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# Variable-Length Coding for Short Packets over a Multiple Access Channel with Feedback

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***Abstract*—We consider a two-user discrete memoryless multiple access channel with a common stop-feedback signal from the receiver to both transmitters. The achievable regions are characterized using joint decoding and successive cancellation decoding and, it is shown that the achievable regions are significantly larger for variable-length stop-feedback codes compared to fixed-blocklength codes. This is analogous to the result by Polyanskiy et al. (2011) for the point-to-point channel. An important conclusion is the following. In the asymptotic case the capacity region can be achieved by joint decoding, but also by successive cancellation decoding and time-sharing. For the case of finite blocklength, joint decoding performs significantly better than successive cancellation decoding, even when aiming for the corners of the achievable regions.**

## I. Introduction

Modern wireless networks are mainly based on reliable transmission of large packets through the use of good channel codes, as dictated by the asymptotic information-theoretic results. On the other hand, many emerging applications that involve machine-to-machine (M2M) communication rely on transmission of very short data packets within strict deadlines, where the asymptotic information-theoretic results are not applicable. The fundamentals of such a communication regime have recently been addressed in [1], where it was shown that the rates achievable by fixed-blocklength codes in point-to-point communication is tightly approximated in terms of the channel capacity and channel dispersion. Interestingly, [2] found that allowing the use of variable-length (VLF) coding based on stop-feedback improves the achievable rates dramatically, since the transmission may be terminated early for favorable noise realizations. In [3], [4], the results for fixed-blocklength codes are extended to the two-user multiple access channel.

We consider a two-user discrete memoryless multiple-access channel with stop-feedback. Each user has a message destined to the receiver and there is a common stop-feedback signal from the receiver to both transmitters after each channel use. This implies that the receiver can terminate the transmission, by feeding back an ACK signal, when it has decoded both messages from the receivers with a certain reliability $1-\epsilon$. Analogously to the achievability bound of VLF codes for the point-to-point channel in [2], we provide achievability bounds for VLF codes for the multiple access channel with stop-feedback based on joint decoding of both users and successive cancellation decoding. Note that variable-length coding for multiple access channel has previously been considered in [5] for another setting, in which the receiver is allowed to decode the messages from both transmitter at different random times.

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Using the achievability results, the achievable regions of the multiple access channels with stop-feedback are computed for specific channels and it is shown that VLF codes improves the achievable rates significantly compared to fixed-blocklength codes in [3]. Moreover, it is shown that joint decoding yields better achievable rates than successive cancellation decoding. This is in contrast to the achievability of the capacity region of the multiple access channel in the asymptotic regime, where one can equally well use either joint decoding or successive cancellation decoding with time sharing.

The paper is organized as follows. Section II describes the system model. Section III contains two achievability theorems, with proofs given in the appendix. Section IV presents numerical results and Section V concludes the paper.

## II. System Model

We consider a two-user discrete memoryless multiple-access channel (DM-MAC) with stop-feedback consisting of finite input alphabets $\mathcal{X}_1$ and $\mathcal{X}_2$, finite output alphabet $\mathcal{Y}$ and a conditional pmf $P_{Y|X_1,X_2}$ defining the channel, whose $n$-th extension is

$$P_{Y^n|X_1^n,X_2^n}(y^n|x_1^n,x_2^n)=\prod_{i=1}^{n}P_{Y|X_1,X_2}(y_i|x_{1i},x_{2i}). \quad (1)$$

There is a common stop-feedback signal from the receiver to both the transmitters after each channel use. That implies that the receiver can terminate the transmission, by feeding back an ACK signal, when it has decoded the messages from the receivers with a certain reliability. The system model is depicted on Fig. 1. An $(l, M_1, M_2, \epsilon)$ VLF code for the DM-MAC is defined as follows.

**Definition 1.** *An $(l, M_1, M_2, \epsilon)$ variable-length stop-feedback (VLF) code, where $l$ is a positive real, $M_1$ and $M_2$ are integers and $0 \le \epsilon \le 1$, consists of*

- *two message sets $\mathcal{M}_i = \{1, \dots, M_i\}$, $i \in \{1, 2\}$,*
- *two sequences of encoders $f_n^{(i)} : \mathcal{M}_i \to \mathcal{X}$, for $i \in \{1, 2\}$, such that the $i$-th encoder transmits $X_{in} = f_n^{(i)}(J_i)$ in the $n$-th channel use and $J_i \in \mathcal{M}_i$ denotes the message encoded by the $i$-th encoder, drawn uniformly from $\mathcal{M}_i$,*
- *a sequence of decoders $g_n : \mathcal{Y}^n \to \mathcal{M}_1 \times \mathcal{M}_2$ that assign estimates $(\hat{J}_1, \hat{J}_2)$ to each received sequence $Y^n$,*

