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AN INEQUALITY OF REARRANGEMENTS ON THE UNIT CIRCLE

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Abstract. We prove that the integral of the product of two functions over a symmetric set in $S^1 \times S^1$, defined as $E = \{(x, y) \in S^1 \times S^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\}$, where $\sigma_1, \sigma_2$ are diffeomorphisms of $S^1$ with certain properties and $d$ is the geodesic distance on $S^1$, increases when we pass to their symmetric decreasing rearrangement. We also give a characterization of these diffeomorphisms $\sigma_1, \sigma_2$ for which the rearrangement inequality holds. As a consequence, we obtain the result for the integral of the function $\Psi(f(x), g(y))$ ($\Psi$ a supermodular function) with a kernel given as $k[d(\sigma_1(x), \sigma_2(y))]$, with $k$ decreasing.

1. Introduction

On a measure space $(X, \mu)$, the Hardy-Littlewood inequality asserts [4]:

$$\int_X f(x)g(x) \, d\mu(x) \leq \int_0^{\mu(X)} f^*(t)g^*(t) \, dt,$$

where $f^*$ and $g^*$ are the decreasing rearrangements of $f$ and $g$, respectively. In what follows, $X = S^1$, or $X = [-\pi, \pi]$, and the above inequality can be written as:

$$\int_{-\pi}^\pi f(x)g(x) \, dx \leq \int_{-\pi}^\pi f^4(x)g^4(x) \, dx,$$

with $f^4, g^4$ the symmetric decreasing rearrangements of $f$ and $g$, given by $f^4(x) = f^*(2|x|)$ and $g^4(x) = g^*(2|x|)$.

These inequalities can be proved using the layer-cake formula [10]: Every measurable function $f : X \to \mathbb{R}_+$ can be written as an integral of the characteristic function of its level sets:

$$f(x) = \int_0^\infty \chi(f \geq t)(x) \, dt.$$

A more general rearrangement inequality on $X = \mathbb{R}^n$ is the Riesz-Sobolev inequality:

$$\int_{\mathbb{R}^{2n}} f(x)g(y)h(x-y) \, dx \, dy \leq \int_{\mathbb{R}^{2n}} f^2(x)g^2(x)h^2(x-y) \, dx \, dy,$$

where $f, g, h$ are non-negative functions which vanish at infinity in a weak sense. The case $n = 1$ is due to Riesz in 1930 (see [12]), and the case $n > 1$ is due to Sobolev in 1938 (see [13]). The proof can be found in the book by Hardy, Littlewood, Pólya [9] which sets the beginning of the systematic study of rearrangement inequalities.

A more general version of this inequality in $\mathbb{R}^n$, involving $n$ functions can be found in [5].

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The equivalent of (1.3) for three non-negative functions on the unit circle was proved by Baernstein [1]:

\[(1.4) \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(e^{i\theta})g(e^{i\theta})h(e^{i(\phi-\theta)}) \, d\theta d\phi \leq \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f^\sharp(e^{i\theta})g^\sharp(e^{i\theta})h^\sharp(e^{i(\phi-\theta)}) \, d\theta d\phi.\]

The proof of this inequality uses a variational principle applied to the convolution of characteristic functions of sets which does not seem to generalize in higher dimensions.

The Riesz-Sobolev inequality (1.3) is equivalent to the Brunn-Minkowski inequality from convex geometry [8, 11, 7] which states that if \(K\) and \(L\) are measurable sets in \(\mathbb{R}^n\), then their Minkowski (pointwise) sum \(K + L\) is related to the measure of the sets \(K\) and \(L\) by

\[V(K + L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n},\]

where \(V\) denotes the \(n\)-dimensional volume. An analog of this inequality for \(S^n\) is not known, and, since the proof of rearrangement inequalities in \(\mathbb{R}^n\) require it, an analog of the Riesz-Sobolev inequality (1.3) is not known in \(S^n\), for \(n > 1\).

However, a partial result in \(S^n\) was proved by Baernstein and Taylor in [2]. They considered a version of the Riesz-Sobolev inequality where one of the functions is symmetric decreasing. They showed that, if \(h = K\) is already symmetric decreasing then

\[\int_{S^n} \int_{S^n} f(x)g(y)K(x \cdot y) \, d\sigma(x)d\sigma(y) \leq \int_{S^n} \int_{S^n} f^\sharp(x)g^\sharp(y)K(x \cdot y) \, d\sigma(x)d\sigma(y),\]

where \(d\sigma\) is the surface measure on the unit sphere \(S^n\) in \(\mathbb{R}^{n+1}\), \(x \cdot y\) is the usual inner product and \(K(t)\) is an increasing function on \([-1,1]\). Since \(x \cdot y = \cos \alpha\), where \(\alpha\) is the angle between the vectors \(x\) and \(y\), we can write \(K(x \cdot y) = k(d(x, y))\), with \(k\) decreasing. Here \(d(x, y)\) is the great circle (geodesic) distance between \(x\) and \(y\). Their proof is based on the polarization technique. They showed first that the inequality holds for the polarizations of \(f\) and \(g\) in any hyperplane and then they passed to the limit for the general case. They were led to this version of the Riesz-Sobolev inequality while trying to generalize a 2-dimensional result stating that \(u\) is subharmonic implies its star function is also subharmonic.

