of continuous structures. The beam elements are rectilinear with the bending centres of all the cross-sections placed along a straight line (the beam axis).

4.1 Equations of Motion for Beam Element

![Diagram of beam element](image)

Fig. 4-1: Bending vibrations of differential beam element.

A local Cartesian $(x, y, z)$-coordinate system is defined for the considered beam element in the static equilibrium state. Origin is placed at one of the end sections and the $x$-axis is placed along the beam axis and orientated against the other end section. The $(x, y)$-coordinate plane is co-existent with the plane of the element. Since the length of the beam element is $l$, the end sections then have the coordinates $(0, 0, 0)$ and $(l, 0, 0)$.

In fig. 4-1 a differential beam element of the length $dx$ is shown in the static equilibrium state and in the deformed state. The element is cut free, and internal stress resultants are applied as external loads in the 2 states. In the following end sections at the abscissas $x$ and $x + dx$ are referred to as the left-hand and right-hand end sections, respectively.

The displacement of the bending centre from the statical equilibrium state at the left-hand end section is $u(x, t)$. The corresponding displacement at the right-hand end section is $u + \frac{\partial u}{\partial x}dx$.

In the statical equilibrium state an axial force $N$ is acting which is considered positive in tension.

In the deformed state a dynamic load $f(x, t)$ per unit length is acting in the $y$-direction.
At the left-hand end section of the differential beam the shear force $Q$ is applied, assumed positive in the negative $y$-direction. Further, a bending moment $M$ is acting, assumed positive in the positive $z$-direction. On the right-hand side, $Q$ and $M$ are changed differentially to $Q + \frac{\partial Q}{\partial x} dx$ and $M + \frac{\partial M}{\partial x} dx$, both assumed positive in the opposite direction of the corresponding quantities on the left-hand end section. The axial force is assumed to be unchanged equal to its referential value $N$ at both end sections, see fig. 4.1.

The equations of equilibrium are next formulated expressing the force equilibrium in the $y$-direction and the moment equilibrium about the $z$-axis for the free differential beam element

$$\frac{\partial Q}{\partial x} + f(x, t) = 0 \quad (4-1)$$

$$\frac{\partial M}{\partial x} - Q + N \frac{\partial u}{\partial x} = 0 \quad (4-2)$$

$(4-2)$ is differentiated with $x$, and $(4-1)$ is used to eliminate $\frac{\partial Q}{\partial x}$, resulting in

$$\frac{\partial^2 M}{\partial x^2} + \frac{\partial}{\partial x} \left( N \frac{\partial u}{\partial x} \right) + f(x, t) = 0 \quad (4-3)$$

The geometrical condition of the Bernoulli-Euler theory that plane sections remain plane during deformations implies that the bending moment is proportional to the curvature, i.e.

$$M(x, t) = -EI(x) \frac{\partial^2 u(x, t)}{\partial x^2} \quad (4-4)$$

$E$ is the elasticity modulus, $I(x)$ is the moment of inertia for bending motions in the $y$-direction, and the product $EI(x)$ is referred to as the bending stiffness. According to the d’Alembert principle the load per unit length $f(x, t)$ is made up of the external dynamic load per unit length $f_d(x, t)$, the inertial load per unit length $-\mu(x) \frac{\partial^2 u(x, t)}{\partial t^2}$ and the linear viscous damping load per unit length $-c(x) \frac{\partial u(x, t)}{\partial t}$, i.e.

$$f(x, t) = f_d(x, t) - \mu(x) \frac{\partial^2 u(x, t)}{\partial t^2} - c(x) \frac{\partial u(x, t)}{\partial t} \quad (4-5)$$

$\mu(x)$ is the mass per unit length at the position $x$ of the differential beam element in the statical equilibrium state, and $c(x) \geq 0$ is the damping constant. Insertion of $(4-4)$ and $(4-5)$ into $(4-3)$ provides the following partial differential equation, relating the displacement $u(x, t)$ from the statical to the external dynamic load per unit length $f_d(x, t)$.
\[
\frac{\partial^2}{\partial x^2} \left( EI(x) \frac{\partial^2 u}{\partial x^2} \right) - \frac{\partial}{\partial x} \left( N \frac{\partial u}{\partial x} \right) + c(x) \frac{\partial u}{\partial t} + \mu(x) \frac{\partial^2 u}{\partial t^2} = f_d(x, t)
\]
\[
t \in [0, \infty[ , \quad x \in [0, l[ \tag{4-6}
\]

It should be noted that \( M(x, t) \) given by (4-4) is the bending moment caused by the dynamic external load \( f_d(x, t) \). Additionally, a bending moment \( M_s(x) \) may be present in the statical equilibrium state caused by the statical external loadings.

Fig. 4-2. Initial and boundary value problem for bending vibrations of plane beam element.

(4-6) must be solved with proper initial values at \( t = 0 \) for all particles in the interval \([0, l[\) and with boundary conditions at \( x = 0 \) and \( x = l \) for all times \( t \geq 0 \). Boundary conditions will be formulated for the beam element shown in fig. 4-2. At the left and right-hand end sections are attached distributed masses of magnitude \( m_0 \) and \( m_1 \) and with the mass moments of inertia around the bending centre \( J_0 \) and \( J_1 \), linear elastic springs with the spring constants \( k_0 \) and \( k_1 \) in translation and with the constants \( r_0 \) and \( r_1 \) in rotation and linear viscous damping elements with the damping constants \( c_0 \) and \( c_1 \) in translation and the constants \( d_0 \) and \( d_1 \) in rotation. The following initial boundary value problem may then be stated:
Differential equation \( \left( \forall \, t \in ]0, \infty[, \, \forall \, x \in ]0, 1[ \right) \):

\[
\frac{\partial^2}{\partial x^2} \left( EI(x) \frac{\partial^2 u}{\partial x^2} \right) - \frac{\partial}{\partial x} \left( N \frac{\partial u}{\partial x} \right) + c(x) \frac{\partial u}{\partial t} + \mu(x) \frac{\partial^2 u}{\partial t^2} = f_d(x, t)
\]

Initial values \( \left( \forall \, x \in ]0, 1[ \right) \):

\[
u(x, 0) = u_0(x), \quad \frac{\partial}{\partial t} u(x, 0) = \dot{u}_0(x)
\]

Geometrical boundary conditions \( \left( \forall \, t \in [0, \infty[ \right) \):

\[
\begin{align*}
\ & u(0, t) = 0, \quad \frac{\partial u(0, t)}{\partial x} = 0 \\
\ & u(1, t) = 0, \quad \frac{\partial u(1, t)}{\partial x} = 0
\end{align*}
\]

Mechanical boundary conditions \( \left( \forall \, t \in [0, \infty[ \right) \):

\[
\begin{align*}
\ & - EI(0) \frac{\partial^2 u(0, t)}{\partial x^2} = -r_0 \frac{\partial u(0, t)}{\partial x} - d_0 \frac{\partial \dot{u}(0, t)}{\partial x} - J_0 \frac{\partial \ddot{u}(0, t)}{\partial x} \\
\ & - EI(l) \frac{\partial^2 u(l, t)}{\partial x^2} = r_1 \frac{\partial u(l, t)}{\partial x} + d_1 \frac{\partial \dot{u}(l, t)}{\partial x} + J_1 \frac{\partial \ddot{u}(l, t)}{\partial x} \\
\ & \frac{\partial}{\partial x} \left( EI(0) \frac{\partial^2 u(0, t)}{\partial x^2} \right) + N \frac{\partial u(0, t)}{\partial x} = k_0 u(0, t) + c_0 \dot{u}(0, t) + m_0 \ddot{u}(0, t) \\
\ & \frac{\partial}{\partial x} \left( EI(l) \frac{\partial^2 u(l, t)}{\partial x^2} \right) + N \frac{\partial u(l, t)}{\partial x} = -k_1 u(l, t) - c_1 \dot{u}(l, t) - m_1 \ddot{u}(l, t)
\end{align*}
\]

For each differential mass element \( \mu(x) dx \) identified by the abscissa \( x \), an initial displacement \( u(x, 0) = u_0(x) \) and an initial velocity \( \frac{\partial}{\partial t} u(x, 0) = \dot{u}_0(x) \) must be formulated as a straightforward generalization of the discrete case.

The boundary conditions are classified as either geometric or mechanical boundary conditions. At each end section exactly 2 boundary conditions (geometric or mechanical) are specified.

Geometric boundary conditions are specified whenever the end section displacements or end section rotations are 0.
Mechanical boundary conditions specify that the bending moments $M(0^+, t), M(l^-, t)$ and the shear forces $Q(0^+, t)$, $Q(l^-, t)$ immediately to the right and the left of the end sections must balance the inertial forces and d'Alembert moments from the distributed masses and the forces and moments in the concentrated dampers and springs, resulting in the following equations of equilibrium, see fig. 4-3.

\[
\begin{align*}
M(0^+, t) &= -r_0 \frac{\partial}{\partial x} u(0, t) - d_0 \frac{\partial^2}{\partial t^2} \left( \frac{\partial}{\partial x} u(0, t) \right) - J_0 \frac{\partial^2}{\partial t^2} \left( \frac{\partial}{\partial x} u(0, t) \right) \\
M(l^-, t) &= r_1 \frac{\partial}{\partial x} u(l, t) + d_1 \frac{\partial^2}{\partial t^2} \left( \frac{\partial}{\partial x} u(l, t) \right) + J_1 \frac{\partial^2}{\partial t^2} \left( \frac{\partial}{\partial x} u(l, t) \right) \\
Q(0^+, t) &= k_0 u(0, t) + c_0 \dot{u}(0, t) + m_0 \ddot{u}(0, t) \\
Q(l^-, t) &= -k_1 u(l, t) - c_1 \dot{u}(l, t) - m_1 \ddot{u}(l, t)
\end{align*}
\]  

(4-8)

The mechanical boundary conditions in (4-7) are obtained from (4-8) upon eliminating the bending moments and shear forces using (4-2) and (4-4). It should be noted that the geometric boundary conditions can be obtained as special cases of the mechanical boundary conditions specifying infinite values of the spring constants $k_0$, $k_1$ or $r_0$, $r_1$. 
Example 4-1: Boundary Conditions for a Simply Supported and Cantilever Beam Element with Compressive Axial Force

a)

\[ P \]

\[ E I \]

\[ u(x, t) \]

\[ l \]


b)

\[ P \]

\[ E I \]

\[ u(x, t) \]

\[ l \]

Fig. 4-4: Boundary conditions for beam elements. a) Simply supported beam. b) Cantilever beam element with axial force.

The boundary conditions of the simply supported beam in fig. 4-4a read:

\[
\begin{align*}
u(0, t) &= u(l, t) = 0 & \text{(geometrical)} \\
\frac{\partial^2}{\partial x^2} u(0, t) &= \frac{\partial^2}{\partial x^2} u(l, t) = 0 & \text{(mechanical)}
\end{align*}
\]

(4-9)

The boundary conditions of the cantilever beam in fig. 4-4b read \((N = -P)\)

\[
\begin{align*}
\frac{\partial}{\partial x} u(0, t) &= 0 & \text{(geometrical)} \\
\frac{\partial^2}{\partial x^2} u(l, t) &= 0 & \text{(mechanical)} \\
\frac{\partial^3}{\partial x^3} u(l, t) + \frac{P}{E I} \frac{\partial u(l, t)}{\partial x} &= 0 & \text{(mechanical)}
\end{align*}
\]

(4-10)

The mechanical boundary conditions in (4-9) and (4-10) specify that the bending moment is 0 at the supports of the simply supported beam and that the bending moment and the shear force are both 0 at the free end of the cantilever beam.
Example 4-2: Dynamic Systems Reducable to the Considered System

The beam element in fig. 4-2 models a broader group of structures than appears at first sight. As an example the dynamic systems in fig. 4-5a and fig. 4-6b can be reduced to the systems shown in figs. 4-5b and 4-6b, respectively, which are special cases of the general system covered by (4-7).
4.2 Undamped Eigenvibrations of Bernoulli-Euler Beam Elements

![Figure 4-7: Undamped eigenvibrations of beam element.](image)

Undamped eigenvibrations of the beam element shown in figure 4-2 follow for \( c(x) \equiv 0 \), \( f_d(x, t) \equiv 0 \). Further, all dissipative elements are extracted from the mechanical boundary conditions, i.e. \( c_0 = c_1 = d_0 = d_1 = 0 \). Undamped eigenvibrations are then obtained as solutions to the boundary value problem (4-11), which is illustrated in figure 4-7.

Differential equation \( (\forall t \in [0, \infty[, \forall x \in [0, l]) \):

\[
\frac{\partial^2}{\partial x^2} \left( EI(x) \frac{\partial^2 u}{\partial x^2} \right) - \frac{\partial}{\partial x} \left( N \frac{\partial u}{\partial x} \right) + \mu(x) \frac{\partial^2 u}{\partial t^2} = 0
\]

Geometric boundary conditions \( (\forall t \in [0, \infty]) \):

\[
\begin{align*}
u(0, t) &= 0, & \frac{\partial}{\partial x} u(0, t) &= 0 \\
u(l, t) &= 0, & \frac{\partial}{\partial x} u(l, t) &= 0
\end{align*}
\]

Mechanical boundary conditions \( (\forall t \in [0, \infty]) \):

\[
\begin{align*}
- EI(0) \frac{\partial^2 u(0, t)}{\partial x^2} &= -r_0 \frac{\partial u(0, t)}{\partial x} - J_0 \frac{\partial \ddot{u}(0, t)}{\partial x} \\
- EI(l) \frac{\partial^2 u(l, t)}{\partial x^2} &= r_1 \frac{\partial u(l, t)}{\partial x} + J_1 \frac{\partial \ddot{u}(l, t)}{\partial x} \\
- \frac{\partial}{\partial x} \left( EI(0) \frac{\partial^2 u(0, t)}{\partial x^2} \right) + N \frac{\partial u(0, t)}{\partial x} &= k_0 u(0, t) + m_0 \ddot{u}(0, t) \\
- \frac{\partial}{\partial x} \left( EI(l) \frac{\partial^2 u(l, t)}{\partial x^2} \right) + N \frac{\partial u(l, t)}{\partial x} &= -k_1 u(l, t) - m_1 \ddot{u}(l, t)
\end{align*}
\]

(4-11)
Guided from the experience gained from the analysis of discrete systems it can be anticipated that all mass particles $\mu(x)dx$ are performing harmonic motions in phase. Consequently, the solution of (4-11) is searched for on the form

$$u(x,t) = \Phi(x) \cos(\omega t) \quad (4-12)$$

$\Phi(x)$ is the real amplitude of the mass particle $\mu(x)dx$, identified by the abscissa $x$ in the statical equilibrium state, and $\omega$ is the circular eigenfrequency. Upon insertion into (4-11) it is seen that (4-12) is a solution if, and only if, $\Phi(x)$ and $\omega$ are solutions to the following linear eigenvalue problem

Differential equation $\forall x \in [0,l]$: \[
\frac{d^2}{dx^2} \left( EI(x) \frac{d^2 \Phi}{dx^2} \right) - \frac{d}{dx} \left( N \frac{d\Phi}{dx} \right) - \omega^2 \mu(x) \Phi(x) = 0
\]

Geometrical boundary conditions:

$$\Phi(0) = 0, \quad \frac{d}{dx} \Phi(0) = 0$$
$$\Phi(l) = 0, \quad \frac{d}{dx} \Phi(l) = 0$$

Mechanical boundary conditions:

$$EI(0) \frac{d^2}{dx^2} \Phi(0) = (r_0 - \omega^2 J_0) \frac{d}{dx} \Phi(0)$$
$$EI(l) \frac{d^2}{dx^2} \Phi(l) = -(r_1 - \omega^2 J_1) \frac{d}{dx} \Phi(l)$$
$$\frac{d}{dx} \left( EI(0) \frac{d^2}{dx^2} \Phi(0) \right) - N \frac{d}{dx} \Phi(0) = -(k_0 - \omega^2 m_0) \Phi(0)$$
$$\frac{d}{dx} \left( EI(l) \frac{d^2}{dx^2} \Phi(l) \right) - N \frac{d}{dx} \Phi(l) = (k_1 - \omega^2 m_1) \Phi(l)$$

 Obviously, (4-13) is fulfilled by the trivial solution $\Phi(x) \equiv 0$. The set of solutions to the eigenvalue problem depends on the parameter $\omega$. For discrete values $\omega = \omega_1, \omega_2, \ldots$ non-trivial solutions $\Phi^{(1)}(x), \Phi^{(2)}(x), \ldots$ to (4-13) do exist. $\omega_1, \omega_2, \ldots$ signify the undamped circular eigenfrequencies of the system. The corresponding solutions $\Phi^{(1)}(x), \Phi^{(2)}(x), \ldots$ are designated as the undamped mode shapes or eigenmodes of the system. The undamped circular eigenfrequencies are supposed to be ordered in ascending order, so that $0 \leq \omega_1 \leq \omega_2 \leq \cdots$.