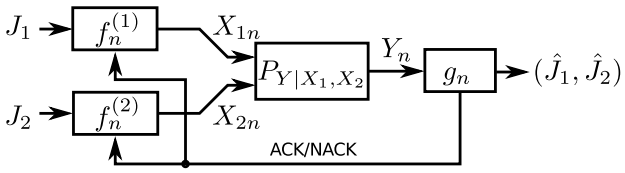



Fig. 1. The multiple access channel with ACK/NACK feedback. $(\hat{J}_1, \hat{J}_2)$ denotes the output of the decoder $g_n$.

- *a stopping time $\tau \in \mathbb{Z}_+$ that depends on $Y^n$, but is independent of $X_{in}$ and satisfies $\mathbb{E}[\tau] \leq l$,*

*such that average probability of error satisfies*

$$Pr\left[g_\tau(Y^\tau) \neq (J_1, J_2)\right] \leq \epsilon. \tag{2}$$

Additionally, a source of common randomness is needed for codebook generation at the users. The above definition implies that the rate pair $(\log(M_1)/l, \log(M_2)/l)$ is achievable with an average blocklength $l$ and average probability of error $\epsilon$ if and only if an $(l, M_1, M_2, \epsilon)$ VLF code exists.

As in [2], the random stopping time corresponds to the time until an ACK signal is fed back and is thereby only allowed to depend on the received symbols $Y^\tau$. In [2], the point-to-point channel with full noiseless feedback is addressed, but their achievable scheme only utilizes stop-feedback. In our work we only allow stop-feedback since if full noiseless feedback is allowed for the DM-MAC, the capacity region is enlarged since the encoders may cooperate [6]. In that case the capacity region is only known for a few channels.

Throughout the paper, the information densities are denoted as

$$\imath(X_1; Y|X_2) = \log \frac{P_{Y|X_1,X_2}(Y|X_1, X_2)}{P_{Y|X_2}(Y|X_2)}, \tag{3}$$

$$\imath(X_2; Y|X_1) = \log \frac{P_{Y|X_1,X_2}(Y|X_1, X_2)}{P_{Y|X_1}(Y|X_1)}, \tag{4}$$

$$\imath(X_1, X_2; Y) = \log \frac{P_{Y|X_1,X_2}(Y|X_1, X_2)}{P_Y(Y)}, \tag{5}$$

where $\log(\cdot)$ denotes the natural logarithm.

For the DM-MAC without stop-feedback, [3] uses the dependency testing bound to show that there exists fixed-blocklength codes with an achievable regions of rate pairs $(R_1(n,\epsilon), R_2(n,\epsilon))$ satisfying

$$\begin{aligned} R_1(n,\epsilon) &\leq I(X_1; Y|X_2) \\ &\quad - \sqrt{V(X_1, Y; X_2)/n}\, Q^{-1}(\lambda_1 \epsilon) + \mathcal{O}(1), \end{aligned} \tag{6}$$

$$\begin{aligned} R_2(n,\epsilon) &\leq I(X_2; Y|X_1) \\ &\quad - \sqrt{V(X_2, Y; X_1)/n}\, Q^{-1}(\lambda_2 \epsilon) + \mathcal{O}(1), \end{aligned} \tag{7}$$

$$\begin{aligned} R_1(n,\epsilon) + R_2(n,\epsilon) &\leq I(X_1, X_2; Y) \\ &\quad - \sqrt{V(X_1, X_2; Y)/n}\, Q^{-1}(\lambda_3 \epsilon) + \mathcal{O}(1). \end{aligned} \tag{8}$$

for some joint distribution $P_{X_1}(x_1)P_{X_2}(x_2)P_{Y|X_1,X_2}(y|x_1,x_2)$ where $Q^{-1}(\cdot)$ is the inverse Q-function, $\lambda_1, \lambda_2, \lambda_3$ are positive constants satisfying $\lambda_1 + \lambda_2 + \lambda_3 = 1$, $I(X_1; Y|X_2)$, $I(X_2; Y|X_1)$ and $I(X_1, X_2; Y)$ denotes the mutual informations and $V(X_1, Y; X_2)$, $V(X_2, Y; X_1)$ and $V(X_1, X_2; Y)$ are given by

$$V(X_1, Y; X_2) = \mathrm{Var}\left[\imath(X_1; Y|X_2)\right], \tag{9}$$

$$V(X_2, Y; X_1) = \mathrm{Var}\left[\imath(X_1; Y|X_1)\right], \tag{10}$$

$$V(X_1, X_2; Y) = \mathrm{Var}\left[\imath(X_1, X_2; Y)\right]. \tag{11}$$

## III. Achievability

In this section, two achievability theorems are presented. The first achievability bound is analogous to the point-to-point channel in [2] and is based on joint decoding, while the second is based on successive cancellation decoding.

