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Abstract
We show that the space of directed paths on the $k$-skeleton of the $n$-cube is homotopy equivalent to the nerve of a certain category of flags of finite sets.

1 Introduction

In [D] Dijkstra introduced the so-called PV-model for linear concurrent programs with semaphores. The space of execution paths of such a program is a directed space. One can study it using the methods of directed algebraic topology introduced by Fajstrup, Goubault, Grandis and Raussen (see [FRG], [G]).

We will consider the following problem: Assume that $s$ is a semaphore (that is a shared resource) of capacity $k$. Run $n$ copies of the program $P_s V_s$ in parallel $P_s V_s || P_s V_s || \ldots || P_s V_s$. What is the homotopy type of the associated space of execution paths?

Let us describe the space of execution paths precisely. Let $I = [0, 1]$ denote the unit interval. In our setting, a directed path is a continuous curve $\gamma : I \to I^n$ with weakly increasing coordinate functions. The semaphore $s$ is modeled by open subintervals $I_i = ]a_i, b_i[ \subseteq I$ for $1 \leq i \leq n$. A point $x$ in $I^n$ is forbidden if the set $\{i | x_i \in I_i\}$ has more than $k$ elements. Thus the state space of the program is

$$X^n_{(k)} = \{ x \in I^n | \# \{ i | x_i \in I_i \} \leq k \}$$

and we want to study the space of directed paths form $\bar{0}$ to $\bar{T}$ in this state space, or its homotopy equivalent trace space consisting of directed paths up to reparametrizations

$$\tilde{\mathcal{P}}(X^n_{(k)})(\bar{0}, \bar{T}) \simeq \tilde{T}(X^n_{(k)})(\bar{0}, \bar{T}).$$

When $k = 0$, the trace space is empty. When $k \geq n$, we have a contractible trace space. So we assume that $1 \leq k \leq n - 1$, since these are the nontrivial cases. Note that for $k = n - 1$ the trace space is homotopy equivalent to the sphere $S^{n-2}$ since this is a special case of [R].
2 A covering of the state space

The $k$-skeleton of the $n$-cube is the special state space where $a_i = 0$ and $b_i = 1$ for all $i$:

$$I^n_k = \{ \bar{x} \in I^n | \# \{ i | 0 < x_i < 1 \} \leq k \}.$$  

By a continuous deformation of each factor of $I^n$ we obtain:

**Proposition 2.1.** The inclusion $I^n_k \hookrightarrow X^n_k$ induces a homotopy equivalence of trace spaces

$$\overline{T}(I^n_k)(0, 1) \simeq \overline{T}(X^n_k)(0, 1).$$

**Proof.** For $0 < a < b < 1$ we define a continuous map $H_{(a, b)} : I \times I \to I$ by

$$H_{(a, b)}(s, x) = \begin{cases} 
0, & 0 \leq x \leq sa, \\
\frac{x - sa}{sb - x + 1}, & sa \leq x \leq sb + 1 - s, \\
1, & sb + 1 - s \leq x \leq 1.
\end{cases}$$

If we fix $s \in I$ to any value we have a weakly increasing map of the variable $x$. So we can write $H_{(a, b)} : I \times I \rightarrow I$. Define $r_{(a, b)} : I \rightarrow I$ by $r_{(a, b)}(x) = H_{(a, b)}(1, x)$ and observe that $H_{(a, b)}(0, x) = x$.

We now form Cartesian products of these types of maps

$$H : I \times \bar{I}^n \to \bar{I}^n; \quad H(s, \bar{x}) = (H_{(a_1, b_1)}(s, x_1), \ldots, H_{(a_n, b_n)}(s, x_n)),
$$

$$r : \bar{I}^n \to \bar{I}^n; \quad r(\bar{x}) = H(1, \bar{x}) = (r_{(a_1, b_1)}(x_1), \ldots, r_{(a_n, b_n)}(x_n)),$$

so that $H$ is a homotopy from $id_{\bar{I}^n}$ to $r$.

Note that $r$ restricts to a map $\tilde{r} : X^n_k \to I^n_k$ and let $\iota : I^n_k \to X^n_k$ denote the inclusion. We also have restrictions $H_1$ and $H_2$ of the homotopy $H$ with the following properties:

$$H_1 : I \times I^n_k \to I^n_k, \quad H_1(0, \bar{x}) = \bar{x}, \quad H_1(1, \bar{x}) = \tilde{r} \circ \iota(x),
$$

$$H_2 : I \times X^n_k \to X^n_k, \quad H_2(0, \bar{x}) = \bar{x}, \quad H_2(1, \bar{x}) = \iota \circ \tilde{r}(x).$$

Thus we have a $d$-homotopy equivalence $I^n_k \simeq X^n_k$. Via precomposition we get a homotopy equivalence of associated directed path spaces and hence a homotopy equivalence of trace spaces.

