# Can different theories of learning work together? Some results from an investigation into pupils' metacognition

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# Can different theories of learning work together?
# Some results from an investigation into pupils' metacognition

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## 1. DIFFERENT THEORIES

> *Nowadays one often hears complains that young people cannot calculate, and furthermore "when I was young, it was different, we really learnt some skills". I do not want to deny that this was exactly what you got in the traditional teaching, one did not waste time letting the pupils themselves find a way to calculate. One calculated one or two examples on the blackboard, made the rule clear, and then one let the pupils do the same kind of exercises until the pupils could do it with proficiency. Then the reversal came, the children should learn to think, this was the sole purpose of mathematics, all the mechanical stuff was an evil.*[1]

This quote is from the Danish female mathematics teacher Johanne Lütken in the year 1905. Personally I was rather shocked when I read it in Hansen (2002, p. 60). It seems that we are right back where we started! Have we really not learn anything during the last century? If she is right in her description, the pedagogical ideas are like a pendulum going back and forth between something we might term "skills" and something else which we might call "understanding". This is also one of the results of Hansen (2002) when he investigate the history of the teaching of mathematics and calculus in the Danish school 1739-1958. He concludes (Conference, November 2003, Trondheim, Norway) that the pendulum more or less has fluctuated as such:

| Skills, basics, content centric | Understanding, child centric |
|---|---|
| | 1814 |
| 1830 | |
| | 1850 |
| 1880 | |
| | 1903 |
| 1920 | |
| | 1958 |
| 1980 | |
| | 1995 |
| 2003 | |

[1] *In Danish: "Man hører ofte i vore Dage Klager over, at de Unge ikke kan regne, og der føjes til: "Da fik vi en anderledes Færdighed i vore Dage". Det kan heller ikke nægtes, at det netop var det, man fik ved den gammeldags Regneundervisning; man spildte ikke Tiden med at lade Børnene selv finde paa en Fremgangsmaade, men regnede et, højst to Stykker paa den store Tavle, slog Reglen fast, og saa lod man den regne Stykke efter Stykke af samme Slags... til de var sikre. Så kom Omslaget. Børnene skulle lære at tænke, og det var Regningens eneste Maal - alt det mekaniske var af den Onde."*

I will not go deeper into a discussion of these fluctuations in the few pages I have for this paper, but instead I would like to use this as yet another example of how we in mathematics education deal with the fact that there are different theories of for instance learning and that some of these theories seem "opposite" and some are even in conflict with one another and build on different epistemological and/or ontological basis. Another example is the relationship between Piaget and Vygotsky. This, and others, has been debated in a number of papers Lerman (1996, 2000, 2001), Steffe & Thompson (2000), Sfard (1998), Kieren (2000), and probably many others. Below I will show some of the debate between Piaget and Vygotsky themselves and then offer an alternative approach. I hope that this rather short paper can initiate a discussion about these issues.

## 2. THE DEBATE BETWEEN PIAGET AND VYGOTSKY

### 2.1 WHAT DOES PIAGET AND VYGOTSKY SAY ABOUT EACH OTHER?

Vygotsky wrote that "Psychology owes a great deal to Jean Piaget. It is not an exaggeration to say that he revolutionized the study of child language and thought" (Vygotsky, 1962, p. 9). Here Vygotsky praised the work of Piaget, but he also had some criticism of Piaget for instance around the concept of egocentrism[2] and egocentric speech. According to Vygotsky (1962, p. 14-15), Piaget's observations made him conclude that children's speech can only fall into two groups, the egocentric and the socialised. The difference between the two groups is mainly in their function since in egocentric speech, the child does only talk about himself and has no interest in others and expects no answers. Socialised speech attempts an exchange with others. According to Vygotsky, Piaget's experiments showed that most of preschool children's talk is egocentric but as the child approaches school age, egocentric speech atrophies. To Vygotsky a main and basic criticism of Piaget is that thinking, to Vygotsky, develops from the social level to the individual, while it is opposite for Piaget. Following the first translation into English, published 1962, of Vygotsky's *Thought and Language* (first published 1934), Piaget wrote in 1962 a "Comment" on Vygotsky's critique of him. Piaget (1962, p. 1) begins by stating:

