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Published in:
ZDM

DOI (link to publication from Publisher):
https://doi.org/10.1007/BF02655907

Publication date:
2006

Document Version
Early version, also known as pre-print

Various theoretical frameworks in concept construction and how to move forward as a field:
A commentary to Pegg and Tall

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Abstract: This paper first summarises and discusses Pegg and Tall's (2005) fundamental cycle model of conceptual construction from action to object and its relationship to other theories. Then the paper compares this with another model of different psychological theories of learning mathematics and discusses how these models can either be merged or complement each other. This leads to a general discussion about the problem of having many different theories and fashions, how knowledge grows and accumulates, and if there is a unifying theory to be found. The paper concludes that the development of meta-theories, such as in the work of Pegg and Tall, is necessary rather than uncritical complementarism.

ZDM Classification: C30

1. Introduction

I have been given the honour of writing a commentary to Pegg and Tall's paper (2005) in a previous issue of ZDM. I will therefore first summarise and discuss some of their main points; then I will compare their model of several different theoretical frameworks with one that I have developed (Dahl 2004a) and try to mingle these two. This will lead to an overall philosophical discussion of the growth of knowledge, particularly within the field of mathematics education research.

2. Fundamental cycles of concept construction underlying various theoretical frameworks

I will not go into all the details of Pegg and Tall's paper but instead focus on what I see as their main agenda, namely their focus on building mathematical concepts over time and their analysis of different theories of concept construction and development. They want to raise the debate above a simple comparison of theories to reveal a fundamental cycle underlying the development of concepts that characterises different ways of thinking during learning mathematics. They distinguish between two kinds of theories of cognitive growth. The first kinds are *global* frameworks of long-term growth of the individual such as the stage-theory of Piaget, van Hiele's theory of geometric development, the long-term development of the enactive-iconic-symbolic modes of Bruner, and the five SOLO modes (Structure of the Observed Learning Outcome). In SOLO, each mode with its operations is included within the next; hence the learner has an ever increasing repertoire of modes of operation (Pegg & Tall 2005, p. 468-469).

Table 1. Global stages of cognitive development

Piaget Stages	van Hiele Levels (Hoffer,1981)	SOLO Modes	Bruner Modes
Sensori Motor	I Recognition	Sensori Motor	Enactive
Pre-operational	II Analysis	Ikonic	Iconic
Concrete Operational	III Ordering	Concrete Symbolic	Symbolic
Formal Operational	IV Deduction	Formal	
	V Rigour	Post-formal	

The second kinds of theories are *local* frameworks of conceptual growth such as Dubinsky's action-process-object-schema (APOS), Sfard's interiorization-condensation-reification, Gray and Tall's procedure-process-procept, and the unistructural-multistructural-relational-unistructural (UMR) levels in the SOLO framework.
The first level of SOLO's local framework is the unistructural (U) which focuses on the problem or domain but only uses one piece of data. The second level, multistructural (M), focuses on two or more pieces of data with no integration among the pieces. The third level, relational (R), focuses on all the data available and each piece is woven into an overall coherent structure. The UMR cycle is a recurring cycle that operates on different levels and modes as well as on the construction of new concepts. The UMR cycle happens in each of the (global) SOLO modes and this cycle in one mode might lead to further abstraction in the next one (Pegg & Tall 2005, p. 469).

Regarding the theories of Dubinsky and Sfard, Pegg and Tall state that there are differences in detail between them. "For instance, Sfard's first stage is referred to as an 'interiorized process', which is the same name given in Dubinsky's second stage" (2005, p. 471). Despite this difference Pegg and Tall find that the overall processes that they describe are broadly similar, namely to begin with actions on known physical or mental objects. These actions are practised until they become routine step-by-step procedures that are seen as a whole as processes. Then they become conceived as independent entities on which the learner can operate on at a higher level to give a further cycle of construction. What is therefore in common of the theories used by Pegg and Tall is that they involve "a shift in focus from *actions* on already known objects to thinking of those actions as manipulable mental *objects*" (2005, p. 471).