In this paper we are interested in the case \(n = 1\) of this inequality with \(K\) replaced by the characteristic function of a symmetric set which does not depend on the distance between two points, but rather on the distance between their images under two diffeomorphisms \(\sigma_1, \sigma_2\) of \(S^1\). We will also obtain a characterization of these diffeomorphisms for which the inequality holds. With the set \(E\) defined as

\[E = \{(x, y) : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\},\]

we will show that

\[(1.5) \int_E f(x)g(y) \, dxdy \leq \int_E f^\sharp(x)g^\sharp(y) \, dxdy,\]

for every \(\alpha > 0\). This result implies the main result of this paper, Theorem 3.6:

\[\int_{S^1} \int_{S^1} \Psi(f(x), g(y))k[d(\sigma_1(x), \sigma_2(y))] \, dxdy \leq \int_{S^1} \int_{S^1} \Psi(f^\sharp(x), g^\sharp(y))k[d(\sigma_1(x), \sigma_2(y))] \, dxdy,\]
with $k$ decreasing and $\Psi$ the distribution function of a measure $\mu$.

The paper is organized as follows: We will first prove (1.5) for $f$ and $g$ replaced by characteristic functions $\chi_A$, $\chi_B$, and $\sigma_2$ the identity. Then we will deduce the result (1.5) mentioned above, and we will show that we can replace the product $f(x)g(y)$ by a function $\Psi(f(x), g(y))$ and that we can replace $\chi_E$ by a decreasing function of the distance between $\sigma_1(x)$ and $\sigma_2(y)$, yielding Theorem 3.6.

2. Preliminaries

Recall that a function $f : I \to \mathbb{R}$, defined on an interval $I \subset \mathbb{R}$, is called convex if, for every $0 < \lambda < 1$ and every $a, b \in I$, the following inequality holds:

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b).$$

A convex function is differentiable almost everywhere on $I$ and its derivative is increasing.

We denote by $S^1$ the unit circle in $\mathbb{R}^2$, i.e., $S^1 = \{z \in \mathbb{C} : |z| = 1\}$, and by $S^1_+$ the upper half unit circle,

$$S^1_+ = \{e^{i\theta} : 0 \leq \theta \leq \pi\}.$$

**Definition 2.1.** A function $\sigma : S^1_+ \to S^1_+$ is called convex if the function $\sigma_1 : [0, \pi] \to [0, \pi]$, defined as:

$$\sigma(e^{i\theta}) = e^{i\sigma_1(\theta)}, \quad 0 \leq \theta \leq \pi,$$

is convex on $[0, \pi]$.

Let $f : S^1 \to \mathbb{R}_+$ be a non-negative measurable function. We define its distribution function:

$$\lambda_f(t) = |\{f > t\}|, \quad t \in [0, \infty),$$

where $\{f > t\} = \{z \in S^1 : f(z) > t\}$ denote the level sets of $f$, and $|A|$ is the linear measure on $S^1$ of $A$. Functions which have the same distribution function are called equimeasurable.

We define the symmetric decreasing rearrangement of $f$ to be the function $f^\sharp : S^1 \to \mathbb{R}_+$, given by:

$$f^\sharp(z) = \inf\{t : \lambda_f(t) \leq 2d(1, z)\},$$

where $d(1, z)$ is the geodesic distance on $S^1$ between $z$ and $1$.

It is clear that $f^\sharp(z) = f^\sharp(\bar{z})$ and that $f^\sharp$ decreases as $d(1, z)$ increases. Also, $f$ and $f^\sharp$ are equimeasurable.

If we write $z = e^{i\theta}$, $-\pi \leq \theta < \pi$, then $d(1, z) = d(1, e^{i\theta}) = |\theta|$, and we can think of $f$ as a function of $\theta$ via the relation

$$\tilde{f}(\theta) = f(e^{i\theta}).$$

For $\tilde{f} : [-\pi, \pi] \to \mathbb{R}_+$, one defines its symmetric decreasing rearrangement as:

$$\tilde{f}^\sharp(\theta) = \inf\{t : \lambda_f(t) \leq 2|\theta|\},$$

where, as before, $\lambda_f(t) = |\{\tilde{f} > t\}|$, and thus, there is a one-to-one correspondence between $f^\sharp$ and $\tilde{f}^\sharp$, given by

$$\tilde{f}^\sharp(\theta) = f^\sharp(e^{i\theta}).$$

Whenever necessary, we will think of a function $f$ defined on $S^1$ as a function on $[-\pi, \pi]$. If $f = \chi_A$ is the characteristic function of a measurable set $A \subset S^1$, then
\( f^2 = \chi_{A^\circ} \), where \( A^\circ \) is the open interval on the unit circle centered at 1, having the same linear measure as \( A \).

