Linear Matrix Inequalities for Analysis and Control of Linear Vector Second-Order Systems

Adegas, Fabiano Daher; Stoustrup, Jakob

Published in:
International Journal of Robust and Nonlinear Control

DOI (link to publication from Publisher):
10.1002/rnc.3242

Publication date:
2015

Document Version
Accepted author manuscript, peer reviewed version

INTERNATIONAL JOURNAL OF ROBUST AND NONLINEAR CONTROL
Int. J. Robust. Nonlinear Control 2010; **00**:1–32
Published online in Wiley InterScience (www.interscience.wiley.com). DOI: 10.1002/rnc

Linear Matrix Inequalities for Analysis and Control of Linear Vector Second-Order Systems †

F. D. Adegas*, J. Stoustrup

Aalborg University, Department of Electronic Systems, Fredrik Bajers Vej 7C, 9220 Aalborg Øst, Denmark

SUMMARY

Many dynamical systems are modeled as vector second order differential equations. This paper presents analysis and synthesis conditions in terms of Linear Matrix Inequalities (LMI) with explicit dependence in the coefficient matrices of vector second-order systems. These conditions benefit from the separation between the Lyapunov matrix and the system matrices by introducing matrix multipliers, which potentially reduce conservativeness in hard control problems. Multipliers facilitate the usage of parameter-dependent Lyapunov functions as certificates of stability of uncertain and time-varying vector second-order systems. The conditions introduced in this work have the potential to increase the practice of analyzing and controlling systems directly in vector second-order form. Copyright © 2010 John Wiley & Sons, Ltd.

Received ...

KEY WORDS: second-order systems, linear matrix inequalities, robust control

1. INTRODUCTION

Many physical systems have dynamics governed by linear time-invariant ordinary differential equations (ODEs) formulated in the vector second order form

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = Ff(t) \tag{1}$$

where $q(t) \in \mathbb{R}^n$, $M \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times n}$, $K \in \mathbb{R}^{n\times n}$, $F \in \mathbb{R}^{n\times n_f}$ and $f(t) \in \mathbb{R}^{n_f}$ is the force input vector. Depending on the type of loads (i.e., conservative, non-conservative), matrices M, C, K have a particular structure. Conservative systems (i.e., pure structural systems) possess symmetric system matrices. Non-conservative systems yielding from the fields of aeroelasticity, rotating machinery, and interdisciplinary system dynamics usually possess non symmetric system matrices. For control

*Correspondence to: Aalborg University, Department of Electronic Systems, Fredrik Bajers Vej 7C, 9220 Aalborg Øst, Denmark. E-mail: fda@es.aau.dk

Contract/grant sponsor: This work is supported by the CASED Project funded by grant DSF-09-063197 of the Danish Council for Strategic Research

purposes, system (1) is often re-written as first-order differential equations

$$\dot{x}(t) = Ax(t) + Bf(t) \tag{2a}$$

commonly referred to as state-space form. The relationship between the physical coordinate description (1) and the state-space description (2) is simply

$$x(t) := \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}, \quad A := \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \quad B := \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix} \tag{3}$$

where a nonsingular matrix M is assumed. Working with the model in physical coordinates has some advantages over the model in state-space form [1], [2], [3]:

- Physical interpretation of the coefficient matrices and insight of the original problem are preserved;
- Natural properties of the coefficient matrices like bandedness, definiteness, symmetry and sparsity are preserved;
- Unlike first-order systems in which the acceleration is composed as a linear combination of position and velocity states by an additional circuitry, the acceleration feedback can be utilized in its original form;
- Physical coordinates favour computational efficiency, because the dimension of the vector $x(t)$ is twice that of the vector $q(t)$;
- Complicating nonlinearities in the parameters introduced by inversion of a parameter-dependent mass matrix are avoided.

The stability of vector second-order systems received considerable interest during the last four decades. In [4] several sufficient conditions for stability and instability using Lyapunov theory are derived. Necessary and sufficient conditions of Lyapunov stability, semistability and asymptotic stability are proposed in [5]. This work also brings a substantial literature survey up to 1995. In [2] the necessary and sufficient conditions of stability are based on the Generalized Hurwitz criteria. A desirable property of these works is the explicit dependence of the conditions on the system coefficient matrices. An undesirable fact is that conditions are particular to systems under different dynamic loadings.

Most of the research on feedback-control design of vector-second order systems has focused on stabilization, pole assignment, eigenstructure assignment and observer design. Identification errors in mechanical systems might be quite large. Therefore, robust stability of the closed-loop system is of utmost importance. The fact that stability of some classes of vector-second order systems can be ensured by qualitative condition on the coefficient matrices has facilitated the design of robust stabilizing controllers. In [6] conditions for robust stabilization via static feedback of velocity and displacement were motivated by the stability condition $M^T = M \succ 0$, $C^T + C \succ 0$, $K^T = K \succ 0$ in the coefficient matrices An extension to dynamic displacement feedback control law is presented in [7]. Dissipative system theory is exploited in [8, 9] for the synthesis of stabilizing controllers. All these approaches result in closed-loop systems inherently insensitive to plant uncertainties. Based on the eigenvalue analysis of real symmetric interval matrices, in [10] the authors propose sufficient conditions for robust stabilizability considering structured uncertainty in the system matrices. A

DOI: 10.1002/rnc

transformation on the system matrices suitable for modal control is proposed in [3]. Partial pole assignment techniques via state feedback control are proposed in [11, 12]. Robustness in the partial pole assignment problem is considered in [13]. An effective method for partial eigenstructure assignment for systems with symmetric mass, damping and stiffness coefficients is presented in [14]. Robust eigenstructure assignment is treated in [15]. Vector second-order observers and their design are addressed in [16, 17, 18].

Despite these efforts, the first-order state-space remains the preferred representation due to the abundance of control techniques and numerical algorithms tailored for such. As far as modern, optimization-based control theories are concerned, the literature lacks on results to handle the systems directly in second-order form. An interesting contribution towards this goal is the stability results of [19] for systems in standard phase-variable canonical form, given in terms of linear matrix inequalities (LMI) extended with multipliers. The numerical tools of modern convex optimization can solve these problems efficiently [20]. The authors of [19] also mentioned the possibility of generalizing these results to systems described by higher order vector differential equations. Also interesting is the work of [21] which associates Lyapunov functions with higher-order derivatives of the state vectors of a state-space system, and proposes a redundant state-space system description to derive a generalization with reduced conservativeness of some of the robust stability results of [19].

The present manuscript extend the results in [19] by presenting conditions for analysis and synthesis of vector second-order systems given in terms of LMI. We believe that the conditions here introduced have the potential to increase the practice in analyzing and controlling mechanical systems explicitly in physical coordinates. Necessary and sufficient LMI criteria for checking stability of vector second-order systems is presented in Section 2. Some of these benefit from the separation between the Lyapunov matrix and the system matrices by introducing Lagrange multipliers, which potentially reduce conservativeness in robust and other hard control problems [22, 19, 23, 24]. The multipliers facilitate the usage of parameter-dependent Lyapunov functions as certificates of stability of uncertain and time-varying systems. They also allow structural constraints on the controller to be addressed less conservatively. Elimination of multipliers is investigated to determine in which circumstances multipliers can be removed without loss of generality. The stabilization problem by a full vector second-order feedback as well as the problem of clustering the closed-loop system poles in a convex region of the complex plane, namely $\mathcal{D}$-Stability, completes the results related to stability. A gradual extension for systems with inputs and outputs during Section 3 leads to the criteria of synthesis subject to Integral Quadratic Constrains (ICQ). Conditions for the design of static state and output feedback controllers in vector second-order form are addressed, with focus on the $\mathcal{L}_2$ to $\mathcal{L}_2$ gain performance measure due to its importance in robust control. Section 5 concludes the paper and suggest topics for future work.

The notation used in this paper is standard. $\mathbb{R}$ and $\mathbb{C}$ denotes the set of real numbers, whereas $\mathbb{S}^n$ indicates the set of symmetric $n \times n$ matrices. For a matrix X, X^T and X^H indicate its transpose and complex conjugate transpose, respectively. The $d \times d$ identity matrix is denoted by I_d, while 1_d represents a column vector composed of 1's with dimension d. For a matrix $X \in \mathbb{S}^n$, $X \prec (\preceq) 0$ indicates that X is negative (semi-)definite. The symbol $\otimes$ denotes the matrix Kronecker product. In long symmetric matrix expressions, the meaning of the symbol $\star$ will be inferred by symmetry. For instance, if X is symmetric, then

DOI: 10.1002/rnc

$$\begin{bmatrix} X+N+(\star) & Q^T \\ \star & Y \end{bmatrix}$$

will be read

$$\begin{bmatrix} X+N+N^T & Q^T \\ Q & Y \end{bmatrix}.$$

2. ASYMPTOTIC STABILITY

Let us recall some concepts of Lyapunov stability for first-order state-space systems before working with vector second-order representation. Consider the dynamics of a continuous-time linear time-invariant (LTI) system governed by the differential equation

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0, \tag{4}$$

where $x(t) : [0, \infty] \to \mathbb{R}^{2n}$ and $A \in \mathbb{R}^{2n \times 2n}$. Define the quadratic Lyapunov function $V : \mathbb{R}^{2n} \to \mathbb{R}$ as

$$V(x) := x(t)^T P x(t) \tag{5}$$

where $P \in \mathbb{S}^{2n}$. According to Lyapunov theory, system (4) is asymptotically stable if there exists $V(x(t)) > 0, \forall x(t) \neq 0$ such that

$$\dot{V}(x(t)) < 0, \quad \dot{x}(t) = Ax(t), \quad \forall x(t) \neq 0. \tag{6}$$

In words, if there exists $P \succ 0$ such that the time derivative of the quadratic Lyapunov function (5) is negative along all trajectories of system (4). Conversely, if the linear system (4) is asymptotically stable then there always exists $P \succ 0$ that satisfies (6). These two affirmatives imply the well known fact that Lyapunov theory with quadratic functions is necessary and sufficient to prove stability of LTI systems. The usual way to obtain a linear matrix inequalities (LMI) condition equivalent to (6) is to explicitly substitute (4) into (5) [20], that is

$$\dot{V}(x(t)) = x(t)^T \left(A^T P + PA\right) x(t) < 0, \quad \forall x(t) \neq 0. \tag{7}$$

The condition (7) is equivalent to the LMI feasibility problem

$$\exists P \in \mathbb{S}^{2n} \ : \ P \succ 0, \quad A^T P + PA \prec 0. \tag{8}$$

The fact that (6) is a set characterized by inequalities subject to dynamic equality constraints is explored in [19] to propose a constrained optimization solution to the stability problem. It is then possible to characterize the set defined by (6) without substituting (4) explicitly into $\dot{V}(x(t)) < 0$ of (6) [19]. The well know Finsler's Lemma [25, 26] is the main mathematical tool to transform the constrained optimization problem into a problem subject to LMI constraints.

Prepared using rncauth.cls
DOI: 10.1002/rnc

Lemma 1 (Finsler)
Let $x(t) \in \mathbb{R}^n$, $\mathcal{Q} \in \mathbb{S}^n$ and $\mathcal{B} \in \mathbb{R}^{m\times n}$ such that $\operatorname{rank}(\mathcal{B}) < n$. The following statements are equivalent.

i. $x(t)^T \mathcal{Q} x(t) < 0, \quad \forall\, \mathcal{B}x(t) = 0, \quad x(t) \neq 0.$
ii. $\mathcal{B}^{\perp T} \mathcal{Q} \mathcal{B}^{\perp} \prec 0.$
iii. $\exists \mu \in \mathbb{R} \ : \ \mathcal{Q} - \mu \mathcal{B}^T \mathcal{B} \prec 0.$
iv. $\exists \mathcal{X} \in \mathbb{R}^{n\times m} : \mathcal{Q} + \mathcal{X}\mathcal{B} + \mathcal{B}^T \mathcal{X}^T \prec 0.$

A similarity between statement i. of the above lemma and (6) can be noticed. In contrast to (7), the space of statement i. is composed of $x(t)$ and $\dot{x}(t)$ that can be seen as an *enlarged space* [19]. Statements iii. and iv. can be seen as an equivalent unconstrained quadratic forms of i. [19]. The equality constraint $\forall \dot{x}(t) = Ax(t)$ is included in the formulation weighted by the Lagrangian scalar multiplier μ or matrix multiplier $\mathcal{X}$.