According to Pegg and Tall, one can apply a SOLO analysis to the theoretical frameworks such as Dubinsky's: The initial action is at a unistructural level of operation where a single procedure is used to solve a particular problem. The multistructural level uses alternative procedures that are not seen as interconnected wherefore they remain at the action level of APOS. The relational level is when different procedures with the same effect are seen as essentially the same process. Hereby the process gets encapsulated as an object (unistructural level) that can be used in the further development of knowledge (2005, p. 472). The transition from process to mental object is different in arithmetic, algebra, trigonometry, calculus, and formal mathematics. Furthermore, Pegg and Tall state that even though there are differences in the various theories of concept construction, they have the same fundamental cycle of concept construction from ‘do-able’ action to ‘think-able’ concept. And there are “corresponding cycles giving increasingly sophisticated conceptions in successive modes of cognitive growth” (2005, p. 473).

Table 2: The fundamental cycle of conceptual construction from action to object (Pegg & Tall 2005, p. 473).

Constructing a Concept via Reflective Abstraction on Actions					
SOLO [Structure of Observed Learning Outcome]	Davis	APOS	Gray & Tall	Fundamental Cycle of Concept Construction	
Unistructural	Visually Moderated Sequence as Procedure	Action	Base Object(s)	Known objects	SCHEMA
			Procedure [as Action on Base Object(s)]	*Procedure as Action* on Known Objects	
Multistructural			[Alternative Procedures]	*[Alternative Procedures]*	
Relational	Process	Process	Process	*Process* [*as Effect* of actions]	
Unistructural [new cycle]	Entity	Object	Procept	Entity as *Procept*	
		Schema			

Pegg and Tall have shown that the theoretical frameworks in their model share the same underlying local cycle of learning (2005, p. 474). It is very interesting that they bring together ‘different’ theories and show that they share this similar cycle of concept development. They, so to speak, build a meta-theory on top of a number of ‘different’ theories. However, their strength might also be their “weakness” since, being the devil’s advocate; they show that rather similar theories are – rather similar. However, it requires an analyse to know which theories do in fact share central features. But what about theories of cognitive development that do not fit the action-object process? I will discuss this again in Section 4.

3. Another framework of cognitive processes

In an attempt to move beyond various dichotomies in the psychology of learning mathematics, I developed a model incorporating a number of different theories focusing on cognitive processes. The theorists were Dubinsky (Asiala et al. 1996), Ernest (1991), Glasersfeld (1995), Hadamard (1945), Krutetskii (1976), Mason (1985), Piaget (1970, 1971), Polya (1971), Sfard (1991), Skemp (1993), and Vygotsky (1962, 1978). I chose theories that have reached a status of being “classics”. The model goes across the theories and sorts the different theories’ statements on various themes; in that sense “mixing” them. There are overlaps and some of the themes interact with each other, but they each have their own identity. CULTIS is therefore a model of various local frameworks even though for instance Piaget also has a global perspective. I used the model as a sorting-tool to analyse a huge amount of interview and observation data taking different theories into account. This revealed that each student in my study referred to elements that are associated with different, sometimes opposing, theories. The model and research results are described in details in Dahl (2004a). Here I will shortly present the model and try to merge it with the fundamental cycle model of Pegg and Tall (2005, p. 473). I named the model ‘CULTIS’, which is an acronym of the first letter in each of the themes. In the table below the six themes are mentioned as binary opposite pairs and there are phrases and keywords for parts of the different theories that falls into the theme in question:

Table 3. Overview of the CULTIS model.