Next, we introduce the Hardy-Littlewood-Pólya preorder relation \( \prec \) for non-negative functions defined on the interval \([-\pi, \pi]\). We say that (see [3, 4]):

\[
\text{if } f \prec F \text{ iff } \int_{-t}^t f^2(s) \, ds \leq \int_{-t}^t F^2(s) \, ds, \text{ for all } 0 \leq t \leq \pi.
\]

This is equivalent to

\[
\int_{-\pi}^{\pi} f^2(s) h^2(s) \, ds \leq \int_{-\pi}^{\pi} F^2(s) h^2(s) \, ds,
\]

for every positive symmetric decreasing function \( h^2 \) defined on \([-\pi, \pi]\). To see this, write \( h^2(s) = \int_0^\infty \chi_{\{h^2 > l\}}(s) \, dl \) (this is the layer cake formula (1.2)), and, using Fubini’s formula and the fact that \( \{h^2 > t\} = (-l(t), l(t)) \) is a symmetric interval,

\[
\int_{-\pi}^{\pi} f^2(s) h^2(s) \, ds = \int_0^\infty \left[ \int_{-l(t)}^{l(t)} f^2(s) \, ds \right] \, dl \\
\leq \int_0^\infty \left[ \int_{-l(t)}^{l(t)} F^2(s) \, ds \right] \, dl = \int_{-\pi}^{\pi} F^2(s) h^2(s) \, ds.
\]

Yet another equivalent characterization is:

\[
f \prec F \iff \int_E f(s) \, ds \leq \int_{E^c} F(s) \, ds, \text{ for every } E \subset [-\pi, \pi].
\]

The next result is well-known and it follows from the proof of the equality case in the Hardy-Littlewood inequality, presented by Lieb and Loss in [10, pp.82]. We will include a proof here for consistency.

**Lemma 2.2.** Let \( f : [-\pi, \pi] \rightarrow \mathbb{R}_+ \) be a measurable function such that

\[
(2.1) \quad \int_{-t}^t f(x) \, dx \geq \int_{-t}^t f^2(x) \, dx, \quad \text{for every } 0 \leq t \leq \pi.
\]

Then \( f = f^2 \) a.e. on \([-\pi, \pi]\).

**Proof.** From (1.1) applied to \( \chi_{(-t,t)} \) and \( f \), it follows that we must have equality in (2.1), i.e.,

\[
(2.2) \quad \int_{-t}^t f(x) \, dx = \int_{-t}^t f^2(x) \, dx.
\]

We will use the layer-cake formula to write \( f(x) = \int_0^\infty \chi_{\{f > s\}}(x) \, ds \), and similarly for \( f^2(x) \).

Using (1.1), we obtain:

\[
(2.3) \quad \int_{-t}^t \chi_{\{f > s\}}(x) \, dx \leq \int_{-t}^t \chi_{\{f^2 > s\}}(x) \, dx, \quad \text{for every } s \geq 0.
\]

Fubini’s theorem and (2.2) imply that:

\[
\int_{-t}^t f(x) \, dx = \int_0^\infty \left[ \int_{-t}^t \chi_{\{f > s\}}(x) \, dx \right] \, ds \\
= \int_0^\infty \left[ \int_{-t}^t \chi_{\{f^2 > s\}}(x) \, dx \right] \, ds = \int_{-t}^t f^2(x) \, dx.
\]
From this equality and (2.3) it follows that, for a fixed \( t \), there exists a set of measure zero \( S_t \), such that
\[
\int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx = \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx, \quad \text{for every } s \in (0, \infty) \setminus S_t.
\]

Next, we choose \( T_N \) a countable dense set in \([0, \pi]\) and we denote by \( S_{T_N} = \cup_{t \in T_N} S_t \).

Then:
\[
(2.4) \quad \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx = \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx, \quad \text{for every } t \in T_N \text{ and } s \in (0, \infty) \setminus S_{T_N}.
\]

Since for every fixed \( s, t \rightarrow \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx \) is a continuous function of \( t \), in fact (2.4) holds for every \( 0 \leq t \leq \pi \).

Thus,
\[
\int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx = \int_{-t}^{t} \chi_{\{f > s\}}(x) \, dx, \quad \text{for all } 0 \leq t \leq \pi \text{ and a.e. } s \in (0, \infty).
\]

Now, let \( t \) be such that \( \{f^2 > s\} = (-t, t) \). Then, it follows that \( \{f > s\} = (-t, t) = \{f^2 > s\} \) a.e., and thus, \( f = f^2 \) by the layer cake formula.

\[ \square \]

The following result shows that \( \int_{-t}^{t} f^2(x) \, dx \) is attained as a supremum. A proof can be found in [4, Theorem 7.5, pp.82].