We will now cover $I^n_k$ by appropriate subsets, such that the nerve lemma can be applied in a similar fashion as in [R]. Let $[1 : m]$ denote the set of integers from 1 to $m$.

**Definition 2.2.** For non-negative integers $j$ and $i_1, i_2, \ldots, i_r \in [1 : n]$ we define the following (possibly empty) subspace of the $n$-cube:

$$X_{i_1, i_2, \ldots, i_r}^{#j} = \{ \bar{x} \in I^n | x_{i_1} = x_{i_2} = \cdots = x_{i_r} = 0 \text{ and } \# \{ i | x_i = 1 \} \geq j \}.$$  

Furthermore, $X_{i_1, i_2, \ldots, i_r}^{#j}$ denotes the subset of the $n$-cube consisting of points with at least $j$ coordinates equal to 1 and $X_{i_1, i_2, \ldots, i_r}^{#0} = X_{i_1, i_2, \ldots, i_r}^{#0}$. 

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Notation 2.3. \( \text{Inj}_m^n \) denotes the set of injective maps from \([1 : m]\) to \([1 : n]\).

**Definition 2.4.** For \( \alpha \in \text{Inj}_{n-k}^n \) we define the subspace \( \mathcal{U}_\alpha \subseteq I_n^{(k)} \) by

\[
\mathcal{U}_\alpha = X_{\alpha(1)}^{\#(n-k)} \cup X_{\alpha(1),\alpha(2)}^{\#(n-k-1)} \cup \cdots \cup X_{\alpha(1),\alpha(2),\ldots,\alpha(n-k)}^{\#(n-k-k)}.
\]

**Proposition 2.5.** Let \( \lor \) denote the coordinate-wise maximum operation on \( \mathbb{R}^n \) and let \( \land \) denote the minimum operation on the integers. For all \( \alpha \in \text{Inj}_{n-k}^n \) and \( r, s \) there is an inclusion

\[
X_{\alpha(1),\ldots,\alpha(r)}^{\#(n-k-r)} \lor X_{\alpha(1),\ldots,\alpha(s)}^{\#(n-k-s)} \subseteq X_{\alpha(1),\ldots,\alpha(r \land s)}^{\#(n-k-r \land s)},
\]

Thus \( \mathcal{U}_\alpha \) is \( \lor \)-closed and so is any intersection of \( \mathcal{U}_\alpha \)’s.

**Proof.** Assume that \( \overline{\pi} \in X_{\alpha(1),\ldots,\alpha(r)}^{\#(n-k-r)} \) and \( \overline{\gamma} \in X_{\alpha(1),\ldots,\alpha(s)}^{\#(n-k-s)} \). Then the point \( \overline{\pi} \lor \overline{\gamma} \) has at least \( (n-k-r) \lor (n-k-s) = n-k-r \land s \) coordinates which are equal to 1 and \( (\overline{\pi} \lor \overline{\gamma})_{\alpha(1)} = \cdots = (\overline{\pi} \lor \overline{\gamma})_{\alpha(r \land s)} = 0 \).

By Proposition 2.8 of [R] we have the following:

**Corollary 2.6.** The trace space

\[
\overline{T}(\bigcap_{\alpha \in A} \mathcal{U}_\alpha)(\overline{0}, \overline{1})
\]

is either empty or contractible for every nonempty family of injections \( A \subseteq \text{Inj}_{n-k}^n \).