The achievability bound based on joint decoding is given in the following theorem.

**Theorem 1.** *For any positive real constants $\gamma_1, \gamma_2, \gamma_{12}$ and a channel $P_{Y|X_1,X_2}$ define the stopping times*

$$\begin{aligned} \tau = \inf\{n \geq 0 : &\imath(X_1^n; Y^n|X_2^n) \geq \gamma_1, \\ &\imath(X_2^n; Y^n|X_1^n) \geq \gamma_2, \imath(X_1^n, X_2^n; Y^n) \geq \gamma_{12}\}, \end{aligned} \tag{12}$$

$$\overline{\tau}_1 = \inf\left\{n \geq 0 : \imath(\overline{X}_1^n; Y^n|X_2^n) \geq \gamma_1\right\}, \tag{13}$$

$$\overline{\tau}_2 = \inf\left\{n \geq 0 : \imath(\overline{X}_2^n; Y^n|X_1^n) \geq \gamma_2\right\}, \tag{14}$$

$$\overline{\tau}_{12} = \inf\left\{n \geq 0 : \imath(\overline{X}_1^n, \overline{X}_2^n; Y^n) \geq \gamma_{12}\right\}, \tag{15}$$

*where $X_{1i}$, $X_{2i}$, $\overline{X}_{1i}$, $\overline{X}_{2i}$, $Y_i$ are distributed according to the joint distribution $P_{X_1\overline{X}_1X_2\overline{X}_2Y}(x_1, \overline{x}_1, x_2, \overline{x}_2, y) = P_{X_1}(x_1)P_{X_1}(\overline{x}_1)P_{X_2}(x_2)P_{X_2}(\overline{x}_2)P_{Y|X_1,X_2}(y|x_1,x_2)$ for $\overline{x}_1 \in \mathcal{X}_1, \overline{x}_2 \in \mathcal{X}_2$. Then, for any pair of positive integers $(M_1, M_2)$, there exists an $(l, M_1, M_2, \epsilon)$ VLF code for the DM-MAC such that*

$$l \leq \mathbb{E}[\tau] \tag{16}$$

$$\begin{aligned} \epsilon \leq\ &(M_1 - 1)(M_2 - 1)Pr\left[\tau \geq \overline{\tau}_{12}\right] \\ &+ (M_1 - 1)Pr\left[\tau \geq \overline{\tau}_1\right] + (M_2 - 1)Pr\left[\tau \geq \overline{\tau}_2\right]. \end{aligned} \tag{17}$$

*Proof:* See Appendix A. ■

In a communication system, $\tau$ represents the time until the transmitted codewords are reliably detected at the receiver, i.e. when the log-likelihood ratios of the codewords, equivalent to the information density terms, surpasses the thresholds $\gamma_1, \gamma_2, \gamma_{12}$. Similarly, $\overline{\tau}_1$, $\overline{\tau}_2$ and $\overline{\tau}_{12}$ represent the time until the receiver would detect a pair of incorrect codewords. An error occurs if an incorrect pair of codewords is reliably detected before the correct pair of codewords. Note that when $\gamma_2 \to -\infty$, $\gamma_{12} \to -\infty$ and $M_2 = 1$, Theorem 1 reduces to Theorem 3 in [2] for the channel $X_1 - Y$ where the codeword from user 2 is known.

Joint decoding may not always feasible due to high complexity. Therefore we also provide a corresponding achievability bound based on successive cancellation decoding, i.e. user 1 is first decoded, then user 2 is decoded knowing the codeword of users 1. In the asymptotic regime the capacity region can be achieved using time-sharing.