Infinitely many undamped circular eigenfrequencies exist corresponding to the infinitely many degrees of freedom (infinitely many differential mass particles $\mu(x)dx$) of the
structural system. This property is characteristic of any system with continuous mass distribution in contrast to discrete systems with finitely many $n$ degrees of freedom.

If $\Phi^{(i)}(x)$ is a non-trivial solution to (4-13), so is $c\Phi^{(i)}(x)$. Hence, the undamped mode shapes are determined within an arbitrary factor $c$.

In what follows (4-13) will be solved for the special case of homogeneous cross-section (constant values of $EI$ and $\mu$). The differential equation of (4-13) can in this case be written in the following way

$$\frac{d^4\Phi}{d\xi^4} - \frac{NI^2}{EI} \frac{d^2\Phi}{d\xi^2} - \frac{\mu\omega^2 l^4}{EI} \Phi = 0$$

(4-14)

where

$$\xi = \frac{x}{l}$$

(4-15)

The complete solution of (4-14) can be written

$$\Phi(x) = A \sin \left( \lambda \frac{x}{l} \right) + B \cos \left( \lambda \frac{x}{l} \right) + C \sinh \left( \nu \frac{x}{l} \right) + D \cosh \left( \nu \frac{x}{l} \right), \quad x \in [0, l]$$

(4-16)

where $A$, $B$, $C$, $D$ are integration constants and $\lambda^2$ and $\nu^2$ are the positive roots of the quadratic equations

$$\begin{align*}
\lambda^4 + \frac{NI^2}{EI} \lambda^2 - \frac{\mu\omega^2 l^4}{EI} &= 0 \\
\nu^4 - \frac{NI^2}{EI} \nu^2 - \frac{\mu\omega^2 l^4}{EI} &= 0
\end{align*}$$

(4-17)

Especially, if $N = 0$ (4-16), (4-17) reduce to

$$\Phi(x) = A \sin \left( \lambda \frac{x}{l} \right) + B \cos \left( \lambda \frac{x}{l} \right) + C \sinh \left( \lambda \frac{x}{l} \right) + D \cosh \left( \lambda \frac{x}{l} \right), \quad x \in [0, l]$$

(4-18)

$$\lambda^4 = \frac{\mu\omega^2 l^4}{EI}$$

(4-19)

(4-16) or (4-18) are inserted into the 4 relevant boundary conditions in (4-13). Then 4 homogeneous linear equations are obtained for the determination of the coefficients $A, B, C, D$, which can always be formulated in the following way

$$K(\lambda, \nu) \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(4-20)
\( \lambda = \lambda(\omega) \) and \( \nu = \nu(\omega) \) are known functions of the circular frequency as follows from (4-17). \( K(\lambda(\omega), \nu(\omega)) = K(\omega) \) is then a known function of \( \omega \). (4-20) always has the solution \([A, B, C, D] = [0, 0, 0, 0]\) which upon insertion into (4-16) provides the trivial solution \( \Phi(x) = 0 \). The necessary condition for non-trivial solutions \([A, B, C, D] \neq [0, 0, 0, 0]\), and hence for non-trivial eigenmodes, is that the determinant of \( K(\omega) \) is 0. This leads to the frequency condition

\[
\det(K(\omega)) = 0 \quad (4 - 21)
\]

Solutions \( \omega_1, \omega_2, \ldots \) to (4-21) determine the undamped circular eigenfrequencies. For each of these a non-trivial solution \([A^{(j)}, B^{(j)}, C^{(j)}, D^{(j)}] \neq [0, 0, 0, 0]\) to (4-20) exist, which together with the corresponding eigenvalues \( \lambda_i, \nu_j \), as determined upon insertion into (4-17) or (4-19), determines the undamped mode shapes of the beam element, \( \Phi^{(i)}(x) \).

Example 4-3: Eigenvibrations of Simply Supported Beam with Homogeneous Cross-sections

![Simply supported beam with homogeneous cross-sections.](image)

The beam is loaded by a compressive axial force \( P \), so \( N = -P \). Then, (4-17) attains the form

\[
\begin{align*}
\lambda^4 - \frac{P^2}{EI} \lambda^2 - \frac{\mu \omega^2 l^4}{EI} &= 0 \\
\nu^4 + \frac{P^2}{EI} \nu^2 - \frac{\mu \omega^2 l^4}{EI} &= 0
\end{align*}
\]

(4 - 22)

The boundary conditions read, cf. (4-9)

\[
\begin{align*}
\Phi(0) &= \frac{d^2}{dx^2} \Phi(0) = 0 \\
\Phi(l) &= \frac{d^2}{dx^2} \Phi(l) = 0
\end{align*}
\]

(4 - 23)

Insertion of (4-16) into the boundary conditions at \( x = 0 \) provides

\[
\begin{align*}
B + D &= 0 \\
-\frac{\lambda^2}{l^2} B + \frac{\nu^2}{l^2} D &= 0
\end{align*}
\] \( \iff \)

\[
\begin{align*}
B + D &= 0 \\
(\nu^2 + \lambda^2) B &= 0
\end{align*}
\]

(4 - 24)
The 2nd equation of (4-24) can either be fulfilled if \( \nu^2 + \lambda^2 = 0 \Leftrightarrow \nu = 0 \wedge \lambda = 0 \) or if \( B = 0 \). Insertion of \( \nu = 0 \wedge \lambda = 0 \) into (4-16) provides \( \Phi(x) = B + D \equiv 0 \), as seen from the 1st equation (4-24). Hence, \( \nu^2 + \lambda^2 = 0 \) implies the trivial solution. Non-trivial solutions are then obtained for \( \nu^2 + \lambda^2 > 0 \wedge B = 0 \). From the 1st equation (4-24) it is then seen that \( D = 0 \). Hence (4-16) reduces to

\[
\Phi(x) = A \sin \left( \frac{\lambda x}{l} \right) + C \sinh \left( \frac{\nu x}{l} \right)
\]  

(4 - 25)

Insertion of (4-25) into the boundary conditions at \( x = l \)

\[
\begin{align*}
A \sin \lambda + C \sinh \nu &= 0 \\
- \frac{\lambda^2}{l^2} A \sin \lambda + C \frac{\nu^2}{l^2} \sinh \nu &= 0
\end{align*}
\]

\Rightarrow

\[
\begin{bmatrix}
\sin \lambda & \sinh \nu \\
- \frac{\lambda^2}{l^2} \sin \lambda & \frac{\nu^2}{l^2} \sinh \nu
\end{bmatrix}
\begin{bmatrix}
A \\
C
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

(4 - 26)

The frequency condition is obtained assigning the determinant of the coefficient matrix to 0. Hence,

\[
\frac{1}{l^2} (\nu^2 + \lambda^2) \sin \lambda \sinh \nu = 0
\]  

(4 - 27)

since \( (\nu^2 + \lambda^2) > 0 \), (4-27) can only be fulfilled if

\[
\sin \lambda = 0 \Rightarrow
\]

\[
\lambda_j = j\pi , \; j = 1, 2, \ldots
\]  

(4 - 28)

(4 - 29)

Insertion of (4-28) into the 1st equation of (4-26) provides \( (\nu_j \neq 0) \)

\[
C \sinh \nu_j = 0 \Rightarrow
\]

\[
C = 0
\]  

(4 - 30)

From (4-29), (4-30) the following eigenmode is obtained from (4-25)

\[
\Phi^{(j)}(x) = A^{(j)} \sin \left( \frac{j\pi x}{l} \right) , \; j = 1, 2, \ldots
\]  

(4 - 31)

(4-31) specifies a sinusoidal curve with \( j \) half waves. Especially, the eigenmodes are independent of the compressive axial force \( P \). The circular eigenfrequencies \( \omega_j \) follow from (4-22) and (4-29)

\[
\omega_j^2 = \frac{j^4 \pi^4 EI}{l^4} \mu \left( 1 - \frac{P l^2}{EI j^2 \pi^2} \right) , \; j = 1, 2, \ldots \Rightarrow
\]

\[
\omega_j = \omega_{j,0} \sqrt{1 - \frac{1}{j^2 \frac{P}{P_E}}} , \; j = 1, 2, \ldots
\]  

(4 - 32)

where
\[ \omega_{j,0} = \frac{j^2 \pi^2}{l^2} \sqrt{\frac{EI}{\mu}} , \quad j = 1, 2, \ldots \]  

(4 - 33)

\[ P_E = \pi^2 \frac{EI}{l^2} \]  

(4 - 34)

For \( P = 0 \) it follows from (4-32) that \( \omega_j = \omega_{j,0} \). Hence, \( \omega_{j,0} \) can be identified as the circular eigenfrequency of a simply supported beam element in absence of any axial force. \( P_E \) is the classical buckling load of the column according to the Euler formula. Especially, for \( j = 1 \) (4-32) provides

\[ \omega_1 = \omega_{1,0} \sqrt{1 - \frac{P}{P_E}} \]  

(4 - 35)

Hence, \( \omega_1 \) follows a parabolic relationship, and \( \omega_1 \downarrow 0 \) for \( P \uparrow P_E \). see fig. 4-9.

![Fig. 4-9: Fundamental circular eigenfrequency as a function of the compressive axial forces.](image)

![Fig. 4-10: Non-destructive test method for determination of buckling load.](image)

A non-destructive determination of the buckling load \( P_E \) can be performed based on (4-35). Assume \( \omega_1 \) is measured for various non-critical values of the axial force \( P \). If \( \omega_1^2 \) is plotted as a function of \( P \) the measurements should group along a straight regression line, which is cutting the abscissa axis at the estimated buckling load \( P_E \), see fig. 4-10.

**Example 4-4: Eigenvibrations of Cantilever Beam with Homogeneous Cross-sections**

![Fig. 4-11: Cantilever beam element with homogeneous cross-sections.](image)
No axial force is considered for simplicity, $N = 0$. The boundary conditions then become, cf. (4-10)

$$
\begin{align*}
\Phi(0) &= \frac{d}{dx}\Phi(0) = 0 \\
\frac{d^2}{dx^2}\Phi(l) &= \frac{d^3}{dx^3}\Phi(l) = 0
\end{align*}
$$

(4 - 36)

Since $N = 0$ the mode shapes are given by (4-18). From the boundary conditions at $x = 0$ the following homogeneous linear equations are obtained

$$
\begin{align*}
B + D &= 0 \\
\lambda l (A + C) &= 0
\end{align*}
$$

(4 - 37)

The second equation of (4-37) is either fulfilled if $\lambda = 0$, or if $(A + C) = 0 \Leftrightarrow A = -C$. Insertion of $\lambda = 0$ in (4-18) provides $\Phi(x) = B + D \equiv 0$, as seen from the 1st equation (4-37). Hence, $\lambda = 0$ implies the trivial solution. Non-trivial solutions are then obtained for $\lambda > 0 \land A = -C$. Using the 1st equation (4-37), (4-18) becomes

$$
\Phi(x) = A \left( \sin \left( \frac{\lambda x}{l} \right) - \sinh \left( \frac{\lambda x}{l} \right) \right) + B \left( \cos \left( \frac{\lambda x}{l} \right) - \cosh \left( \frac{\lambda x}{l} \right) \right)
$$

(4 - 38)

Next, (4-38) is inserted into the boundary conditions at $x = l$, resulting in the homogeneous linear equations

$$
\begin{align*}
- \frac{\lambda^2}{l^2} (\sin \lambda + \sinh \lambda) A - \frac{\lambda^2}{l^2} (\cos \lambda + \cosh \lambda) B &= 0 \\
- \frac{\lambda^3}{l^3} (\cos \lambda + \cosh \lambda) A - \frac{\lambda^3}{l^3} (-\sin \lambda + \sinh \lambda) B &= 0
\end{align*}
$$

(4 - 39)

The frequency condition becomes

$$
\begin{align*}
\sinh^2 \lambda - \sin^2 \lambda - \cos^2 \lambda - 2 \cos \lambda \cosh \lambda - \cosh^2 \lambda &= 0 \\
\cosh \lambda \cos \lambda + 1 &= 0
\end{align*}
$$

(4 - 40)

where the identities $\cosh^2 \lambda - \sinh^2 \lambda = 1$, $\cos^2 \lambda + \sin^2 \lambda = 1$ have been used. The 3 lowest solutions of (4-40) become

$$
\begin{align*}
\lambda_1 &= 1.8751041 \\
\lambda_2 &= 4.6940911 \\
\lambda_3 &= 7.8547574
\end{align*}
$$

(4 - 41)

Next, the circular eigenfrequencies follow from (4-19)

$$
\omega_j = \frac{\lambda_j^2}{l^2} \sqrt{\frac{EI}{\mu}} , \quad j = 1, 2, \ldots
$$

(4 - 42)
From the first equation (4-39) follows that the eigenmodes can be written

\[ \Phi^{(i)}(x) = K \left[ \left( \cos \lambda_i + \cosh \lambda_i \right) \left( \sin \left( \frac{x}{l} \right) - \sinh \left( \frac{x}{l} \right) \right) - \\
\left( \sin \lambda_i + \sinh \lambda_i \right) \left( \cos \left( \frac{x}{l} \right) - \cosh \left( \frac{x}{l} \right) \right) \right] \]

(4 - 43)

The 3 lowest eigenmodes have been sketched in fig. 4-12.

![Eigenmodes](image)

**Fig. 4-12: Eigenmodes of clamped beam with homogeneous cross-section.**

### 4.3 Orthogonality Property of Eigenmodes

For the eigenmodes obtained as solution to the eigenvalue problem (4-13) the following theorem can be proved.