**Theorem 2.3.** (J. V. Ryff) For every measurable function \( f \) as in Lemma 2.2, there exists a measure preserving transformation \( T \) such that \( f = f^2 \circ T \). This guarantees, for every \( t \), the existence of a set \( A \subset [-\pi, \pi] \) of measure \( 2t \) such that \( \int_{A} f(x) \, dx = \int_{-t}^{t} f^2(x) \, dx \).

### 3. Main results: inequalities on the circle

**Notation.** As before, \( d \) is the geodesic distance, also called the arclength, on the unit circle \( S^1 \). We have:
\[
(3.1) \quad d(u, v) = d(\bar{w}, 1), \quad \text{for all } u, v \in S^1,
\]
where \( \bar{v} \) denotes the complex conjugate of \( v \).

We define, for \( \alpha > 0 \), the function:
\[
\chi_{\alpha}(u, v) = \begin{cases} 
1, & \text{if } d(u, v) \leq \alpha, \\
0, & \text{otherwise}
\end{cases}
\]
and we observe that \( \chi_{\alpha}(u, v) = \chi_{\alpha}(\bar{w}, 1) \), by (3.1).

We introduce a new function, which we call again \( \chi_{\alpha} : S^1 \rightarrow \mathbb{R}_+ \), given by \( \chi_{\alpha}(z) = \chi_{\alpha}(z, 1) \), which is the characteristic function of the closed interval on \( S^1 \) of linear length \( 2\alpha \), centered at 1.

We will make use, in what follows, of the relation:
\[
(3.2) \quad \chi_{\alpha}(\bar{w}) = \chi_{\alpha}(u, v), \quad \text{for all } u, v \in S^1.
\]

Given two positive measurable functions \( f, g : S^1 \rightarrow \mathbb{R}_+ \), their convolution, \( f * g \), is defined to be the function:
\[
(f * g)(z_0) = \int_{S^1} f(z_0 \bar{z}) g(z) \, dz = \int_{-\pi}^{\pi} f(e^{i\theta}) g(e^{i\theta}) \, d\theta,
\]
with \( z_0 = e^{i\theta_0} \) and \( dz \) represents the arclength element on \( S^1 \), usually denoted by \( |dz| \).

Given three positive functions \( f, g, h \) defined on \( S^1 \), we can write
\[
\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(e^{i(\theta-t)})g(e^{it})h(e^{i\theta}) \, dt \, d\theta = (f \ast g \ast h^-)(1),
\]
where \( h^-(z) = h(\bar{z}) \), i.e., \( h^-(e^{i\theta}) = h(e^{-i\theta}) \).

**Theorem 3.1.** Let \( \sigma : S^1 \to S^1 \) be a \( C^1 \) diffeomorphism such that \( \sigma(1) = 1 \) and \( \sigma(-1) = -1 \). Additionally, we assume that \( \sigma(S^1_+) \subseteq S^1_+ \) and \( \sigma(S^1_-) \subseteq S^1_- \). Let \( d \) be the geodesic distance on the unit circle, \( \alpha \) be a positive real number, and we define the set \( E = \{(x,y) \in S^1 \times S^1 : d(\sigma(x),y) \leq \alpha\} \). For \( A,B \subset S^1 \) measurable sets, let
\[
I_\alpha(A,B) = \int_{S^1 \times S^1} \chi_A(x)\chi_B(y)\chi_E(x,y) \, dx \, dy.
\]

Then, for any \( A, B \) measurable subsets of \( S^1 \), and \( \alpha > 0 \),
\[
I_\alpha(A,B) \leq I_\alpha(A^1,B^2),
\]
if and only if, \( \sigma \) is symmetric (i.e. \( \sigma(z) = \sigma(\bar{z}) \), for every \( z \in S^1 \)) and convex on \( S^1_+ \).

**Proof.** Sufficiency. We define \( \sigma_1 : [-\pi, \pi) \to [-\pi, \pi) \) by \( e^{\sigma_1(\theta)} := \sigma(e^{i\theta}) \) and we assume that \( \sigma_1 \) is convex on \( (0, \pi) \). Using change of variables, \( (\sigma(x),y) = (u,v), \) the integral \( I_\alpha \) becomes:
\[
I_\alpha(A,B) = \int_{S^1 \times S^1} \chi_{\sigma(A)}(u)\chi_{B}(v)\chi_{\sigma^{-1}}(u)(u') \, dudv.
\]

With \( \chi_{\sigma}(u,v) = \chi_{\alpha}(uv) \), as in (3.2), the above expression becomes:
\[
I_\alpha(A,B) = \int_{S^1 \times S^1} \chi_{\sigma(A)}(u)\chi_{B}(v)\chi_{\sigma^{-1}}(uv) \psi(u) \, dudv,
\]
where \( \psi(e^{i\theta}) = \tau_1(\theta) \) and \( \tau_1 \) is defined by \( \sigma^{-1}(e^{i\theta}) = e^{i\tau_1(\theta)} \), and is the inverse of \( \sigma_1 \).