**Theorem 2.** *For any positive real constants $\gamma_1, \gamma_2$ and a channel $P_{Y|X_1,X_2}$ define the stopping times*

$$\tau = \inf\{n \geq 0 : \imath(X_1^n; Y^n) \geq \gamma_1, \imath(X_2^n; Y^n|X_1^n) \geq \gamma_2\} \tag{18}$$

$$\overline{\tau}_1 = \inf\left\{n \geq 0 : \imath(\overline{X}_1^n; Y^n) \geq \gamma_1\right\} \tag{19}$$

$$\overline{\tau}_2 = \inf\left\{n \geq 0 : \imath(\overline{X}_2^n; Y^n|X_1^n) \geq \gamma_1\right\} \tag{20}$$

*where $X_{1i}$, $X_{2i}$, $\overline{X}_{1i}$, $\overline{X}_{2i}$, $Y_i$ are distributed according to the joint distribution $P_{X_1\overline{X}_1X_2\overline{X}_2Y}(x_1, \overline{x}_1, x_2, \overline{x}_2, y) = P_{X_1}(x_1)P_{X_1}(\overline{x}_1)P_{X_2}(x_2)P_{X_2}(\overline{x}_2)P_{Y|X_1,X_2}(y|x_1, x_2)$ for $\overline{x}_1 \in \mathcal{X}_1, \overline{x}_2 \in \mathcal{X}_2$. Then, for any pair of positive integers $(M_1, M_2)$, there exists an $(l, M_1, M_2, \epsilon)$ VLF code for the DM-MAC such that*

$$l \leq \mathbb{E}[\tau] \tag{21}$$

$$\epsilon \leq (M_1 - 1)Pr[\tau \geq \overline{\tau}_1] + (M_2 - 1)Pr[\tau \geq \overline{\tau}_2]. \tag{22}$$

*Proof:* The procedure is similar to the proof of Theorem 1. ■

A bound similar to Theorem 2 can be found for $\imath(X_2^n; Y^n)$ and $\imath(X_1^n; Y^n|X_2^n)$, and hence, as in traditional network information theory literature, time-sharing can be used to achieve any rate pairs in the convex hull of the two achievable regions [6]. Note that Theorem 1 and Theorem 2 can readily be extended to include a time-sharing variable as in [3].

## IV. Numerical Results

In this section, the evaluation of the bounds in Theorem 1 and Theorem 2 is discussed, and the achievable regions of a specific channel are computed for a range of average blocklengths $l$.

### A. Evaluation

In Theorem 1, each triple $(\gamma_1, \gamma_2, \gamma_{12})$ yields an achievable region with a certain average blocklength. To evaluate the full achievable region of the DM-MAC with stop-feedback, we need to compute the union of achievable regions obtained for all choices of triples $(\gamma_1, \gamma_2, \gamma_{12})$ with average blocklength less than or equal $l$. This is depicted in Fig. 2, where the achievable regions obtained from Theorem 1 are plotted for a range of triples $(\gamma_1, \gamma_2, \gamma_{12})$ with average blocklength less than or equal 100. To evaluate each point on the border of the achievable region, a constant $\alpha \in (0, 1)$ is fixed and $\alpha \log_2(M_1 - 1) + (1 - \alpha)\log_2(M_2 - 1)$ is maximized with respect to $\gamma_1, \gamma_2$ and $\gamma_{12}$ under the constraint $\mathbb{E}[\tau] \leq l$

The main difficulty in the computation of the achievable region of Theorem 1 is to compute the average blocklength and the distribution of $\tau$. For these operations, we need to deal with the joint distribution of $\imath(X_1^n; Y^n|X_2^n)$, $\imath(X_2^n; Y^n|X_1^n)$ and $\imath(X_1^n, X_2^n; Y^n)$ conditioned on $\tau \geq n$. In this work, we have done this by discretizing the joint pmf of $\imath(X_1; Y|X_2)$, $\imath(X_2; Y|X_1)$ and $\imath(X_1, X_2; Y)$. In the $n$-th step, we use a three dimensional convolution to obtain the joint pmf of the $(n+1)$-th step. The probability mass corresponding to the decoding event in the $(n + 1)$-th step is then subtracted.

As in [2], the evaluation of the probability of error in (17) can be simplified if $\imath(\overline{X}_1^n, \overline{X}_2^n; Y^n)$, $\imath(\overline{X}_1^n; Y^n|X_2^n)$ and $\imath(\overline{X}_2^n; Y^n|X_1^n)$ are independent of $(X_1^n, X_2^n, Y^n)$ using the identity