In order to obtain a stability condition for an unforced system of the form (1) ($f(t) = 0$), define the quadratic Lyapunov function $V : \mathbb{R}^{2n} \to \mathbb{R}$ as

$$V(q(t), \dot{q}(t)) := \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}^T P \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} := \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} \tag{9}$$

where $P \in \mathbb{S}^{2n}$ is conveniently partitioned into $P_1,\ P_3 \in \mathbb{S}^n$, $P_2 \in \mathbb{R}^n$. Resorting to Lyapunov theory once again, system (1) is asymptotically stable if, and only if, there exists $V(q(t), \dot{q}(t)) > 0$, $\forall q(t), \dot{q}(t) \neq 0$ such that

$$\dot{V}(q(t), \dot{q}(t)) < 0, \quad \forall M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = 0, \quad \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} \neq 0 \tag{10}$$

with the time derivative of the quadratic function as

$$\dot{V}(q(t), \dot{q}(t)) = \begin{pmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix} + \begin{pmatrix} q(t) \\ \dot{q}(t) \end{pmatrix}^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \begin{pmatrix} \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix} < 0. \tag{11}$$

Let an enlarged state space vector be defined as $x(t) := \begin{pmatrix} q(t)^T, & \dot{q}(t)^T, & \ddot{q}(t)^T \end{pmatrix}^T$. For this enlarged space, the constrained Lyapunov stability problem becomes

$$\begin{gathered} \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix}^T \begin{bmatrix} 0 & P_1 & P_2 \\ P_1 & P_2 + P_2^T & P_3 \\ P_2^T & P_3 & 0 \end{bmatrix} \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix} < 0, \\ \forall M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = 0, \quad \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix} \neq 0. \end{gathered} \tag{12}$$

An LMI stability condition for vector second-order systems results from the direct application of Finsler's Lemma 1 to the problem above.

DOI: 10.1002/rnc

Theorem 1
System (1) is asymptotically stable if, and only if, one of the following equivalent conditions holds:

i. $\exists P_2 \in \mathbb{R}^{n\times n},\ P_1,\ P_3 \in \mathbb{S}^n\ :$

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \\ E_{31} & E_{32} \end{bmatrix}^T \begin{bmatrix} 0 & P_1 & P_2 \\ P_1 & P_2 + P_2^T & P_3 \\ P_2^T & P_3 & 0 \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \\ E_{31} & E_{32} \end{bmatrix} \prec 0, \tag{13a}$$

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \\ E_{31} & E_{32} \end{bmatrix} := \begin{bmatrix} K & C & M \end{bmatrix}^{\perp},$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0. \tag{13b}$$

ii. $\exists P_2 \in \mathbb{R}^{n\times n},\ P_1,\ P_3 \in \mathbb{S}^n,\ \lambda \in \mathbb{R}\ :$

$$\begin{bmatrix} -\lambda K^T K & P_1 - \lambda K^T C & P_2 - \lambda K^T M \\ \star & P_2 + P_2^T - \lambda C^T C & P_3 - \lambda C^T M \\ \star & \star & -\lambda M^T M \end{bmatrix} \prec 0, \tag{14a}$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0. \tag{14b}$$

iii. $\exists \Phi,\ \Gamma,\ \Lambda,\ P_2 \in \mathbb{R}^{n\times n}$ and $P_1,\ P_3 \in \mathbb{S}^n\ :$

$$\begin{bmatrix} K^T\Phi^T + \Phi K & P_1 + K^T\Gamma^T + \Phi C & P_2 + K^T\Lambda^T + \Phi M \\ \star & P_2 + P_2^T + C^T\Gamma^T + \Gamma C & P_3 + C^T\Lambda^T + \Gamma M \\ \star & \star & M^T\Lambda^T + \Lambda M \end{bmatrix} \prec 0, \tag{15a}$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0. \tag{15b}$$

Proof
Assign

$$x(t) \leftarrow \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \end{pmatrix},\ \mathcal{Q} \leftarrow \begin{bmatrix} 0 & P_1 & P_2 \\ P_1 & P_2 + P_2^T & P_3 \\ P_2^T & P_3 & 0 \end{bmatrix},\ \mathcal{B}^T \leftarrow \begin{bmatrix} K^T \\ C^T \\ M^T \end{bmatrix},\ \mathcal{X} \leftarrow \begin{bmatrix} \Phi \\ \Gamma \\ \Lambda \end{bmatrix}$$

and apply Lemma 1 to the constrained Lyapunov problem (12) with $P \succ 0$. □

Notice the diagonal entries of the first inequality of statement ii., i.e., $\lambda K^T K \succ 0$ and $\lambda M^T M \succ 0$, which implies that $\lambda > 0$ and $K,\ M$ nonsingular. The condition reflects that *asymptotic* stability of mechanical systems requires that no eigenvalues of matrix K should lie at the origin, i.e., no rigid body modes. At last, notice that the condition does not enforce any specific requirement on the structure of the damping matrix C (except $C \neq 0$). Thus, applicable to system under different loadings.

DOI: 10.1002/rnc

2.1. *Elimination of Multipliers*

The matrix inequality (15) is a function of the multipliers $\Phi,\ \Gamma,\ \Lambda$. It is worth questioning if all degrees of freedom introduced by the multipliers are really necessary. Would it be possible to constrain or eliminate multipliers without loss of generality? The Elimination Lemma [20, 26] will serve for the purpose of removing multipliers without adding conservatism to the solution.

Lemma 2 (Elimination Lemma)
Let $\mathcal{Q} \in \mathbb{S}^n$, $\mathcal{B} \in \mathbb{R}^{m\times n}$, $\mathcal{C} \in \mathbb{R}^{n\times k}$. The following statements are equivalent.

i. $\exists \mathcal{X} \in \mathbb{R}^{n\times m} : \mathcal{Q} + \mathcal{C}^T\mathcal{X}\mathcal{B} + \mathcal{B}^T\mathcal{X}^T\mathcal{C} \prec 0$

ii. $$\mathcal{B}^{\perp T}\mathcal{Q}\mathcal{B}^{\perp} \prec 0 \quad \text{(16a)} \qquad \mathcal{C}^{\perp T}\mathcal{Q}\mathcal{C}^{\perp} \prec 0 \quad \text{(16b)}$$

iii. $\exists \mu \in \mathbb{R}\ :\ \mathcal{Q} - \mu\mathcal{B}^T\mathcal{B} \prec 0, \quad \mathcal{Q} - \mu\mathcal{C}^T\mathcal{C} \prec 0.$

Notice that Elimination Lemma reduces to the Finsler's Lemma when particularized with $\mathcal{C} = I$. In such a case $\mathcal{C}^{\perp} = \{0\}$ and (16b) is removed from the statement. A discussion on the relation between these two lemmas can be found in [20, 26]. The elimination of multipliers on LMI conditions for systems in first-order form was studied in [24]. In general terms, the idea is to select a suitable $\mathcal{C}$ such that (16b) does not introduce conservatism to the original problem while reducing the size of the multiplier $\mathcal{X}$. The next theorems result from a similar rationale.

Theorem 2
System (1) is asymptotically stable if, and only if, $\exists \Phi,\ \Lambda,\ P_2 \in \mathbb{R}^{n\times n}$ and $P_1,\ P_3 \in \mathbb{S}^n\ :$

$$\begin{bmatrix} \Phi K + K^T\Phi^T & P_1 + K^T(\alpha\Phi+\Lambda)^T + \Phi C & P_2 + \alpha K^T\Lambda^T + \Phi M \\ \star & P_2 + (\alpha\Phi+\Lambda)C + (\star) & P_3 + \alpha C^T\Lambda^T + (\alpha\Phi+\Lambda)M \\ \star & \star & \alpha\left(\Lambda M + M^T\Lambda^T\right) \end{bmatrix} \prec 0, \tag{17a}$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0. \tag{17b}$$

for an arbitrary scalar $\alpha > 0$.

Proof
Assign

$$\mathcal{Q} \leftarrow \begin{bmatrix} 0 & P_1 & P_2 \\ P_1 & P_2 + P_2^T & P_3 \\ P_2^T & P_3 & 0 \end{bmatrix}, \quad \mathcal{B}^T \leftarrow \begin{bmatrix} K^T \\ C^T \\ K^T \end{bmatrix}, \quad \mathcal{C}^{\perp} \leftarrow \begin{bmatrix} \alpha^2 I \\ -\alpha I \\ I \end{bmatrix},$$

$$\mathcal{C}^T \leftarrow \begin{bmatrix} I & 0 \\ \alpha I & I \\ 0 & \alpha I \end{bmatrix}^T, \quad \mathcal{X} \leftarrow \begin{bmatrix} \Phi \\ \Lambda \end{bmatrix}.$$

DOI: 10.1002/rnc

and apply the Elimination Lemma with $P \succ 0$. The chosen $\mathcal{C}^{\perp}$ does not introduce conservativeness to the condition. To see this expand (16b)

$$\mathcal{C}^{\perp T}\mathcal{Q}\mathcal{C}^{\perp} = -\alpha^3 P_1 - \alpha P_3 + \alpha^2 P_2 + \alpha^2 P_2^T \prec 0 \tag{18a}$$

$$\Updownarrow$$

$$\alpha^3 P_1 + \alpha P_3 - \alpha^2 P_2 - \alpha^2 P_2^T \succ 0. \tag{18b}$$

Notice that the following support inequality

$$-NW^{-1}N^T \preceq W - N - N^T \tag{19}$$

with $W \in \mathbb{S}^n$, $N \in \mathbb{R}^{n\times n}$ holds whenever $W \succ 0$. Resorting to the support inequality with $N := \alpha^2 P_2$, $W := \alpha P_3 \succ 0$, (18b) is satisfied whenever

$$\alpha^3(P_1 - P_2 P_3^{-1} P_2^T) \succ 0. \tag{20}$$

$P_1 - P_2 P_3^{-1} P_2^T \succ 0$ is equivalent to (17b) by a Schur complement argument and thus positive definite. Therefore (20) and consequently (18b) holds for an arbitrary real scalar $\alpha > 0$. □

A similar, equivalent characterization of the Theorem above can be derived by assigning

$$\mathcal{C}^{\perp} \leftarrow \begin{bmatrix} I \\ -\alpha I \\ \alpha^2 I \end{bmatrix}, \quad \mathcal{C}^T \leftarrow \begin{bmatrix} \alpha I & 0 \\ I & \alpha I \\ 0 & I \end{bmatrix}^T$$

and following the same steps presented on the proof.

The number of multipliers can be further reduced by constraining $\Phi := \mu\Lambda$ in (17a) where $\mu > 0$ is a real scalar. The idea to introduce line search parameters is exploited in the LMI literature [27, 28, 24]. Unfortunately, this constraint introduce conservativeness leading to a sufficient condition.

Theorem 3

System (1) is asymptotically stable if $\exists P_2,\ \Lambda \in \mathbb{R}^{n\times n},\ P_1,\ P_3 \in \mathbb{S}^n,\ \mu, \in \mathbb{R}\ :$

$$\begin{bmatrix} \mu(\Lambda K + K^T\Lambda^T) & P_1 + (1+\alpha\mu)K^T\Lambda^T + \mu\Lambda C & P_2 + \alpha K^T\Lambda^T + \mu\Lambda M \\ \star & P_2 + (1+\alpha\mu)\Lambda C + (\star) & P_3 + \alpha C^T\Lambda^T + (1+\alpha\mu)\Lambda M \\ \star & \star & \alpha(\Lambda M + M^T\Lambda^T) \end{bmatrix} \prec 0 \tag{21a}$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0, \quad \alpha > 0, \quad \mu > 0. \tag{21b}$$

for an arbitrary scalar $\alpha > 0$.

A source of conservatism is the appearance of a single multiplier on the block diagonal entries of (21a). For $M \succ 0$, usual property of the mass matrix, the (3,3) block $\alpha(M^T\Lambda^T + \Lambda M) \prec 0$ with $\alpha > 0$ is satisfied only if $\Lambda \prec 0$. As a consequence, stability cannot be certified when $M \succ 0$ and K is indefinite because $\mu(K^T\Lambda^T + \Lambda K) \prec 0$ never holds when $\mu > 0$ and $\Lambda \prec 0$. Numerical

DOI: 10.1002/rnc

experiments suggest that a similar situation is encountered when the matrix C is indefinite and M or K are positive definite. The condition was unable to attest stability of randomly generated stable systems (M, C, K) in which C had at least one negative eigenvalue and $M,\ K \succ 0$. $P_2 + P_2^T \succ 0$ holds whenever the condition was able to find a certificate of stability, another contributing fact to why the (2-2) block cannot be verified as negative definite when C is indefinite.