[2] *The notion of egocentrism in Piaget's work is "quite unrelated to the common meaning of the term, hypertrophy of the consciousness of self. Cognitive egocentrism, as I have tried to make clear, stems from a lack of differentiation between*

> *It is not without sadness that an author discovers, twenty-five years after its publication, the work of a colleague who has died in the meantime, when that work contains so many points of immediate interest to him which should have been discussed personally and in detail … I was never able to read his [Vygostsky] writings or to meet him in person, and in reading his book today, I regret this profoundly, for we could have come to an understanding on a number of points. … on certain points I find myself more in agreement with Vygotsky that I would have been in 1934, while on other points I believe I now have better arguments for answering him.*

As Vygotsky, also Piaget honoured the work of the other. As a response to Vygotsky's critique of Piaget's previous view of egocentric speech, Piaget (1962, p. 7) argued:

> *he [Vygotsky] proposed a new hypothesis: that egocentric speech is the point of departure for the development of inner speech, which is found at a later stage of development, and that this interiorised language can serve both autistic ends and logical thinking. I find myself in complete agreement with these hypotheses. On the other hand, what I think Vygotsky still failed to appreciate fully is egocentrism itself as the main obstacle to the co-ordination of viewpoints and to co-operation.*

Above, Piaget declared that he agrees completely with Vygotsky in that, for instance, egocentric speech is the point of departure for the development of inner speech and that it is this inner speech that can serve logical thinking. Vygotsky emphasised that language is not just a means of expression; it is an instrument of thought. A place where they do still disagree is, according to Piaget, that Vygotsky still fails to understand that egocentrism itself can be a main obstacle for learning. This means that language can also hamper learning.

Piaget and Vygotsky do also disagree on other issues for instance the known duality of individual and social, but the abstract above from their discussion of the role of egocentric speech is an example to show that different theories can move closer to each other after some careful research and considerations. Had Vygotsky lived longer, or not been forbidden by Stalin, a mutual understanding of the two might have been achieved. Piaget believed that this would be possible. Vejleskov (1998, p. 117) has suggested to build a bridge between the two, by naming in co-constructivism and Vejleskov does also quote Bruner for, at a Piaget-Vygotsky congress in 1996, having said that Piaget owes us an explanation of how the self-regulation is taking place and Vygotsky owes us an explanation to why we do not all become a copy of the socio-cultural context in which we grow up. Mellin-Olsen (1989, p. 18) furthermore argues that the relationship between Vygotsky and Piaget can be interpreted as being dialectical. It is not either-or. Instead it is about,

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*one's own point of view and the other possible ones, and not at all from an individualism that precedes relations with others" (Piaget, 1962, p. 4).*

while teaching, to have these two theories in one's mind (as well as others) and then balance wisely. Also Sfard (1998, p. 4) wrote that "Strenuous attempts of many authors to come to terms with the change by forging theoretical bridges between the competing outlooks." She describes an acquisition metaphor and a participation metaphor that runs across the distinctions between the individualist-social perspective (Sfard, 1998, p. 7).

## 2.2 SIMILAR PROBLEMS IN SOCIOLOGY AND NATURAL SCIENCE

It is not only within psychology that one sees different theories competing as well as attempts to reconcile the theories. For instance in sociology Giddens puts the actor at the centre and with the notion of structure-duality he tries to transgress the sociology's traditional structure/actor dualism. With this notion Giddens wants to emphasise that social system's structural characteristics at once is a medium for and a result of the individual actors actions (Giddens, 1986, p. 25). Giddens' theory has however been criticised for being so abstract that it could not be employed in empirical research in practice (Gregson, 1989). For natural sciences physicists are faced with the problem of light. Is it a wave or a particle? The theories are mutually exclusive, but still physicists use both. "Niels Bohr's Principle of Complementarity … states that each description [of light] excludes the other, but both are necessary - they complement each other" (Marshall & Zohar, 1997, p. 101). Bohr applied the *Principle of Complementarity* in fields such as thought and action, subjectivity and objectivity, feeling and reasoning, male and female, the truths and values of one culture and those of another (Marshall & Zohar, 1997, p. 102). Researchers in mathematics education also use the concept of complementarity. For instance Sfard (1991, p. 4) wrote that "operational and structural conceptions of the same mathematical notion are not mutual exclusive. Although ostensibly incompatible … they are in fact complementary".