Theme I: Consciousness	*Theme II: Unconscious*
<u>Polya:</u> 4 phases: understand the problem and desire the solution, devise a plan, carry out the plan, look back and discuss. Imitation and practice important. <u>Mason:</u> 3 phases: entry, attack, and review. Practice with reflection important. <u>Sfard:</u> 3 stages: interiorization, condensation (operational understanding), reification (structural understanding). <u>Dubinsky</u>: Action, process, object, schema (APOS). <u>Skemp</u>: Do automatic manipulation with minimal attention.	<u>Hadamard:</u> 4 phases: preparation, incubation, illumination, verification. Illuminations cannot be produced without unconscious mental processes. The incubation gets rid of false leads to be able to approach the problem with an open mind <u>Krutetskii:</u> A sudden inspiration results from previously acquired experience, skills, and knowledge. <u>Polya:</u> Conscious effort and tension needed to start the subconscious work. Mason: Time is necessary.
Theme III: Language	*Theme IV: Tacit*
<u>Polya:</u> Good ideas are often connected with a well-turned sentence or question. <u>Vygotsky</u>: Language is the logical and analytical thinking-tool. Thoughts are created through words. <u>Skemp</u>: 2 principles: Higher order concepts cannot be communicated by a definition but only through examples. All concepts except the primary are derived from other concepts. <u>Piaget</u>: Assimilation takes in new data, accommodation modifies the cognitive structure. To know an object is not to copy it but to act upon it. To know reality is to construct systems of transformations that correspond to reality.	<u>Polya:</u> A student who behaves the right way often does not care to express this in words and, possibly, he cannot express it. <u>Hadamard:</u> Thoughts die the moment they are embodied by words but signs are necessary support of thought. <u>Piaget</u>: The root of logical thought is not the language but is to be found in the coordination of actions, which is the basis of reflective abstraction. An abstraction is not drawn from the object that is acted upon, but from the action itself. <u>Skemp:</u> Primary concepts can be formed without the use of language.
Theme V: Individual	*Theme VI: Social*
<u>Glasersfeld:</u> Knowledge is in the heads of persons and the thinking subject has no alternative but to construct	<u>Vygotsky:</u> 2 levels in internalisation: interpsychological, intrapsychological. First a teacher

knowledge from own experience. All experience is subjective. Piaget: Reflective abstractions are the basis of mathematical abstraction. They are based on coordinated actions such as joint actions, actions succeeding each other etc. The individual is therefore active and learns as he manipulates with the objects and reflects on this manipulation. Skemp: Some have strong visual imaginations. To communicate visual pictures one needs a drawing. Individuals have to construct a conceptual system themselves.	guides, then they share the problem solving, at last the learner is in control and the teacher supports. The potential for learning is in the zone of proximal development (ZPD). Ernest: Reconstruct objective knowledge as subjective knowledge through negotiation with teachers, books, or other students. Skemp: Talking aloud brings ideas into consciousness more fully than sub-vocal speech. One can solve a problem after talking aloud about it even if the listener has not interfered. In a discussion this effect is on both sides.

4. Merging the frameworks

CULTIS can be refined on particularly the first theme where CULTIS uses both Sfard and Dubinsky, which also Pegg and Tall do. Instead of using these two theorists in CULTIS one could add the whole model of Pegg and Tall into CULTIS's theme 1, as their model is a kind of meta-theory on Dubinsky, Sfard, etc. Pegg and Tall also state that:

> It is not claimed that this is the *only* way in which concepts grow. ... there are different ways in which concepts can be constructed, including constructions from *perceptions of* objects, *actions on* objects and *properties of* objects. ... Significantly, all of these can be categorised so that the learning outcomes can be analysed in terms of the SOLO UMR cycle (2005, p. 473-474).

This links to the discussion at the end of Section 2. Pegg and Tall might also want to discuss Skemp who also focused on concept creation. Does Skemp's theory fit the SOLO UMR cycle? From a tentative look, it does not seem so. Skemp (1993, p. 29-42) argued that in learning mathematical concepts there are two basic principles (see Theme 3 in CULTIS). The examples that the learner should experience must be alike in the features that should be abstracted, and different in the ways that are irrelevant for the particular concept. But these examples do not seem to have to be actions or processes, like in the APOS and SOLO theory. However, *if and when* these examples *are* such processes, the change from experiences a number of examples to create the concept might have resemblance with the transition from the multistructural level to the relational level. It also fits with Skemp's second principle that the actions are performed on already known objects. Skemp's theory also works with the concept of a schema that is also part of APOS, which might then provide a link to Pegg and Tall's model. Skemp furthermore argues that naming an object classifies it and when an object is classified, then we know how to behave in relation to it. This is almost the opposite of Pegg and Tall's model where the object is constructed as a result of the actions. Lastly, Skemp criticised the use of rote-memorised rules to manipulate symbols and arithmetic. He states that it will create unconnected rules that are harder to remember than an integrated structure. Hence both Skemp and Pegg and Tall have as a goal an integrated conceptual structure, a schema, but they seem to argue for it from two different angles. Skemp's critique of the manipulations of symbols could seem like a potential critique of the action-object model of Pegg and Tall. It needs further analysis but it would be interested to see how they, and other theories, differ and if it is possible to create another meta-theory in top of these and if this analysis reveals any white spots. As Pegg and Tall state themselves, their focus is to raise the debate beyond a simple comparison of details in different theories. This analysis and discussion will in my opinion create a kind of meta-theory that moves the knowledge and discussion to a higher level. This is also the purpose of CULTIS.