**Theorem 4-1:** The eigenmodes \( \Phi^{(i)}(x) \) and \( \Phi^{(j)}(x) \) to the eigenvalue problem (4-13) belonging to different circular eigenfrequencies \( \omega_i \) and \( \omega_j \) fulfil the orthogonality properties

\[ \int_0^l m_0 \Phi^{(i)}(x) \Phi^{(j)}(x) \, dx + J_0 \frac{d}{dx} \Phi^{(i)}(0) \frac{d}{dx} \Phi^{(j)}(0) + \\
m_1 \Phi^{(i)}(l) \Phi^{(j)}(l) + J_1 \frac{d}{dx} \Phi^{(i)}(l) \frac{d}{dx} \Phi^{(j)}(l) = \begin{cases} 0 & , \ i \neq j \\ M_i & , \ i = j \end{cases} \]

(4 - 44)

\[ \int_0^l \left( EI(x) \frac{d^2}{dx^2} \Phi^{(i)}(x) \frac{d^2}{dx^2} \Phi^{(j)}(x) + N \frac{d}{dx} \Phi^{(i)}(x) \frac{d}{dx} \Phi^{(j)}(x) \right) \, dx + \\
k_0 \Phi^{(i)}(0) \Phi^{(j)}(0) + r_0 \frac{d}{dx} \Phi^{(i)}(0) \frac{d}{dx} \Phi^{(j)}(0) + k_1 \Phi^{(i)}(l) \Phi^{(j)}(l) + r_1 \frac{d}{dx} \Phi^{(i)}(l) \frac{d}{dx} \Phi^{(j)}(l) = \\
\begin{cases} 0 & , \ i \neq j \\ \omega_i^2 M_i & , \ i = j \end{cases} \]

(4 - 45)

where \( M_i \) is the modal mass in the \( i \)th eigenvibration defined by

\[ M_i = \int_0^l m_0 (\Phi^{(i)}(x))^2 \, dx + m_1 (\Phi^{(i)}(l))^2 + J_0 \left( \frac{d}{dx} \Phi^{(i)}(0) \right)^2 + \\
m_1 (\Phi^{(i)}(l))^2 + J_1 \left( \frac{d}{dx} \Phi^{(i)}(l) \right)^2, \ i = 1, 2, \ldots \]

(4 - 46)
Proof: In order to prove (4.44), (4.45) the differential equation in (4.13) is written for the ith eigenmode $\Phi^{(j)}(x)$ and is multiplied by $\Phi^{(i)}(x)$, followed by an integration over $[0, l]$. Partial integration and application of the mechanical boundary conditions of (4.13) provides

$$
\int_0^l \Phi^{(i)}(x) \left( \frac{d^2}{dx^2} (EI(x) \frac{d^2}{dx^2} \Phi^{(j)}(x)) - \frac{d}{dx} \left( N \frac{d}{dx} \Phi^{(j)}(x) \right) \right) \, dx = \\
\omega_j^2 \int_0^l \mu(x) \Phi^{(i)}(x) \Phi^{(j)}(x) \, dx \Rightarrow \\
\Phi^{(i)}(l) \left( k_1 - \omega_j^2 m_1 \right) \Phi^{(j)}(l) - \Phi^{(i)}(0) \left( -(k_0 - \omega_j^2 m_0) \Phi^{(j)}(0) \right) - \\
\int_0^l \frac{d}{dx} \Phi^{(i)}(x) \left( \frac{d}{dx} (EI(x) \frac{d^2}{dx^2} \Phi^{(j)}(x)) - N \frac{d}{dx} \Phi^{(j)}(x) \right) \, dx = \\
\omega_j^2 \int_0^l \mu(x) \Phi^{(i)}(x) \Phi^{(j)}(x) \, dx \Rightarrow \\
\Phi^{(i)}(0) \Phi^{(j)}(0) (k_0 - \omega_j^2 m_0) + \Phi^{(i)}(l) \Phi^{(j)}(l) (k_1 - \omega_j^2 m_1) - \\
\frac{d}{dx} \Phi^{(i)}(l) \left( -(r_1 - \omega_j^2 t_1) \frac{d}{dx} \Phi^{(j)}(l) \right) + \frac{d}{dx} \Phi^{(i)}(0) \left( r_0 - \omega_j^2 J_0 \right) \frac{d}{dx} \Phi^{(j)}(0) + \\
\int_0^l EI(x) \frac{d^2}{dx^2} \Phi^{(i)}(x) \frac{d^2}{dx^2} \Phi^{(j)}(x) \, dx + \int_0^l N \frac{d}{dx} \Phi^{(i)}(x) \frac{d}{dx} \Phi^{(j)}(x) \, dx = \\
\omega_j^2 \int_0^l \mu(x) \Phi^{(i)}(x) \Phi^{(j)}(x) \, dx \Rightarrow \\
\int_0^l \left( EI(x) \frac{d^2}{dx^2} \Phi^{(i)}(x) \frac{d^2}{dx^2} \Phi^{(j)}(x) + N \frac{d}{dx} \Phi^{(i)}(x) \frac{d}{dx} \Phi^{(j)}(x) \right) \, dx + \\
k_0 \Phi^{(i)}(0) \Phi^{(j)}(0) + r_0 \frac{d}{dx} \Phi^{(i)}(0) \frac{d}{dx} \Phi^{(j)}(0) + k_1 \Phi^{(i)}(l) \Phi^{(j)}(l) + r_1 \frac{d}{dx} \Phi^{(i)}(l) \frac{d}{dx} \Phi^{(j)}(l) = \\
\omega_j^2 \left( \int_0^l \mu(x) \Phi^{(i)}(x) \Phi^{(j)}(x) \, dx + m_0 \Phi^{(i)}(0) \Phi^{(j)}(0) + J_0 \frac{d}{dx} \Phi^{(i)}(0) \frac{d}{dx} \Phi^{(j)}(0) + \\
m_1 \Phi^{(i)}(l) \Phi^{(j)}(l) + J_1 \frac{d}{dx} \Phi^{(i)}(l) \frac{d}{dx} \Phi^{(j)}(l) \right) \\
(4.47)
$$

On the left-hand side of (4.47) the subscripts $i$ and $j$ can be interchanged without changing the value of the expression. Then, this must also be the case on the right-hand side. Hence,

$$
\omega_j^2 \left( \int_0^l \mu(x) \Phi^{(i)}(x) \Phi^{(j)}(x) \, dx + m_0 \Phi^{(i)}(0) \Phi^{(j)}(0) + J_0 \frac{d}{dx} \Phi^{(i)}(0) \frac{d}{dx} \Phi^{(j)}(0) + \\
m_1 \Phi^{(i)}(l) \Phi^{(j)}(l) + J_1 \frac{d}{dx} \Phi^{(i)}(l) \frac{d}{dx} \Phi^{(j)}(l) \right) = \\
\omega_i^2 \left( \int_0^l \mu(x) \Phi^{(j)}(x) \Phi^{(i)}(x) \, dx + m_0 \Phi^{(j)}(0) \Phi^{(i)}(0) + J_0 \frac{d}{dx} \Phi^{(j)}(0) \frac{d}{dx} \Phi^{(i)}(0) + \\
m_1 \Phi^{(j)}(l) \Phi^{(i)}(l) + J_1 \frac{d}{dx} \Phi^{(j)}(l) \frac{d}{dx} \Phi^{(i)}(l) \right) \Rightarrow \\
(\omega_i^2 - \omega_j^2) \left( \int_0^l \mu(x) \Phi^{(i)}(x) \Phi^{(j)}(x) \, dx + m_0 \Phi^{(i)}(0) \Phi^{(j)}(0) + J_0 \frac{d}{dx} \Phi^{(i)}(0) \frac{d}{dx} \Phi^{(j)}(0) + \\
m_1 \Phi^{(i)}(l) \Phi^{(j)}(l) + J_1 \frac{d}{dx} \Phi^{(i)}(l) \frac{d}{dx} \Phi^{(j)}(l) \right) = 0
$$
Since \( \omega_i \neq \omega_j \), the 2nd factor of (4-48) must be 0, which proves (4-44). If (4-44) is valid, (4-45) follows directly from the last statement of (4-47). This ends the proof of the theorem.

The left-hand side of (4-45) indicates for \( i = j \) twice the strain energy in case the structure is deformed with the eigenmode \( \Phi^{(j)}(x) \). This is always positive (the differential operator is positive definite) if only the statical equilibrium state is stable, i.e. the compressive axial force \(-N\) is below the Euler buckling load.

Fig. 4-13: Undamped eigenvibrations of compound beam.

Fig. 4-13 shows a rectilinear compound beam composed of \( n \) sub-beams. The end sections of the sub-beams are placed at the abscissas \( x_0, x_1, \ldots, x_n \). \( x_0 = 0 \) and \( x_n = l \) indicate the end sections of the compound beam. At the abscissa \( x_i, i = 0, \ldots, n, \) is placed a mass of magnitude mass \( m_i \) and with the mass moment of inertia \( J_i \) around the beam axis. Further, a linear elastic spring \( k_i \) in the \( y \)-direction and a rotational spring \( r_i \) active against rotations in the \( z \)-axis are attached. The bending stiffness \( EI(x) \), the axial force \( N(x) \) in the statical equilibrium state and the mass per unit length \( \mu(x) \) are composed by piecewise sufficiently smooth contributions from each sub-beam. Equally, the eigenmodes \( \Phi^{(i)}(x) \) are composed of partial solutions from the sub-beams, which fulfil at the boundaries \( x = 0 \) and \( x = l \) the geometrical or mechanical boundary conditions in (4-13), and at the interfaces at \( x = x_i, i = 1, \ldots, n - 1 \) relevant \textit{transition conditions}, expressing the continuity of displacements and rotations, as well as the equilibrium of the bending moment and shear force at these sections.

The orthogonality conditions for the eigenmodes of the system defined in fig. 4-13 can now be shown to be

\[
\int_0^l \mu(x) \Phi^{(i)}(x) \Phi^{(j)}(x) \, dx + \sum_{k=0}^n m_k \Phi^{(i)}(x_k) \Phi^{(j)}(x_k) + \sum_{k=0}^n J_k \frac{d}{dx} \Phi^{(i)}(x_k) \frac{d}{dx} \Phi^{(j)}(x_k) =
\begin{cases}
0, & i \neq j \\
M_i, & i = j
\end{cases}
\]
\[
\int_0^l \left( EI(x) \frac{d^2}{dx^2} \Phi^{(i)}(x) - N(x) \frac{d}{dx} \Phi^{(i)}(x) + \frac{d}{dx} \Phi^{(j)}(x) - \Phi^{(j)}(x) \right) dx + \\
\sum_{k=0}^n k_k \Phi^{(i)}(x_k) \Phi^{(j)}(x_k) + \sum_{k=0}^n r_k \frac{d}{dx} \Phi^{(i)}(x_k) \frac{d}{dx} \Phi^{(j)}(x_k) = \\
\begin{cases}
0, & i \neq j \\
\omega_i^2 M_i, & i = j
\end{cases}
\] (4-50)

\[
M_i = \int_0^l \mu(x) \left( \Phi^{(i)}(x) \right)^2 dx + \sum_{k=0}^n m_k \left( \Phi^{(i)}(x_k) \right)^2 \left( \frac{d}{dx} \Phi^{(i)}(x_k) \right)^2
\] (4-51)

The formal proof of (4-49), (4-50), which will be omitted, is performed in the same way as for theorem 4-1 using the mentioned transition and boundary conditions after partial integration for each sub-beam. Note that (4-44), (4-45) are obtained as special cases of (4-49), (4-50) for \( n = 1 \).

In case of multiple eigenvalues it can be shown that it is always possible to define a complete sequence of eigenmodes \( \Phi^{(1)}(x), \Phi^{(2)}(x), \ldots \), which fulfils (4-44), (4-45) or (4-49), (4-50).

### 4.4 Forced Vibrations of Bernoulli-Euler Beam Elements

In this section an analytical solution to the initial and boundary value problem (4-7) in terms of an expansion of undamped eigenmodes. Guided by (3-170), the displacement field is written on the form

\[
u(x, t) = \sum_{j=1}^{\infty} \Phi^{(j)}(x) q_j(t)
\] (4-52)

As previous the coefficients \( q_j(t), j = 1, 2, \ldots \) are referred to as the undamped modal coordinates. These are obtained as solutions to the following coupled ordinary differential equations

\[
\ddot{q}_i + 2\omega_i \left( \zeta_i \dot{q}_i + \sum_{j=1, j \neq i}^{\infty} \frac{M_j \omega_j}{M_i \omega_i} \zeta_{ij} \dot{q}_j \right) + \omega_i^2 q_i = \frac{1}{M_i} F_i(t), \quad i = 1, 2, \ldots, t > 0
\] (4-53)

where

\[
F_i(t) = \int_0^l \Phi^{(i)}(x) f_2(x, t) dx
\] (4-54)
The modal mass $M_i$ is given by (4-46), $F_i(t)$ signifies the modal load, $\zeta_i$ is the modal damping ratio and $\zeta_{ij}$ are the modal coupling coefficients.