**Theorem 2.7.**

\[
\overline{T}(I_n^{(k)})(\overline{0}, \overline{1}) = \bigcup_{\alpha \in \text{Inj}_{n-k}^n} \overline{T}(\mathcal{U}_\alpha)(\overline{0}, \overline{1})
\]

**Proof.** For \( K, L \subseteq [1 : n] \) we introduce the notation

\[
\mathcal{U}_K^L = \{ \overline{\pi} \in I_n^L \mid \forall i \in K : x_i = 0, \forall j \in L : x_j = 1 \}.
\]

Let \( \gamma \in \overline{T}(I_n^{(k)})(\overline{0}, \overline{1}) \). Choose \( t_1 > 0 \) minimal such that \( \exists j : \gamma_j(t_1) = 1 \). Define sets

\[
J_1 = \{ j \in [1 : n] \mid \gamma_j(t_1) = 1 \},
\]

\[
K_1 = \{ \ell \in [1 : n] \mid \gamma_\ell(t) = 0 \text{ for } t < t_1 \}.
\]

We have

\[
n - k \leq |K_1|, \quad 1 \leq |J_1| \leq k, \quad K_1 \cap J_1 = \emptyset.
\]

Inductively, choose \( t_i > t_{i-1} \) minimal such that \( \gamma_j(t_i) = 1 \) for some \( j \in [1 : n] \) with \( j \notin J_1 \cup \cdots \cup J_{i-1} \). Define sets

\[
J_i = \{ j \in [1 : n] \mid \gamma_j(t_i) = 1, \quad j \notin J_1 \cup \cdots \cup J_{i-1} \},
\]

\[
K_i = \{ \ell \in [1 : n] \mid \gamma_\ell(t) = 0 \text{ for } t < t_i \}.
\]

We have

\[
n - k \leq |K_i| + |J_1 \cup \cdots \cup J_{i-1}|, \quad 1 \leq |J_i| \leq k, \quad K_i \cap (J_1 \cup \cdots \cup J_i) = \emptyset.
\]
Stop at \( t_m > t_{m-1} \) when \( \gamma_i(t) = 1 \) for all \( t \geq t_m \) and all \( j \in [1 : n] \). Note that \( m \leq n \).

We now have \( K_1 \supseteq K_2 \supseteq \cdots \supseteq K_m = \emptyset \) and

\[
\gamma([0, t_1]) \subseteq U_{K_1}^0, \quad \gamma([t_1, t_2]) \subseteq U_{K_2}^1, \ldots, \gamma([t_{i-1}, t_i]) \subseteq U_{K_i}^{J_{i-1} \cup J_{i-1}^j}, \ldots,
\]

\[
\gamma([t_{m-1}, t_m]) \subseteq U_{K_m}^{J_{m-1} \cup J_{m-1}^j}, \quad \gamma([t_m, 1]) \subseteq U_{Q}^{[1 : m]}.
\]

Next, we form an appropriate subsequence. Choose \( i_1 \) such that \( n - k \leq |K_{i_1}| \) but \( n - k > |K_{i_1+1}| \). Put \( A_1 = K_{i_1} \) and \( B_1 = \emptyset \). We have

\[
A_1 \cap B_1 = \emptyset, \quad n - k \leq |A_1| + |B_1|.
\]

Inductively, choose \( i_j > i_{j-1} \) such that \( K_{i_j} = K_{i_{j-1}+1} \) but \( K_{i_{j+1}} \neq K_{i_{j-1}+1} \). Put \( A_j = K_{i_j} \) and \( B_j = J_1 \cup \cdots \cup J_{i_{j-1}} \). For some \( s \) we have \( K_{i_{s+1}+1} = \emptyset \). Here we finish by \( A_s = \emptyset, B_s = J_1 \cup \cdots \cup J_{i_{s-1}} \). We now have a strictly decreasing sequence \( A_1 \supset A_2 \supset \cdots \supset A_s = \emptyset \) and a sequence \( \emptyset = B_1 \subseteq B_2 \subseteq \cdots \subseteq B_s \) such that

\[
A_i \cap B_i = \emptyset, \quad n - k \leq |A_i| + |B_i|
\]

for all \( i \). Furthermore, we have

\[
\gamma([0, t_{i_1}]) \subseteq X_{A_1}^{B_1}, \ldots, \gamma([t_{i_{j-1}}, t_{i_j}]) \subseteq X_{A_j}^{B_j}, \ldots, \gamma([t_s, 1]) \subseteq X_{A_s}^{B_s},
\]

such that

\[
\gamma([0, 1]) \subseteq \bigcup_{j=1}^s X_{A_j}^{B_j}.
\]

Put \( n_i = |A_i| \) and choose \( \alpha \in \text{Inj}_{n-k}^n \) such that

\[
\{\alpha(1), \ldots, \alpha(n_i)\} = A_i, \quad \text{for } 2 \leq i \leq s - 1
\]

and \( \{\alpha(1), \ldots, \alpha(n-k)\} \subseteq A_1 \). Then \( \gamma([0, 1]) \subseteq U_\alpha \). \( \square \)

As in [R], we need a covering of \( I^n_{(k)} \) by open subsets of \( I^n \). Choose \( 0 < \epsilon < \frac{1}{3} \).