One of the questions that the Pegg and Tall paper raises, as well as the CULTIS model, is, how do we handle the fact that there are these different theories? Some of them are very different, even opposing, others are rather similar. Vygotsky stated that psychology ought not to be divided into different schools: "As long as we lack a generally accepted system incorporating all the available psychological knowledge, any important factual discovery inevitably leads to the creation of a new theory to fit the newly observed facts" (1962, p. 10). We do not have a generally accepted system incorporating all the available knowledge in mathematics education research. Do we want it? Are we accumulating knowledge or do we sit by ourselves and continue on our tangent making the octopus of research of mathematics education grow ever bigger, or do we actually fight each other or simply just never meet? What Pegg and Tall are doing in their paper is in fact the opposite – namely drawing various slightly different theories together and theorise on them. Pegg and Tall's model is a step towards a system that can integrate existing knowledge as it already now integrates a number of slightly different theories. But also more different theories need to be considered and analysed to enhance the knowledge of concept construction. CULTIS, on the other hand, includes, but not always integrates, many different theories. This might not in the long run create a single theory, but it might create a structure from which one can identify areas, white spots, that need more research. Mewborn argues: "Moving toward predictive frameworks is not going to come from doing more studies alone; it will come from thoughtful analysis of a large collection of existing studies" (2005, p. 5). I will devote the rest of the paper to discuss this.

5. Where are we as a field?

During at least the last decade there has been discussions among researchers in mathematics education about where we are as a field. One of the problems with not having a generally accepted system to incorporate different theories is that it affects the education system, which is part of the reason why the history of mathematics education has been characterised by pendulum swings (Dahl 2005).

Pendulum swings are also seen within the research of mathematics education. At the tenth International Congress on Mathematics Education (ICME-10) in 2004, a Discussion Group was entitled: "Different perspectives, positions, and approaches in mathematics education research". The report from this states that it is difficult to accumulate knowledge in mathematics education research due to various approaches to mathematics education research that sometimes appear as fashion waves. The diversity might be an advantage if it gives a more complete picture but it also causes fragmentation, which hinders that the field can be recognized as a discipline with a coherent body of knowledge. Further, the diversity makes communication complicated. Advantages are however that by focusing on a single aspect, this aspect can be thoroughly examined. But often when the fashion fades, the results obtained during this period are forgotten wherefore it becomes difficult to lay a strong and lasting foundation of research for understanding educational phenomena. Furthermore there is not a common knowledge base on which to refute claims made and we do not focus enough on knowledge accumulation or on saying which theories and studies are important. "The group agreed that we need to respect different schools of thought for what each has to offer and take the best of each one" (English & Sierpinska 2004).

In an ICMI (International Commission on Mathematical Instruction) Study from 1998, Brown stated that mathematics education research has expanded from psychometrics in the 1950s into inquiry that draws eclectically on theories and methodologies from science, social science, and the humanities. "As this field of inquiry has become more diffuse, both its rationale and it scope of potential utility has become more diffuse" (p. 263).

In this respect Pegg and Tall's (2005, p. 471) work actually build on previous work and disciplines, hence accumulate, when they for instance explain how the view of cycles of cognitive development is consistent both with the tradition of Piaget and neurophysiology and also in the fact that they build on existing theories. But how does science (not just *natural* science) progress and how can one integrate knowledge over time?

6. How do we move forward?

How does knowledge in general grow? How *ought* knowledge to grow? How do the models of Pegg and Tall (2005) and of Dahl (2004a) described above fit into this?