2.2. *Stabilization by Static State Feedback*

The dependence of the stability condition to a single multiplier Λ is particularly interesting in the context of feedback stabilization. The vector second order system is augmented with a controllable input $u(t) \in \mathbb{R}^{n_u}$

$$M\ddot{q}(t) + C\dot{q}(t) + K\dot{q}(t) = F_u u(t), \qquad q(0),\ \dot{q}(0) = 0 \tag{22}$$

where $F_u \in \mathbb{R}^{n \times n_u}$. Consider a static state feedback controller of the form

$$u(t) = -G_a\ddot{q}(t) - G_v\dot{q}(t) - G_p q(t) \tag{23}$$

where $G_a\ ,G_v,\ G_p \in \mathbb{R}^{n \times n}$ are static feedback gains from acceleration, velocity and position, respectively. The plant (22) in closed-loop with the controller (23) yields the equations of motion

$$\mathbf{M}\ddot{q}(t) + \mathbf{C}\dot{q}(t) + \mathbf{K}q(t) = 0 \tag{24a}$$

$$\mathbf{M} := (M + F_u G_a)\,, \quad \mathbf{C} := (C + F_u G_v)\,, \quad \mathbf{K} := (K + F_u G_p) \tag{24b}$$

Conditions for controller synthesis often involve products between controller gains and Lyapunov matrices or multipliers, resulting in nonlinear matrix inequalities. The nonlinear terms can be linearized by resorting to the change-of-variables, firstly introduced in [29] in which only the Lyapunov variable is involved, and later in the context of conditions extended with multipliers [22]. Define the following nonlinear change-of-variables

$$\hat{G}_a := G_a\Lambda, \quad \hat{G}_v := G_v\Lambda, \quad \hat{G}_p := G_p\Lambda. \tag{25}$$

Notice from (21) that the matrix Λ multiplies the system matrices in a position not suitable for linearization of the nonlinear terms, i.e., $\Lambda\,(K + F_u G_p)$. A dual transformation of the closed-loop system

$$\mathbf{M} \leftarrow \mathbf{M}^T, \quad \mathbf{C} \leftarrow \mathbf{C}^T, \quad \mathbf{K} \leftarrow \mathbf{K}^T \tag{26}$$

makes the linearizing change of variables possible. A discussion on algebraic duality of vector second-order system with inputs and outputs is given in the appendix. It is worth mentioning that the above dual transformation preserves the eigenvalues of the system cast in first-order form, that is

Int. J. Robust. Nonlinear Control (2010)
DOI: 10.1002/rnc

$$\lambda\left(\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}\right) = \lambda\left(\begin{bmatrix} 0 & I \\ -M^{-T}K^{T} & -M^{-T}C^{T} \end{bmatrix}\right).$$

With these definitions at hand, the stabilizability conditions by static feedback can now be stated.

Theorem 4

System (22) is stabilizable by a static feedback law of the form (23) if $\exists\, \Lambda,\ P_2 \in \mathbb{R}^{n\times n},\ P_1,\ P_3 \in \mathbb{S}^n,\ \hat{G}_a,\ \hat{G}_v,\ \hat{G}_p \in \mathbb{R}^{n_u\times n},\ \alpha,\ \mu \in \mathbb{R}\ :$

$$\begin{bmatrix} \mu(K\Lambda + F_u\hat{G}_p + \Lambda^T K^T + \hat{G}_p^T F_u^T) & P_1 + (1+\alpha\mu)(K\Lambda + F_u\hat{G}_p) + \mu(C\Lambda + F_u\hat{G}_v)^T & P_2 + \alpha(K\Lambda + F_u\hat{G}_p) + \mu(M\Lambda + F_u\hat{G}_a)^T \\ \star & P_2 + (1+\alpha\mu)(C\Lambda + F_u\hat{G}_v) + (\star) & P_3 + \alpha(C\Lambda + F_u\hat{G}_v) + (1+\alpha\mu)(M\Lambda + F_u\hat{G}_a)^T \\ \star & \star & \alpha(M\Lambda + F_u\hat{G}_a + \Lambda^T M^T + \hat{G}_a^T F_u^T) \end{bmatrix} \prec 0, \tag{27a}$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0, \quad \mu > 0, \tag{27b}$$

and Λ is nonsingular.

Proof

The LMI (27) results from a direct application of Proposition 2 to the dual of closed-loop system (24), together with a dual transformation $\Lambda \leftarrow \Lambda^T$ of the multiplier and the change-of-variables (25). The change-of-variables are without loss of generality when Λ is nonsingular thus invertible. The original controller gains can then be recovered by the inverse map

$$G_a = \hat{G}_a\Lambda^{-1}, \quad G_v = \hat{G}_v\Lambda^{-1}, \quad G_p = \hat{G}_p\Lambda^{-1}. \tag{28}$$

which characterizes the stabilizable control law. □

A nonsingular multiplier Λ is not implied by inequality (27). This fact contrasts with stability criteria for systems in first-order form where nonsingularity of multipliers is a direct consequence of the structure of the LMI [23, 30]. With some restrictions imposed on the problem formulation, it is possible to ensure a nonsingular Λ. For instance, the multiplier can be confined to the positive cone of symmetric matrices, i.e., $\Lambda \in \mathbb{S}^n,\ \Lambda \succ 0$, or to the negative cone of symmetric matrices $\Lambda \in \mathbb{S}^n,\ \Lambda \prec 0$. In these cases, extra conservativeness is brought into the condition.

An assumption that facilitates a nonsingular Λ without adding conservativeness is to exclude the acceleration feedback, i.e., $G_a = 0$. In this case, the lower right block $M\Lambda + \Lambda^T M^T \prec 0$ of the LMI (27) with M nonsingular implies a nonsingular multiplier Λ. Therefore, a stabilizing controller can be computed according to (28) whenever (27) is feasible. Note that the control law (23) with $G_a = 0$ is a full state feedback in the first-order state-space sense.

As mentioned in the introduction section, acceleration feedback is often desirable due to practical reasons. When position feedback is excluded from the control law ($G_p = 0$) the multiplier is assured to be nonsingular. The entry $\mu(K\Lambda + \Lambda^T K^T) \prec 0$ located in the upper left of (27) with

DOI: 10.1002/rnc

K nonsingular and $\mu > 0$ implies Λ nonsingular. Therefore, once a solution for the LMI problem above is found, the controller gains can be reconstructed according to (28).

2.3. $\mathcal{D}$-Stability

Performance specifications like time response and damping in closed-loop can often be achieved by clustering the closed-loop poles into a suitable subregion of the complex plane. The subclass of *convex regions* of the complex plane can be characterized in terms of LMI constraints [31]. A class of convex subregions representable as LMI conditions extended with multipliers was proposed in [32]. Let $R_{11},\ R_{22} \in \mathbb{S}^d$, $R_{12} \in \mathbb{R}^d$, $R_{22} \succeq 0$. The $\mathcal{D}_R$ region of the complex plane is defined as the set [32]

$$\mathcal{D}_R(s) := \left\{ s \in \mathbb{C} \ :\ R_{11} + R_{12}s + R_{12}^T s^H + R_{22}s^H s \prec 0 \right\} \tag{29}$$

where s is the Laplace operator. An LMI characterization for $\mathcal{D}_R$-stability of vector second-order systems can be derived from $\mathcal{D}_R$-stability condition of a system in first-order form. The autonomous system (4) is $\mathcal{D}_R$-stable if and only if $\exists P \in \mathbb{S}^{2n}$: [32]

$$R_{11} \otimes P + R_{12} \otimes (PA) + R_{12}^T \otimes \left(A^T P\right) + R_{22} \otimes \left(A^T PA\right) \prec 0 \tag{30}$$

A relation between regions of the complex plane and a particular Lyapunov constrained problem can be deduced from the above LMI. First define the d-stacked system as

$$x_d(t) := 1_d \otimes x(t), \quad A_d := I_d \otimes A \quad \Rightarrow \quad \dot{x}_d(t) = I_d \otimes A x_d(t) \tag{31}$$

where 1_d represents a column vector composed of 1's and I_d is the identity matrix both with dimension d. For example, the d-stacked system for $d = 2$ yields

$$\begin{pmatrix} \dot{x}(t) \\ \dot{x}(t) \end{pmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} \begin{pmatrix} x(t) \\ x(t) \end{pmatrix}.$$

The time derivative of the Lyapunov function tailored for $\mathcal{D}_R$-stability analysis is defined as

$$\begin{aligned} \dot{V}(x_d(t), \dot{x}_d(t)) :&= x_d(t)^T R_{11} \otimes P x_d(t) + \dot{x}_d(t)^T R_{12} \otimes P x_d(t) \\ &+ x_d(t)^T R_{12}^T \otimes P \dot{x}_d(t) + \dot{x}_d(t)^T R_{22} \otimes P \dot{x}_d(t) < 0 \end{aligned} \tag{32}$$

Substitute (31) into (32) and expand to arrive at (30). The usual time derivative of a quadratic Lyapunov function, i.e., $\dot{V}(x(t)) = \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t)$ is recovered from the above by choosing $R_{11} = R_{22} = 0,\ R_{12} = 1$. The set of solutions of the $\mathcal{D}$-stability problem in time-domain is defined as

$$\mathcal{D}_R(x(t)) := \left\{ x(t) \in \mathbb{R}^n \ :\ \dot{V}(x(t), \dot{x}(t)) < 0,\ \dot{V}(x(t), \dot{x}(t)) \text{ as (32)},\ P \succ 0 \right\}. \tag{33}$$

DOI: 10.1002/rnc

For the sake of $\mathcal{D}_R$-stability of vector second-order systems the d-stacked system is defined

$$q_d(t) := 1_d \otimes q(t), \quad \dot{q}_d(t) := 1_d \otimes \dot{q}(t), \quad \ddot{q}_d(t) := 1_d \otimes \ddot{q}(t), \tag{34a}$$

$$M_d := I_d \otimes M, \quad C_d := I_d \otimes C, \quad K_d := I_d \otimes K \tag{34b}$$

$$M_d\ddot{q}_d(t) + C_d\dot{q}_d(t) + K_d q_d(t) = 0. \tag{34c}$$

Let the constrained Lyapunov problem in the enlarged space be formalized

$$\begin{pmatrix} q_d(t) \\ \dot{q}_d(t) \\ \ddot{q}_d(t) \end{pmatrix}^T \left[\begin{matrix} R_{11}\otimes P_1 & R_{11}\otimes P_2 + R_{12}\otimes P_1 \\ \star & R_{11}\otimes P_3 + R_{12}^T\otimes P_2 + R_{12}\otimes P_2^T + R_{22}\otimes P_1 \\ \star & \star \end{matrix}\right.$$
$$\left.\begin{matrix} R_{12}\otimes P_2 \\ R_{12}\otimes P_3 + R_{22}\otimes P_2 \\ R_{22}\otimes P_3 \end{matrix}\right] \begin{pmatrix} q_d(t) \\ \dot{q}_d(t) \\ \ddot{q}_d(t) \end{pmatrix} < 0, \tag{35a}$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0, \quad \forall M_d\ddot{q}_d(t) + C_d\dot{q}_d(t) + K_d q_d(t) = 0, \quad \begin{pmatrix} q_d(t) \\ \dot{q}_d(t) \\ \ddot{q}_d(t) \end{pmatrix} \neq 0. \tag{35b}$$

The $\mathcal{D}$-stability condition for vector second-order systems is stated in the next theorem.

Theorem 5

System (1) is $\mathcal{D}_R$-stable if, and only if, $\exists\, P_1,\ P_3 \in \mathbb{S}^n,\ P_2 \in \mathbb{R}^n,\ \Phi,\ \Gamma,\ \Lambda \in \mathbb{R}^{dn\times dn}$:

$$\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0, \quad \mathcal{H} := \begin{bmatrix} \Phi(I_d\otimes K) & \Phi(I_d\otimes C) & \Phi(I_d\otimes M) \\ \Gamma(I_d\otimes K) & \Gamma(I_d\otimes C) & \Gamma(I_d\otimes M) \\ \Lambda(I_d\otimes K) & \Lambda(I_d\otimes C) & \Lambda(I_d\otimes M) \end{bmatrix}, \tag{36a}$$

$$\mathcal{J} := \left[\begin{matrix} R_{11}\otimes P_1 & R_{11}\otimes P_2 + R_{12}\otimes P_1 \\ \star & R_{11}\otimes P_3 + R_{12}^T\otimes P_2 + R_{12}\otimes P_2^T + R_{22}\otimes P_1 \\ \star & \star \end{matrix}\right.$$
$$\left.\begin{matrix} R_{12}\otimes P_2 \\ R_{12}\otimes P_3 + R_{22}\otimes P_2 \\ R_{22}\otimes P_3 \end{matrix}\right], \tag{36b}$$

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0. \tag{36c}$$

The proof follows similarly to Theorem 1 and is omitted for brevity. Multipliers need to be eliminated to make the above condition suitable for computing stabilizing controllers. The same choice of $\mathcal{C}^{\perp}$ and $\mathcal{C}$ of Theorems 2 and 3 serve this purpose. However, conservativeness when eliminating multipliers depend on the particular $\mathcal{D}_R$ region. Taking $(\mathcal{C}^{\perp}, \mathcal{C})$ similarly to Theorem 2, $\mathcal{C}^{\perp T}\mathcal{Q}\mathcal{C}^{\perp} \prec 0$ after expansion and some algebraic manipulations yields

DOI: 10.1002/rnc

$$
\begin{aligned}
&R_{11} \otimes \left(\alpha^4 P_1 - \alpha^3 P_2 - \alpha^3 P_2^T + \alpha^2 P_3\right) + R_{12} \otimes \left(-\alpha^3 P_1 + \alpha^2 P_2 + \alpha^2 P_2^T - \alpha P_3\right) \\
&+ R_{12}^T \otimes \left(-\alpha^3 P_1 + \alpha^2 P_2 + \alpha^2 P_2^T - \alpha P_3\right) + R_{22} \otimes \left(\alpha^2 P_1 - \alpha P_2 - \alpha P_2^T + P_3\right) \prec 0
\end{aligned} \tag{37}
$$

This inequality depends on the matrices R_{11}, R_{12}, R_{22} that defines the stability region. Although it is not trivial to state non-conservativeness independently of the chosen region, one can attest if the elimination of a multiplier brings any conservativeness for a particular $\mathcal{D}_R$-region. To do so, first note that all of the addends of the above inequality are similar in structure. A correspondence with the Lyapunov matrix P can be established via a congruence transformation involving α, and multiplications with a matrix and its transpose, e.g.

$$
Y^T H^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} HY \succ 0, \quad H := \text{diag}(\alpha I, I), \quad Y := \begin{bmatrix} I & -I \end{bmatrix}^T.
$$

Whenever $H^T P H \succ 0$ holds, which is always the case because of (36c), $\alpha^2 P_1 - \alpha P_2 - \alpha P_2^T + P_3 \succ 0$ also holds. Let us take some typical regions as examples. The continuous-time stability region is determined by $R_{11} = R_{22} = 0,\ R_{12} = 1$, rendering non-conservativeness as shown in Theorem 2. For a region with minimum decay rate $\beta > 0$ set with $R_{11} = 2\beta,\ R_{22} = 0,\ R_{12} = 1$, if

$$
2\alpha H^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} H \succ 2\alpha^2 \beta H^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} H \succ 0 \tag{38}
$$

holds, then (37) also holds. Indeed, multiply the inequality above with $Y := \begin{bmatrix} I & -I \end{bmatrix}^T$ from the right and Y^T from the left to obtain (37). A set of values of α which does not introduce conservativeness to the condition can be inferred from (38), that is, $\{\alpha : \alpha - \alpha^2\beta > 0,\ \alpha > 0,\ \beta > 0\}$. For the discrete-time stability region, represented as a circle centred at the origin of the complex plane with $R_{11} = -1,\ R_{22} = 1,\ R_{12} = 0$, if

$$
\alpha^2 H^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} H \succ H^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} H \succ 0
$$

is satisfied than inequality (37) is satisfied. Therefore, any $\alpha > 1$ does not bring conservativeness.