Replace the conditions \( x_i = 0 \) and \( x_j = 1 \) by \( x_i < \epsilon \) and \( x_j > 1 - \epsilon \) respectively in the above definition of the covering. Then one gets an open covering and an analogue of Proposition 2.12 in [R] holds.

## 3 A simplicial complex model of the trace space

**Definition 3.1.** Let \( C^n_k \) denote the poset

\[
C^n_k = \{A \subseteq \text{Inj}_{n-k}^n | \ A \neq \emptyset, \ \tilde{T}(\bigcap_{\alpha \in A} U_\alpha)(\overline{0, 1}) \neq \emptyset\}.
\]

The partial order is inclusion.

For an object \( A \in C^n_k \) we have a simplex in \( \mathbb{R}^{|\text{Inj}_{n-k}^n|} \simeq \mathbb{R}^{n! / k!} \) as follows:

\[
\Delta^{|A|-1} = \{(x_\alpha) | \ x_\alpha \geq 0 \text{ for all } \alpha, \ \sum x_\alpha = 1, \ x_\alpha = 0 \text{ if } \alpha \notin A\}.
\]
We define functors

\[ D : (C^n_k)^{op} \to \text{Top}; \quad A \mapsto \vec{T}(\bigcap_{\alpha \in A} U_{\alpha})(0, 1), \]

\[ E : C^n_k \to \text{Top}; \quad A \mapsto \Delta_{|A|-1}. \]

An argument similar to the one given [R], proof of Theorem 3.5, shows that the trace space of \( I^n_{(k)} \) is homotopy equivalent to the colimit of \( E \) and to the nerve of \( C^n_k \). Thus, we have the following result:

**Theorem 3.2.** There is a homotopy equivalence

\[ \vec{T}(I^n_{(k)})(0, 1) \simeq N(C^n_k). \]

We will now give a better description of the category \( C^n_k \). An intersection of subsets can be described as follows:

**Lemma 3.3.** Let \( \alpha_1, \alpha_2, \ldots, \alpha_r \in \text{Inj}^n_{n-k} \). Then one has

\[ \bigcap_{j=1}^r U_{\alpha_j} = \bigcup_{i=0}^{n-k} X^\#(n-k-i)_{\alpha_1(1), \ldots, \alpha_1(i), \alpha_2(1), \ldots, \alpha_2(i), \ldots, \alpha_r(1), \ldots, \alpha_r(i)}. \]

**Proof.** We insert the expression defining \( U_{\alpha_j} \) and obtain

\[ \bigcap_{j=1}^r U_{\alpha_j} = \bigcup_{j=1}^r X^\#(n-k)_{\alpha_j} \cup X^\#(n-k-1)_{\alpha_j(1)} \cup X^\#(n-k-2)_{\alpha_j(1), \alpha_j(2)} \cup \cdots \cup X_{\alpha_j(1), \alpha_j(2), \ldots, \alpha_j(n-k)} \]

\[ = X^\#(n-k) \cup \bigcup_{j=1}^r X^\#(n-k-1)_{\alpha_j(1)} \cup X^\#(n-k-2)_{\alpha_j(1), \alpha_j(2)} \cup \cdots \cup X_{\alpha_j(1), \alpha_j(2), \ldots, \alpha_j(n-k)} \]

since the term \( X^\#(n-k) \) appears in every factor and when we intersect it by another summand, we get a subset of \( X^\#(n-k) \). By a similar argument, we can rewrite the above as

\[ X^\#(n-k) \cup X^\#(n-k-1)_{\alpha_1(1), \alpha_2(1), \ldots, \alpha_r(1)} \cup \bigcap_{j=1}^r X^\#(n-k-2)_{\alpha_j(1), \alpha_j(2)} \cup \cdots \cup X_{\alpha_j(1), \alpha_j(2), \ldots, \alpha_j(n-k)}. \]

We continue this way, and get the desired result. \( \square \)

We can now formulate a useful criterion for whether the trace space of an intersection of subsets is empty or not.