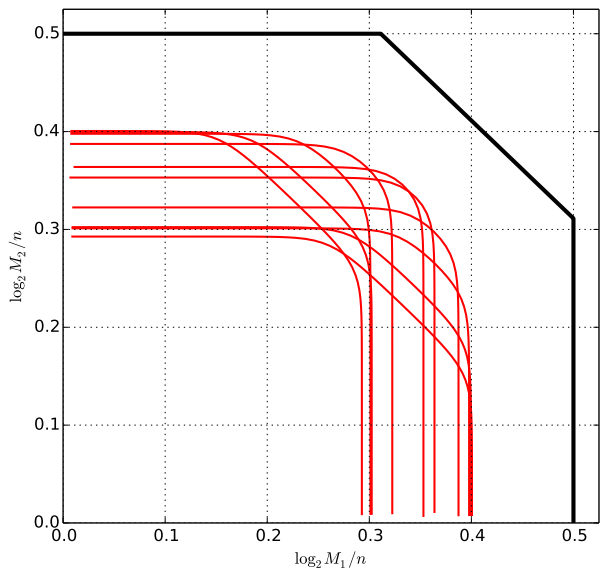



Fig. 2. Achievable regions of Theorem 1 with the channel in (31) for $\epsilon = 10^{-3}$ and $\delta = 0.5$. Each (red) region corresponds to a specific choice of $\gamma_1, \gamma_2, \gamma_{12}$ with average blocklength less than or equal 100. The full achievable region is given by the union of all such regions.

$$\mathbb{E}\left[f(\overline{A}^n, B^n)\right] = \mathbb{E}\left[f(A^n, B^n)\exp\{-\imath(A^n; B^n)\}\right], \tag{23}$$

which is valid for any measurable function $f$ [2], where $A$, $\overline{A}$ and $B$ are random variables with joint pmf $P_{A,\overline{A},B}(a, \overline{a}, b) = P_A(a)P_A(\overline{a})P_{B|A}(b|a)$. In that case (17) can be written as

$$\begin{aligned}\epsilon \leq\ & (M_1 - 1)(M_2 - 1)\mathbb{E}[f_{12}(\tau)] \\ & + (M_1 - 1)\mathbb{E}[f_1(\tau)] + (M_2 - 1)\mathbb{E}[f_2(\tau)]\end{aligned} \tag{24}$$

where

$$f_{12}(n) = \mathbb{E}\left[\mathbb{1}\{\tau_{12} \leq n\}\exp\{-\imath(X_1^{\tau_{12}}, X_2^{\tau_{12}}; Y^{\tau_{12}})\}\right] \tag{25}$$

$$f_1(n) = \mathbb{E}\left[\mathbb{1}\{\tau_1 \leq n\}\exp\{-\imath(X_1^{\tau_1}; Y^{\tau_1}|X_2^{\tau_1})\}\right] \tag{26}$$

$$f_2(n) = \mathbb{E}\left[\mathbb{1}\{\tau_2 \leq n\}\exp\{-\imath(X_2^{\tau_2}; Y^{\tau_2}|X_1^{\tau_2})\}\right], \tag{27}$$

where (25) follows by setting $A = (X_1, X_2)$ and $B = Y$ in (23), (26) by setting $A = X_1$ and $B = (X_2, Y)$, and (27) by setting $A = X_2$ and $B = (X_1, Y)$.

When independence does not hold, the terms of the probability of error $\Pr[\overline{\tau}_{12} \leq \tau]$, $\Pr[\overline{\tau}_1 \leq \tau]$, $\Pr[\overline{\tau}_2 \leq \tau]$ can be loosened as following [2, Proof of Theorem 32]

$$\Pr[\overline{\tau}_{12} \leq \tau] \leq \exp\{-\gamma_{12}\}, \tag{28}$$

$$\Pr[\overline{\tau}_1 \leq \tau] \leq \exp\{-\gamma_1\}, \tag{29}$$

$$\Pr[\overline{\tau}_2 \leq \tau] \leq \exp\{-\gamma_2\}. \tag{30}$$

### B. Results

The achievable regions of Theorem 1 and Theorem 2 are evaluated for the following channel (as in [3])

$$Y = X_1 + X_2 + Z, \tag{31}$$

where $+$ denotes the real addition and $X_1, X_2 \in \mathcal{X} = \{0,1\}$ are binary channel inputs distributed as Bern$(\frac{1}{2})$, and $Z$ is binary noise distributed as Bern$(\delta)$, and hence $Y \in \{0,1,2,3\}$. For this channel $(\imath(X_1^n;Y^n|X_2^n), \imath(X_2^n;Y^n|X_1^n), \imath(X_1^n,X_2^n;Y^n))$ becomes a three dimensional random walk. Moreover, $(\imath(\overline{X}_1^n;Y^n|X_2^n), \imath(\overline{X}_2^n;Y^n|X_1^n), \imath(\overline{X}_1^n,\overline{X}_2^n;Y^n))$ does depend on $(X_1^n, X_2^n, Y^n)$, thus the probability of error is computed using the loosened bounds (28)-(30).