In order to prove (4-53) the differential equation of (4-7) is multiplied by $\Phi^{(j)}(x)$ followed by an integration over $[0, l]$. Application of partial integration and the mechanical boundary conditions of (4-7) provides

$$
\int_0^l \left( \frac{\partial^2 u}{\partial x^2} (EI(x)) \Phi^{(j)}(x) \right) dx + \int_0^l \Phi^{(j)}(x) \mu(x) \ddot{u}(x, t) dx = \int_0^l \Phi^{(j)}(x) f_d(x, t) dx \Rightarrow
$$

$$
\Phi^{(i)}(0) (k_0 u(0, t) + c_0 \ddot{u}(0, t) + m_0 \dddot{u}(0, t)) + \Phi^{(j)}(l) (k_1 u(l, t) + c_1 \ddot{u}(l, t) + m_1 \dddot{u}(l, t)) +
$$

$$
\frac{d}{dx} \Phi^{(i)}(0) \left( r_0 \frac{\partial}{\partial x} u(0, t) + d_0 \frac{\partial}{\partial x} \ddot{u}(0, t) + J_0 \frac{\partial}{\partial x} \dddot{u}(0, t) \right) +
$$

$$
\frac{d}{dx} \Phi^{(j)}(l) \left( r_1 \frac{\partial}{\partial x} u(l, t) + d_1 \frac{\partial}{\partial x} \ddot{u}(l, t) + J_1 \frac{\partial}{\partial x} \dddot{u}(l, t) \right) +
$$

$$
\int_0^l \left( EI(x) \frac{d^2 u}{dx^2} (\Phi^{(j)}(x)) \frac{\partial^2}{\partial x^2} (\Phi^{(j)}(x)) \right) dx +
$$

$$
\int_0^l \Phi^{(i)}(x) c(x) \ddot{u}(x, t) dx + \int_0^l \Phi^{(i)}(x) \mu(x) \dddot{u}(x, t) dx = \int_0^l \Phi^{(i)}(x) f_d(x, t) dx \quad (4-57)
$$

Next, (4-52) is inserted on the left-hand side of (4-57). Then

$$
\sum_{j=1}^{\infty} q_j(t) \left( \int_0^l \mu(x) (\Phi^{(i)}(x) \Phi^{(j)}(x)) dx + m_0 \Phi^{(i)}(0) \Phi^{(j)}(0) + J_0 \frac{d}{dx} \Phi^{(i)}(0) \frac{d}{dx} \Phi^{(j)}(0) +
$$

$$
m_1 \Phi^{(i)}(l) \Phi^{(j)}(l) + J_1 \frac{d}{dx} \Phi^{(i)}(l) \frac{d}{dx} \Phi^{(j)}(l) \right) +
$$

$$
\sum_{j=1}^{\infty} q_j(t) \left( \int_0^l c(x) (\Phi^{(i)}(x) \Phi^{(j)}(x)) dx + c_0 \Phi^{(i)}(0) \Phi^{(j)}(0) + d_0 \frac{d}{dx} \Phi^{(i)}(0) \frac{d}{dx} \Phi^{(j)}(0) +
$$

$$
c_1 \Phi^{(i)}(l) \Phi^{(j)}(l) + d_1 \frac{d}{dx} \Phi^{(i)}(l) \frac{d}{dx} \Phi^{(j)}(l) \right) +
$$
The orthogonality condition (4-44), (4-45) is next applied for the 1st and 3rd term on the left-hand side of (4-58), leading immediately to (4-53). (4-53) is identical to (3-181), aside from the number of modal coordinates and modal coordinates are infinite. (4-53) has to be solved with proper initial conditions on the modal coordinates. In terms of the initial value fields $u_0(x)$ and $u_0(x)$ these become

$$
q_i(0) = \frac{1}{M_i} \left( \int_0^l \mu(x) \Phi^{(i)}(x) u_0(x) dx + m_0 \Phi^{(i)}(0) u_0(0) + 
J_0 \frac{d}{dx} \Phi^{(i)}(0) \frac{d}{dx} u_0(0) + m_1 \Phi^{(i)}(l) u_0(l) + J_1 \frac{d}{dx} \Phi^{(i)}(l) \frac{d}{dx} u_0(l) \right) 
$$

$$
\dot{q}_i(0) = \frac{1}{M_i} \left( \int_0^l \mu(x) \Phi^{(i)}(x) \dot{u}_0(x) dx + m_0 \Phi^{(i)}(0) \dot{u}_0(0) + 
J_0 \frac{d}{dx} \Phi^{(i)}(0) \frac{d}{dx} \dot{u}_0(0) + m_1 \Phi^{(i)}(l) \dot{u}_0(l) + J_1 \frac{d}{dx} \Phi^{(i)}(l) \frac{d}{dx} \dot{u}_0(l) \right) 
$$

From (4-52) it follows that

$$
u_0(x) = \sum_{j=1}^{\infty} \Phi^{(j)}(x) q_j(0) 
$$

$$\dot{u}_0(x) = \sum_{j=1}^{\infty} \Phi^{(j)}(x) \dot{q}_j(0) 
$$

Insertion of (4-60) on the right-hand side of (4-59) and application of the orthogonality condition (4-44) proves the validity of (4-59).

The differential equations (4-53) decouple if $\zeta_{ij} = 0$. In this case (4-53) becomes identical to (3-187). The solutions are then given by (3-188), (3-189), (3-190).

If the modal coordinates are determined, the displacement field follows upon insertion into (4-52). Other response quantities such as the bending moment $M(x, t)$ and the shear force $Q(x, t)$ follow next by partial differentiation

$$
M(x, t) = \sum_{j=1}^{\infty} M^{(j)}(x) q_j(t) 
$$

$$M^{(j)}(x) = -EI \frac{d^2}{dx^2} \Phi^{(j)}(x) 
$$
\[ Q(x, t) = \sum_{j=1}^{\infty} Q^{(j)}(x) q_j(t) \]
\[
Q^{(j)}(x) = -\frac{d}{dx} \left( EI(x) \frac{d^2 \Phi^{(j)}(x)}{dx^2} \right) + N \frac{d}{dx} \Phi^{(j)}(x) \]
\]

(4 – 62)

\[ M^{(j)}(x) \text{ and } Q^{(j)}(x) \text{ signify the bending moment and the shear force at the abscissa } x, \text{ if the beam element is deformed corresponding to } u(x) = \Phi^{(j)}(x). \text{ Note that (4-61) and (4-62) signify the dynamic incremental bending moment and shear force from the dynamic load } f_d(x, t). \text{ To these must be added the contributions to the bending moment and shear force from the statical load.} \]

**Example 4-5: Simply Supported Homogeneous Beam with Moving Load**

![Simply supported beam with moving load.](image)

**Fig. 4-14: Simply supported beam with moving load.**

Fig. 4-14 shows a simply supported Bernoulli-Euler beam with the length \( l \). The cross-section is assumed constant with the bending stiffness \( EI \) and the mass per unit length \( \mu \). The damping load is assumed linearly viscous with the constant damping constant \( c \). The beam is assumed to be at rest at the time \( t = 0 \), where a force \( P \) enters on the beam with the constant velocity \( v \). The motion of the beam is to be determined.

The undamped circular eigenfrequencies of the beam \( \omega_j \) are given by (4-33). The eigenmodes are given by (4-31). These are assumed on the form

\[ \Phi^{(j)}(x) = \sin \left( \frac{\pi x}{l} \right), \quad j = 1, 2, \ldots \]

(4 – 63)

The orthogonality condition (4-46) becomes

\[ \int_0^l \mu \Phi^{(i)}(x) \Phi^{(j)}(x) \, dx = \begin{cases} 0 & , \ i \neq j \\ M_i & , \ i = j \end{cases} \]

(4 – 64)

\[ M_j = \frac{1}{2} \mu l \quad , \quad j = 1, 2, \ldots \]

(4 – 65)

In this case the decoupling condition is fulfilled, since

\[ \int_0^l c \Phi^{(i)}(x) \Phi^{(j)}(x) \, dx = \frac{c}{\mu} \int_0^l \mu \Phi^{(i)}(x) \Phi^{(j)}(x) \, dx = \begin{cases} 0 & , \ i \neq j \\ 2 \zeta \omega_i M_i & , \ i = j \end{cases} \]

(4 – 66)
From (4-55), (4-64) and (4-66) it follows that

\[ 2\zeta_j \omega_j = \frac{c}{\mu}, \quad j = 1, 2, \ldots \]  

(4-67)

The load per unit length can formally be written as

\[ f_d(x, t) = \begin{cases} P \delta(x - vt), & 0 \leq t \leq \frac{l}{v} \\ 0, & \frac{l}{v} < t < \infty \end{cases} \]  

(4-68)

\[ [0, \frac{l}{v}] \] indicates the time interval, for which the force acts on the beam. Using (4-54) and (4-63) the modal loads become

\[ F_j(t) = \begin{cases} P \Phi^{(j)}(vt), & 0 \leq t \leq \frac{l}{v} \\ 0, & \frac{l}{v} < t < \infty \end{cases} \]  

(4-69)

The beam is at rest at the time \( t = 0 \). Hence

\[ u_0(x) = \dot{u}_0(x) \equiv 0 \]  

(4-70)

Using (4-59) one then has

\[ q_j(0) = \dot{q}_j(0) = 0, \quad j = 1, 2, \ldots \]  

(4-71)

The modal coordinates follow from (3-188)

\[ q_j(t) = \int_0^{\min\left(t, \frac{l}{v}\right)} h_j(t - \tau)P \Phi^{(j)}(vt) \, d\tau = P \int_0^{\min\left(t, \frac{l}{v}\right)} h_j(t - \tau) \sin\left(j\pi \frac{vt}{l}\right) \, d\tau, \quad j = 1, 2, \ldots \]  

(4-72)

The impulse response function \( h_i(t) \) follows from (3-190), (4-65), (4-67)

\[ h_j(t) = \begin{cases} 0, & t < 0 \\ \frac{2}{\mu \omega_{d,j}} e^{-\frac{\omega_{d,j}}{2} t} \sin\left(\omega_{d,j} t\right), & t > 0 \end{cases} \]  

(4-73)

(4-72) can be evaluated analytically, although the result will not be given here. Finally, the movement of the beam follows from insertion of (4-72) into (4-52)

\[ u(x, t) = \frac{2P}{\mu l} \sum_{j=1}^{\infty} \sin\left(j\pi \frac{x}{l}\right) \int_0^{\min\left(t, \frac{l}{v}\right)} e^{-\frac{\omega_{d,j}}{2} (t - \tau)} \sin\left(\omega_{d,j} (t - \tau)\right) \sin\left(j\pi \frac{vt}{l}\right) \, d\tau \]  

(4-74)

Using (4-4) and (4-61) the bending moment at the middle of the beam becomes

\[ M\left(\frac{l}{2}, t\right) = \frac{1}{4} P l \cdot D(t) \]  

(4-75)

\[ D(t) = 8 \sum_{j=1}^{\infty} \frac{\pi^2 j^2}{\omega_{d,j} \mu l} \sin\left(j\pi \frac{x}{2}\right) \int_0^{\min\left(t, \frac{l}{v}\right)} e^{-\frac{\omega_{d,j}}{2} (t - \tau)} \sin\left(\omega_{d,j} (t - \tau)\right) \sin\left(j\pi \frac{vt}{l}\right) \, d\tau \]
\[
M_s = \frac{1}{2} P L
\]
denotes the maximum statical moment. It should be noted that the indicated analysis only concentrates on one part of the dynamic amplification factor. The second part, which is far the most important for small bridges and modest velocities, stems from the vehicle dynamics. Obviously, this latter part is totally ignored, when the vehicle is modelled as merely a moving force.

**Example 4-6: Simply Supported Homogeneous Beam with Harmonic Varying Load**

The stationary dynamic bending moment at the midpoint \( B \) from the external dynamic load \( f_d(x,t) = f_0 \cos(\omega t) \) is to be determined, when this load has been acting for such a period of time that the response from the initial conditions has been dissipated. Otherwise, the effect of any damping mechanism on the system is ignored.

The undamped circular eigenfrequencies, the eigenmodes and the modal mass are given at (4-33), (4-63), (4-65). Since the system is undamped the decoupled modal coordinate differential equations (4-53) become

\[
\ddot{q}_j + \omega_n^2 q_j = \frac{1}{M_j} F_j(t) = \frac{2}{\mu l} F_j(t), \quad j = 1, 2, \ldots
\]

The modal load follows from (4-54)

\[
F_j(t) = \int_0^l \Phi_j(x) f_d(x,t) \, dx = \\
\int_0^l \sin \left( \frac{j \pi x}{l} \right) \cdot f_0 \cos(\omega t) \, dx = \frac{f_0 l}{j \pi} \left( 1 - (-1)^j \right) \cos(\omega t), \quad j = 1, 2, \ldots
\]

With \( F_j(t) \) given by (4-78), the stationary solution of (4-77) becomes

\[
q_j(t) = Q_j \cos(\omega t)
\]
\[ Q_j = \frac{2}{j\pi} \left(1 - (-1)^j\right) f_0 \frac{1}{\mu} \frac{1}{\omega_j^2 - \omega^2}, \quad j = 1, 2, \ldots \quad (4 - 80) \]

As seen the amplitudes \( Q_j = 0 \) for \( j \) even. Inserting (4-79) into (4-52) the stationary dynamic displacement field can next be determined. With the sign as defined in fig. 4-15, the stationary dynamic bending moment at the midpoint finally becomes, see (4-4)

\[ M_B(t) = -EI \frac{\partial^2}{\partial x^2} u \left(\frac{l}{2}, t\right) = M_{B,0}(\omega) \cos(\omega t) \quad (4 - 81) \]

\[ M_{B,0}(\omega) = -EI \sum_{j=1}^{\infty} \frac{d^2}{dx^2} \Phi(j) \left(\frac{l}{2}\right) : Q_j = \]

\[ EI \sum_{j=1}^{\infty} \frac{\left(\frac{j\pi}{l}\right)^2}{2} \sin \left(\frac{j\pi}{2}\right) \cdot \frac{2}{j\pi} \left(1 - (-1)^j\right) \frac{f_0}{\mu} \frac{1}{\omega_j^2 - \omega^2} = \]

\[ \pi^3 EI \frac{4}{\mu l^4} \sum_{j=1,3,5,\ldots} j \frac{l^2}{\pi^3} \left(-1\right)^{j-1} \frac{j^4}{4\omega^2 l^2 - \omega^2} = \frac{1}{8} f_0 l^2 D(\omega) \quad (4 - 82) \]

\[ D(\omega) = \frac{8}{\pi^3} \sum_{j=1,3,5,\ldots} \left(-1\right)^{j-1} \frac{j^4}{4\omega^2 l^2 - \omega_1^2} = \frac{32}{\pi^3} \sum_{n=1}^{\infty} \frac{(2n-1)^{n-1}}{(2n-1)^4 - \omega_1^2} \quad (4 - 83) \]

where \( \omega_1^2 = \pi^4 \frac{EI}{\mu l^4} \) has been used to normalize the circular excitation frequency \( \omega \). As seen from (4-83), \( D(\omega) \) is a dynamic amplification factor to the quasi-static bending moment \( M_B(t) = \frac{1}{2} f_0 l^2 \cos(\omega t) \). Hence, one would expect \( D(0) = 1 \), which also follows from (4-83) by application of the series

\[ 1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots = \frac{\pi^3}{32} \quad (4 - 84) \]

In the present case a solution to the stationary displacement field and the stationary bending moment can be obtained on closed form by direct integration of the partial differential equation for the beam element, without any resort to series solutions, which result from the application of modal expansion. The partial differential equation and associated boundary conditions become, cf. (4-7)

\[ \begin{aligned} E I \frac{\partial^4 u}{\partial x^4} + \mu \frac{\partial^2 u}{\partial t^2} &= f_0 \cos(\omega t), \quad x \in [0, l] \\ u(0, t) &= \frac{\partial^2 u(0, t)}{\partial x^2} = u(l, t) = \frac{\partial^2 u(l, t)}{\partial x^2} = 0 \end{aligned} \quad (4 - 85) \]

The stationary displacement field must be harmonic. Because the system is free of damping all mass particles must be in phase and in phase with the excitation. Hence, the stationary displacement field is given on the form

\[ u(x, t) = U(x) \cos(\omega t) \quad (4 - 86) \]
where $U(x)$ is a real amplitude function. Upon insertion of (4-86) into (4-85) $U(x)$ is seen to fulfil the boundary value problem

$$
EI \frac{d^4}{dx^4} U(x) - \omega^2 \mu U(x) = f_0 \quad x \in [0, l]
$$

$$
U(0) = \frac{d^2}{dx^2} U(0) = U(l) = \frac{d^2}{dx^2} U(l) = 0
$$

(4-87)

A particular solution of the inhomogeneous differential equation (4-87) is $U(x) = -\frac{f_0}{\omega^2 \mu}$. Hence, the complete solution of (4-87) can be written, cf. (4-18) and (4-19)

$$
U(x) = -\frac{f_0}{\omega^2 \mu} + A \sin \left( \frac{\lambda x}{l} \right) + B \cos \left( \frac{\lambda x}{l} \right) + C \sinh \left( \frac{\lambda x}{l} \right) + D \cosh \left( \frac{\lambda x}{l} \right)
$$