Thus, we can write using convolution and (3.3):
\[
I_\alpha(A,B) = [(\chi_{\sigma(A)} \cdot \psi) \ast \chi_{\alpha} \ast \chi_{B}](1),
\]
where we used the fact that \( \chi_{\alpha} \) is a symmetric function.

It was proved in [1] (see also (1.4)) by Baernstein that, for any three positive measurable functions \( f, g, h \) on \( S^1 \), the following inequality holds:
\[
(f \ast g \ast h^-)(1) \leq (f^2 \ast g^2 \ast h^2)(1).
\]
One can replace \( h^- \) in the inequality above by \( h \) since they are equimeasurable functions. Thus, based on (3.6) and the fact that \( \chi_{\alpha} \) is symmetric decreasing, we conclude that:
\[
I_\alpha(A,B) \leq [(\chi_{\sigma(A)} \cdot \psi)^2 \ast \chi_{\alpha} \ast \chi_{B}](1).
\]

**Fact:** If \( F \) is a positive symmetric decreasing function and if \( f \prec F \) in the sense of Hardy-Littlewood-Pólya (i.e. \( \sup_{|G| = 2\theta} \int_G f \leq \int_{-\theta}^{\theta} F \)), then \( f^2 \) in inequality (3.6) can be replaced by \( F \). Indeed, \( f \prec F \) is equivalent to \( \int_{S^1} f^2(z)g^2(z) \, dz \leq \int_{S^1} F(z)g^2(z) \, dz \),
for all positive symmetric decreasing functions \( g^* \). Now, since \( g^* \) is symmetric decreasing and since the convolution \((f^* \ast g^* \ast h^*)(1)\) can be written as the integral of the product \( f^*(z)(g^* \ast h^*)(z)\), we conclude that:

\[
(f^* \ast g^* \ast h^*)(1) \leq (F \ast g^* \ast h^*)(1).
\]

Therefore, using (3.7) and the Fact, we can prove (3.4) if we show that \( \chi_{\sigma(A)} \psi < \chi_{\sigma(A^\#)} \psi \), i.e.

\[
\int_E \chi_{\sigma(A)} \psi \leq \int_{E^\#} \chi_{\sigma(A^\#)} \psi.
\]

Let \( E' = \sigma^{-1}(E) \), and \( E'' = \sigma^{-1}(E^\#) \). With these notations, inequality (3.8) becomes:

\[
\int_{A \cap E'} dx \leq \int_{A \cap E''} dx,
\]

or equivalently, \(|A \cap E'| \leq |A^\# \cap E''|\), which is true if \(|E'| \leq |E''|\), since \( E'' \) is symmetric. Since \( \psi \) is symmetric decreasing, we have that \( \int_E \psi(u) du \leq \int_{E^\#} \psi(u) du \), which is equivalent to \( \int_{\sigma^{-1}(E)} dx \leq \int_{\sigma^{-1}(E^\#)} dx \), using change of variables. The latter inequality simply states that \(|E'| \leq |E''|\), and the proof of the sufficiency is now complete.

**Necessity.** Dividing (3.5) by \( 2\alpha \), and letting \( \alpha \) tend to zero, we obtain:

\[
I_0(A, B) = \int_{S^1} \chi_{\sigma(A)}(u) \chi_B(u) \psi(u) du,
\]

and inequality (3.4) implies that:

\[
I_0(A, B) \leq I_0(A^\#, B^\#).
\]

With the notation \( \tau = \sigma^{-1} \), the Jacobian of \( \tau \), and \( x = \tau(u), I_0 \) becomes:

\[
(3.10) \quad I_0(A, B) = \int_{S^1} \chi_A(x) \chi_{\tau(B)}(x) dx = |A \cap \tau(B)|.
\]

First, we will show that the symmetry condition is necessary. Suppose \( \tau \) is not symmetric. Then, there exists a point \( x = e^{i\theta} \) in \( S^1 \), such that \( \tau(x) \neq \tau(\bar{x}) \). If we consider \( A = \tau(\{e^{it} : |t| < \theta\}) \) and \( B = \{e^{it} : |t| < \theta\} \), then we have: \(|A \cap \tau(B)| = |\tau(B)| > |A^\# \cap \tau(B^\#)|\), since \( \tau(B^\#) \) is not symmetric and \(|A| = |\tau(B)|\). But this contradicts (3.9) and therefore (3.4).

Suppose now that \( \tau_1 \) is symmetric, but not concave (or, equivalently, \( \sigma_1 \) is symmetric, but \( \sigma_1 \) is not convex on \((0, \pi)\)). Then, there exist \( e^{ib}, e^{ic} \in S^1_+ \) with \( b, c \in (0, \pi) \) such that:

\[
(3.11) \quad \frac{\tau_1(b) + \tau_1(c)}{2} > \tau_1 \left( \frac{b + c}{2} \right).
\]

Without loss of generality we can assume that \( b > c \) and let us denote by \( a = \frac{b+c}{2} \). Letting \( B = \{e^{it} : -c < t < b\} \), it follows that \( B^\# = \{e^{it} : -a < t < a\} \). We calculate \(|\tau(B)| = \tau_1(b) - \tau_1(-c) = \tau_1(b) + \tau_1(c) \) and \(|\tau(B^\#)| = 2\tau_1(a) \).