In Fig. 3(a) and Fig. 3(b), the achievable regions characterized by Theorem 1 and Theorem 2 are plotted for $\delta = 0.5$ and $\delta = 0.11$. The regions are compared to the capacity regions and the rate regions achievable by fixed-blocklength codes given in (6)-(8) from [3] disregarding the $\mathcal{O}(1)$ terms.

It is seen that rates significantly higher than for fixed-blocklength codes are achievable and around 90% of the capacity regions is achieved at blocklengths of $100 - 200$ channel uses for the channel at hand. Finally, we observe that there is a significant gain in using joint decoding as opposed to successive cancellation decoding, even when aiming for the corner points of the achievable region.

## V. Discussion and Conclusions

We have put forth rate regions achievable for the multiple access channel with stop-feedback which are significantly better than those achievable for fixed-blocklength codes. Moreover, we have found that there is a significant penalty of using successive cancellation decoding instead of joint decoding.

The results show that the achievable regions of Theorem 1 and Theorem 2 are considerable more rounded than for fixed-blocklength codes. A possible explanation for this observation is that the average blocklength $\mathbb{E}[\tau]$ is dominated by the information density term reaching the corresponding threshold last. Note that $\tau$ in (12) can also be written as

$$\tau = \max(\tau_1, \tau_2, \tau_{12}), \tag{32}$$

where

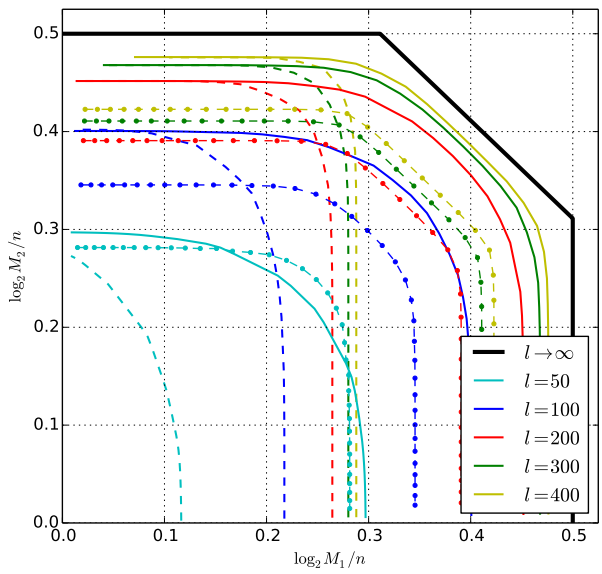
$$\tau_1 = \inf\{n \geq 0 : \imath(X_1^n;Y^n|X_2^n) \geq \gamma_1\} \tag{33}$$
$$\tau_2 = \inf\{n \geq 0 : \imath(X_2^n;Y^n|X_1^n) \geq \gamma_2\} \tag{34}$$
$$\tau_{12} = \inf\{n \geq 0 : \imath(X_1^n,X_2^n;Y^n) \geq \gamma_{12}\}. \tag{35}$$

The expected stopping times of (33)-(35) can be shown to satisfy [2, Proof of Theorem 2]

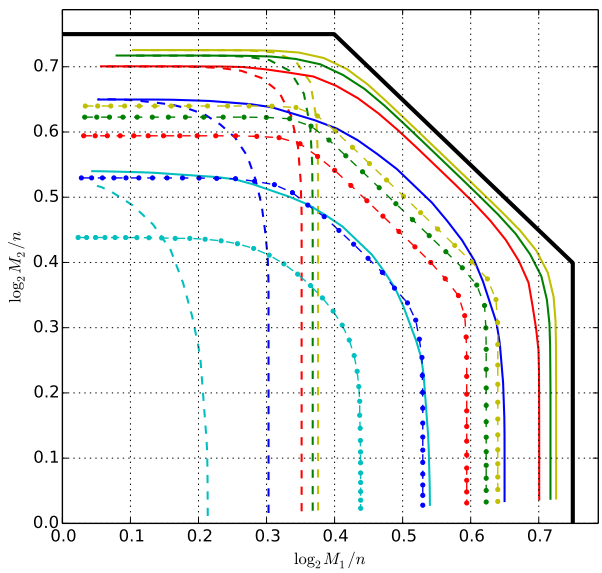
$$\mathbb{E}[\tau_1] \leq \frac{\gamma_1}{I(X_1;Y|X_2)} + a_1 \tag{36}$$
$$\mathbb{E}[\tau_2] \leq \frac{\gamma_2}{I(X_2;Y|X_1)} + a_2 \tag{37}$$
$$\mathbb{E}[\tau_{12}] \leq \frac{\gamma_{12}}{I(X_1,X_2;Y)} + a_{12} \tag{38}$$