6.1 Discontinuity and lack of progress

According to Kuhn, a new theory does not have to be in conflict with old theories if it deals with phenomena not previously known or if it is on a higher level. If this was always the case, "scientific development would be genuinely cumulative" (1996, p. 95). Such a description seems to fit the work of Pegg and Tall (2005) who further theorise on the differences of existing theories and create a model of these theories. However, Kuhn states that history shows "that science does not tend toward the ideal that our image of its cumulativeness has suggested" (1996, p. 96). This historical picture of science seems to fit with the actual history of mathematics education research as discussed above. However, mathematics education research does accumulate to some extent within a fashion, which might fit Kuhn's view of normal science, which is research based upon previous scientific achievements (1996, p. 10) and which is cumulative (1996, p. 96). Kuhn argues that paradigms give all phenomena except anomalies a place in a theory. A new theory that can resolve anomalies in relation to an existing theory is making different predictions than those derived from the present theory. This difference would not happen if the two theories were logically compatible, hence the new theory must displace the old (1996, p. 97).

6.2 Is complementarity the solution?

Following Kuhn, the argument that different theories can complement each other is illogical. This fits with Lerman's statement: "Vygotsky's and Piaget's programs have fundamentally different orientations, the former placing the social life as primary and the latter placing the individual as primary … the assumption of complementarity leads to incoherence" (1996, p. 133). However, many researchers in mathematics education seem to favour the concept of complementarity. For instance Goldin who "addressed the need for a synthetic and eclectic approach that includes rather than excludes the many different important constructs that have previously been viewed as mutually exclusive" (in English & Sierpinska 2004), Pegg and Tall (2005, p. 472): "the embodied mode of operation is complemented by the use of symbols in arithmetic, algebra, ...", Piaget (1970, p. 14): "I shall begin by making a distinction between two aspects of thinking that are different, although complementary", and finally Sfard:

> It seems that the most powerful research is the one that stands on more than one metaphorical leg … giving full exclusivity to one conceptual framework would be hazardous. Dictatorship of a single metaphor ... may lead to theories that serve the interests of certain groups to the disadvantage of others (1998, p. 11).

This means that a lot of the research practice disagrees with Kuhn's rejection of the possibility of simultaneously having a number of incompatible theories. But is it possible to live with this? There is a parallel to this discussion in the philosophy of mathematics. The Formalists define mathematical truth relative to a formal system that should be consistent so that it is not possible both to prove a theorem and its negation. But both a theorem and its negation can be proven if it takes place in two different systems; for instance Euclidean and non-Euclidean geometry. A Platonist would regard this as competing theories but a Formalist has no ambition of achieving a "global" truth and it is not necessary that a system can be used "on the entire reality" for it to be acceptable (Dahl et al. 1992, p. 16-17). To live with a number of theories that are incompatible seems like a paradox but "a paradox is not a conflict within reality. It is a conflict between reality and your feeling of what reality should be like" (Feynman in Marshall & Zohar 1997, p. 387). This calls for a principle of complementarity, which also Bohr used: "To accept that light is both a wave and particle, is one of the creative leaps quantum physics calls upon us ... Seeing the truth of all tells us something more profound about the situation" (Marshall & Zohar 1997, p. 102). Transferring this to a discussion about different theories of learning mathematics, it might mean that all the theories are 'right' if they are consistent within themselves and can explain data and do good predictions.

6.3 Are we shooting with a shotgun then?

The danger of accepting a paradigm that acknowledges many truths is that we lack an instrument to help us chose between these truths; what to take from each of them, and when. Pegg and Tall also seem to accept many truths for instance when they state that : "It is not claimed that this is the *only* way in which concepts grow" (2005, p. 473), without explicitly stating if there are ways to construct concepts that do not fit the SOLO UMR cycle, and if so, when to believe what theory. Without a system to help chose between the theories, the choice might become a matter of luck, taste, believing everything all the time – like shooting with a shotgun – hoping to at least hit something or be partly right - sometimes. It can also become a question of power, as the Sfard (1998) quote above might suggest when she writes about the dictatorship of a single metaphor. More research might make it possible to make informed decisions based on evidence (whatever that is). I think that we as a field need higher ambitions than just saying that we need the insight from all approaches – or take the best of each one, which I will call 'uncritical complementarism'. In the lack of an overall theory, we at least need a meta-theory to help us make these informed decisions. Perhaps we should "forget" Kuhn, and instead follow Poppers ideas of scientific growth and deliberately try to falsify theories that so far have been put forward.