3. QUADRATIC PERFORMANCE

The following linear time-invariant vector second-order system with inputs and outputs

$$
M\ddot{q}(t) + D\dot{q}(t) + K\dot{q}(t) = F_w w(t), \qquad q(0),\ \dot{q}(0) = 0 \tag{39a}
$$

$$
z(t) = U\ddot{q}(t) + V\dot{q}(t) + Xq(t) + D_{zw} w(t) \tag{39b}
$$

DOI: 10.1002/rnc

is considered in this section, where $w(t) \in \mathbb{R}^{n_w}$ and $z(t) \in \mathbb{R}^{n_z}$ are the disturbance input and performance output vectors, respectively, U, V, $X \in \mathbb{R}^{n_z \times n}$. The presence of input signals $w(t)$ requires a definition of stability.

3.1. $\mathcal{L}_2$ to $\mathcal{L}_2$ Stability

The notion of stability of a system with inputs it related to the characteristics of the input signal $w(t)$. Assume $w(t) : [0, \infty) \to \mathbb{R}^{n_w}$ a piecewise continuous function in the Lebesgue function space $\mathcal{L}_2$

$$\|w(t)\|_{\mathcal{L}_2} := \left(\int_0^\infty w(\tau)^T w(\tau) d\tau \right)^{1/2} < \infty.$$

In the control literature, the quantity $\|w(t)\|_{\mathcal{L}_2}$ is often referred to as the energy of signal $w(t)$. The system (39) is said to be $\mathcal{L}_2$ stable if the output signal $z(t) \in \mathcal{L}_2$ for all $w(t) \in \mathcal{L}_2$. Define the $\mathcal{L}_2$ to $\mathcal{L}_2$ gain as the quantity

$$\gamma_\infty := \sup_{w(t) \in \mathcal{L}_2} \frac{\|z\|_2}{\|w\|_2}. \tag{40}$$

This quantity can serve as a certificate of $\mathcal{L}_2$ stability. If the $\mathcal{L}_2$ to $\mathcal{L}_2$ gain of a system is finite, i.e., $0 < \gamma_\infty < \infty$, then one can conclude that the system is $\mathcal{L}_2$ stable. Because γ_∞ is bounded by below, it suffices to find an upper bound γ such that $0 < \gamma_\infty < \gamma < \infty$. Consider the modified Lyapunov stability condition

$$\dot{V}\left(q(t), \dot{q}(t), \ddot{q}(t)\right) < 0, \quad z(t)^T z(t) \geq \gamma^2 w(t)^T w(t) \tag{41a}$$
$$\forall (q(t), \dot{q}(t), \ddot{q}(t), w(t), z(t)) \text{ satisfying (39)}, \tag{41b}$$
$$(q(t), \dot{q}(t), \ddot{q}(t), w(t), z(t)) \neq 0. \tag{41c}$$

where $\gamma > 0$ is a given scalar. Invoking the S-procedure [20] produces a necessary and sufficient equivalent condition [19]

$$\dot{V}\left(q(t), \dot{q}(t), \ddot{q}(t)\right) < \gamma^2 w(t)^T w(t) - z(t)^T z(t), \tag{42a}$$
$$\forall (q(t), \dot{q}(t), \ddot{q}(t), w(t), z(t)) \text{ satisfying (39)}, \tag{42b}$$
$$(q(t), \dot{q}(t), \ddot{q}(t), w(t), z(t)) \neq 0. \tag{42c}$$

To realize that (42) implies an $\mathcal{L}_2$ to $\mathcal{L}_2$ gain less than γ, integrate both sides of (42a) over time $t > 0$ to get

$$\int_0^t \dot{V}(q(\tau), \dot{q}(\tau), \ddot{q}(\tau))\, d\tau < \int_0^t \gamma^2 w(\tau)^T w(\tau) - z(\tau)^T z(\tau)\, d\tau \tag{43}$$

For $t \to \infty$, the resulting Lyapunov function

$$\int_0^t \dot{V}(q(\tau), \dot{q}(\tau), \ddot{q}(\tau))\, d\tau = V(q(\tau), \dot{q}(\tau), \ddot{q}(\tau)) > 0 \tag{44}$$

DOI: 10.1002/rnc

is positive by definition. From the above and (43) it can be inferred that

$$\|z(t)\|_{\mathcal{L}_2}^2 < \gamma^2 \|w(t)\|_{\mathcal{L}_2}^2. \tag{45}$$

which compared to (40) implies $\gamma > \gamma_\infty$.

Synthesis of controllers are usually attached to some performance indicator or measure of a system. The $\mathcal{L}_2$ gain also serve as a system performance measure.

3.2. *Integral Quadratic Constraints*

The notion of system performance can be further generalized by enforcing an integral quadratic constraint on the input and output signals [33, 34]

$$\int_0^t \begin{pmatrix} z(\tau) \\ w(\tau) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(\tau) \\ w(\tau) \end{pmatrix} d\tau \geq 0 \tag{46}$$

where $Q \in \mathbb{S}^{n_z}$, $R \in \mathbb{S}^{n_w}$, $S \in \mathbb{R}^{n_z \times n_w}$. Similarly to (43), pose the inequality

$$\int_0^t \dot{V}(q(\tau), \dot{q}(\tau), \ddot{q}(\tau))\, d\tau < -\int_0^t \begin{pmatrix} z(\tau) \\ w(\tau) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(\tau) \\ w(\tau) \end{pmatrix} d\tau \tag{47}$$

The right hand side of the above inequality can be seen as a quadratic constraint on the Lyapunov quadratic function $V(q(t), \dot{q}(t), \ddot{q}(t))$. The modified Lyapunov problem then becomes

$$\dot{V}\,(q(t), \dot{q}(t), \ddot{q}(t)) < -\begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}, \tag{48a}$$

$$\forall (q(t), \dot{q}(t), \ddot{q}(t), w(t), z(t)) \text{ satisfying (39)}, \tag{48b}$$

$$(q(t), \dot{q}(t), \ddot{q}(t), w(t), z(t)) \neq 0, \tag{48c}$$

ready to be transformed in an LMI condition by Finsler's Lemma.

Theorem 6 (Integral Quadratic Constraints)
The following statements are equivalent.

i. The set of solutions of the Lyapunov problem (48) with $\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0$ is not empty.

ii. $\exists P_1,\ P_3 \in \mathbb{S}^n,\ \ \Phi_1,\ \Gamma_1,\ \Lambda_1,\ P_2 \in \mathbb{R}^n,\ \ \Pi_1 \in \mathbb{R}^{n_z \times n},\ \ \Xi_1 \in \mathbb{R}^{n_w \times n},\ \ \Phi_2,\ \Gamma_2,\ \Lambda_2, \in \mathbb{R}^{n \times n_z},$ $\Pi_2 \in \mathbb{R}^{n_z \times n_z}$, $\Xi_2 \in \mathbb{R}^{n_w \times n_z}$:

DOI: 10.1002/rnc

$$\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0, \quad \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0, \quad \text{where} \tag{49a}$$

$$\mathcal{J} := \begin{bmatrix} 0 & P_1 & P_2 & 0 & 0 \\ P_1 & P_2 + P_2^T & P_3 & 0 & 0 \\ P_2^T & P_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & Q & S \\ 0 & 0 & 0 & S^T & R \end{bmatrix}, \tag{49b}$$

$$\mathcal{H} := \begin{bmatrix} \Phi_1 K - \Phi_2 X & \Phi_1 C - \Phi_2 V & \Phi_1 M - \Phi_2 U & \Phi_2 & -\Phi_1 F_w - \Phi_2 D_{zw} \\ \Gamma_1 K - \Gamma_2 X & \Gamma_1 C - \Gamma_2 V & \Gamma_1 M - \Gamma_2 U & \Gamma_2 & -\Gamma_1 F_w - \Gamma_2 D_{zw} \\ \Lambda_1 K - \Lambda_2 X & \Lambda_1 C - \Lambda_2 V & \Lambda_1 M - \Lambda_2 U & \Lambda_2 & -\Lambda_1 F_w - \Lambda_2 D_{zw} \\ \Pi_1 K - \Pi_2 X & \Pi_1 C - \Pi_2 V & \Pi_1 M - \Pi_2 U & \Pi_2 & -\Pi_1 F_w - \Pi_2 D_{zw} \\ \Xi_1 K - \Xi_2 X & \Xi_1 C - \Xi_2 V & \Xi_1 M - \Xi_2 U & \Xi_2 & -\Xi_1 F_w - \Xi_2 D_{zw} \end{bmatrix}. \tag{49c}$$

Proof
Assign

$$x(t) \leftarrow \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \\ z(t) \\ w(t) \end{pmatrix}, \quad \mathcal{Q} \leftarrow (49b), \ \mathcal{B}^T \leftarrow \begin{bmatrix} K^T & -X^T \\ C^T & -V^T \\ M^T & -U^T \\ 0 & I \\ -F_w^T & -D_{zw}^T \end{bmatrix}, \ \mathcal{X} \leftarrow \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Gamma_1 & \Gamma_2 \\ \Lambda_1 & \Lambda_2 \\ \Pi_1 & \Pi_2 \\ \Xi_1 & \Xi_2 \end{bmatrix} \tag{50}$$

and apply Finsler's lemma to the constrained Lyapunov problem (48) with $P \succ 0$. □

The above condition yields specialized quadratic performance criterias depending on the choice of Q, S, R. Assign

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \leftarrow \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}.$$

to verify the $\mathcal{L}_2$ performance criteria

$$\begin{aligned} \int_0^t \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \geq 0 &\Leftrightarrow \int_0^t z(t)^T z(t)\, dt < \gamma^2 \int_0^t w(t)^T w(t)\, dt \\ &\Leftrightarrow \|z(t)\|_{\mathcal{L}_2}^2 < \gamma^2 \|w(t)\|_{\mathcal{L}_2}^2 \\ &\Leftrightarrow \|H_{zw}(j\omega)\|_{\mathcal{H}_\infty}^2 < \gamma^2 \end{aligned}$$

also known as *bounded real lemma*. To check *passivity* of a vector second order system, select

DOI: 10.1002/rnc

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \leftarrow \begin{bmatrix} 0 & -I \\ -I & 0 \end{bmatrix}.$$

reducing the integral quadratic constraint to

$$\int_0^t \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \geq 0 \Leftrightarrow -2 \int_0^t z(t)^T w(t)\, dt < 0$$
$$\Updownarrow$$
$$\int_0^t z(t)^T w(t)\, dt > 0 \Leftrightarrow H_{zw}(j\omega) + H_{zw}(j\omega)^* \succ 0 \;\Leftrightarrow\; H_{zw}(j\omega) \text{ is passive,}$$

condition also known as *positive real lemma*. Sector bounds on the signals $z(t)$ and $w(t)$ can be enforced by choosing

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \leftarrow \begin{bmatrix} I & -\frac{1}{2}(\alpha+\beta)I \\ -\frac{1}{2}(\alpha+\beta)I & -\alpha\beta I \end{bmatrix}.$$

The integral quadratic constraint yields

$$\int_0^t \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix} \geq 0 \Leftrightarrow \int_0^t (z(t) - \alpha w(t))^T (z(t) - \beta w(t))\, dt > 0$$
$$\Leftrightarrow (z(t), w(t)) \in \text{ sector}(\alpha, \beta).$$

Similarly to the stability case, in (49) the product of the multipliers with the system matrices occurs in a position that does not facilitate possible change-of-variables. One would be tempted to invoke algebraic duality of the vector second-order system once again. However, as discussed in the appendix, the presence of outputs bring complicating issues making such an approach not trivial. Add to this, the multipliers involved in the ICQ condition have different dimensions. An ICQ condition dependent on a single, square, invertible and well located multiplier is desirable for synthesis purposes.