for some positive constants $a_1, a_2, a_{12}$. Hence $\mathbb{E}[\tau]$ is equal to $\max(\mathbb{E}[\tau_1], \mathbb{E}[\tau_2], \mathbb{E}[\tau_{12}])$ plus some penalty term that depends on the correlation between the stopping times $\tau_1, \tau_2, \tau_{12}$. This penalty term is expected to be most significant when the corner points are approached.

In [7], it was found that rate-compatible convolutional codes combined with reliablity-based decoding achieved rates that surpassing the achievability bound for point-to-point VLF codes in [2, Theorem 3]. An interesting extension is therefore to consider the achievable rates of rate-compatible convolutional codes with joint decoding, e.g. based on message passing algorithms, for the multiple access channel with stop-feedback.




(a) $\delta = 0.5$




(b) $\delta = 0.11$. Legend is the same as for (a).

Fig. 3. Achievable regions with $\epsilon = 10^{-3}$. Solid (colored): Theorem 1, dashed: Theorem 2, dashed-dotted: fixed-blocklength codes, (6)-(8).

## Appendix A
## Proof of Theorem 1

*Proof:* The proof extends the proof of Theorem 3 in [2] to the multiple access channel.
*Codebook generation:* Based on common randomness, gener-

ate infinite dimensional vectors $\mathbf{c}_{j_1}^{(1)} \in \mathcal{X}^\infty$ and $\mathbf{c}_{j_2}^{(2)} \in \mathcal{X}^\infty$, $j_1 \in \mathcal{M}_1$ and $j_2 \in \mathcal{M}_2$, according to the pmfs $P_{X_1}$ and $P_{X_2}$, respectively.

*Encoders:* The $i$-th user uses the encoder

$$X_{in} = f_n^{(i)}(j_i) = (\mathbf{c}_{j_i}^{(i)})_n, \tag{39}$$

where $(\mathbf{c}_{j_i}^{(i)})_n$ is the $n$-th entry in $\mathbf{c}_{j_i}^{(i)}$. This is done until an ACK signal is received.

*Decoder:* At the $n$-th channel use, the decoder computes the $M_1 M_2$ information densities

$$S_{j_1,j_2}^{(1)}(n) = \imath(\mathbf{c}_{j_1}^{(1)}(n); Y^n | \mathbf{c}_{j_2}^{(2)}(n)) \tag{40}$$
$$S_{j_1,j_2}^{(2)}(n) = \imath(\mathbf{c}_{j_1}^{(2)}(n); Y^n | \mathbf{c}_{j_2}^{(1)}(n)) \tag{41}$$
$$S_{j_1,j_2}^{(12)}(n) = \imath(\mathbf{c}_{j_1}^{(1)}(n), \mathbf{c}_{j_1}^{(2)}(n); Y^n), \tag{42}$$

where $\mathbf{c}_{j_i}^{(i)}(n)$ denotes the first $n$ entries of $\mathbf{c}_{j_i}^{(i)}$. Define the stopping times

$$\tau_{j_1,j_2} = \inf\{n \geq 0 : S_{j_1,j_2}^{(12)}(n) \geq \gamma_1, S_{j_1,j_2}^{(2)}(n) \geq \gamma_2 , S_{j_1,j_2}^{(12)}(n) \geq \gamma_{12}\} \tag{43}$$

for $j_1 \in \mathcal{M}_1, j_2 \in \mathcal{M}_2$. The receiver feds back an ACK signal as soon as $S_{j_1,j_2}^{(12)}(n) \geq \gamma_1, S_{j_1,j_2}^{(2)}(n) \geq \gamma_2$ and $S_{j_1,j_2}^{(12)}(n) \geq \gamma_{12}$ for some pair $(j_1, j_2)$. The decoding time is hence given by

$$\tau^* = \min_{j_1 \in \mathcal{M}_1, j_2 \in \mathcal{M}_2} \tau_{j_1,j_2} \tag{44}$$

The output of the decoder is given by

$$g(Y^{\tau^*}) = \max\{(j_1, j_2) : \tau_{j_1,j_2} = \tau^*\}, \tag{45}$$

where the maximum is in calligraphic order.