(4-88)

$$
\lambda^4 = \frac{\mu \omega^2 l^4}{EI}
$$

(4-89)

In contrast to the eigenvalue problems, where $\lambda$ is the unknown searched for, $\lambda$ is now a known quantity given by (4-89). Inserting (4-88) into the boundary conditions of (4-87) provides the following linear equations for the determination of the expansion coefficients $A, B, C, D$ of the complementary solution

$$
B + D = \frac{f_0}{\omega^2 \mu}
$$

$$
\frac{\lambda^2}{l^2} (-B + D) = 0
$$

$$
A \sin \lambda + B \cos \lambda + C \sinh \lambda + D \cosh \lambda = \frac{f_0}{\omega^2 \mu}
$$

$$
\frac{\lambda^2}{l^2} \left( -A \sin \lambda - B \cos \lambda + C \sinh \lambda + D \cosh \lambda \right) = 0
$$

(4-90)

From (4-90) it follows that

$$
B = D = \frac{f_0}{2 \omega^2 \mu}
$$

$$
A = \frac{f_0}{2 \omega^2 \mu} \left( \frac{1 - \cos \lambda}{\sin \lambda} \right)
$$

$$
C = \frac{f_0}{2 \omega^2 \mu} \left( \frac{1 - \cosh \lambda}{\sinh \lambda} \right)
$$

(4-91)

(4-88) can then be written

$$
U(x) = \frac{f_0}{2 \omega^2 \mu} \left[ -2 + \left( \frac{1 - \cos \lambda}{\sin \lambda} \right) \sin \left( \frac{\lambda x}{l} \right) + \cos \left( \frac{\lambda x}{l} \right) + \left( \frac{1 - \cosh \lambda}{\sinh \lambda} \right) \sinh \left( \frac{\lambda x}{l} \right) + \cosh \left( \frac{\lambda x}{l} \right) \right]
$$

(4-92)

$M_B(t)$ is still given by (4-81). The amplitude $M_{B,0}(\omega)$ can now be written

$$
M_{B,0}(\omega) = -EI \frac{d^2}{dx^2} U \left( \frac{l}{2} \right) =
$$
\[-EI \frac{p_0 \lambda^2}{2 \omega^2 \mu^2} \left( -\left( \frac{1 - \cos \lambda}{\sin \lambda} \right) \sin \frac{\lambda}{2} - \cos \frac{\lambda}{2} + \left( \frac{1 - \cosh \lambda}{\sinh \lambda} \right) \sinh \frac{\lambda}{2} + \cosh \frac{\lambda}{2} \right) = \frac{1}{8} p_0 l^2 D(\omega) \] 

\[D(\omega) = -\frac{8EI\lambda^2}{2\omega^2 \mu^4} \left( -\frac{\sin^2 \lambda}{\cos \frac{\lambda}{2}} - \cos \frac{\lambda}{2} - \frac{\sinh^2 \frac{\lambda}{2}}{\cosh \frac{\lambda}{2}} + \cosh \frac{\lambda}{2} \right) = \]

\[\frac{4}{\lambda^2} \left( \frac{1}{\cos \frac{\lambda}{2}} - \frac{1}{\cosh \frac{\lambda}{2}} \right), \quad \lambda = \sqrt{\frac{\mu \omega^2 I^4}{EI}} \] 

(4 - 93)

(4 - 94)

At the derivation of (4-94), (4-89) has been applied as well as the trigonometric and hyperbolic identities \( \cos(2x) = 1 - 2 \sin^2 x, \sin(2x) = 2 \sin x \cos x, \cosh(2x) = 1 + 2 \sinh^2 x, \sinh(2x) = 2 \sinh x \cosh x \). (4-83) is merely a convergent series expansion of the closed form solution (4-94) for the dynamic amplification factor.

Example 4-7: Axial Vibrations of Beam Elements

Fig. 4-16 shows a differential beam element of the length \( dx \) in the static equilibrium state. The displacements of the left and right end sections in the local \( z \)-direction are designated \( u(x,t) \) and \( u(x,t) + \frac{\partial}{\partial z} u(x,t) dx \), respectively. In the static equilibrium state there may be a normal force, which is carried out unchanged to the deformed state as described subsequent to figure 4-1. With this in mind, \( N(x,t) \) now signify the additional dynamic normal force caused by an axial loading \( f(x,t) \) per unit length. Hence, \( N(x,t) = 0 \) in the referential state and is only acting in the deformed state as shown in fig. 4-16. The equation of equilibrium then reads

\[ \frac{\partial}{\partial z} N(x,t) + f(x,t) = 0 \]  

(4 - 95)

The constitutive equation, assuming linear elasticity for axial deformations, reads

\[ N(x,t) = AE(x) \frac{\partial u(x,t)}{\partial z} \]  

(4 - 96)
$AE(x)$ signifies the axial stiffness of the beam sections. Using d’Alembert’s principle the loading per unit length is written in the form (4-5), where $f_d(x,t)$ now signifies an external dynamic loading per unit length in the $x$-direction, and $c(x)$ is the damping constant of the applied linear viscous damping. The damping constants in bending and axial deformations are generally different, whereas the mass per unit length $\mu(x)$ is the same as for both types of vibrations. Insertion of (4-96) and (4-5) into (4-95) provides

$$-\frac{\partial}{\partial x} \left( AE(x) \frac{\partial u}{\partial x} \right) + c(x) \frac{\partial u}{\partial t} + \mu(x) \frac{\partial^2 u}{\partial t^2} = f_d(x,t) \quad , \quad t \in ]0,\infty[ , \quad x \in ]0,l[ \quad (4-97)$$

The partial differential equation (4-97) must be solved with proper initial values at $t = 0$, and with boundary conditions at $x = 0$ and $x = l$. The boundary conditions will be formulated for the beam element shown in fig. 4-17. At the left and right end-sections are attached concentrated masses $m_0$ and $m_1$, concentrated linear elastic springs with spring constants $k_0$ and $k_1$ and linear viscous damping elements with the damping constants $c_0$ and $c_1$.

**Fig. 4-17: Initial and boundary value problem for axial vibrations of beam element.**

**Differential equation** \( \forall t \in ]0,\infty[ , \ x \in ]0,l[ \) :

$$-\frac{\partial}{\partial x} \left( AE(x) \frac{\partial u}{\partial x} \right) + c(x) \frac{\partial u}{\partial t} + \mu(x) \frac{\partial^2 u}{\partial t^2} = f_d(x,t)$$

**Initial values** \( \forall x \in ]0,l[ \) :

$$u(x,0) = u_0(x) \quad , \quad \frac{\partial u(x,0)}{\partial t} = \dot{u}_0(x)$$

**Geometrical boundary conditions** \( \forall t \in [0,\infty[ \) :

$$u(0,t) = 0 \quad , \quad u(l,t) = 0$$

**Mechanical boundary conditions** \( \forall t \in [0,\infty[ \) :

$$AE(0) \frac{\partial u(0,t)}{\partial x} = k_0 u(0,t) + c_0 \dot{u}(0,t) + m_0 \ddot{u}(0,t)$$

$$AE(l) \frac{\partial u(l,t)}{\partial x} = -k_1 u(l,t) - c_1 \dot{u}(l,t) - m_1 \ddot{u}(l,t)$$

\( (4-98) \)
Fig. 4-18: Specification of mechanical boundary conditions due to concentrated masses, dampers and springs. a) End section at $x = 0$. b) End section at $x = l$.

The mechanical boundary conditions specify that the normal forces $N(0^+, t), N(l^-, t)$ immediately to the right and to the left of the end sections must balance the inertial forces of the concentrated masses and the forces in the dampers and springs. Hence, see fig. 4-18a and fig. 4-18b

$$
N(0^+, t) = k_0 u(0, t) + c_0 \dot{u}(0, t) + m_0 \ddot{u}(0, t)
$$

$$
N(l^-, t) = -k_1 u(l, t) - c_1 \dot{u}(l, t) - m_1 \ddot{u}(l, t)
$$

(4-99)

The mechanical boundary conditions of (4-98) are then obtained upon eliminating $N(0^+, t)$ and $N(l^-, t)$ from (4-99) by means of (4-96). Exactly one boundary condition (either geometrical or mechanical) must be fulfilled at $x = 0$ and $x = l$.

Undamped eigen vibrations are obtained setting $c_0 = c_1 = 0$, $c(x) \equiv 0$, $f_d(x, t) \equiv 0$ in (4-98). Using the assumption (4-12) also for axial eigen vibrations $u(x, t)$ the following linear eigenvalue problem is obtained for the determination of the eigenmodes $\Phi(x)$ and associated circular eigenfrequencies.

Differential equation $(\forall x \in [0, l])$:

$$
\frac{d}{dx} \left( AE(x) \frac{d\Phi}{dx} \right) + \mu(x) \omega^2 \Phi(x) = 0
$$

Geometrical boundary conditions:

$$
\Phi(0) = 0, \; \Phi(l) = 0
$$

(4-100)

Mechanical boundary conditions:

$$
AE(0) \frac{d\Phi(0)}{dx} = (k_0 - \omega^2 m_0) \Phi(0)
$$

$$
AE(l) \frac{d\Phi(l)}{dx} = -(k_1 - \omega^2 m_1) \Phi(l)
$$
5. DYNAMIC MODELLING OF CONTINUOUS SYSTEMS

Linear discrete systems are defined by their mass matrix $M$, damping matrix $C$, stiffness matrix $K$ and load-vector $f(t)$. A continuous system, such as a beam element possesses infinitely many degrees of freedom. However, at numerical treatments for almost all problems in practice, reductions to a finite number of degree of freedom become necessary. In this section it is demonstrated how this reduction is performed based on the finite element method (FEM) of structural analysis. In section 3.9 it was demonstrated how the damping matrix can be modelled to fit measured modal damping ratios. Hence, this section concentrates on the modelling of the mass and stiffness matrices and the load vector. A straightforward finite element modelling will often give an excessive number of dynamic degrees of freedom for a real structure. Hence, it may become necessary to combine such a procedure with a system reduction scheme as described in section 3.9 in order to keep the dynamic degrees of freedom sufficiently low. In section 5.1 is outlined how a modelling of a continuous system by a single dynamic degree-of-freedom system may be performed. Finally, in section 5.2 the modelling of continuous systems by multi degrees-of-freedom systems is treated.

5.1 Dynamic Modelling of Continuous Systems by Means of Single Degree-of-Freedom Systems

![Diagram](image)

Fig. 5-1: a) Examples of real continuous structures. b) Equivalent single degree-of-freedom system.

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1 In parts this section has been based on S. Krenk and J.D. Sørensen: Dynamic Modelling and Analysis of Structures. Aalborg University, 1990.
In fig. 5-1a examples of continuous structures are shown. These are characterized by continuous distribution of the bending stiffness $EI(x)$, the mass per unit length $\mu(x)$ and the external dynamic loading per unit length $f_d(x, t)$.

In fig. 5-1b is shown a linear viscous damped system of a single degree of freedom with the spring constant $k$ the damping constant $c$ and the mass $m$. The dynamic loading on the mass is $f(t)$. The dynamic modelling problem deals with the selection of the parameter $k, c, m$ and $f(t)$, so the single degree-of-freedom system at best describes the motion of the real system.

The equation of motion of the single degree-of-freedom system is given by, cf. (2-32), (2-33)

$$\begin{align*}
m\ddot{q} + c\dot{q} + kq &= f(t), \quad t > 0 \\
q(0) &= q_0, \quad \dot{q}(0) = \dot{q}_0
\end{align*}
$$

(5-1)

Alternatively, (5-1) can be formulated in terms of a power balance equation, cf. (2-19), (2-20), (2-26)

$$\frac{d}{dt} (T(t) + U(t)) = P(t) - cq^2(t)$$

(5-2)

$$T(t) = \frac{1}{2}m\dot{q}^2(t)$$

(5-3)

$$U(t) = \frac{1}{2}kq^2(t)$$

(5-4)

$$P(t) = f(t)\dot{q}(t)$$

(5-5)

$T(t)$ is the kinetic energy, $U(t)$ is the potential energy and $P(t)$ is the power supplied by the external dynamic force $f(t)$. The power balance equation (5-2) can be used for estimation of the parameters of the single degree-of-freedom system, if the shape $\Phi(x)$ of the displacement field can be estimated. In principle this means that the displacement field can be written as a product of a shape function $\Phi(x)$ and a time-varying amplitude, $q(t)$, i.e.

$$u(x, t) \simeq q(t)\Phi(x)$$

(5-6)

For the damped beam in fig. 5-1a the kinetic energy $T(t)$, the potential energy $U(t)$ and the supplied power $P(t)$ become

$$T(t) = \int_0^l \frac{1}{2}\mu(x) \left( \frac{\partial}{\partial t} u(x, t) \right)^2 dx = \frac{1}{2} \left( \int_0^l \mu(x) (\Phi(x))^2 dx \right) \dot{q}^2(t)$$

(5-7)
\[ U(t) = \int_0^l \frac{1}{2} EI(x) \left( \frac{\partial^2}{\partial x^2} u(x,t) \right)^2 \, dx = \frac{1}{2} \left( \int_0^l EI(x) \left( \frac{d^2}{dx^2} \Phi(x) \right)^2 \, dx \right) q^2(t) \quad (5-8) \]

\[ P(t) = \int_0^l f_d(x,t) \frac{\partial}{\partial t} u(x,t) \, dx = \left( \int_0^l \Phi(x) f_d(x,t) \, dx \right) \dot{q}(t) \quad (5-9) \]

\( \Phi(x) \) must fulfil all prescribed geometrical boundary conditions, \( (\Phi(0) = \frac{d}{dx} \Phi(0) = \Phi(l) = \frac{d}{dx} \Phi(l) = 0) \), and the 2nd derivative \( \frac{d^2}{dx^2} \Phi(x) \) must exist and be piecewise continuous. Upon comparison of (5-3), (5-4), (5-5) with (5-7), (5-8), (5-9) follows

\[ m = \int_0^l \mu(x) (\Phi(x))^2 \, dx \quad (5-10) \]

\[ k = \int_0^l EI(x) \left( \frac{d^2}{dx^2} \Phi(x) \right)^2 \, dx \quad (5-11) \]

\[ f(t) = \int_0^l \Phi(x) f_d(x,t) \, dx \quad (5-12) \]

Figure 5-2: Axial vibrations of beam element.