A modification on the constrained Lyapunov problem is the first step towards a condition with such properties. The integral quadratic constraint may depend explicitly on positions, velocities and accelerations by substituting $z(t) = U\ddot{q}(t) + V\dot{q}(t) + Xq(t) + D_{zw}w(t)$ into (46) yielding

$$\int_0^t \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \\ w(t) \end{pmatrix}^T \begin{bmatrix} Z^T Q Z & Z^T(S + QD_{zw}) \\ \star & R + D_{zw}^T Q D_{zw} + D_{zw}^T S + S^T D_{zw} \end{bmatrix} \begin{pmatrix} q(t) \\ \dot{q}(t) \\ \ddot{q}(t) \\ w(t) \end{pmatrix} \geq 0, \tag{51a}$$

$$Z := \begin{bmatrix} X & V & U \end{bmatrix} \tag{51b}$$

DOI: 10.1002/rnc

The new constrained Lyapunov problem

$$\dot{V}\left(q(t),\dot{q}(t),\ddot{q}(t)\right) < -\begin{pmatrix} q(t)\\ \dot{q}(t)\\ \ddot{q}(t)\\ w(t)\end{pmatrix}^T \begin{bmatrix} Z^TQZ & Z^T(S+QD_{zw})\\ (S+QD_{zw})^TZ & \bar{R}\end{bmatrix}\begin{pmatrix} q(t)\\ \dot{q}(t)\\ \ddot{q}(t)\\ w(t)\end{pmatrix}, \tag{52a}$$

$$\bar{R} := R + D_{zw}^TQD_{zw} + D_{zw}^TS + S^TD_{zw}$$

$$\forall (q(t),\dot{q}(t),\ddot{q}(t),w(t)) \text{ satisfying } M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = F_w w(t), \tag{52b}$$

$$(q(t),\dot{q}(t),\ddot{q}(t),w(t)) \neq 0, \tag{52c}$$

is not dependent explicitly on the output vector $z(t)$. Sufficient conditions with reduced number of multipliers can be derived from the above Lyapunov problem by applying the Elimination Lemma. They become also necessary if the acceleration vector (or position vector) is absent in $z(t)$ i.e., $U = 0$ (or $X = 0$).

Theorem 7

The set of solutions of the Lyapunov problem (52) with $P \succ 0$ is not empty if $\exists\, P_1,\, P_3 \in \mathbb{S}^n,\; P_2,\; \Phi,\; \Lambda \in \mathbb{R}^{n\times n},\; \alpha \in \mathbb{R}$:

$$\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0, \quad \text{where} \tag{53a}$$

$$\mathcal{J} := \begin{bmatrix} 0 & P_1 & P_2 & 0\\ P_1 & P_2+P_2^T & P_3 & 0\\ P_2^T & P_3 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} + \begin{bmatrix} X^TQX & X^TQV & X^TQU & X^T(S+QD_{zw})\\ \star & V^TQV & V^TQU & V^T(S+QD_{zw})\\ \star & \star & U^TQU & U^T(S+QD_{zw})\\ \star & \star & \star & \bar{R}\end{bmatrix}, \tag{53b}$$

$$\mathcal{H} := \begin{bmatrix} \Phi K & \Phi C & \Phi M & -\Phi F_w\\ (\alpha\Phi+\Lambda)K & (\alpha\Phi+\Lambda)C & (\alpha\Phi+\Lambda)M & -(\alpha\Phi+\Lambda)F_w\\ \alpha\Lambda K & \alpha\Lambda C & \alpha\Lambda M & -\alpha\Lambda F_w\\ 0 & 0 & 0 & 0\end{bmatrix}, \quad \alpha > 0, \tag{53c}$$

$$\begin{bmatrix} P_1 & P_2\\ P_2^T & P_3\end{bmatrix} \succ 0. \tag{53d}$$

This is necessary and sufficient whenever $U = 0$ in (52).

Proof

Assign

DOI: 10.1002/rnc

$$\mathcal{Q} \leftarrow \begin{bmatrix} 0 & P_1 & P_2 & 0 \\ P_1 & P_2 + P_2^T & P_3 & 0 \\ P_2^T & P_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} X^TQX & X^TQV & X^TQU & X^T(S+QD_{zw}) \\ \star & V^TQV & V^TQU & V^T(S+QD_{zw}) \\ \star & \star & U^TQU & U^T(S+QD_{zw}) \\ \star & \star & \star & \bar{R} \end{bmatrix}$$

$$\mathcal{B}^T \leftarrow \begin{bmatrix} K^T \\ C^T \\ M^T \\ -F_w^T \end{bmatrix}, \quad \mathcal{C}^\perp \leftarrow \begin{bmatrix} \alpha^2 I & 0 \\ -\alpha I & 0 \\ I & 0 \\ 0 & I \end{bmatrix}, \quad \mathcal{C}^T \leftarrow \begin{bmatrix} I & 0 \\ \alpha I & I \\ 0 & \alpha I \\ 0 & 0 \end{bmatrix}, \quad \mathcal{X} \leftarrow \begin{bmatrix} \Phi \\ \Lambda \end{bmatrix}$$

and apply the Elimination Lemma with $P \succ 0$. This lemma renders a condition without extra conservatism whenever $\mathcal{C}^{\perp T}\mathcal{Q}\mathcal{C}^\perp \prec 0$, that is

$$\begin{aligned} &\begin{bmatrix} F_{11}(P_1,\, P_2,\, P_3) & F_{12} \\ F_{12}^T & -\bar{R} \end{bmatrix} \succ 0, \quad \bar{R} \prec 0, \\ F_{11}(\alpha,\, P_1,\, P_2,\, P_3) :=& 2(\alpha^3 P_1 - \alpha^2(P_2 + P_2^T) + \alpha P_3) - \big(\alpha^4 X^TQX - \alpha^3 X^TQV + \alpha^2 X^TQU \\ &+ \alpha^2 V^TQV - \alpha V^TQU + U^TQU\big) + \star, \\ F_{12}(\alpha) :=& -\alpha^2 X^T(S+QD_{zw}) + \alpha V^T(S+QD_{zw}) - U^T(S+QD_{zw}). \end{aligned} \tag{54}$$

Use the support inequality (19) with $N := \alpha^2 P_2^T$, $W := \alpha^3 P_1 \succ 0$ and a Schur complement with respect to $\bar{R}$ to show that (54) is equivalent to

$$\begin{aligned} P_3 - P_2^T P_1^{-1} P_2 \succ \frac{1}{2}\big(&\alpha^3 X^TQX - \alpha^2 X^TQV + \alpha X^TQU + \alpha V^TQV - V^TQU \\ &+ \alpha^{-1}U^TQU\big) - \frac{1}{2}(\alpha^3 X^T\bar{R}^{-1}X - \alpha^2 X^T\bar{R}^{-1}V + \alpha(X^T\bar{R}^{-1}U - V^T\bar{R}^{-1}V) \\ &+ \alpha^{-1}(U^T\bar{R}^{-1}U - U^T\bar{R}^{-1}V)) + \star \succeq 0. \end{aligned} \tag{55}$$

Note that $P_3 - P_2^T P_1^{-1} P_2 \succ 0$ implies $P \succ 0$ due to a Schur complement argument. When $U = 0$ the right hand side of (55) is polynomial in α with no constant term. Thus, there exists a sufficiently small α such that (55) holds which implies no added conservatism as long as $\alpha > 0$ is considered a variable in the formulation. □

For a constant α the above constraint is an LMI. However, the condition requires a line search in α. When $X = 0$ the same rationale with slightly modified $\mathcal{C}^\perp$ and $\mathcal{C}$

$$\mathcal{C}^\perp \leftarrow \begin{bmatrix} I & 0 \\ -\alpha I & 0 \\ \alpha^2 I & 0 \\ 0 & I \end{bmatrix}, \quad \mathcal{C}^T \leftarrow \begin{bmatrix} 0 & \alpha I \\ \alpha I & I \\ I & 0 \\ 0 & 0 \end{bmatrix}$$

also yields a necessary and sufficient condition.

DOI: 10.1002/rnc

The second step towards an ICQ condition for synthesis is to define a nonlinear change-of-variables between the Lyapunov matrices and a multiplier. Let a congruence transformation be $Y := \operatorname{diag}(\Gamma, \Gamma)$, $\Gamma := \Lambda^{-T}$ where Λ is assumed invertible. Apply it to the partitioned Lyapunov variable, leading to the change-of-variables

$$Y^T \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} Y := \begin{bmatrix} \hat{P}_1 & \hat{P}_2 \\ \hat{P}_2^T & \hat{P}_3 \end{bmatrix} \succ 0 \tag{56a}$$

$$\hat{P}_1 := \Gamma^T P_1 \Gamma, \quad \hat{P}_2 := \Gamma^T P_2 \Gamma, \quad \hat{P}_2^T := \Gamma^T P_2^T \Gamma \quad \hat{P}_3 := \Gamma^T P_3 \Gamma. \tag{56b}$$

The original Lyapunov matrices can be reconstructed by the inverse congruence transformation

$$\begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix} = Y^{-T} \begin{bmatrix} \hat{P}_1 & \hat{P}_2 \\ \hat{P}_2^T & \hat{P}_3 \end{bmatrix} Y^{-1} \succ 0 \tag{57}$$

With the results of Theorem 7 and the previously defined nonlinear change-of-variables at hand, an ICQ criteria suitable for synthesis can be stated.

Theorem 8

The set of solutions of the Lyapunov problem (52) with $P \succ 0$ is not empty if $\exists\, \hat{P}_1,\, \hat{P}_3 \in \mathbb{S}^n,\ \hat{P}_2,\ \Gamma \in \mathbb{R}^{n\times n},\ \alpha,\ \mu \in \mathbb{R}$:

$$\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0, \qquad \begin{bmatrix} \hat{P}_1 & \hat{P}_2 \\ \hat{P}_2^T & \hat{P}_3 \end{bmatrix} \succ 0, \quad \text{where} \tag{58a}$$

$$\begin{aligned} \mathcal{J} := &\begin{bmatrix} 0 & \hat{P}_1 & \hat{P}_2 & 0 \\ \hat{P}_1 & \hat{P}_2 + \hat{P}_2^T & \hat{P}_3 & 0 \\ \hat{P}_2^T & \hat{P}_3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \\ &\begin{bmatrix} \Gamma^T X^T Q X \Gamma & \Gamma^T X^T Q V \Gamma & \Gamma^T X^T Q U \Gamma & \Gamma^T X^T (S + Q D_{zw}) \\ \star & \Gamma^T V^T Q V \Gamma & \Gamma^T V^T Q U \Gamma & \Gamma^T V^T (S + Q D_{zw}) \\ \star & \star & \Gamma^T U^T Q U \Gamma & \Gamma^T U^T (S + Q D_{zw}) \\ \star & \star & \star & \bar{R} \end{bmatrix}, \\ &\bar{R} := R + D_{zw}^T Q D_{zw} + D_{zw}^T S + S^T D_{zw} \end{aligned} \tag{58b}$$

$$\mathcal{H} := \begin{bmatrix} \mu K \Gamma & \mu C \Gamma & \mu M \Gamma & -\mu F_w \\ (1+\alpha\mu) K \Gamma & (1+\alpha\mu) C \Gamma & (1+\alpha\mu) M \Gamma & -(1+\alpha\mu) F_w \\ \alpha K \Gamma & \alpha C \Gamma & \alpha M \Gamma & -\alpha F_w \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \alpha > 0,\ \mu > 0. \tag{58c}$$

Proof

To derive (58) from (53), first introduce the constraint $\Phi = \mu\Lambda$ where $\mu > 0$. Apply the congruence transformation $Y_a := \operatorname{diag}(\Gamma, \Gamma, \Gamma, I)$, $\Gamma := \Lambda^{-T}$ to (53a), congruence transformation $Y_d := \operatorname{diag}(\Gamma, \Gamma)$ to (53d), and the change-of-variables (56). Notice that the upper left entry of $\mathcal{J} + \mathcal{H} + \mathcal{H}^T \prec 0$ in (58), i.e., $K\Gamma + \Gamma^T K^T + \Gamma^T X^T Q X \Gamma \prec 0$ with K nonsingular implies Γ

DOI: 10.1002/rnc

nonsingular. This fact corroborates the assumption of an invertible Γ in the change-of-variables (56). □

The condition from Theorem 8 benefits from some convenient properties. It depends on a single multiplier Γ in products with M, C, K matrices as well as U, V, X matrices. Moreover the product occurs at the "right side" of the matrices. Both properties facilitate change-of-variables involving the controller data, as will become clear later in this manuscript.

Synthesis of controllers is the subject of the reminder of this paper. It will be given focus to the design of controllers with guaranteed $\mathcal{L}_2$-gain performance for clarity and its practical relevance. Synthesis conditions considering other ICQ criterias can be derived similarly by particularizing Q, R, S and appropriate Schur complements involving these matrices.