From (3.11) we obtain that \(|\tau(B)| > |\tau(B^\#)|\) which shows that \( I_0(S^1, B) > I_0(S^1, B^\#) \) and contradicts (3.4). Therefore, \( \tau \) must also be concave. \( \square \)
Theorem 3.2. Suppose we have two functions $\sigma_1, \sigma_2$ satisfying the conditions of $\sigma$ in Theorem 3.1 and define $E = \{(x, y) \in S^1 \times S^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\}$, for $\alpha \in \mathbb{R}_+$. Let

$$I_\alpha(A, B) = \int_{S^1 \times S^1} \chi_A(x)\chi_B(y)\chi_E(x, y)dxdy.$$  

Then, for any $A, B$ subsets of $S^1$ and $\alpha > 0$,  

$$(3.12) \quad I_\alpha(A, B) \leq I_\alpha(A^\sharp, B^\sharp),$$

if and only if $\sigma_1, \sigma_2$ are symmetric and convex on $S^1_+$. 

Proof. Sufficiency. Very similar to Theorem 3.1. Using change of variables, $(\sigma_1(x), \sigma_2(y)) = (u, v)$, the integral becomes:

$$I_\alpha(A, B) = \int_{S^1 \times S^1} \chi_{\sigma_1(A)}(u)\chi_{\sigma_2(B)}(v)\chi_\alpha(uv)\psi_1(u)\psi_2(v)dudv,$$

where $\psi_1, \psi_2$ are defined similarly to $\psi$ in Theorem 3.1 (see (3.5)). Using convolution, this integral can be written as:

$$I_\alpha(A, B) = \{(\chi_{\sigma_1(A)} \cdot \psi_1) * \chi_\alpha * (\chi_{\sigma_2(B)} \cdot \psi_2^-)(1)\}.$$

We have already proven that $\chi_{\sigma_1(A)} \psi_1 \prec \chi_{\sigma_1(A)} \psi_1$ and $\chi_{\sigma_2(B)} \psi_2 \prec \chi_{\sigma_2(B)} \psi_2$, from which it follows that $I_\alpha(A, B) \leq I_\alpha(A^\sharp, B^\sharp)$.

Necessity. Using change of variable $v = \sigma_2(y)$, $I_\alpha$ becomes:

$$I_\alpha(A, B) = \int_{S^1 \times S^1} \chi_A(x)\chi_{\{x, v\} \in S^1 \times S^1 : d(\sigma_1(x), v) \leq \alpha} \chi_{\sigma_2(B)}(v)\psi_2(v)dxdv.$$

Dividing by $\alpha$ and letting $\alpha \to 0$, we obtain:

$$I_0(A, B) = \int_{S^1} \chi_A(x)\chi_{\sigma_2(B)}(\sigma_1(x))\psi_2(\sigma_1(x))dx.$$

Inequality (3.12) of the theorem implies the following inequality:

$$(3.13) \quad I_0(A, B) \leq I_0(A^\sharp, B^\sharp),$$

for all subsets $A$ and $B$ of $S^1$.

Now let $B = S^1$ in the above identity. Then:

$$I_0(A, S^1) = \int_{S^1} \chi_A(x)\psi_2(\sigma_1(x))dx \leq \int_{S^1} \chi_{A^\sharp}(x)\psi_2(\sigma_1(x))dx,$$

or equivalently,

$$\int_A \psi_2(\sigma_1(x))dx \leq \int_{A^\sharp} \psi_2(\sigma_1(x))dx,$$

for every measurable set $A \subset S^1$. Since the inequality is true for every measurable set $A$, we conclude by Lemma 2.2 and Theorem 2.3 that $\psi_2 \circ \sigma_1$ is symmetric (i.e., $\psi_2(\sigma_1(z)) = \psi_2(\sigma_1(\bar{z}))$) and decreasing, which implies that $\psi_2$ is decreasing on $S^1_+$. Likewise, $\psi_1 \circ \sigma_2$ is symmetric and decreasing on $S^1_+$, implying that $\psi_1$ is decreasing on $S^1_+$. Thus, $\sigma_1^{-1}$ and $\sigma_2^{-1}$ are concave on $S^1_+$ and therefore, $\sigma_1$ and $\sigma_2$ are convex on $S^1_+$. 