The average transmission length satisfies

$$\mathbb{E}[\tau^*] \overset{(a)}{\leq} \frac{1}{M_1 M_2} \sum_{j_1=1}^{M_1} \sum_{j_2=1}^{M_2} \mathbb{E}[\tau_{j_1,j_2} | J_1 = j_1, J_2 = j_2] \tag{46}$$
$$\overset{(b)}{=} \mathbb{E}[\tau_{j_1,j_2} | J_1 = j_1, J_2 = j_2] \tag{47}$$
$$\overset{(c)}{=} \mathbb{E}[\tau], \tag{48}$$

where (a) follows since $\min_{j_1,j_2} \tau_{j_1,j_2} \leq \tau_{J_1,J_2}$, (b) by symmetry of the codebook and (c) by the definition of $\tau$ in (12).

The average probability of error is bounded as following

$$\Pr\left[g(Y^{\tau^*}) \neq (J_1, J_2)\right]$$
$$\overset{(a)}{\leq} \Pr\left[g(Y^{\tau^*}) \neq (1,1) | J_1 = 1, J_2 = 1\right] \tag{49}$$
$$\overset{(b)}{\leq} \Pr[\tau_{1,1} \geq \tau^* | J_1 = 1, J_2 = 1] \tag{50}$$
$$\leq \Pr\left[\bigcup_{\substack{M_1 \in \mathcal{M}_1, M_2 \in \mathcal{M}_2 \\ (M_1,M_2) \neq (1,1)}} \{\tau_{j_1,j_2} \leq \tau_{1,1}\} \big| J_1 = 1, J_2 = 1\right] \tag{51}$$
$$\leq \Pr\left[\bigcup_{j_1=2}^{M_1} \bigcup_{j_2=2}^{M_2} \{\tau_{j_1,j_2} \leq \tau_{1,1}\} \cup \bigcup_{j_1=2}^{M_1} \{\tau_{j_1,1} \leq \tau_{1,1}\} \cup \bigcup_{j_2=2}^{M_2} \{\tau_{1,j_2} \leq \tau_{1,1}\} \big| J_1 = 1, J_2 = 1\right] \tag{52}$$
$$\overset{(c)}{\leq} (M_1 - 1)(M_2 - 1)\Pr[\tau_{2,2} \leq \tau_{1,1} | (J_1, J_2) = (1,1)] + (M_1 - 1)\Pr[\tau_{2,1} \leq \tau_{1,1} | (J_1, J_2) = (1,1)] + (M_2 - 1)\Pr\left[\tau_{1,2} \leq \tau_{1,1} \big| (J_1, J_2) = (1,1)\right], \tag{53}$$

where (a) is by (45), i.e. the highest error probability is for the $J_1 = 1, J_2 = 1$, (b) follows from (44), (c) is by the union bound. Note that the random stopping times $\tau_{2,1}, \tau_{1,2}, \tau_{2,2}$ conditioned on $J_1 = 1, J_2 = 1$ have the same distributions as

$$\overline{\tau}_1' = \inf(n \geq 0 : \imath(\overline{X}_1^n; Y^n | X_2^n) \geq \gamma_1, \imath(X_2^n; Y^n | \overline{X}_1^n) \geq \gamma_2 , \imath(\overline{X}_1^n, X_2^n; Y^n) \geq \gamma_{12}) \tag{54}$$
$$\overline{\tau}_2' = \inf(n \geq 0 : \imath(X_1^n; Y^n | \overline{X}_2^n) \geq \gamma_1, \imath(\overline{X}_2^n; Y^n | X_1^n) \geq \gamma_2 , \imath(X_1^n, \overline{X}_2^n; Y^n) \geq \gamma_{12}) \tag{55}$$
$$\overline{\tau}_{12}' = \inf(n \geq 0 : \imath(\overline{X}_1^n; Y^n | \overline{X}_2^n) \geq \gamma_1, \imath(\overline{X}_2^n; Y^n | \overline{X}_1^n) \geq \gamma_2 , \imath(\overline{X}_1^n, \overline{X}_2^n; Y^n) \geq \gamma_{12}), \tag{56}$$

respectively. We obtain (17) from the fact that $\overline{\tau}_1' \geq \overline{\tau}_1$,$\overline{\tau}_2' \geq \overline{\tau}_2$ and $\overline{\tau}_{2,2}' \geq \overline{\tau}_{12}$, where $\overline{\tau}_1$, $\overline{\tau}_2$ and $\overline{\tau}_{12}$ are given in (13)-(15).

■

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