In relation to the possibility of a *grand theory*, Vygotsky thought that psychology ought not to be divided into different schools; he states: "As long as we lack a generally accepted system incorporating all the available psychological knowledge, any important factual discovery inevitably leads to the creation of a new theory to fit the newly observed facts" (Vygotsky, 1962, p. 10). He did not argue for complementarity, but for finding an overall theory or system that incorporates all available knowledge.

# 3. CAN PUPILS' METACOGNITION CAST LIGHT ON THE "WAR ON THEORIES"?

The discussion above shows an attempt to discuss these theories in relation to each other and hence what this tells us of how pupils learn. I would now like to present an "opposite" approach, namely one where I have (Dahl, 2004) interviewed some pupils about how they learn a mathematical concept that is new to them, and then investigated what knowledge this might give us of the relationship between the theories. It is a kind of grounded theory approach but where the explorative interviews are compared with a number of existing theories. If Mellin-Olsen, quoted above, is right, which I believe he is, teachers may need a tool to help them "balance wisely" in their teaching. To find out what to balance between, I will now investigate which "themes" various psychological theories of learning mathematics find are important for the cognitive side of learning mathematics. The way I will do this is to investigate *inter alia* the following authors' main work: von Glasersfeld, Hadamard, Krutetskii, Mason, Piaget, Polya, Schoenfeld, Skemp, and Vygotsky. These theories are classics that have survived critique through a longer period of time and I therefore assume that they express some central and valid points (Popper, 1979). I therefore develop the 'CULTIS model for analysis' (henceforth CULTIS) consisting of six themes: Consciousness - Unconsciousness; Language - Tacit; Individual - Social. The themes "go across" the theories and are not types or styles of learning, but one type of learning might have special characteristics within several themes. I divide the six themes into three binary opposite-pair. The themes are overall boxes to sort various areas and topics the students might mention for instance during an interview on the topic of learning.

I developed model through firstly to read the authors, wrote resumes of essential parts of their work, put all these resumes in one big document and then use word processing and comparison to find the general themes. I noticed for instance that several mentioned the topic of language and therefore I created a subsection titled "language" and put everything said about the effect of language under this headline. In the remaining document I then noticed that the unconscious was also mentioned a lot and subsequently created a new subsection with this title. And so forth. To some extent the themes interact and overlap each other but each have their own identity.

## 3.1 FIRST PAIR; THEME 1: CONSCIOUSNESS

According to Polya (1971) and Mason (1985), working with mathematics has three phases. First: 'enter' the problem, understand the problem and device a plan. Second: carry out the plan, and the third is a revision of the whole process. A plan is "based on past experience and formerly acquired knowledge" (Polya, 1971, p. 9). The pupil must furthermore understand the problem before starting to work on it, and the pupil should "desire its solution" (Polya, 1971, p. 6), or in other words, be motivated. In Activity Theory motivation plays the major role (Mellin-Olsen, 1989, p. 16-17), but is here mentioned as one item of many. According to Polya, it is a practical skill to be able to solve problems and since we acquire all practical skills by imitation and practice, this also applies for solving mathematical problems (Polya, 1971, p. 4-5). Mason writes that practice is important but without reflection it may leave no permanent mark, and that it also needs time. Mason also states that to support mathematically thinking one needs a questioning, challenging, and reflective atmosphere (Mason, 1985, p. 153).