3.3. *Static Full Vector Feedback*

The proposed ICQ condition offers the possibility of synthesizing controllers. Consider the vector second-order system with disturbance and controllable inputs

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = F_w w(t) + F_u u(t) \tag{59a}$$

$$z(t) = U\ddot{q}(t) + V\dot{q}(t) + Xq(t) + D_{zw}w(t) + D_{zu}u(t) \tag{59b}$$

in loop with the static full vector feedback (23) yielding the closed-loop system denoted $\mathbf{H}_{zw}$:

$$\mathbf{M}\ddot{q}(t) + \mathbf{C}\dot{q}(t) + \mathbf{K}q(t) = F_w w(t) \tag{60a}$$

$$z(t) = \mathbf{U}\ddot{q}(t) + \mathbf{V}\dot{q}(t) + \mathbf{X}q(t) + D_{zw}w(t) \tag{60b}$$

$$\mathbf{M} := (M + F_u G_a)\,, \quad \mathbf{C} := (C + F_u G_v)\,, \quad \mathbf{K} := (K + F_u G_p) \tag{60c}$$

$$\mathbf{U} := (U - D_{zu} G_a)\,, \quad \mathbf{V} := (V - D_{zu} G_v)\,, \quad \mathbf{X} := (X - D_{zu} G_p) \tag{60d}$$

The same issues regarding the nonsingularity of the multiplier in the stabilizability case have also to be consider here. Therefore, the next theorem states the existence of a static controller in which the acceleration feedback is absent ($G_a = 0$). This controller structure corresponds to a full state feedback in the first-order state-space sense.

Theorem 9

There exists a controller of the form (23) with $G_a = 0$ such that $\|\mathbf{H}_{zw}\|_{\mathcal{L}_2} < \gamma^2$ if $\exists\, \hat{P}_1, \hat{P}_3 \in \mathbb{S}^n,\ \hat{P}_2,\ \Gamma \in \mathbb{R}^n,\ \hat{G}_v,\ \hat{G}_p \in \mathbb{R}^{n_u \times n},\ \alpha, \mu \in \mathbb{R}\ :$

DOI: 10.1002/rnc

$$\begin{bmatrix} \mu(K\Gamma + F_u\hat{G}_p + \Gamma^T K^T + \hat{G}_p^T F_u^T) & \hat{P}_1 + \mu(C\Gamma + F_u\hat{G}_v) + (1+\alpha\mu)(K\Gamma + F_u\hat{G}_p)^T & \hat{P}_2 + \mu M\Gamma + \alpha(K\Gamma + F_u\hat{G}_p)^T & -\mu F_w & \Gamma^T X^T - \hat{K}_p^T D_{zu}^T \\ \star & \hat{P}_2 + (1+\alpha\mu)(C\Gamma + F_u\hat{G}_v) + (\star) & \hat{P}_3 + (1+\alpha\mu)M\Gamma + \alpha(C\Gamma + F_u\hat{G}_v)^T & -(1+\alpha\mu)F_w & \Gamma^T V^T - \hat{K}_v^T D_{zu}^T \\ \star & \star & \alpha(M\Gamma + \Gamma^T M^T) & -\alpha F_w & \Gamma^T U^T \\ \star & \star & \star & -\gamma^2 I & D_{zw}^T \\ \star & \star & \star & \star & -I \end{bmatrix} \prec 0. \tag{61a}$$

$$\alpha > 0, \quad \mu > 0, \quad \begin{bmatrix} \hat{P}_1 & \hat{P}_2 \\ \hat{P}_2^T & \hat{P}_3 \end{bmatrix} \succ 0 \tag{61b}$$

Proof

In order to obtain the above inequalities from (58), first particularize it with $Q = I$, $R = -\gamma^2 I$ and apply a Schur complement with respect to Q. A direct application of the resulting inequalities to the closed-loop system (24) together with a change-of-variables of the form (25) involving the multiplier Γ and the controller data G_v, G_p yields (61). Nonsingularity of Γ is implied by the entry $M\Lambda + \Lambda^T M^T \prec 0$ with M nonsingular. Once a solution to the problem above is found, invertibility of Γ assures the reconstruction of the controller gains from the auxiliary ones according to $G_v = \hat{G}_v\Gamma^{-1}$ and $G_p = \hat{G}_p\Gamma^{-1}$. □

The acceleration feedback was removed from the feedback law for theoretical reasons: ensure a nonsingular Γ. As discussed in the stabilizability section, a nonsingular Γ could also be enforced by neglecting the position feedback G_p. If all feedback gains are desired, in practice the LMI above could be augmented with the acceleration gain and solved. The multiplier Γ could be invertible. In case this happens, the acceleration, velocity and position gains can all be recovered from the auxiliary controller gains.

Working with the closed-loop system in vector form facilitates the feedback of only the position or the velocity vector without introducing extra conservatism to the presented formulation. These controller structures would correspond to partial state feedback in the first-order state-space sense, to which convex reformulations without loss of generality are not known to exist.

3.4. Static Output Feedback

The acceleration, velocity or position vectors are often partially available for feedback. In such a case, the vector second order system

$$M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = F_w w(t) + F_u u(t) \tag{62a}$$

$$z(t) = U\ddot{q}(t) + V\dot{q}(t) + Xq(t) + D_{zw}w(t) + D_{zu}u(t) \tag{62b}$$

$$y(t) = R\ddot{q}(t) + S\dot{q}(t) + Tq(t) + D_{yw}w(t) \tag{62c}$$

DOI: 10.1002/rnc

is augmented with a measurement vector $y(t) \in \mathbb{R}^{n_y}$. The interest lies on the synthesis of a static output feedback controller of the form

$$u(t) = -G_y y(t) \tag{63}$$

where $G_y \in \mathbb{R}^{n_u \times n_y}$. To facilitate the derivations that follows, the measurement vector is not corrupted by noise ($D_{yw} = 0$). Assume, without loss of generality, that the output matrices R, S and T are of full row rank. Then, there exist nonsingular transformation matrices $W_a,\ W_v,\ W_p \in \mathbb{R}^{n \times n}$ such that

$$RW_a = \begin{bmatrix} I & 0 \end{bmatrix}, \quad SW_v = \begin{bmatrix} I & 0 \end{bmatrix}, \quad TW_p = \begin{bmatrix} I & 0 \end{bmatrix}. \tag{64}$$

For any given triple (R, S, T), the corresponding (W_a, W_v, W_p) are not unique in general. A particular (W_a, W_v, W_p) can be obtained by

$$W_a := \begin{bmatrix} R^T(RR^T)^{-1} & R^{\perp} \end{bmatrix}, \ W_v := \begin{bmatrix} S^T(SS^T)^{-1} & S^{\perp} \end{bmatrix}, \ W_p := \begin{bmatrix} T^T(TT^T)^{-1} & T^{\perp} \end{bmatrix}.$$

The feedback of a single quantity, that is either accelerations, velocities or positions are addressed here. Let the measurement vector be composed of position feedback only, i.e., $y(t) = Tq(t)$. From the coordinate transformation defined as $\ddot{q} := W_p \ddot{\tilde{q}}$, $\dot{q} := W_p \dot{\tilde{q}}$ and $q := W_p \tilde{q}$, the system matrices of (62) are substituted according to

$$\begin{aligned} &M \leftarrow MW_p, \quad C \leftarrow CW_p, \quad K \leftarrow KW_p \\ &U \leftarrow UW_p, \quad V \leftarrow VW_p, \quad X \leftarrow XW_p \\ &R \leftarrow RW_p, \quad S \leftarrow SW_p, \quad T \leftarrow \begin{bmatrix} I & 0 \end{bmatrix}. \end{aligned}$$

The closed-loop matrices of the transformed system related to positions are then

$$\mathbf{K} := \left(K + F_u \begin{bmatrix} G_y & 0 \end{bmatrix} \right), \quad \mathbf{X} := \left(X - D_{zu} \begin{bmatrix} G_y & 0 \end{bmatrix} \right)$$

while the other closed-loop matrices are the same as the open-loop ones. A static output-feedback gain can be obtained by imposing on the auxiliary controller gain $\hat{G}$ and the multiplier Γ the structure

$$\hat{G} := \begin{bmatrix} \hat{G}_y & 0 \end{bmatrix}, \quad \Gamma := \begin{bmatrix} \Gamma_1 & 0 \\ \Gamma_3 & \Gamma_4 \end{bmatrix}. \tag{65}$$

This kind of controller/multiplier constraint was firstly proposed in [23] in the context of first-order state-space systems. This structure is merged in Theorem 9 by imposing the structural constraints $\hat{G}_a := \hat{G}_v := \hat{G}_p := \hat{G}$ and Γ as (65). Supposing Γ nonsingular, and consequently the upper left block Γ_1 invertible, the original controller data can be recovered by the inverse change-of-variables

$$G = \begin{bmatrix} G_y & 0 \end{bmatrix} = \begin{bmatrix} \hat{G}_y \Gamma_1^{-1} & 0 \end{bmatrix}. \tag{66}$$

DOI: 10.1002/rnc

Thus, the structure imposed to the state feedback gain matrix G facilitates the output feedback law $u(t) = G_y y(t)$. The same procedure can be made when the measurement vector is $y(t) = R\ddot{q}(t)$ or $y(t) = S\dot{q}(t)$.

3.5. Robust Control

The inherent decoupling of the Lyapunov and system matrices occasioned by the introduction of multipliers facilitates the usage of parameter-dependent Lyapunov functions [35]. This decoupling property was firstly exploited under the context of robust stability of first-order state-space systems in [22] and latter extended to performance specifications [23]. Assume that the matrices of system (59) are uncertain but belong to a convex and bounded set. This set is such that the matrix

$$\mathcal{S} := \begin{bmatrix} M & C & K & F_u & F_w \\ U & V & X & D_{zu} & D_{zw} \end{bmatrix}$$

takes values in a domain defined as a polytopic combination of N given matrices $\mathcal{Q}_1, \ldots, \mathcal{Q}_N$, that is,

$$\mathsf{S} := \left\{ \mathcal{S}(\alpha) : \mathcal{S}(\alpha) := \sum_{i=1}^{N} \mathcal{S}_i \alpha_i, \quad \sum_{i=1}^{N_\alpha} \alpha_i = 1, \quad \alpha_i \geq 0. \right\}$$

The operator $\text{Vert}(\mathsf{S}) := \{\mathcal{S}_1, \ldots, \mathcal{S}_N\}$ reduces the infinite dimensional set S to the vertex $\mathcal{S}_i$, $i = 1, \ldots, N$. The LMI conditions for vector second-order systems presented here can turn into sufficient conditions for robust analysis and synthesis by defining a parameter-dependent Lyapunov matrix

$$P(\alpha) := \sum_{i=1}^{N_\alpha} P_i \alpha_i \tag{67}$$

and maintaining the multipliers as parameter-independent. In this case, the LMIs are infinite-dimensional functions of the uncertain vector α. A finite-dimensional problem arises with Vert $(F(x, \alpha) \prec 0)$. Consider the robust stability problem as an example. System (22) is robustly

DOI: 10.1002/rnc

stabilizable by a static feedback law of the form (23) with $G_a = 0$, for all $\mathcal{S} \in \mathsf{S}$, if $\exists\, \Lambda,\ P_{2,i} \in \mathbb{R}^{n\times n},\ P_{1,i},\ P_{3,i} \in \mathbb{S}^n,\ \hat{G}_v,\ \hat{G}_p \in \mathbb{R}^{n_u\times n},\ \alpha,\ \mu \in \mathbb{R}$:

$$\mathcal{J}_i + \mathcal{H}_i + \mathcal{H}_i^T \prec 0, \quad \mathcal{J}_i := \begin{bmatrix} 0 & P_{1,i} & P_{2,i} \\ P_{1,i} & P_{2,i} + P_{2,i}^T & P_{3,i} \\ P_{2,i}^T & P_{3,i} & 0 \end{bmatrix}, \tag{68a}$$

$$\mathcal{H}_i := \begin{bmatrix} \mu(K_i\Lambda + F_{u,i}\hat{G}_p) & \mu(C_i\Lambda + F_{u,i}\hat{G}_v) & \mu(M_i\Lambda) \\ (1+\alpha\mu)(K_i\Lambda + F_{u,i}\hat{G}_p) & (1+\alpha\mu)(C_i\Lambda + F_{u,i}\hat{G}_v) & (1+\alpha\mu)(M_i\Lambda) \\ \alpha(K_i\Lambda + F_{u,i}\hat{G}_p) & \alpha(C_i\Lambda + F_{u,i}\hat{G}_v) & \alpha(M_i\Lambda) \end{bmatrix} \tag{68b}$$

$$\begin{bmatrix} P_{1,i} & P_{2,i} \\ P_{2,i}^T & P_{3,i} \end{bmatrix} \succ 0, \quad \alpha > 0, \quad \mu > 0, \tag{68c}$$

for $i = 1, \ldots, N$.

4. NUMERICAL EXAMPLES

4.1. Three-Mass System

The simplicity of a three-mass system depicted in Fig. 1 allows an easy analysis and straightforward interpretation of the results. In this figure, m_1, m_2 and m_3 are system masses, k_1, k_2, k_3 and k_4 are stiffness coefficients, while d_1, d_2, d_3 and d_4 are damping coefficients.