6.4 Scientific growth through falsifications

According to Popper, knowledge is always conjectural and hypothetical and not any positive outcome of a test can verify a theory. Popper writes that our understanding of the universe seems to improve over time. This happens through an evolution like process where a tentative theory, made in response to a problem situation, systematically is tested through a process of error elimination. The theories that survive this process are not more true, but more applicable to the problem situations (1979, p. 241-244). Pegg and Tall (2005, p. 469) state that the SOLO framework is a neo-Piagetian model that evolved as a reaction to observed inadequacies in Piaget's framework around the issue of 'décalage'. This is therefore an example of a theory that gets refined through partly refutation.
According to Popper, a theory is scientific only if it is refutable by a conceivable event. A theory is therefore prohibitive since it forbids particular events or occurrences. In this regard, it would be interesting if Pegg and Tall would state what particular event their model forbids. Popper distinguishes between the logic of falsifiability and its methodology. Methodologically, no observation is free from the possibility of error. A single counter-example is therefore not enough to falsify a theory (1979, p. 17). This might fit with Garrison's epistemological conservatism that argues against quickly abandoning principles which have worked in the past simply because they do not work on a single occasion: "It is better to adjust the theory elsewhere and in such a way as to least disturb our most basic beliefs" (1986, p. 13).
But falsifying or adjusting theories require that one in fact can compare the theories. Garrison here writes that "theory-ladenness ... raises the possibility of finding no neutral empirical ground of shared facts on which to judge between competing theories" (1986, p. 16). This is also called the incommensurability of theories. Feyerabend (in Motterlini 1999, p. 177-118) states: "Neither Lakatos nor anybody else has shown that science is better than witchcraft and that science proceeds in a rational way. Taste, not argument, guides our choice of science". Arbib and Hesse state that for any given data set, several theoretical models might fit these data relatively well while still presuppose different ontologies (1986, p. 6).

Popper, however, stated that "*we do justify our preferences by an appeal to the idea of truth*: truth plays the role of a regulative idea. *We test for truth*, by eliminating falsehood" (1979, p. 29-30). Kuhn might here argue that this type of falsification is still within the same paradigm. However, Hollis argues that "the difference is a matter of degree of entrenchment, with normal science more willing to question its core theories than Kuhn recognised" (1994, p. 88). I would agree with Pring:

> Once one loses one's grip on 'reality', or questions the very idea of 'objectivity', or denies a knowledge-base for policy and practice, or treats facts as mere invention or construction, then the very concept of research seems unintelligible. There is a need, therefore, once again to plug educational research into that perennial (and pre-modern) philosophical tradition, and not be seduced by the postmodern embrace (2000, p. 159).

6.5 Unified theory and truth

Can we reach the truth about learning mathematics? Eisner stated: "Insofar as our understanding of the world is our own making, what we consider true is ... the product of our own making" (1993, p. 54). This is, however, a circle argument since "the social items that are claimed to generate social facts must themselves be understood to be generated by other social items, and so on *'ad infinitum'*" (Collin 1997, p. 78). Furthermore it is inconsistent to reject 'truth' while replacing it with a new 'truth', namely that the 'truth' does not exist. Even if it is true that truth does not exists, it does, in my view, not change the fact that there are more or less true descriptions of how one learns mathematics. For instance, is Behaviourism as good/bad as any other theory? I would follow Phillips who argues that truth exists independently of us but we can never reach it. Objectivity and truth are not synonyms but through criticism we can approach truth and the, at any time, most rational theory is therefore the most objective (1993, p. 61). We can never reach the truth, but this does not mean that any theory is as good as any other.
But is the truth then a unified theory or do we have to live with a situation as the one of the Formalist mathematicians? Hawking writes that "we might be near finding a complete theory that would describe the universe and

everything in it" (1994, p. 29). This fits with the modern ideal that there is "a 'grand narrative' ... namely, the 'enlightenment' view that reason, in the light of systematically researched evidence, will provide the solution to the various problems we are confronted with" (Pring 2000, p. 110). It also relates to Naturalism that has the assumption "that there is only space-time Reality and that this reality is sufficiently understandable in terms of scientific methods" (Arbib & Hesse 1986, p. 3).