Axial vibrations \( u(x,t) \) of a linear elastic beam with axial stiffness \( AE(x) \) and mass per unit length \( \mu(x) \), caused by the external dynamic loading per unit length \( f_d(x,t) \) unidirectional to \( u(x,t) \), can be reduced to a single degree-of-freedom system in the same way, i.e. (5-1) is assumed to be valid also for the axial vibrations. Of course, \( u(x,t) \) and \( q(t) \) now have a different meaning. \( q(t) \) should now be interpreted as a degree of freedom defining the axial vibrations. The shape function still need to fulfil the geometrical boundary condition, which now has the form \( \Phi(0) = 0 \) or \( \Phi(l) = 0 \), and only the 1st derivative \( \frac{d\Phi(x)}{dx} \) needs to exist piecewise throughout the beam. The kinetic energy \( T(t) \) and the supplied power \( P(t) \) for axial vibrations are unchanged given by (5-7) and (5-9) and the equivalent mass \( m \) and the loading \( f(t) \) are given by (5-10) and (5-12). The potential energy \( U(t) \) and the equivalent spring constant now become

\[ U(t) = \int_0^l \frac{1}{2} AE(x) \left( \frac{\partial}{\partial x} u(x,t) \right)^2 \, dx = \frac{1}{2} \left( \int_0^l AE(x) \left( \frac{d}{dx} \Phi(x) \right)^2 \, dx \right) q^2(t) \quad (5-13) \]
\[ k = \int_{0}^{l} AE(x) \left( \frac{d}{dx} \Phi(x) \right)^2 \, dx \]  \hspace{1cm} (5 - 14)

In contrast to the mass, the spring constant and the loading, the damping constant \( c \) cannot be estimated in a similar simple way. Instead, the damping constant must be specified from the modal damping ratio in the first mode \( \zeta_1 \), which is estimated or measured on the real structure. \( \zeta_1 \) should then also be the damping ratio of the equivalent single degree-of-freedom system. Consequently, \( c \) should be selected as follows, cf. (2-39)

\[ c = \zeta_1 \cdot 2\sqrt{km} \]  \hspace{1cm} (5 - 15)

Example 5-1: Eigenvibrations of Cantilever Beam Supporting a Concentrated Mass at the Free End

A Cartesian \((x, y, z)\)-coordinate system is defined as shown in the figure. The eigenvalue problem for bending vibrations becomes, cf. (4-13)

\[
\begin{align*}
\frac{d^4 \Phi_y}{dx^4} - \frac{\lambda^4}{l^4} \Phi_y(x) &= 0, & \lambda^4 &= \frac{\mu \omega^2 l^4}{EI}, \quad x \in [0, l], \\
\Phi_y(0) &= \frac{d}{dx} \Phi_y(0) = \frac{d^2}{dx^2} \Phi_y(l) = 0, \\
\frac{d^3}{dx^3} \Phi_y(l) &= -\frac{\omega^2}{EI} m_1 \Phi_y(l) = -\frac{\lambda^4}{l^3} \frac{m_1}{\mu} \Phi_y(l)
\end{align*}
\]  \hspace{1cm} (5 - 16)
Mode shapes \( \Phi^{(i)}_y (x) \) are given by (4-43), where the eigenvalues \( \lambda_i \) fulfil the following equation, obtained upon insertion of (4-43) into the shear force boundary condition of (5-16)

\[
\frac{\lambda^3}{l^3} \left( (\cos \lambda_i + \cosh \lambda_i) (-\cos \lambda_i - \cosh \lambda_i) - (\sin \lambda_i + \sinh \lambda_i) (\sin \lambda_i - \sinh \lambda_i) \right) = \frac{\lambda^4}{l^3} m_1 \mu \left( (\cos \lambda_i + \cosh \lambda_i) (\sin \lambda_i - \sinh \lambda_i) - (\sin \lambda_i + \sinh \lambda_i) (\cos \lambda_i - \cosh \lambda_i) \right) = \\
1 + \cos \lambda_i \cosh \lambda_i = \lambda_i \frac{m_1}{\mu l} \left( \sin \lambda_i \cosh \lambda_i - \cos \lambda_i \sinh \lambda_i \right)
\]  

(5 - 17)

From the solutions \( \lambda_i \) of (5-17) the circular bending eigenfrequency is obtained from \( \omega_{y,1} = \frac{\lambda_i^2}{\mu l^2} \), cf. (4-19). In table 5-1, \( \omega_{y,1} \) has been shown in the 2nd column as a function of the ratio of the mass \( m_1 \) to the total beam mass \( \mu l \).

<table>
<thead>
<tr>
<th>( m_1/\mu l )</th>
<th>Analytical ( \omega_{y,1} / \sqrt{\frac{E I}{\mu l^4}} )</th>
<th>Numerical ( \omega_{y,1} / \sqrt{\frac{E I}{\mu l^4}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>3.516015</td>
<td>3.567530</td>
</tr>
<tr>
<td>1.0</td>
<td>1.557297</td>
<td>1.558123</td>
</tr>
<tr>
<td>2.0</td>
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<td>1.158383</td>
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<td>0.962888</td>
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<tr>
<td>4.0</td>
<td>0.841546</td>
<td>0.841584</td>
</tr>
<tr>
<td>5.0</td>
<td>0.756937</td>
<td>0.756960</td>
</tr>
</tbody>
</table>

Table 5-1: Cantilever beam supporting a concentrated mass at the free end. Analytical and approximate numerical solutions for fundamental circular bending eigenfrequencies.

The corresponding eigenvalue problem for axial vibration reads, cf. (4-100)

\[
\frac{d^2 \Phi_x}{dx^2} + \lambda^2 \Phi_x (x) = 0 \ , \ \lambda^2 = \frac{\mu \omega^2 l^2}{AE} \ , \ x \in [0, l] \\
\Phi_x (0) = 0 \ , \ \frac{d \Phi_x (l)}{dx} = \frac{\omega^2}{AE} m_1 \Phi_x (l) = \frac{\lambda^2}{l} \frac{m_1}{\mu l} \Phi_x (l) 
\]  

(5 - 18)

with the solution for the mode shapes and the frequency condition

\[
\Phi^{(i)}_x (x) = \sin \left( \lambda_i \frac{x}{l} \right)
\]  

(5 - 19)

\[
\cos \lambda_i = \lambda_i \frac{m_1}{\mu l} \sin \lambda_i
\]  

(5 - 20)

From the fundamental solution \( \lambda_1 \) of (5-20) the fundamental circular eigenfrequency in axial vibrations is obtained from \( \omega_{x,1} = \lambda_1 \sqrt{\frac{AE}{\mu l^2}} \). In table 5-2 \( \omega_{x,1} \) has been shown in the 2nd column as a function of the mass ratio \( \frac{m_1}{\mu l} \).
Table 5-2: Cantilever beam supporting a concentrated mass at the free end. Analytical and approximate numerical solutions for fundamental circular eigenfrequencies in axial vibrations.

Next, an approximate solution based on the following shape functions will be derived, valid for axial and bending vibrations, respectively

\[ \Phi_x(x) = \frac{x}{l} \]  

(5 - 21)

\[ \Phi_y(x) = \frac{1}{2} \left( \frac{x}{l} \right)^2 \left( 3 - \frac{x}{l} \right) \]  

(5 - 22)

\( \Phi_x(x) \) fulfils the geometrical boundary condition \( \Phi_x(0) = 0 \). \( \Phi_y(x) \) fulfils as well the geometrical boundary condition \( \Phi_y(0) = 0 \) as the mechanical boundary condition \( \frac{d^2 \Phi_y}{dx^2}(l) = 0 \). It is not mandatory that the shape function fulfils any of the mechanical boundary conditions at \( x = l \). However, the more of the conditions specified in (5-16) and (5-18) that can be fulfilled, the better will be the obtained solution. Both \( \Phi_x(x) \) and \( \Phi_y(x) \) have been normalized to 1 at \( x = l \). Then \( q(t) \) can be interpreted as the end-section motion in either the axial or transverse direction according to (5-6).

Formally, the concentrated mass may be written as the equivalent distributed mass \( \mu(x) = m_1 \delta(x - l) \). (5-10) then provides

\[
m_x = \int_0^l \mu(\Phi_x(x))^2 dx + \int_{l-}^{l+} m_1 \delta(x - l) \left( \Phi_x(x) \right)^2 dx = \\
\int_0^l \frac{(x)}{l}^2 dx + m_1 \left( \Phi(l) \right)^2 = \frac{1}{3} \mu l + m_1
\]  

(5 - 23)

\[
m_y = \int_0^l \mu(\Phi_y(x))^2 dx + \int_{l-}^{l+} m_1 \delta(x - l) \left( \Phi_y(x) \right)^2 dx = \\
\int_0^l \frac{1}{4} \left( \frac{x}{l} \right)^4 \left( 3 - \frac{x}{l} \right)^2 dx + m_1 \left( \Phi_y(l) \right)^2 = \frac{33}{140} \mu l + m_1
\]  

(5 - 24)

The spring constants follow from (5-11) and (5-14)

\[
k_x = \int_0^l AE \left( \frac{d\Phi_x(x)}{dx} \right)^2 dx = \int_0^l AE \left( \frac{1}{l} \right)^2 dx = \frac{AE}{l}
\]  

(5 - 25)

\[
k_y = \int_0^l EI \left( \frac{d^2\Phi_y(x)}{dx^2} \right)^2 dx = \int_0^l EI \left( \frac{3}{l^2} - \frac{3x}{l^3} \right)^2 dx = 3 \frac{EI}{l^3}
\]  

(5 - 26)
These spring stiffnesses are seen to be identical to the statical spring stiffness of the beam when applying a unit force at the end-section in the x- and y-directions. This result is obtained because the shape functions (5-21), (5-22) are proportional to the corresponding statical displacement fields caused by unit forces applied at end-section in the relevant coordinate directions. Actually, (5-21) and (5-22) fulfill the differential equations in (5-16) and (5-18) in the statical case where \( \lambda = 0 \).

Finally, the following approximate solutions are obtained for the fundamental eigenfrequencies, cf. (2-7)

\[
\omega_{x,1} = \sqrt[3]{\frac{AE}{1 + 3 \frac{m_1}{\mu l}}} = \sqrt[3]{\frac{3}{1 + 3 \frac{m_1}{\mu l}}} \sqrt{\frac{AE}{\mu l^2}} \tag{5-27}
\]

\[
\omega_{y,1} = \sqrt{\frac{3 \frac{EI}{120 \mu l} + m_1}{33 + 140 \frac{m_1}{\mu l}}} = \sqrt{\frac{420 \frac{EI}{120 \mu l}}{33 + 140 \frac{m_1}{\mu l}}} \sqrt{\frac{EI}{\mu l^4}} \tag{5-28}
\]

(5-27) and (5-28) have been shown as the 3rd columns in table 5-1 and table 5-2 as a function of the mass ratio. As seen the approximate numerical solutions provide an upper bound for \( \omega_1 \). The reason for this is that (5-27) and (5-28) represent the results obtained by the Rayleigh fraction method in section 3.10 generalized to continuous systems. Any shape function fulfilling the kinematical boundary conditions and the necessary differentiability requirements for evaluation the integral of the spring constant, will give an upper bound to the fundamental circular eigenfrequency. The approximation for \( \omega_{x,1} \) is rather poor for the case \( m_1 = 0 \), which is due to the poor resemblance in this case between the exact mode shape \( \Phi_{x,1}^{(1)}(x) = \sin \left( \frac{x}{l} \right) \) and the assumed mode shape \( \Phi_x(x) = \frac{x}{l} \).

### 5.2 Dynamic Modelling of Continuous Systems by Means of Multi Degrees-of-Freedom Systems

Conventionally, multi degree of freedom models are formulated by the finite element method. Below the basic principles of this approach will be outlined.

The equations of motion of an \( n \) degrees-of-freedom linear system read, cf. (3-35), (3-36)

\[
\begin{align*}
M \ddot{q} + C \dot{q} + Kq &= f(t) , \quad t > 0 \\
q(0) &= q_0 , \quad \dot{q}(0) = \dot{q}_0
\end{align*}
\tag{5-29}
\]

where the degrees of freedom have been assembled in the vector \( q^T(t) = [q_1(t), q_2(t), \ldots, q_n(t)] \). \( M, C \) and \( K \) are the mass matrix, the damping matrix and the stiffness matrix, which are all supposed to be positive definite. Besides \( M \) and \( K \) are symmetric matrices so a kinetic energy \( T(t) = \frac{1}{2} q^T(t)Mq(t) \) and a potential energy \( U(t) = \frac{1}{2} q^T(t)Kq(t) \) may be defined, cf. (3-12) and (3-17). \( f(t) \) signifies a vector of external dynamic loadings conjugated to the selected degrees of freedom, \( q(t) \). For a linear viscously damped system the power balance equation becomes, cf. (3-16), (3-22)

\[
\frac{d}{dt} (T(t) + U(t)) = P(t) - q^T(t)Cq(t) \tag{5-30}
\]
The finite element method is based on a division of the structure into a finite number \( m \) of structural elements. The kinetic energy \( T(t) \), the potential energy \( U(t) \) and the supplied power \( P(t) \) of the structural assemblage can then be calculated as a sum of contributions \( T_j(t), U_j(t), P_j(t) \) from all \( m \) element of the structure.

\[
P(t) = \mathbf{q}^T(t) f(t) \tag{5-31}
\]

\[
T(t) = \sum_{j=1}^{m} T_j(t) \tag{5-32}
\]

\[
U(t) = \sum_{j=1}^{m} U_j(t) \tag{5-33}
\]

\[
P(t) = \sum_{j=1}^{m} P_j(t) \tag{5-34}
\]

\[
q_1 \quad q_2 \quad q_3 \quad q_4 \quad q_5 \quad q_6
\]

\[
\begin{bmatrix}
U_x(x, t) \\
U_y(x, t)
\end{bmatrix} = \mathbf{N}(x) \mathbf{q}_j(t) \tag{5-35}
\]

Fig. 5-4: Plane Bernoulli-Euler beam element with homogeneous sectional properties.