Next, we denote by $\tau = \sigma_1^{-1} \circ \sigma_2$. With this notation, $I_0$ becomes:

$$I_0(A, B) = \int_{\mathbb{S}^1} \chi_A(x) \chi_{\sigma_2(B)}(\sigma_1(x)) [\psi_2 \circ \sigma_1](x) \, dx$$

$$= \int_{\mathbb{S}^1} \chi_A(x) \chi_{\tau(B)}(x) [\psi_2 \circ \sigma_1](x) \, dx = \int_{A \cap \tau(B)} [\psi_2 \circ \sigma_1](x) \, dx.$$  

We will show that $\tau$ is symmetric, i.e., $\tau(\bar{x}) = \overline{\tau(x)}$, for every $x \in \mathbb{S}^1$. Suppose this is not the case. Then there exists $x = e^{i\theta}$, with $\theta \in (0, \pi)$, such that $\tau(e^{i\theta}) = \tau(e^{-i\theta})$.

Let $B = \{e^{it} : |t| < \theta\} = B^\circ$ and $A = \tau(B) \neq A^\circ$. Then, we have that $A^\circ \cap \tau(B^\circ) \subset A \cap \tau(B) = A$ and $|A \cap \tau(B)| > |A^\circ \cap \tau(B^\circ)|$. Since $\psi_2 \circ \sigma_1$ is positive, it follows that $I_0(A, B) > I_0(A^\circ, B^\circ)$, which contradicts (3.13). Thus, $\sigma_1^{-1} \circ \sigma_2$ is symmetric.

We have shown before that $\psi_1 \circ \sigma_2$ is also symmetric.

Claim: $\sigma_1^{-1} \circ \sigma_2$ and $\psi_1 \circ \sigma_2$ symmetric imply $\sigma_2$ is symmetric.

Proof of claim: We define $f_2$ on the interval $[-\pi, \pi]$ as follows:

$$\sigma_2(e^{i\theta}) = e^{if_2(\theta)}.$$  

Since $\psi_1 \circ \sigma_2$ is symmetric and $[\psi_1 \circ \sigma_2](e^{i\theta}) = \psi_1(e^{if_2(\theta)}) = \tau_1'(f_2(\theta))$, as in (3.5), it follows that $\tau_1'(f_2)$ is even. Since $[\sigma_1^{-1} \circ \sigma_2](e^{i\theta}) = e^{i\tau_1(f_2(\theta))}$ is symmetric, it follows that $\tau_1 \circ f_2$ is odd.

Now, $(\tau_1 \circ f_2)'(\theta) = (\tau_1' \circ f_2) \cdot f_2'$ is even and $\tau_1' \circ f_2$ is also even (as we have previously shown) and nonzero, so that $f_2'$ is even and thus $f_2$ is odd. Therefore $\sigma_2$ is symmetric and the proof of the claim is now complete.

Following exactly the same steps, we can show that $\sigma_1$ is symmetric. We have shown that $\sigma_1, \sigma_2$ are symmetric and convex on $\mathbb{S}^1_+$. □

Corollary 3.3. With $\sigma, \alpha$ and $E = \{(x, y) \in \mathbb{S}^1 : d(\sigma(x), y) \leq \alpha\}$, as in Theorem 3.1, we have the following result: For every $f, g : \mathbb{S}^1 \rightarrow \mathbb{R}_+$ positive measurable functions, and every $\alpha > 0$,

$$\int_E f(x)g(y) \, dx \, dy \leq \int_E f^\circ(x)g^\circ(y) \, dx \, dy,$$

if and only if, $\sigma$ is symmetric, and convex on $\mathbb{S}^1_+$.

To sketch the proof, we write $f$ and $g$ as the integrals of their level sets, using the layer-cake representation formula (1.2):

$$f(x) = \int_0^\infty \chi_{\{f > t\}}(x) \, dt \quad \text{and} \quad g(y) = \int_0^\infty \chi_{\{g > t\}}(y) \, dt,$$

and we notice that $\{f > t\}^\circ = \{f^\circ > t\}$ and $\{g > t\}^\circ = \{g^\circ > t\}$ so that inequality (3.14) reduces to the case where $f$ and $g$ are characteristic functions, and thus, Theorem 3.1 applies.

Corollary 3.4. Let $\sigma_1, \sigma_2$ and $E = \{(x, y) \in \mathbb{S}^1 \times \mathbb{S}^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\}$ be as in Theorem 3.2. For every $f, g : \mathbb{S}^1 \rightarrow \mathbb{R}_+$ positive measurable functions, and every $\alpha > 0$,

$$\int_E f(x)g(y) \, dx \, dy \leq \int_E f^\circ(x)g^\circ(y) \, dx \, dy,$$

if and only if, $\sigma_1$ and $\sigma_2$ are symmetric, and convex on $\mathbb{S}^1_+$. 
The proof of Corollary 3.4 is indeed very similar to the proof of Corollary 3.3, in which one represents \( f \) and \( g \) as integrals of the characteristic functions of their level sets.