### 3.2 FIRST PAIR; THEME 2: UNCONSCIOUSNESS

Hadamard (1945, p. 56) states that there are four stages in learning: preparation, incubation, illumination, and verification. Conscious work is preparatory to the illuminations. Polya states that "only such problems come back improved whose solution we passionately desire ... conscious effort and tension seem to be necessary to set the subconscious work going" (Polya, 1971, p. 198). The illumination is generally preceded by an incubation stage where the solving of the problem is completely interrupted (Hadamard, 1945, p. 16). The first stage in solving a problem is therefore to work in a very concentrated manner on it. What is experienced as sudden inspiration "is the result of previous protracted thinking, of previous acquired experience, skills, and knowledge" (Krutetskii, 1976, p. 305).

### 3.3 SECOND PAIR; THEME 3: LANGUAGE

Different theorists discuss the indispensable role of language, words, and concept formation in learning. Vygotsky describes language as the logical and analytical thinking-tool (Vygotsky, 1962, p. viii) and that thoughts are not just merely expressed in words but come into existence through the words (Vygotsky, 1962, p. 125). Furthermore, as mathematics in itself is a language (Pimm, 1990, p. 2; Dahl, 1996), it becomes important also to discuss concept formations. In relation to the learning of mathematical concepts, a basic principle is that all concepts, except the primary ones,

are derived from other concepts and they take part in the formation of other concepts (Skemp, 1993, p. 35). This conceptual structure is called a schema, and a schema is therefore a tool for learning as it integrates existing knowledge (Skemp, 1993, p. 37). Similarly, Tall (1991, p. 9) refers to the notions of *assimilation,* a process by which an individual adopts new information and *accommodation,* which signifies that the individual's cognitive structure must be changed. Thus, it seems that language is not essential for the creation of the basic concepts. But higher order concepts build on the basic concepts and to learn the higher order concepts, other concepts are necessary. Furthermore (Skemp, 1993, p. 29-30), an integrated conceptual structure is easier to remember than unconnected rules.

### 3.4 SECOND PAIR; THEME 4: TACIT

There are also more negative views of the language as a tool for learning. To Hadamard, "thoughts die the moment they are embodied by word" (1945, p. 75) however, "signs are necessary support of thought" (Hadamard, 1945, p. 96). Piaget (1970, p. 18-19) states that "the roots of logical thought are not to be found in language alone, even though language coordinations are important, but are to be found more generally in the coordination of actions, which are the basis of reflective abstraction". In relation to tacit knowledge, one can observe that a person has a certain kind of knowledge, but "on questioning, it appeared that he did not know he was doing this. Here the subject got to know a practical operation, but could not tell how he worked it" (Polanyi, 1967, p. 8). The "negative" arguments are thus centred on the general uselessness of words in thinking and learning, that language merely "supports thinking", and the inability to describe what one is doing.

### 3.5 THIRD PAIR; THEME 5: INDIVIDUAL

Glasersfeld's epistemology is that "knowledge, no matter how it is defined, is in the heads of persons, and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience. … all kinds of experience are essentially subjective" (Glasersfeld, 1995, p. 1). The basis of the abstractions that leads to mathematical-logical knowledge is the action itself, not the object (Piaget, 1970, p. 16-18). The individual who learns is therefore active and the acknowledgement comes as the individual manipulates with the objects and reflects on this manipulation. Piaget talks here about reflective abstractions that are based on coordinated actions. This therefore means that (1) language is not the main thinking-tool, (2) both individual

actions and the individual performs coordinated ones and they both lead to abstraction, but it is the latter that leads to reflective abstractions and then to logical-mathematical knowledge. Piaget therefore finds that logical-mathematical abilities do not arise from language or linguistic competency, but from the ability to coordinate actions and operate with objects.

### 3.6 THIRD PAIR; THEME 6: SOCIAL

Social interaction plays a fundamental role in shaping pupils' internal cognitive structure (Schoenfeld, 1985, p. 141). This process has two levels: "first between people … and then inside the child ... All higher functions originate as actual relations between human individual" (Vygotsky, 1978, p. 56-57). This process of internalisation is gradual. In the beginning a teacher controls and guides the pupil's activity, but later they begin to share the problem-solving functions, and here it is the pupil who takes the initiative while the teacher corrects and guides. At last, the pupil is in control and the teacher's role is mainly supportive. According to Vygotsky, the potential for learning is limited to the "zone of proximal development (ZPD)" (Vygotsky, 1978, p. 86). ZPD is the area between the tasks a pupil can do without assistance, and those that require help. It is therefore essential that pupils are active and have the opportunity to be guided by a knowledgeable person. Verbal thinking is an example of a social activity. When the pupil speaks aloud, the "audible speech brings ideas into consciousness more clearly and fully than does sub-vocal speech" (Skemp, 1993, p. 91-92). Vision is therefore individual, while hearing is collective (Skemp, 1993, p. 104).