\( T_j(t), U_j(t), P_j(t) \) will be calculated for a plane Bernoulli-Euler beam element with linear elastic axial deformations. A local Cartesian \((x, y, z)\)-coordinate system for the beam element \( j \) is defined as shown in fig. 5-4. The degrees of freedom of the element are assembled in the vector \( \mathbf{q}_j^T(t) = [q_1(t), q_2(t), q_3(t), q_4(t), q_5(t), q_6(t)] \), where \( q_1(t), q_4(t) \) and \( q_2(t), q_5(t) \) signify the components of the end section displacements relative to the local \((x, y, z)\)-coordinate system, and \( q_3(t), q_6(t) \) signify the end section rotations in the local \( z \)-direction. The basic assumption of the finite element method states that the displacement fields \( u_x(x, t), u_y(x, t) \) in the \( x \) - and \( y \) -directions can be estimated by interpolation between the displacements and rotations of the end-sections, i.e.

\[
\mathbf{u}_j(x, t) = \begin{bmatrix} u_x(x, t) \\ u_y(x, t) \end{bmatrix} = \mathbf{N}(x) \mathbf{q}_j(t)
\]
The shape functions $N_1(x), N_4(x)$ are linearly varying, ensuring continuity of the axial displacements at the element nodes, whereas $N_2(x), N_3(x), N_5(x), N_6(x)$ are cubic interpolation functions, ensuring continuity of as well transverse displacements as rotations at the element nodes. The displacement components $u_x(x, t), u_y(x, t)$ and cross-sectional rotations $\frac{\partial}{\partial x} u_y(x, t)$ of the structural system are then made up of assemblages of corresponding contributions from all structural elements, and such contributions are continuous at the element nodes. The shape functions are given as

\[
\begin{align*}
N_1(x) &= 1 - \xi \\
N_2(x) &= 2\xi^3 - 3\xi^2 + 1 \\
N_3(x) &= \xi^3 - 2\xi^2 + \xi \\
N_4(x) &= \xi \\
N_5(x) &= -2\xi^3 + 3\xi^2 \\
N_6(x) &= \xi^3 - \xi^2
\end{align*}
\]

where

\[
\xi = \frac{x}{l}
\]

From (5-35) follows that the kinetic energy, the potential energy and the supplied power from the external dynamic loading per unit length $\mathbf{f}_j^T(x, t) = [f_x(x, t), f_y(x, t)]$ become

\[
T_j(t) = \frac{1}{2} \int_0^l \mu(x)\mathbf{u}^T(x, t)\mathbf{u}(x, t)dx = \frac{1}{2} \mathbf{q}_j^T(t)\mathbf{m}_j\mathbf{q}_j(t)
\]

\[
U_j(t) = \frac{1}{2} \int_0^l \left[ AE(x) \left( \frac{\partial u_x(x, t)}{\partial x} \right)^2 + EI(x) \left( \frac{\partial^2 u_y(x, t)}{\partial x^2} \right)^2 \right] dx = \frac{1}{2} \mathbf{q}_j^T(t)k_j\mathbf{q}_j(t)
\]

\[
P_j(t) = \int_0^l \left( f_x(x, t)\dot{u}_x(x, t) + f_y(x, t)\dot{u}_y(x, t) \right) dx
\]
\[ \int_0^l \left( f_x(x, t)(N_x(x)\dot{q}_j(t)) + f_y(x, t)(N_y(x)\dot{q}_j(t)) \right) dx = \dot{q}_j^T(t)f_j(t) \] (5 - 41)

\( m_j, k_j \) and \( f_j(t) \) signify the element mass matrix, element stiffness matrix and element nodal load vector with components in the local coordinate. In case \( AE, EI, \mu \) are all constant these are defined by

\[
k_j = \int_0^l \left( AE \frac{dN^T_x}{dx} \frac{dN_x}{dx} + EI \frac{d^2N^T_y}{dx^2} \frac{d^2N_y}{dx^2} \right) dx = \\
\begin{bmatrix}
AE/4 & 0 & 0 & -AE/4 & 0 & 0 \\
0 & 12EI/3 & 6EI/2 & 0 & -12EI/3 & 6EI/2 \\
0 & 6EI/4 & 4EI & 0 & -6EI/4 & 2EI \\
-AE/4 & 0 & 0 & AE/4 & 0 & 0 \\
0 & -12EI/3 & -6EI/2 & 0 & 12EI/3 & -6EI/2 \\
0 & 6EI/4 & 2EI & 0 & -6EI/4 & 4EI
\end{bmatrix} 
(5 - 42)
\]

\[
m_j = \int_0^l \mu N^T(x)N(x)dx = \\
\frac{\mu l}{420} \begin{bmatrix}
140 & 0 & 0 & 70 & 0 & 0 \\
0 & 156 & 22l & 0 & 54 & -13l \\
0 & 22l & 4l^2 & 0 & 13l & -3l^2 \\
70 & 0 & 0 & 140 & 0 & 0 \\
0 & 54 & 13l & 0 & 156 & -22l \\
0 & -13l & -3l^2 & 0 & -22l & 4l^2
\end{bmatrix} 
(5 - 43)
\]

\[
f_j(t) = \int_0^l \left( f_x(x, t)N^T_x(x) + f_y(x, t)N^T_y(x) \right) dx 
(5 - 44)
\]

\( m_j \) is alternatively designated as the consistent element mass matrix, because the same shape functions are used at the discretization of the kinetic and the potential energy of the beam element.

The element degrees of freedom \( q_j(t) \), with components specified in the local coordinate system, is related to system degrees of freedom \( q(t) \), with components specified in a global coordinate system, by the topological transformation

\[
q_j(t) = A_j q(t) 
(5 - 45)
\]
The transformation matrix $A_j$ has the dimension $n_e \times n$, where $n_e$ is the number of element degrees of freedom (6 for a beam element with axial flexibility), and $n$ is the number of global degrees of freedom. $A_j$ is very sparsely populated, since only $n_e$ columns contain non-zero elements.

Insertion of (5-45) into (5-39), (5-40), (5-41) provides

$$
T_j(t) = \frac{1}{2} \dot{q}^T(t) M_j \dot{q}(t) \\
U_j(t) = \frac{1}{2} \dot{q}^T(t) K_j q(t) \\
P_j(t) = \dot{q}^T(t) F_j(t)
$$

where

$$
M_j = A_j^T m_j A_j \\
K_j = A_j^T k_j A_j \\
F_j(t) = A_j^T f_j(t)
$$

$M_j$, $K_j$ and $F_j(t)$ signify the consistent element mass matrix, the element stiffness matrix and the element loading vector in global coordinates. Notice that the matrices have the dimension $n \times n$ and the load vector the dimension $n$.

Next, insertion of (5-46), (5-47), (5-48) into (5-32), (5-33), (5-34) provides

$$
T(t) = \sum_{j=1}^{m} \frac{1}{2} \dot{q}^T(t) M_j \dot{q}(t) = \frac{1}{2} \dot{q}^T(t) \left( \sum_{j=1}^{m} M_j \right) \dot{q}(t) \\
U(t) = \sum_{j=1}^{m} \frac{1}{2} \dot{q}^T(t) K_j q(t) = \frac{1}{2} \dot{q}^T(t) \left( \sum_{j=1}^{m} K_j \right) q(t) \\
P(t) = \sum_{j=1}^{m} \dot{q}^T(t) F_j(t) = \dot{q}^T(t) \left( \sum_{j=1}^{m} F_j(t) \right)
$$

Comparison with (3-12), (3-17) and (5-31) finally provides the following expression for the consistent global mass matrix $M$, the global stiffness matrix $K$ and the global load vector $f(t)$

$$
M = \sum_{j=1}^{n} M_j = \sum_{j=1}^{n} A_j^T m_j A_j \\
K = \sum_{j=1}^{n} K_j = \sum_{j=1}^{n} A_j^T k_j A_j \\
f(t) = \sum_{j=1}^{n} F_j(t) = \sum_{j=1}^{n} A_j^T f_j(t)
$$

The correction for geometrical boundary conditions in $M$ (and $C$) is performed in the same way as in $K$. 
Example 5-2: 2 DOF Model to Bending Vibrations of Cantilever Beam Supporting a Concentrated Mass at the Free End

![Diagram](image)

Fig. 5-5: 2 degrees of freedom model of cantilever beam supporting a concentrated mass at the free end.

The beam element is modelled by a single finite element with the local degrees of freedom $q_1(t), q_2(t), q_5(t), q_6(t)$. Since $q_2(t) = q_5(t) = 0$, the equations of motion for undamped eigenvibrations of the system become

$$M\ddot{q} + Kq = 0$$

$$q(t) = \begin{bmatrix} q_5(t) \\ q_6(t) \end{bmatrix}$$

(5-58)

(5-59)

where, cf. (5-36), (5-37)

$$M = \int_0^l \left( \mu + m_1 \delta(l-x) \right) \begin{bmatrix} N_x^2(x) & N_6(x)N_6(x) \\ N_5(x)N_6(x) & N_6^2(x) \end{bmatrix} dx =$$

$$\begin{bmatrix} \mu l/420 & -22l \\ -22l & 4l^2 + m_1 \end{bmatrix} =$$

$$\begin{bmatrix} 156 & -22l \\ -22l & 4l^2 + m_1 \end{bmatrix}$$

(5-60)

$$K = \int_0^l EI \left[ \left( \frac{d^2N_5(x)}{dx^2} \right)^2 \frac{d^2N_6(x)}{dx^2} \right] dx = \frac{EI}{l^3} \begin{bmatrix} 12 & -6l \\ -6l & 4l^2 \end{bmatrix}$$

(5-61)

As seen, the concentrated mass at $x = l$ has formally been taken into account in the same way as in example 5-1. The last statement of (5-60) follows, since $N_5(l) = 1, N_6(l) = 0$, cf. (5-37) and fig. 5-5. (5-60) indicates that a concentrated mass at a node may be taken into account simply by adding the mass to the component in the main diagonal of the global mass matrix at the corresponding degrees of freedom. This observation is generally true.

The undamped circular eigenfrequencies $\omega_j$ and the eigenmodes $\Phi_{(j)} = \begin{bmatrix} \phi_{1j} \\ \phi_{2j} \end{bmatrix}$ are obtained as nonlinear solutions of the homogeneous equations, cf. (3-42)
The frequency condition becomes

\[
\begin{bmatrix}
12 - (156 + 420 \frac{m_1}{\mu_1}) \lambda_j & -6l + 22l \lambda_j \\
-6l + 22l \lambda_j & 4l^2 - 4l^2 \lambda_j
\end{bmatrix}
\begin{bmatrix}
\Phi_1^{(j)} \\
\Phi_2^{(j)}
\end{bmatrix} = \begin{bmatrix} 0 \\
0 \end{bmatrix}
\]

(5-62)

\[
\lambda_j = \frac{1}{420} \omega^2 \frac{\mu t^4}{EI}
\]

(5-63)

The frequency condition becomes

\[
\left(12 - (156 + 420 \frac{m_1}{\mu_1}) \lambda_j \right) \left(4l^2 - 4l^2 \lambda_j \right) - (6l - 22l \lambda_j)^2 = 0 \Rightarrow
\]

\[
\lambda_j = \left\{ \begin{array}{ll}
\frac{51+210\gamma-\sqrt{2496+20160\gamma+44100\gamma^2}}{36+420\gamma}, & j = 1 \\
\frac{151+210\gamma+\sqrt{2496+20160\gamma+44100\gamma^2}}{36+420\gamma}, & j = 2
\end{array} \right.
\]

(5-64)

where \( \gamma = \frac{m_1}{\mu_1} \) signifies the mass ratio. The circular eigenfrequencies then follows from (5-63)

\[
\omega_{y,j} = \sqrt{\frac{420}{\mu t^4}} \lambda_j, \quad j = 1, 2
\]

(5-65)

<table>
<thead>
<tr>
<th>( \frac{m_1}{\mu_1} )</th>
<th>Analytical solutions</th>
<th>Numerical solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \omega_{y,1} / \sqrt{\frac{EI}{\mu t^4}} )</td>
<td>( \omega_{y,2} / \sqrt{\frac{EI}{\mu t^4}} )</td>
</tr>
<tr>
<td>0.0</td>
<td>3.516015</td>
<td>22.034491</td>
</tr>
<tr>
<td>1.0</td>
<td>1.557297</td>
<td>16.250085</td>
</tr>
<tr>
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<tr>
<td>4.0</td>
<td>0.841546</td>
<td>15.646863</td>
</tr>
<tr>
<td>5.0</td>
<td>0.756937</td>
<td>15.602346</td>
</tr>
</tbody>
</table>

Table 5-3: Cantilever beam supporting a concentrated mass at the free end. Analytical and numerical solutions using 2 degrees of freedom for circular bending eigenfrequencies.

In Table 5-3 the approximate numerical solutions for the 1st and 2nd circular eigenfrequency in bending based on (5-60), (5-61) have been compared with the corresponding analytical solutions based on (5-17) and (5-20). As seen the numerical solution for \( \omega_{y,1} \) provides a sharper upper bound to the analytical solution, compared to the numerical solution in Table 5-1 as a consequence of increasing the number of degrees of freedom from 1 to 2. Whereas all approximations to \( \omega_{y,1} \) are rather acceptable, the approximations to \( \omega_{y,2} \) are not acceptable (deviations \( \simeq 33\% \)). In order to obtain better estimates for the 2nd circular eigenfrequency, the beam elements should be divided into a few artificial sub-beams corresponding to the introduction of extra degree of freedoms between the end-sections.
Example 5-3: Eigenfrequencies of Continuous Beam

Fig. 5-6: Continuous beam with 2 sub-beams. a) Definition of structural data. b) Finite element model. Global node numbering and definition of global degrees of freedom.

Eigenfrequencies of continuous beam consisting of 2 sub-beams shown in fig. 5-6a is to be calculated. The global degrees of freedom have been indicated in fig. 5-6b. Global and local coordinate systems are co-directional. Below, the local element stiffness and consistent mass matrices have been indicated in global coordinates.

Element 1:

\[
k_1 = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} = \begin{bmatrix} k_{1,AA} & k_{1,AB} \\ k_{1,AB}^T & k_{2,BB} \end{bmatrix}
\]  
(5 - 66)

\[
m_1 = \frac{\mu l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ -22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} = \begin{bmatrix} m_{1,AA} & m_{1,AB} \\ m_{1,AB}^T & m_{1,BB} \end{bmatrix}
\]  
(5 - 67)

Element 2:

\[
k_2 = \frac{4EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} = \begin{bmatrix} k_{2,BB} & k_{2,BC} \\ k_{2,BC}^T & k_{2,CC} \end{bmatrix}
\]  
(5 - 68)

\[
m_2 = \frac{2\mu l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ -22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix} = \begin{bmatrix} m_{2,BB} & m_{2,BC} \\ m_{2,BC}^T & m_{2,CC} \end{bmatrix}
\]  
(5 - 69)

\(k_{1,AA}\) and \(k_{1,AB}\) represent the sub-matrices belonging to stiffness matrix \(k_1\) for node \(A\) and the coupling between nodes \(A\) and \(B\) respectively. The global stiffness and mass matrices with no correction...
for geometrical boundary conditions, $K_0$ and $M_0$, are assembled according to the topology of the system

\[
K_0 = \begin{bmatrix}
    k_{1,AA} & k_{1,AB} & 0 \\
    k_{1,AB}^T & k_{1,BB} + k_{2,BB} & k_{2,BC} \\
    0 & k_{2,BC}^T & k_{2,CC}
\end{bmatrix}
\]

(5 - 70)

\[
M_0 = \begin{bmatrix}
    m_{1,AA} & m_{1,AB} & 0 \\
    m_{1,AB}^T & m_{1,BB} + m_{2,BB} & m_{2,BC} \\
    0 & m_{2,BC}^T & m_{2,CC}
\end{bmatrix}
\]

(5 - 71)

Introduction of the geometrical boundary conditions $\theta_A = u_A = \theta_C = u_C = 0$ leads to the following global stiffness matrix and consistent mass matrix

\[
K = k_{1,BB} + k_{2,BB} = \frac{EI}{I^3} \begin{bmatrix}
    60 & 18l \\
    18l & 20l^2
\end{bmatrix}
\]

\[
M = m_{1,BB} + m_{2,BB} = \frac{\mu l}{420} \begin{bmatrix}
    468 & 22l \\
    22l & 12l^2
\end{bmatrix}
\]

(5 - 72)

The frequency condition becomes, cf. (3-43)

\[
\det \left( \begin{bmatrix}
    30 & -234\lambda_j & (9 - 11\lambda_j)l \\
    (9 - 11\lambda_j)l & (10 - 6\lambda_j)l^2
\end{bmatrix} \right) = 0 \quad j = 1, 2
\]

(5 - 73)

\[
\lambda_j = \frac{1}{420} \frac{\mu l^4 \omega_j^2}{EI} \quad j = 1, 2
\]

(5 - 74)

(5-73) gives

\[
1283\lambda_j^2 - 2322\lambda_j + 219 = 0 \quad \Rightarrow \\
\lambda_j = \begin{cases}
    \frac{11611 - \sqrt{1056944}}{1283}, & j = 1 \\
    \frac{11611 + \sqrt{1056944}}{1283}, & j = 2
\end{cases}
\]

(5 - 75)

\[
\omega_j = \begin{cases}
    6.47493 \sqrt{\frac{EI}{\mu l^4}}, & j = 1 \\
    26.79925 \sqrt{\frac{EI}{\mu l^4}}, & j = 2
\end{cases}
\]

(5 - 76)

Example 5-4: Eigenvibrations of 1 Storey Framed Structure

\[
\begin{align*}
\lambda_j = & \begin{cases}
    \frac{11611 - \sqrt{1056944}}{1283}, & j = 1 \\
    \frac{11611 + \sqrt{1056944}}{1283}, & j = 2
\end{cases} \\
\omega_j = & \begin{cases}
    6.47493 \sqrt{\frac{EI}{\mu l^4}}, & j = 1 \\
    26.79925 \sqrt{\frac{EI}{\mu l^4}}, & j = 2
\end{cases}
\end{align*}
\]

Fig. 5-7: Plane 1 storey framed structure.
Eigenfrequencies of the 1 storey framed structure shown in fig. 5-7 is to be calculated. All beam are linear elastic Bernoulli-Euler beams with the indicated lengths, bending stiffness and masses per unit length, and are all assumed to be infinitely stiff against axial deformations.