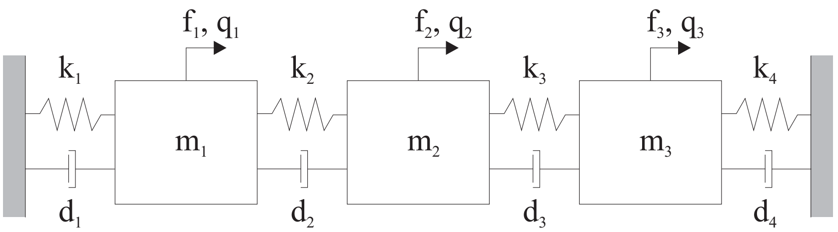


Figure 1. Three-mass mechanical system.

The control input $u(t)$ acts at mass 2 and mass 3 in opposite directions. The first disturbance $w_1(t)$ acts at mass 2 and mass 3 in opposite directions, with an amplification factor of 3, and the second disturbance $w_2(t)$ acts at mass 2. The controlled outputs ($z_1(t)$, $z_2(t)$, $z_3(t)$) are the displacement of mass 2 with an amplification factor of 3, the velocity of mass 3, and the input $u(t)$, respectively. The motion of this mechanical system is described by the differential equations

$$\begin{aligned} &\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \ddot{q}(t) + \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix} \dot{q}(t) \\ &+ \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} q(t) = \begin{bmatrix} 0 & 0 \\ 3 & 1 \\ -3 & 0 \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u(t), \end{aligned} \tag{69a}$$

$$z(t) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \dot{q}(t) + \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} q(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t). \tag{69b}$$

For this system, $m_1 = 3$, $m_2 = 1$, $m_3 = 2$, $k_1 = 30$, $k_2 = 15$, $k_3 = 15$, $k_4 = 30$, and $C = 0.004K + 0.001M$. Magnitude plots of the open-loop transfer functions from disturbances (w_1, w_2) to outputs (z_1, z_2) are depicted in Fig. 3a. The lightly damped characteristics of the system modes are noticeable.

$\mathcal{H}_\infty$ control will be used to reject oscillatory response of these modes in face of disturbances. Full vector feedback gains of positions and velocities are synthesized using Theorem 9 for different values of the scalars α, μ. The upper bound γ of the $\mathcal{H}_\infty$-norm for various (α, μ) is illustrated in Fig. 2. The minimum achieved upper bound $\gamma^* = 7.679$ occurs at $(\alpha, \mu) = (0.0060, 0.0820)$ with corresponding position and velocity feedback gains

$$G_p = \begin{bmatrix} 0.2501 & 0.0774 & -0.0786 \end{bmatrix}, \quad G_v = \begin{bmatrix} 5.2757 & 1.9574 & -1.6351 \end{bmatrix}.$$

Improved vibration performance is corroborated by magnitude plots and impulse responses of the closed-loop system (Fig. 3a and 3b).

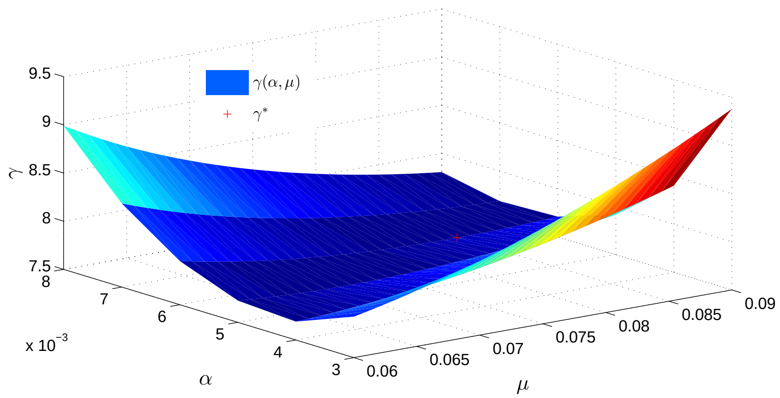


Figure 2. Upper bound on the $\mathcal{H}_\infty$-norm of the closed-loop three-mass system with full position and velocity feedback, obtained by Theorem 9.

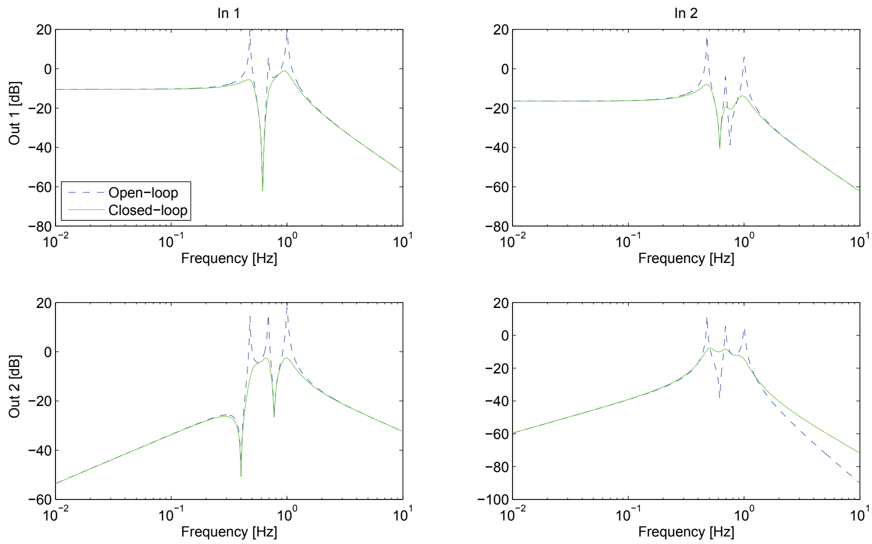
4.2. *Model Matching Control of Wind Turbines*

A different perspective to modern control of wind turbines is given here by considering the design model in its natural form. For clarity, the turbine model contains only the two structural degrees of freedom with lowest frequency contents: rigid body rotation of the rotor and fore-aft tower bending described by the axial nacelle displacement. The simplified dynamics of a wind turbine can be described by the nonlinear differential equations

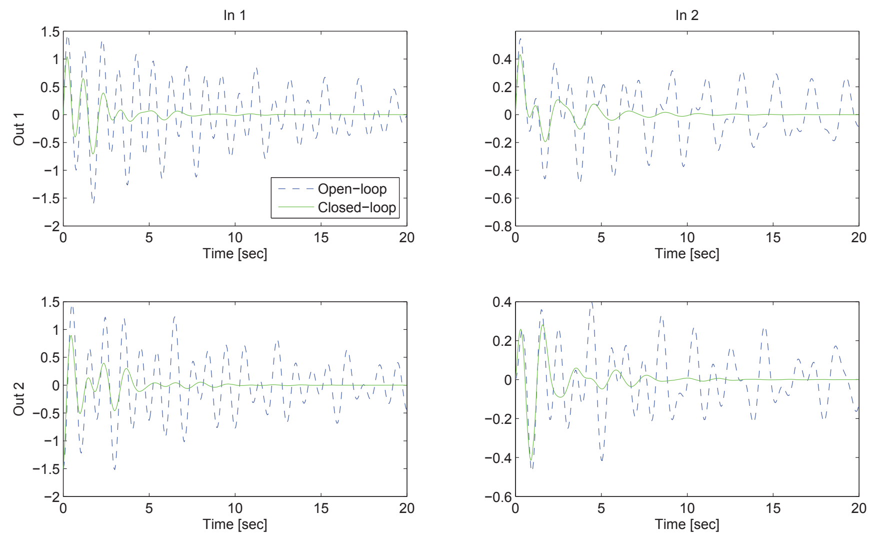
$$J\ddot{\psi} = Q_a(v - \dot{q}_1, \dot{\psi}, \beta)(t) - Q_g(t) \tag{70}$$

$$M_1\ddot{q}_1 + K_1 q_1 = T_a(v - \dot{q}_1, \dot{\psi}, \beta) \tag{71}$$

DOI: 10.1002/rnc



(a) Magnitude Plot



(b) Impulse Response

Figure 3. Comparison of open-loop three-mass system and closed-loop with full position and velocity feedback (Theorem 9).

where the aerodynamic torque $Q_a(t)$ and thrust $T_a(t)$ are nonlinear functions of the relative wind speed $v(t) - q_1(t)$ with $v(t)$ being the mean wind speed over the rotor disk, the rotor speed $\dot{\psi}(t)$, and the collective pitch angle $\beta(t)$. Linearization of (70) around an equilibrium point θ yields

$$(J_r + N_g^2 J_g)\ddot{\psi}(t) = \left.\frac{\partial Q_a}{\partial \dot{\psi}}\right|_\theta \dot{\psi}(t) + \left.\frac{\partial Q_a}{\partial V}\right|_\theta (v(t) - \dot{q}_1(t)) + \left.\frac{\partial Q_a}{\partial \beta}\right|_\theta \beta(t) - \eta^{-1} N_g Q_g(t) \tag{72}$$

$$M_1 \ddot{q}_1(t) + K_1 q_1(t) = \left.\frac{\partial T_a}{\partial \dot{\psi}}\right|_\theta \dot{\psi}(t) + \left.\frac{\partial T_a}{\partial V}\right|_\theta (v(t) - \dot{q}_1(t)) + \left.\frac{\partial T_a}{\partial \beta}\right|_\theta \beta(t) \tag{73}$$

where J_r and J_g are the rotational inertia of the rotor (low speed shaft part) and the generator (high speed shaft part), K_1 is the stiffness for axial nacelle motion $q_1(t)$ due to fore-aft tower bending, M_1

DOI: 10.1002/rnc

is the modal mass of the first fore-aft tower bending mode, η is the total electrical and mechanical efficiency, and N_g is the gearbox ratio.

The primary control objective of pitch controlled wind turbines operating at rated power is to regulate power generation despite wind speed disturbances. To accomplish this, rotor speed is controlled using the collective blade pitch angle, and generator torque is maintained constant ($Q_g(t) = 0$ in (72)). Tower fore-aft oscillations are induced by the wind turbulence hitting the turbine as well as changes in the thrust force due to pitch angle variations. The collective blade pitch angle can be controlled to suppress these oscillations without degrading rotor speed regulation. The vector second-order system

$$\begin{aligned}\begin{bmatrix} J_r + N_g^2 J_g & 0 \\ 0 & M_1 \end{bmatrix} \begin{pmatrix} \ddot{\psi}(t) \\ \ddot{q}_1(t) \end{pmatrix} + \begin{bmatrix} \left.\dfrac{\partial Q_a}{\partial \dot{\psi}}\right|_\theta & -\left.\dfrac{\partial Q_a}{\partial V}\right|_\theta \\ \left.\dfrac{\partial T_a}{\partial \dot{\psi}}\right|_\theta & -\left.\dfrac{\partial T_a}{\partial V}\right|_\theta \end{bmatrix} \begin{pmatrix} \dot{\psi}(t) \\ \dot{q}_1(t) \end{pmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_1 \end{bmatrix} \begin{pmatrix} \psi(t) \\ q_1(t) \end{pmatrix} \\ = \begin{bmatrix} \left.\dfrac{\partial Q_a}{\partial v}\right|_\theta \\ \left.\dfrac{\partial T_a}{\partial v}\right|_\theta \end{bmatrix} v(t) + \begin{bmatrix} \left.\dfrac{\partial Q_a}{\partial \beta}\right|_\theta \\ \left.\dfrac{\partial T_a}{\partial \beta}\right|_\theta \end{bmatrix} \beta(t)\end{aligned} \tag{74}$$

arise from re-arranging expression (72). In the above, the disturbance vector is $w(t) := v(t)$ and control input is $u(t) := \beta(t)$. The open-loop system (74) has a singular stiffness matrix due to the rigid-body mode of the rotor, which at first may seem inadequate for a direct application of the conditions presented in this work. However, the closed-loop stiffness matrix is non-singular because the position of the rotor is part of the feedback law. Feedback of rotor position is analogous to the inclusion of integral action on rotor speed regulation, usual scheme in wind turbine control.

Controller design follows an $\mathcal{H}_\infty$ model matching criteria, which has an elegant structure when considered in vector second-order form. The performance of the system in closed-loop should approximate a given a reference model

$$M_r \ddot{q}_r(t) + C_r \dot{q}_r(t) + K_r q_r(t) = F_{wr} w(t) \tag{75a}$$

$$z_r(t) = U_r \ddot{q}_r(t) + V_r \dot{q}_r(t) + X_r q_r(t) \tag{75b}$$

in an $\mathcal{H}_\infty$-norm sense. The matrices of the reference model are chosen to enforce a desired second-order closed-loop sensitivity function from wind speed disturbance $v(t)$ to rotor speed $\dot{\psi}(t)$. The augmented system for synthesis is

$$\begin{aligned}\begin{bmatrix} M & 0 \\ -M_{(1,:)} & M_r \end{bmatrix} \begin{pmatrix} \ddot{\psi} \\ \ddot{q}_1 \\ \ddot{\psi}_r \end{pmatrix} \begin{bmatrix} C & 0 \\ -C_{(1,:)} & C_r \end{bmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{q}_1 \\ \dot{\psi}_r \end{pmatrix} \begin{bmatrix} K & 0 \\ -K_{(1,:)} & K_r \end{bmatrix} \begin{pmatrix} \psi \\ q_1 \\ \psi_r \end{pmatrix} \\ = \begin{bmatrix} F_w \\ 0 \end{bmatrix} w(t) + \begin{bmatrix} F_u \\ F_{u\,(1,:)} \end{bmatrix} u(t)\end{aligned} \tag{76a}$$

$$z(t) = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{\psi} \\ \dot{q}_1 \\ \dot{\psi}_r \end{pmatrix} + \begin{bmatrix} 0 \\ D_{zu} \end{bmatrix} u(t) \tag{76b}$$

where $\dot{\psi}_r(t)$ is the reference model velocity and $(\cdot)_{(1,:)}$ stands for the first line of matrix $(\cdot)$. The reference filter in (76a) is forced indirectly by the the open-loop system (74), which is convenient for implementation purposes. In this example, $M_r = 6.0776 \cdot 10^6$, $C_r = 6.1080 \cdot 10^6$, and $K_r = 3.9346 \cdot 10^6$ characterizes a reference system with damped natural frequency $\omega_d = 0.628$ rad/s and damping $\xi = 0.625$.