### 3.7 THE SIX THEMES IN RELATION TO EACH OTHER

The four last themes seem to be connected since the language (Theme 3) and social (Theme 6) have some intersection and the same can be said about the tacit (Theme 4) and the individual (Theme 5). Another link between the themes is between the first and third theme: one could argue that to get a conscious planning, it is necessary with metaknowledge which again might presuppose that this knowledge has been made linguistic - unless metaknowledge can be tacit.

The third pair deals with the dichotomy individual-versus-social. However, Piaget used 'social' in the context of ontogenesis in relation to techniques of learning. To Vygotsky, social was not meant to only denote the mechanism of social interaction, which is essential for the individual acquisition of knowledge, but also to denote the social phylogenesis of knowledge. One might

therefore argue that there are in fact two dichotomies. One is *individual - collective* and the other is *natural - social*. The first dichotomy regards the nature of activities that lead to learning as well as the creation of knowledge. The other dichotomy regards the ontological-epistemological status of knowledge. In the Piaget tradition knowledge has its sources in the world itself and could, in principle, be learnt directly from 'the nature'. In the Vygotskian sociocultural school, what we learn is inherently a product of human communication and even though it is constraint by reality, it would not exist for us, if we were not part of the human community. However, since the term 'social' is generally used also denoting 'collective', I will stick with this word in the present paper. An advantage of having a model for analysis such as CULTIS is that it provides an easier tool for spotting differences and alikeness between the theories.

### 3.8 CULTIS SUMMING UP, KEYWORDS

There are therefore six themes in the CULTIS model for analysis to which the students' narratives can belong. To some extent they overlap and interact with each other. The six themes are:

- Theme I: Consciousness
  Keywords: *practice, planning, reflection, control, self-confidence, positive atmosphere, motivation.*
- Theme II: Unconsciousness
  Keywords: *preparatory work, incubation, illumination*.
- Theme III: Language
  Keywords: *language as the prime thinking-tool, basis is important, schematic understanding, rote-learning, assimilation, accommodation.*
- Theme IV: Tacit
  Keywords: *words can hamper learning, words only support thinking, cannot tell but only show.*
- Theme V: Individual
  Sub theme/keywords: *construction, self-activity, visualisation.*
- Theme VI: Social
  Keywords: *internalisation, guiding, interaction, discussion, ZPD, hearing, verbalisation.*

## 4. EXAMPLES

I did some interviews with some high-achieving high school pupils in Denmark and England. The pupils were between the age of 17 and 20 and I performed in-depth explorative interviews with them focusing on getting their metacognition around how they learn a mathematical concept that is new to them. Schoenfeld (1985, 1992) discussed the notion of metacognition, which can be understood both as knowledge about and regulation of cognition (Schoenfeld, 1992, p. 334). Knowledge about cognition means that one has relatively stable information about one's own

cognitive processes. This knowledge develops with age and "performance on many tasks is positively correlated with the degree of one's metaknowledge" (Schoenfeld, 1985, p. 138). The pupils' conceptual models influence the problem solving behaviour: "Expert behavior, in which the appropriate resources are routinely accessed, is a result of the experts' possession of stable conceptual models. Conversely, many students' difficulties are due to the fact that their conceptual models are unstable" (Schoenfeld, 1985, p. 139). One can therefore assume that successful pupils know how they learn mathematics. The English pupils also got a sheet with some basic know theory to initiate the discussion. Below I will show a few examples of what the pupils said.