First a 3 element model of the frame is adopted. Because axial deformations of the beam elements have been disregarded, the structure has but 3 global degrees of freedom, \( \mathbf{q}^T(t) = [q_1(t), q_2(t), q_3(t)] \) where \( q_1(t) \) and \( q_2(t) \) signify the nodal rotations of points B and C, and\( q_3(t) \) indicates the horizontal displacement of the storey beam, all with signs defined in fig. 5-8. The global stiffness and consistent mass matrices become

\[
\mathbf{K} = \begin{bmatrix}
4EI & 2EI & -6EI \\
2EI & 4EI & -6EI \\
-6EI & -6EI & 12EI + 12EI/\ell^3
\end{bmatrix}
\]

\[
\mathbf{M} = \frac{1}{420} \begin{bmatrix}
4\mu l^3 + 4 \cdot 5\mu (2l)^3 & -3 \cdot 5\mu (2l)^3 & -22\mu l^2 \\
-3 \cdot 5\mu (2l)^3 & 4\mu l^3 + 4 \cdot 5\mu (2l)^3 & -22\mu l^2 \\
-22\mu l^2 & -22\mu l^2 & 156\mu l + 156\mu l + 420 \cdot 5\mu \cdot 2l
\end{bmatrix}
\]

The stiffness matrix is obtained by the usual procedure of the finite element method (or the deformation method), taking the geometric boundary conditions at the points A and D into consideration. The mass matrix is assembled in the same way. The components of \( \mathbf{K} \) and \( \mathbf{M} \) are pair-wise related as indicated by the element matrices (5-42) and (5-43) except from the element \( M_{33} \), where the mass \( 5\mu \cdot 2l \) of beam \( BC \) must be added to the contribution \( \frac{1}{420}(156\mu l + 156\mu l) \) from the columns \( AB \) og \( CD \). Only the latter contribution is the equivalence of the shear stiffness \( K_{33} = 12\frac{EI}{l^3} + 12\frac{EI}{l^3} \). The circular eigenfrequencies of the system (5-77) may be presented in the following way

\[
\omega_i = \lambda_i^2 \sqrt{\frac{EI}{\mu l^5}}, \quad i = 1, 2, 3
\]

Below the results for the frequency parameter as well as the associated mode shapes have been shown

\[
\begin{align*}
\lambda_1 &= 1.119583 \quad \text{(anti-symmetric)} \\
\lambda_2 &= 1.725920 \quad \text{(symmetric)} \\
\lambda_3 &= 3.125869 \quad \text{(anti-symmetric)}
\end{align*}
\]

Fig. 5-8: 3 element model of framed structure.
Table 5-4: Eigenvalue $\lambda_i$ for plane 1 storey frame. Analytical and numerical solutions for various levels of discretization.

The solutions (5-79) have been compared with exact analytical solution of the problem in table 5-4. As seen the eigenvalue $\lambda_1$ corresponding to the 1st anti-symmetric eigenmode is very accurately predicted, whereas higher order eigenvalues are badly estimated. In the table is also shown the convergence of the numerical solution as the beam elements are divided into smaller auxiliary artificial beams. With 3 elements the 1st symmetric eigenfrequency can be estimated with sufficiently accuracy, whereas the prediction of the 2nd anti-symmetric eigenfrequency requires 12 elements.
6. APPENDICES

6.1 Appendix A: Fourier Series and Fourier Transforms

$x(t) = x(t + T)$ signifies a periodic function with the period $T$ which is piecewise differentiable in the periodic interval $[0, T]$. Then the following Fourier series is valid

$$\frac{1}{2}a_0 + \sum_{m=1}^{\infty} a_m \cos(\omega_m t) + b_m \sin(\omega_m t) = \frac{x(t^+) + x(t^-)}{2}$$  \hspace{1cm} (A - 1)

$$\omega_m = m \frac{2\pi}{T}, \quad m = 1, 2, \ldots$$ \hspace{1cm} (A - 2)

$$a_m = \frac{2}{T} \int_{0}^{T} x(t) \cos(\omega_m t) dt, \quad m = 0, 1, 2, \ldots$$

$$b_m = \frac{2}{T} \int_{0}^{T} x(t) \sin(\omega_m t) dt, \quad m = 1, 2, \ldots$$ \hspace{1cm} (A - 3)

(A-1) indicates that the Fourier series converges to $x(t)$ at the continuity points of $x(t)$, whereas the series converges to the mean value of the limits $x(t^+)$ and $x(t^-)$ from the right- and left-hand sides at the discontinuity points.

By use of Euler's formulas $\cos(\omega_m t) = \frac{1}{2}(e^{i\omega_m t} + e^{-i\omega_m t})$ and $\sin(\omega_m t) = -\frac{i}{2}(e^{i\omega_m t} - e^{-i\omega_m t})$, (A-1) can be written in the following equivalent complex form

$$x(t) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} \frac{1}{2}(a_m - ib_m)e^{i\omega_m t} + \sum_{m=1}^{\infty} \frac{1}{2}(a_m + ib_m)e^{-i\omega_m t} = \sum_{m=-\infty}^{\infty} A_me^{i\omega_m t}$$ \hspace{1cm} (A - 4)

where

$$A_m = \begin{cases} \frac{1}{2}(a_m - ib_m) & , \ m > 0 \\ \frac{1}{2}a_0 & , \ m = 0 \\ \frac{1}{2}(a_m + ib_m) & , \ m < 0 \end{cases}$$ \hspace{1cm} (A - 5)

In (A-5) the definition has been used that $\omega_m = -\omega_m$, cf. (A-2). It follows from (A-5) that $A_{-m} = A_{-m}^*$. $A_m$ for $m > 0$ can be calculated directly from (A-3) and (A-5) as follows

$$A_m = \frac{1}{2} \left( \frac{2}{T} \int_{0}^{T} x(t) \cos(\omega_m t) dt - \frac{2}{T} \int_{0}^{T} x(t) \sin(\omega_m t) dt \right) = \frac{1}{T} \int_{0}^{T} x(t)e^{-i\omega_m t} dt$$ \hspace{1cm} (A - 6)

(A-1) may be written in the following form

$$x(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} c_m \cos(\omega_m t - \Psi_m)$$ \hspace{1cm} (A - 7)
(A-7) shows how an arbitrary periodic motion can be synthesized as a superposition of harmonic component motions. According to (A-3) $\frac{1}{2}a_0$ is equal to the time average

$$x = \frac{1}{T} \int_0^T x(t) \, dt$$

(A-10)

(A-1) is multiplied by $x(t)$, and integration is performed over the interval $[0, T]$. From (A-1) and (A-3) it then follows that

$$\int_0^T x^2(t) \, dt = \int_0^T \left( \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_m \cos(\omega_m t) + b_n \sin(\omega_m t) \right) x(t) \, dt \quad \Rightarrow$$

$$\frac{1}{T} \int_0^T x^2(t) \, dt = \frac{1}{4}a_0^2 + \frac{1}{2} \sum_{m=1}^{\infty} (a_m^2 + b_m^2)$$

(A-11)

(A-11) is denoted *Parceval’s theorem*. The right-hand side of (A-11) converges to the left-hand side under weaker conditions than required for (A-1). Actually, the convergence of (A-11) is insured, if $x(t)$ is piecewise continuous. The timeaverage on the left-hand side of (A-11) is denoted the *mean-square value*. From (A-8) and (A-11) it is seen that $\frac{1}{4}a_0^2$ and $\frac{1}{2}(a_n^2 + b_n^2)$ may be interpreted as the mean-square values of the time average $\bar{x} = \frac{a_0}{2}$ and of the harmonic component $c_m \cos(\omega_m t - \Psi_m)$. Parceval’s theorem then states that the mean-square values of a harmonic motion are equal to the mean-square value of the sum of the time average and of all involved harmonic components.

(A-4) and (A-6) may be written in the following form

$$x(t) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} X_m e^{i\omega_m t} \Delta\omega$$

$$X_m = \int_0^T x(t)e^{-i\omega_m t} \, dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-i\omega_m t} \, dt$$

(A-12)
where

\[ X_m = TA_m , \quad \Delta \omega = \frac{2\pi}{T} \]  \hspace{1cm} (A – 13)

Since \( \exp(i\omega_m T) = \exp(i\frac{2\pi}{T} mT) = \exp(i2\pi m) = 1 \) and \( x(t) = x(t - T) \), the last statement of (A-12) follows from the identity

\[
\int_{\frac{T}{2}}^{T} x(t)e^{-i\omega_m t}dt = \int_{\frac{T}{2}}^{T} x(t - T)e^{-i\omega_m (t - T)}dt = \int_{-\frac{T}{2}}^{0} x(u)e^{-i\omega_m u}du
\]  \hspace{1cm} (A – 14)

As \( T \to \infty \) equation (A-12) attains the form

\[
x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega
\]

\[
X(\omega) = \left\{ \int_{-\infty}^{\infty} x(t)e^{-i\omega_m t}dt \right\}
\]  \hspace{1cm} (A – 15)

At present (A-15) should be considered as merely formal identities, i.e. the convergence of \( \frac{1}{2\pi} \int_{-\omega_0}^{\omega_0} X(\omega)e^{i\omega t}d\omega \) for \( \omega_0 \to \infty \) and of \( \int_{T/2}^{T} x(t)e^{-i\omega_m t}dt \) for \( T \to \infty \) to the indicated limit values are not guaranteed. However, if \( x(t) \) is differentiable with piecewise continuous derivative in any finite subinterval of \( ]-\infty, \infty[ \), and if \( x(t) \) is absolutely integrable in \( ]-\infty, \infty[ \), such a convergence can be proved. Absolute convergence of \( x(t) \) means

\[
\lim_{T \to \infty} \int_{-T/2}^{T/2} |x(t)|dt < \infty
\]  \hspace{1cm} (A – 16)

The pair of identities (A-15) expresses Fourier's integral theorem, and \( X(\omega) \) is denoted the Fourier transform of \( x(t) \). Generally, \( X(\omega) \) is a complex function of \( \omega \) even if \( x(t) \) is real. The first of the identities (A-15) merely holds at continuity points of \( x(t) \). At discontinuity points the convergence will be to the mean value of the limits \( x(t^+) \) and \( x(t^-) \) from the right- and left-hand sides of the discontinuity point similar to (A-1), i.e.

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega = \frac{x(t^+) + x(t^-)}{2}
\]  \hspace{1cm} (A – 17)

It is assumed that

\[
\lim_{|t| \to \infty} \left| \frac{d^r x}{dt^r} \right| = 0, \quad r = 0, 1, \ldots, n - 1
\]  \hspace{1cm} (A – 18)

Use of multiple partial integrations provides the following Fourier transform \( X^{(n)}(\omega) \) of \( \frac{d^n x(t)}{dt^n} \)

\[
X^{(n)}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} \frac{d^n x(t)}{dt^n}dt = (i\omega)^n \int_{-\infty}^{\infty} e^{-i\omega t} x(t)dt = (i\omega)^n X(\omega), \quad n \geq 1
\]
Assume that $x(t)$ is defined by the integral
\[ x(t) = \int_{-\infty}^{\infty} h(t-\tau)f(\tau)d\tau = \int_{-\infty}^{\infty} f(t-\tau)h(\tau)d\tau \]  
(A - 20)

The Fourier transform of $x(t)$ then follows from (A-15) and (A-20)
\[ X(\omega) = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} h(t-\tau)f(\tau)d\tau \right) e^{-i\omega t} dt = \int_{-\infty}^{\infty} e^{-i\omega \tau} f(\tau) \left( \int_{-\infty}^{\infty} e^{-i\omega(t-\tau)} h(t-\tau)dt \right) d\tau = \int_{-\infty}^{\infty} e^{-i\omega \tau} f(\tau) H(\omega) d\tau = H(\omega) F(\omega) \]
(A - 21)

where $H(\omega)$ and $F(\omega)$ signify the Fourier transforms of the functions $h(t)$ and $f(t)$
\[ H(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} h(t) dt \]  
(A - 22)
\[ F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t} f(t) dt \]  
(A - 23)

In the 2nd statement of (A-21) the sequence of integration has been exchanged. The right-hand side of (A-20) is denoted a convolution integral. (A-21) states the so-called convolution theorem of Fourier transforms that the Fourier transform of a convolution integral is equal to the product of the Fourier transform of the functions $h(t)$ and $f(t)$ involved. In dynamics $h(t)$ signifies the impulse response function and $f(t)$ is the external dynamic load. In this case (A-21) states that the Fourier transform $X(\omega)$ of the response $x(t)$ is equal to the product of the frequency response function $H(\omega)$ and the Fourier transform $F(\omega)$ of the external loading. Under this interpretation the result (A-21) is merely a generalization of the result (2-96) from a discrete to a continuous spectrum of circular frequencies in the excitation.

6.2 Appendix B: Flexibility Coefficients for Statically Determinate Rectilinear Bernoulli-Euler Beams with Constant Cross-Section

Below 5 examples of flexibility coefficients for rectilinear Bernoulli-Euler beams are shown with constant cross-sections loaded in the symmetry plane of the beam. The bending stiffness is denoted $EI$.

\[ \text{Reproduced from C. Dyrbye: Bygningsdynamik I, Polyteknisk Forlag, Lyngby 1973.} \]
\[
\delta_{ij} = \frac{x(l - \xi)(2\xi l - x^2 - \xi^2)}{6lEI} , \quad x \leq \xi 
\]  
(B - 1)

\[
\delta_{ij} = -\frac{x\xi(l^2 - x^2)}{6lEI} 
\]  
(B - 2)

\[
\delta_{ij} = \frac{x(2l\xi + 3x\xi - x^2)}{6EI} , \quad x \leq \xi 
\]  
(B - 3)
\[ \delta_{ij} = \frac{x \xi l}{6EI} \]  \hspace{1cm} (B - 4)

\[ \delta_{ij} = \frac{x^2 (3 \xi - x)}{6EI}, \quad x \leq \xi \]  \hspace{1cm} (B - 5)
7. SUBJECT INDEX

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