The next theorem is a generalization of the previous results, where one replaces the product by a function \( \Psi \) defined as follows:
\[
\Psi : \mathbb{R}^2_+ \to \mathbb{R} \text{ vanishes on the boundary of } \mathbb{R}^2_+, \text{ i.e., } \Psi|_{\{x_1=0\}} = \Psi|_{\{x_2=0\}} = 0, \text{ and }
\]
\[
\Psi(x_1, x_2) + \Psi(y_1, y_2) \leq \Psi(x_1 \wedge x_2, y_1 \wedge y_2) + \Psi(x_1 \vee x_2, y_1 \vee y_2).
\]
If \( \Psi \) is twice continuously differentiable, then the above inequality is equivalent to
\[
\partial_{\Sigma_2} \Psi \geq 0.
\]
Crowe, Zweibel and Rosenbloom [6] noticed that a continuous such \( \Psi \) is the distribution function of a Borel measure \( \mu \) on \( \mathbb{R}^2_+ \), i.e.,
\[
(3.16) \quad \Psi(s, t) = \mu([0, s) \times [0, t)),
\]
and using Fubini’s theorem:
\[
(3.17) \quad \int \Psi(f(x), g(y)) \, dx \, dy = \int_{\mathbb{R}^2_+} \left[ \int_0^s \chi_{(f \leq s)}(x) \chi_{(g > t)}(y) \, dx \right] \, d\mu(s, t).
\]

We are now ready to state our next result.

**Theorem 3.5.** With \( \sigma_1, \sigma_2 \) and \( E = \{(x, y) \in S^1 \times S^1 : d(\sigma_1(x), \sigma_2(y)) \leq \alpha\} \) as in Theorem 3.2, and \( \Psi \) the distribution function of a Borel measure \( \mu \) on \( \mathbb{R}^2_+ \) as in (3.16), the following inequality holds for every \( \alpha > 0 \):
\[
\int_E \Psi(f(x), g(y)) \, dx \, dy \leq \int_E \Psi(f^2(x), g^2(y)) \, dx \, dy,
\]
if and only if, \( \sigma_1 \) and \( \sigma_2 \) are symmetric on \( S^1 \), and convex on \( S^1_+ \).

Again, we can reduce \( \Psi(f(x), g(y)) \) to a product of characteristic functions, using (3.17), and the result follows from Theorem 3.2.

The next theorem shows that we can replace the characteristic function of the set \( E \) by a decreasing function of the distance between \( \sigma_1(x) \) and \( \sigma_2(y) \), call it \( k[d(\sigma_1(x), \sigma_2(y))] \).

**Theorem 3.6.** Let \( \sigma_1, \sigma_2 \) be as in Theorem 3.2 and let \( k : [0, \infty) \to [0, \infty) \) be a decreasing function, and \( \Psi \) the distribution function of a Borel measure \( \mu \) on \( \mathbb{R}^2_+ \) as in (3.16). Then, the following inequality holds for every decreasing function \( k \),
\[
\int_{S^1} \int_{S^1} \Psi(f(x), g(y)) k[d(\sigma_1(x), \sigma_2(y))] \, dx \, dy \leq \int_{S^1} \int_{S^1} \Psi(f^2(x), g^2(y)) k[d(\sigma_1(x), \sigma_2(y))] \, dx \, dy,
\]
if and only if, \( \sigma_1 \) and \( \sigma_2 \) are symmetric on \( S^1 \), and convex on \( S^1_+ \).

**Proof.** Using (1.2), we can write:
\[
k(\tau) = \int_0^\infty \chi_{(k > t)}(\tau) \, dt = \int_0^\infty \chi_{[0, k(t)]} (\tau) \, dt,
\]
and substituting \( d(\sigma_1(x), \sigma_2(y)) \) for \( \tau \) in the above formula, we have
\[
(3.18) \quad k[d(\sigma_1(x), \sigma_2(y))] = \int_0^\infty \chi_{[0, k(t)]} [d(\sigma_1(x), \sigma_2(y))] \, dt.
\]
We define the set $E_{l(t)}$ as follows:

$$E_{l(t)} = \{(x, y) : d(\sigma_1(x), \sigma_2(y)) \leq l(t)\}.$$

Then

$$\chi_{[0,l(t)]} [d(\sigma_1(x), \sigma_2(y))] = 1 \iff (x, y) \in E_{l(t)}.$$

Using this fact, (3.18), Fubini’s theorem and Theorem 3.5 we obtain the conclusion of Theorem 3.6 by:

$$\int_{S^1} \int_{S^1} \Psi(f(x), g(y))k[d(\sigma_1(x), \sigma_2(y))] \, dx \, dy = \int_0^\infty \int_{S^1} \int_{S^1} \Psi(f(x), g(y)) \chi_{E_{l(t)}}(x, y) \, dx \, dy \, dt \leq \int_0^\infty \int_{S^1} \int_{S^1} \Psi(f^*(x), g^*(y)) \chi_{E_{l(t)}}(x, y) \, dx \, dy \, dt = \int_{S^1} \int_{S^1} \Psi(f^*(x), g^*(y))k[d(\sigma_1(x), \sigma_2(y))] \, dx \, dy.$$

□

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