Example 1 - from an interview with two English boys:

*C You know, it all boils down to the teaching method and the teacher not just about, it's a two-way thing you see, it's more about you learning, you being able, no, you learning as well you being taught properly, if you are taught in a way that you can fit in, you know, then it is good.*

Example 2 - from an interview with four Danish girls:[3]

*Æ But, I think that it is not only about the teacher, it is something [inaudible] a combination of that the teacher comes with some inputs which one has to work with oneself, and then one can return to the teacher to get new ones, right. But the way these input comes, that I think [inaudible] plays a role for how one can understand it further on.*

Here one, in brief, sees that these two pupils discuss what might be termed to be a complementarity between Theme 5 and 6 - individual-social. They use words such as "two-way thing" and "combination" to describe the relationship between the personal activity and the external guidance. Burton has done research on how mathematicians say they learn mathematics and her investigation was based on interviews. One of her conclusions was that: "Learning is *neither* wholly individual *nor wholly* social" (Burton, 1999, p. 139).

Example 3 - from an interview with four Danish girls:[4]

[3] *In Danish: "Æ: Jamen, jeg, jeg tror, jeg tror ikke kun at det drejer sig ikke kun om læreren, det er noget [inaudible] en kombination af at læreren kommer med nogle inputs, som man selv så må arbejde med, og så kan man egentlig vende tilbage til læreren og få nogle nye, ikke. Men den der måde som de der input så kommer på, den tror jeg [inaudible], det, det spiller en rolle på, altså, hvordan det kommer for at man videre kan forstå det, altså."*

[4] *In Danish: "Z: Det er jo også vigtigt, altså den der motivation ikke, som jeg også tror der ligger i det, den der forvirring, der ligger også en eller anden form for motivation ikke. Man VIL sgu gerne lære det. (I: mmm) Og det bliver man så nød til at have, fordi at det hjælper ikke noget bare at skulle sidde og lære hele det der matematikSPROG først vel. Jeg tror også det vi [inaudible] må holde fast i det hvor du [Ø] sagde, at det du sagde, det kunne være en person der li'som havde flair for det ikke, kunne se den her ting for sig på en anden måde. Og jeg tror måske også det, det er der skellet ligger mellem og så begynde at tænke SELV og kreativt, på en eller anden måde ikke, fordi hvis man virkelig*

*Z* *It is also important, this motivation, which I also thinks lies in the confusing, there is a kind of motivation there, right. One really WANTS to learn it. (I: mmm) And this you need to have, because it does not help just sitting and learning this mathematics LANGUAGE first does it. I also think that we [inaudible] have to stick to what you [Ø] said, that perhaps a person has a kind of flair for it and is able to see this thing in another way. And I also think that this is where the distinction is between starting to thing FOR YOURSELF and creatively, in some kind of way, because if one really has this flair [inaudible] one can just see this spatial geometry, one can see it inside one's head the moment one is told about it. And then one does not need all these concepts, if one can see it for oneself. (Someone else: mmm), and then I think that one begins to work in quite a different way with it (Someone else: yes) [a few seconds silence]. One can also first [inaudible] concepts and examples.*

*I* *Do you think perhaps that there is a ping-pong between that one in a way can see it and then the language?*

*Z* *Yes.*

Here one sees that the pupils argue for a complementary relationship between the language and visual side. Hence a complementarity between Theme 3 and 5.

Example 4 - from an interview with a boy and a girl from England:

*E* *Depends who they [authors to the book about knot theory] are targeting it at. Don't use such big words, they are aiming to people who don't understand it and use basic. It's the way they approach it, the language, it's just too, people would struggling with the language when they are suppose to be learning the maths.*

*I* *So is there a diff, I mean, er, so maths has nothing to do with the language? Or, can you learn maths without language.*

*D* *Yea.*

*E* *No, but you can use different language, simple language to convey a point.*

*D* *Cause the maths in it is quite easy, I think, well, it's not. It is nothing really difficult what it is saying is this is what a knot is, this is (E: Yea) what a link is, and, OK, that really really simplistic. It took me a long time to work out what they were trying (E: Yea what they were explaining) whereas the fact as soon as I, kind of translated it, I thought oh well, that's what a knot is, find that's easy.*