Full vector feedback gains of positions and velocities are synthesized using Theorem 9 with $\alpha = 0.9$ and $\mu = 1$, yielding a guaranteed upper bound $\gamma = 1.462$. The true upper bound of the augmented system in closed-loop computed using Theorem 7 is $\gamma = 0.1058$. Controller gains are

$$G_v = \begin{bmatrix} -0.3734 & -0.1702 & 0.0028 \end{bmatrix}, \quad G_p = \begin{bmatrix} -0.1951 & -0.1029 & -0.0096 \end{bmatrix}$$

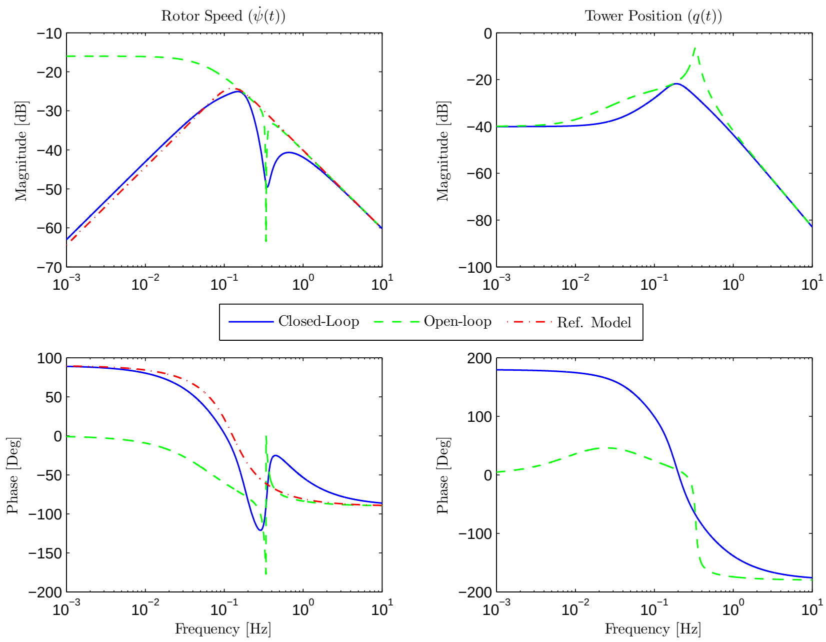
Bode plots of the closed-loop, open-loop and reference systems are depicted in Fig.4a. A good agreement between the closed-loop and reference model is noticeable. The chosen reference model indirectly impose some damping of the tower fore-aft displacement by trying to reduce the difference in magnitude between open-loop and reference model at the tower natural frequency. Step responses of the controlled and reference systems are compared in Fig.4b, showing a good correspondence.

5. CONCLUSIONS

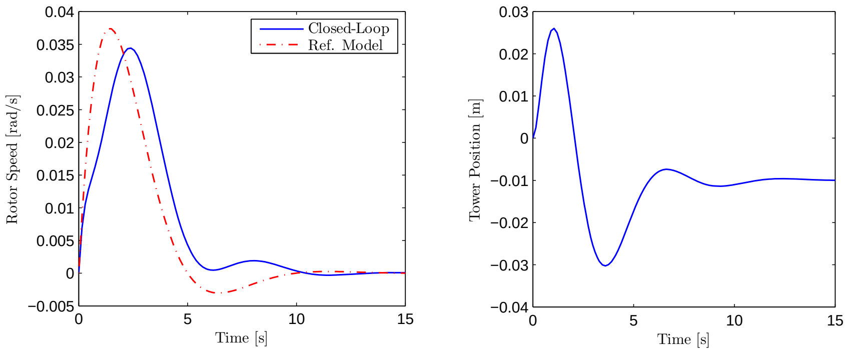
The analysis and synthesis conditions of vector second-order systems obtained during our studies have the potential to increase the practice of working with systems directly in vector second-order form. LMI conditions for verifying asymptotic stability and quadratic performance were shown to be necessary and sufficient, irrespective of the type of dynamic loading. Due to their linear dependence in the coefficient matrices and the inclusion of multipliers on the formulation, the conditions are appropriate to robust analysis of systems with structured uncertainty. Synthesis of vector second-order controllers with guaranteed stability and quadratic performance are also formulated as LMI problems. Unfortunately, the synthesis conditions are only sufficient to the existence of full state-feedbacks. This is the major drawback when compared to synthesis in state-space first-order form, to which necessary and sufficient LMI conditions are available in the literature. However, when structural constraints are imposed on the controller gains, the design in vector second-order form may render less conservative results.

REFERENCES

1. Skelton RE. Adaptive orthogonal filters for compensation of model errors in matrix second-order systems. *J. Guidance, Contr. Dynam.* 1979; :pp. 214221.
2. Diwekar AM, Yedavalli RK. Stability of matrix second-order systems: New conditions and perspectives. *IEEE Transactions on Automatic Control* 1999; **Vol. 44, No. 9, Sept. 1999**:1773–1777.



(a) Bode Plots



(b) Step Response

Figure 4. $\mathcal{H}_\infty$ model matching control of a simplified wind turbine model.

3. Inman DJ. Active modal control for smart structures. *doi: 10.1098/rsta.2000.0721 Phylosophical Transactions of the London Society A, 15 2001 no. 1778 205-219* 2001; **Vol. 359 No.1778, DOI 10.1098/rsta.2000.0721**:205–219.
4. Shieh L, Mehio M, Dib H. Stability of the second-order matrix polynomial. *Automatic Control, IEEE Transactions on* mar 1987; **32**(3):231 – 233, doi:10.1109/TAC.1987.1104572.
5. Bernstein D, Bhat SP. Lyapunow stability, semistability, and asymptotic stability of matrix second-order systems. *Transactions of the ASME* 1995; **Vol. 117**:145–153.
6. Joshi SM. Robustness properties of collocated controllers for flexible spacecraft. *J. Guidance Control* 1986; **vol. 9, no. 1**:8591.
7. Fujisaki Y, Ikeda M, Miki K. Robust stabilization of large space structures via displacement feedback. *Automatic Control, IEEE Transactions on* dec 2001; **46**(12):1993 –1996, doi:10.1109/9.975507.
8. Gardiner JD. Stabilizing control for second order models and positive real systems. *Journal of Guidance, Control and Dynamics* 1992; **15**:280–282.
9. Morris KA, Juang JN. Dissipative controller designs for second-order dynamic systems. *IEEE Transactions on Automatic Control* 1994; **39**:1056–1063.

DOI: 10.1002/rnc

10. Diwekar A, Yedavalli R. Robust controller design for matrix second-order systems with structured uncertainty. *IEEE Transactions on Automatic Control* 1999; **Vol. 41, No. 2, DOI 10.1109/9.746276**:401–405.
11. Datta BN, Elhay S, Ram YM. Orthogonality and partial pole assignment for the symmetric definite quadratic pencil. *Linear Algebra and its Applications* 1997; **257**(0):29 – 48, doi:10.1016/S0024-3795(96)00036-5. URL `http://www.sciencedirect.com/science/article/pii/S0024379596000365`.
12. Datta B, Sarkissian D. Multi-input partial eigenvalue assignment for the symmetric quadratic pencil. *American Control Conference, 1999. Proc. of the*, vol. 4, 1999; 2244 –2247 vol.4, doi:10.1109/ACC.1999.786401.
13. Datta B, Lin WW, Wang JN. Robust partial pole assignment for vibrating systems with aerodynamic effects. *Automatic Control, IEEE Transactions on* dec 2006; **51**(12):1979 –1984, doi:10.1109/TAC.2006.886543.
14. Datta B, Elhay S, Ram Y, Sarkissian D. Partial eigenstructure assignment for the quadratic pencil. *Journal of Sound and Vibration* 2000; **230**(1):101 – 110, doi:10.1006/jsvi.1999.2620. URL `http://www.sciencedirect.com/science/article/pii/S0022460X99926202`.
15. Nichols NK, Kautsky J. Robust eigenstructure assignment in quadratic matrix polynomials. *SIAM J. Matrix Anal. Applicat* 2001; **vol. 23**:77102.
16. Kwak S, Yedavalli R. Observer designs in matrix second order system framework: measurement conditions and perspectives. *Proc. of the American Control Conference*, Chicago, Illinois, 2000; 2316–2320.
17. Kwak S, Yedavalli R. New approaches for observer design in linear matrix second order systems. *Proc. of the American Control Conference*, Chicago, Illinois, 2000; 2316–2320.
18. Duan G, Wu Y. Generalized luenberger observer design for matrix second-order linear systems. *Control Applications, 2004. Proc. of the IEEE Int. Conf. on*, vol. 2, 2004; 1739 – 1743 Vol.2, doi:10.1109/CCA.2004.1387628.
19. Oliveira M, Skelton R. Stability tests for constrained linear systems. *Perspectives in Robust Control - Lecture Notes in Control and Information Sciences*. Springer, 2001.
20. Boyd S, Ghaoui L, Feron E, Balakrishnan V. *Linear Matrix Inequalities in System and Control Theory*. SIAM: London, 1994.
21. Ebihara Y, Hagiwara T, Peaucelle D, Arzelier D. Robust performance analysis of linear time-invariant uncertain systems by taking higher-order time-derivatives of the state. *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC '05. 44th IEEE Conference on*, 2005; 5030–5035, doi:10.1109/CDC.2005.1582959.
22. Oliveira MD, Bernussou J, Geromel J. A new discrete-time robust stability condition. *Systems & Control Letters* 1999; **37**:261–265.
23. Oliveira MD, Geromel J, Bernussou J. Extended H_2 and H_∞ norm characterizations and controller parametrizations for discrete-time systems. *International Journal of Control* 2002; **75:9**:666–679.
24. Pipeleers G, Demeulenaere B, Swevers J, Vandenbergh L. Extended LMI characterizations for stability and performance of linear systems. *Systems & Control Letters* 2009; **58**:510–518.
25. Finsler P. Uber das vorkommen definiter und semidefiniter formen in scharen quadratischer formem. *Commentarii Mathematici Helvetici*, 1937; **9**:188–192.
26. Skelton R, Iwasaki T, Grigoriadis K. *An Unified Algebraic Approach to Linear Control Design*. Taylor and Francis, 1999.
27. Stoustrup J, Iwasaki T, Skelton R. Mixed h_2 / h_∞ state feedback control with an improved covariance bound. *Proceedings of the IFAC World Congress*, vol. 5, Sydney, Australia, 1993; 235–238. URL `http://www.control.aau.dk/~jakob/selPubl/papers1993/ifac_wc_1993.pdf`, **I**nvited paper.
28. Xie W. An equivalent lmi representation of bounded real lemma for continuous-time systems. *Journal of Inequalities and Applications* 2008; **2008**(1):672 905.
29. Bernussou J, Geromel J, Peres PLD. A linear programming oriented procedure for quadratic stabilization of uncertain systems. *Systems and Control Letters* 1989; **13**:65–72.
30. Apkarian P, Tuan H, Bernussou J. Continuous-time analysis, eigenstructure assignment, and H_2 synthesis with enhanced linear matrix inequalities (LMI) characterizations. *IEEE Transactions on Automatic Control* 2001; **Vol. 46, No. 12**:1941–1946.
31. Chilali M, Gahinet P. H_∞ design with pole placement constrains: An LMI approach. *IEEE Transactions on Automatic Control* 1996; **Vol. 41, No. 3**:358–367.
32. Peaucelle D, Arzelier D, Bachelier O, Bernussou J. A new robust D-stability condition for real convex polytopic uncertainty. *Systems & Control Letters* 2000; **40**:21–30.
33. Megretski A, Rantzer A. System analysis via integral quadratic constraints. *IEEE Transactions on Automatic Control* 1997; **Vol. 42, No. 6**:819–830.

DOI: 10.1002/rnc

34. Fu M, Dasgupta S, Sohc YC. Integral quadratic constraint approach vs. multiplier approach. *Automatica* 2005; **Vol. 41, Issue 2**:281287.
35. Feron E, Apkarian P, Gahinet P. Analysis and synthesis of robust control systems via parameter-dependent Lyapunov functions. *IEEE Transaction on Automatic Control* 1996; **Vol. 41, No. 7**:1041–1046.

DOI: 10.1002/rnc