Johnson states that it is disputed if a unified theory can be achieved (1995, p. 55). Furthermore a theory of everything is self-referential since science assumes that human beings (scientists) are rational beings who through accurate observation and logical reasoning can understand the reality behind the phenomena. A theory of everything would also include the laws that determine the thoughts and actions of the scientists who discover the theory itself. But can the scientists trust that these laws pemit that their power of reasoning would discover the laws? The validation of the mind's reasoning power is the principle of natural selection; however, this is also a theory that itself is a product of the human mind (Johnson 1995, p. 61-61). Furthermore, "[E]ven if we had perfect knowledge we could not perfectly articulate it, was a commonplace for ancient sceptics ... but was forgotten in the Enlightenment" (Lakatos 1976, p. 52). However, Skinner argued that the sceptical stands against a grand theory actually contribute to a return of grand theory since, while they have argued for repudiating the activity of theorising, they have at the same time been engaged in theorising and articulating a general view of the nature of knowledge (2000, p. 12-14). The existence of a unifying theory is therefore disputed and Pegg and Tall do not seem to subscribe to this view either since they state that there are other ways of creating concepts.

7. Conclusions

It seems that the idea of a unifying grand theory ultimately is self-referential but also the idea that there is no 'grand narrative' is problematic. But do we not "know" that there are theories that give a better description of the learning of mathematics than others? My suggestion is that we follow Phillips and the ancient sceptics and acknowledge that there is 'truth', perhaps even a grand unifying theory, but we will never get there (partly because of that language will not let us), but we can try to get as close as possible through using a Popper-inspired approach. Before we get 'close' we can regard various theories as being complementary, using the Formalist approach, but it should not be an uncritical complementarism that simply puts everything in the pot. If we do that – of course 'something' will be 'right', since *everything* is there. We can and must do better than that. This also follows Pring who states that "the pursuit of truth makes sense without the guarantee of ever attaining it. The belief in rationality is compatible with the provisional and fallible nature of one's conclusions" (2000, p. 114).

There are several ways to move forward. One idea comes from research on the effectiveness of computer-based instructions. This revealed that the: "final feature that was significantly related to study outcome was publication source. Results found in journal articles were clearly more positive than results from dissertations and technical documents" (Kulik & Kulik 1991, p. 89-90). Is the situation similar in mathematics education? In any case, we should put effort into writing and publishing papers that has "negative" results unless, of course, the lack of result is due to poor quality research. A way to move forward is also to try and falsify present theories.

A second idea is to revisit some of yesteryear's scholars (Mewborn, 2005, p. 7), and a third suggestion is that to ask people from outside out research community to provide criticism on fundamental issues. "The system of peer review has important virtues, but it means that even a very esteemed scientist who goes too far in criticizing fundamental assumptions can be effectively excluded from the research community" (Johnson 1995, p. 95-96).

Fourthly, we may also need a series of "State of the Art" articles pulling together present research in an attempt to create/discover a meta theory. As Mewborn (2005, p. 7) state: "Authors of handbook chapters typically do a sort of meta-analysis of the findings of studies in a subfield, but this same type of analysis of frameworks could be quite instructive". Therefore papers like Pegg and Tall's (2005) are very important since they gather and display the state of knowledge within a particular subfield of mathematics education research. Their paper also illustrates the development of a meta-theory and the adding to the body of knowledge when they compare the theories of Dubinsky, Sfard, etc. and built on their commonalties.

A fifth suggestion could be drawn from Boero and Szendrei: "We need to invent new kinds of scientific meeting where different schools compare their results – especially their vocabulary and methodology regarding the same subject" (1998, p. 207). Lastly, we should not just discuss the theories in relation to each other but also perform new research that is directly aimed at casting light on the areas of disagreement (Dahl 2004b, p. 12).

Finally, while perhaps complaining that we do not yet have an accumulating body of knowledge that see through everything regarding learning mathematics, we might get comfort from Lewis: "If you see through everything, then everything is transparent. But a wholly transparent world is an invisible world. To 'see through' all things is the same as not to see" (1996, p. 87).

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