*I* *What did you translated it.*

*D* *Into simple language [laughs] er, it er.*

*I* *You translate it before you understand it, er, so if you have understand, then, you don't need to translate it.*

*E* *I think it here would be easier if the author translated (D: Yea [laughs]) rather than er leaving the reader to do it, I mean.*

*D* *You have you have do the two together, you have to translate while you're trying to understand*

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*har den der flair, [inaudible] man bare kan se det der rumgeometri, det kan man bare se, inde i hovedet så snart man får det at vide. Og så behøver man ikke at have alle de der betegnelser egentlig vel, hvis man kan se det for sig (En anden: mmm), og da tror jeg måske også, da begynder man også at kunne arbejde på en anden måde med det. (En anden: ja) [et par sekunders stilhed] Man kan også godt først [inaudible] betegnelser og eksempler.*
*I: Synes I det måske er ping-pong mellem at man på en måde kan se det og så også sproget?*

This shows that the pupils are aware of the dual nature of language, both as one that can create thinking and also the one that destroys thinking. Hence, a complementarity of Theme 3 and 4.

## 5. CONCLUSIONS

The four examples from my own research as well as the example from Burton shows that we might learn something from drawing on people's metacognition - together with gathering information from many other sources. In reality we really do not know the cognitive processes and we will always be faced with the problem of how to investigate something we cannot see or measure. This is the problem of all research on mental processes. Therefore we have to "sneak into" these processes for instance through metacognitive dialogues - but not only that. This type of research must be compared with research in for instance pupils' diagrammes (Dörfler, 2004), observation of pupils etc. Then, when we add all these information, and sit and "compare notes" we might use the various results as bricks building a wall where some of the bricks may have to be reshaped a bit to make the wall looks right. Anyway, one could also ask why there are so many different theories. I think that one answer might be that the different theories build on different empirical work. Krutetskii's study is based on 201 pupils (primary and lower secondary) during the years 1955-1966 while Schoenfeld investigates university students. Piaget and Vygotsky often write about "children", and some of Hadamard's examples are taken from famous mathematicians. Perhaps it is not surprising that the theories express different things. But my main point is that to get wiser on the cognitive side of learning, we must not only discuss the theories in relation to each other, such as the debate between Lerman and others, but equally important is it to collect more empirical data that might cast light on these discussions. This paper is an attempt to start such a debate.

Finally, going back to where the paper started, the discussion of the pendulum effect, Sfard (1998, p.11) wrote something which I find might begin the "cure" of the pendulum:

> *Because no two students have the same needs and no two teachers arrive at their best performance in the same way, theoretical exclusivity and didactic single-mindedness can be trusted to make even the best of educational ideas fail. What is true about educational practice also holds for theories of learning. It seems that the most powerful research is the one that stands on more than one metaphorical leg ... giving full exclusivity to one conceptual framework would be hazardous. Dictatorship of a single metaphor, like a dictatorship of a single ideology, may lead to theories that serve the interests of certain groups to the disadvantage of others.*

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*Z: Ja."*

# REFERENCES

Burton, L. (1999). The practices of mathematicians: What do they tell us about coming to know mathematics? *Educational Studies in Mathematics, 37,* 121-143.

Dahl, B. (1996). Læring som sprogspilsoverskridelse [Learning Mathematics as Language-Games Transitions]. *Nordic Studies in Mathematics Education, 4*(1), 7-24.

Dahl, B. (2004). Analysing cognitive learning processes through group interviews of successful high school pupils: Development and use of a model. *Educational Studies in Mathematics*. Forthcoming.

Dörfler, W. (2004). Mathematical Reasoning and Observing Transformations of Diagrams. *Paper presented at MADIF-4 The Fourth Swedish Mathematics Education Research Seminar, Malmö, Sweden, January 21-22, 2004.*

Giddens, A. (1986). *The Constitution of Society. Outline of the Theory of Structuration*. Cambridge: Polity.

Glasersfeld, E. v. (1995). *Radical Constructivism: A Way of Knowing and Learning*. London: Falmer.