**Corollary 3.4.** Let \( \alpha_1, \alpha_2, \ldots, \alpha_r \in \text{Inj}^n_{n-k} \). Then one has

\[ \vec{T}(\bigcap_{j=1}^r U_{\alpha_j})(0, 1) \neq \emptyset \iff \forall i : \#\{\alpha_1(1), \ldots, \alpha_1(i), \alpha_2(1), \ldots, \alpha_2(i), \ldots, \alpha_r(1), \ldots, \alpha_r(i)\} \leq i + k - 1. \]
Proof. The end point $\bar{1}$ lies in $X^{#(n-k)}$ and not in any other summand. The starting point $\bar{0}$ lies only in $X^{\alpha_1(1),\ldots,\alpha_1(n-k),\ldots,\alpha_r(1),\ldots,\alpha_r(n-k)}$. There exists a directed path from $\bar{0}$ which moves further ahead precisely when

$$\#\{\alpha_1(1),\ldots,\alpha_1(n-k),\ldots,\alpha_r(1),\ldots,\alpha_r(n-k)\} \leq n-1.$$  

Assuming that, we can arrive at a point in $X^{\alpha_1(1),\ldots,\alpha_1(n-k-1),\ldots,\alpha_1(n-k),\ldots,\alpha_r(1),\ldots,\alpha_r(n-k)}$. There exists a directed path further ahead from this point precisely when

$$1 + \#\{\alpha_1(1),\ldots,\alpha_1(n-k-1),\ldots,\alpha_r(1),\ldots,\alpha_r(n-k-1)\} \leq n-1.$$  

Assuming that, we can get to a point in $X^{\alpha_1(1),\ldots,\alpha_1(n-k-2),\ldots,\alpha_1(n-k),\ldots,\alpha_r(1),\ldots,\alpha_r(n-k-2)}$ and so on. If we have arrived at a point in $X^{#(n-k)}$, we can always move further to the end point $\bar{1}$. □

Definition 3.5. For $A \subseteq \text{Inj}^n_{n-k}$ we let

$$\text{Im}_i(A) = \bigcup_{\alpha \in A} \{\alpha(1), \alpha(2), \ldots, \alpha(i)\}.$$  

Theorem 3.6. The poset $C^n_k$ has the following purely combinatorial description:

$$C^n_k = \{A \subseteq \text{Inj}^n_{n-k} \mid \forall i : i \leq |\text{Im}_i(A)| \leq i + k - 1\}.$$  

4 The flag-category model

Definition 4.1. The flag category is the poset

$$F^n_k = \{(F_1 \subseteq F_2 \subseteq \cdots \subseteq F_{n-k}) \mid F_{n-k} \subseteq [1 : n], \forall i : i \leq |F_i| \leq i + k - 1\}.$$  

The partial order is given by $F \subseteq G \Leftrightarrow \forall i : F_i \subseteq G_i$.

Note that there is a forgetful functor

$$U : C^n_k \rightarrow F^n_k; \quad A \mapsto (\text{Im}_1(A), \text{Im}_2(A), \ldots, \text{Im}_{n-k}(A)).$$

We now define a functor, which turns out to be the left adjoint of $U$.

Definition 4.2. The functor $V : F^n_k \rightarrow C^n_k$ is defined by

$$V(F) = \{\alpha \in \text{Inj}^n_{n-k} \mid U(\{\alpha\}) \subseteq F\}.$$  

Note that $V(F)$ is a final among the objects $A$ with property $U(A) = F$.

Theorem 4.3. $V : F^n_k \rightarrow C^n_k$ is the left adjoint of the forgetful functor $U : C^n_k \rightarrow F^n_k$. Thus there is a homotopy equivalence

$$\mathcal{N}(C^n_k) \xrightarrow{\mathcal{N}(U)} \mathcal{N}(F^n_k), \quad \mathcal{N}(C^n_k) \xleftarrow{\mathcal{N}(V)} \mathcal{N}(F^n_k).$$
Proof. The composite $UV$ is the identity on $F^n_k$, so the counit of the adjunction is simply the identity

$$UV \xrightarrow{\text{Id}} \text{Id}_{F^n_k}.$$ 

By definition of $V$ we have an inclusion $A \hookrightarrow VU(A)$ for all objects $A \in C^n_k$. This defines the unit of the adjunction

$$\text{Id}_{C^n_k} \xrightarrow{} VU.$$ 

A functor induces a map of nerves and a natural transformation of functors induces a homotopy between such maps. Thus an adjoint pair of functors induces a homotopy equivalence.

By combining Theorems 3.2 and 4.3 we get our main result:

**Theorem 4.4.** There is a homotopy equivalence

$$\bar{T}(I^n_k)((0,1)) \simeq N(F^n_k).$$

**References**