Gregson, N. (1989). On the (ir)relevance of Structuration Theory to empirical research. In D. Held & J. Thompson (Eds.), *Social Theory of Modern Society - Anthony Giddens and his Critics* (pp. 235-248). Cambridge: Cambridge University.

Hansen, H.C. (2002). *Fra forstandens slibesten til borgerens værktøj: Regning og matematik i folkets skole 1739-1958* (Dansk Center for Naturvidenskabsdidaktik, DCN (Center for Educational Development in University Science) Papers from DCN, No. 16, Aalborg University, Denmark).

Hadamard, J. (1945). *An Essay on The Psychology of Invention in the Mathematical Field*. New York: Dover.

Krutetskii, V. A. (1976). *The Psychology of Mathematical Abilities in Schoolchildren*. Chicago: The University of Chicago.

Lerman, S. (1996). Intersubjectivity in Mathematics Learning: A Challenge to the Radical Constructivist Paradigm? *Journal for Research in Mathematics Education, 27*(2), 211-223.

Lerman, S. (2000). A case of interpretations of social: A response to Steffe and Thompson. *Journal for Research in Mathematics Education, 31*, 210-227.

Lerman, S. (2001). Cultural discursive psychology: A sociocultural approach to studying the teaching and learning of mathematics. *Educational Studies in Mathematics, 46,* 87-113.

Marshall, I. & Zohar, D. (1997). *Who's Afraid of Schrödinger's Cat?*. London: Bloomsbury.

Mason, J. (1985). *Thinking Mathematically* (with L. Burton & K. Stacy). Amsterdam: Addison-Wesley.

Mellin-Olsen, S. (1987). *The Politics of Mathematics Education*. Dordrecht: Kluwer.

Mellin-Olsen, S. (1989). *Kunnskapsformidling. Virksomhetsteoretiske perspektiver*. Rådal: Caspar.

Piaget, J. (1962). *Comments on Vygotsky's critical remarks concerning 'The Language and Thought of the Child', and 'Judgment and Reasoning in the Child'*. Cambridge Massachusetts: The M.I.T.

Piaget, J. (1970). *Genetic Epistemology*. New York: Columbia University.

Pimm, D. (1987). *Speaking Mathematically - Communication in Mathematics Classroom*. London: Routledge.

Polya, G. (1971). *How To Solve It - A New Aspect of Mathematical Method*. Princeton: Princeton University.

Polanyi, M. (1967). *The Tacit Dimension*. London, Routledge.
Popper, K. (1979). *Objective Knowledge - An Evolutionary Approach*. Oxford, Clarendon.
Schoenfeld, A. H. (1985). *Mathematical Problem Solving*. London: Academic.
Schoenfeld, A. H. (1992). Learning to Think Mathematically: Problem Solving, Metacognition, and Sense Making in Mathematics. In D. A. Grouws (Ed.) *Handbook of Research on Mathematics Teaching and Learning* (pp. 334-370). New York: MacMillan.
Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1-36.
Sfard, A. (1998). On two metaphors for learning and the danger of choosing just one. *Educational Researcher, 27*(2), 4-13.
Skemp R. R. (1993). *The Psychology of Learning Mathematics*. London: Penguin.
Steffe, L. P., & Thompson, P. W. (2000). Interaction or subjectivity? A reply to Lerman. *Journal for Research in Mathematics Education, 31*, 191-209.
Tall, D. (1991). The Psychology of Advanced Mathematical Thinking. In D. Tall, D. (Ed.), *Advanced Mathematical Thinking* (pp. 3-21). Dordrecht: Kluwer.
Vejleskov, H. (1998). Teorier om kognitiv udvikling som inspiration for pædagogikken. In N. J. Bisgaard (Ed.), *Pædagogiske teorier* (pp. 94-119). Værløse: Billesø & Baltzer.
Vygotsky, L. S. (1962). *Thought and Language*. Cambridge Massachusetts: The M.I.T.
Vygotsky, L. S. (1978). *Mind in Society - The Development of Higher Psychological Processes*. Cambridge Massachusetts: Harvard University.