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Broadcasting a Common Message with Variable-Length Stop-Feedback Codes

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***Abstract*—We investigate the maximum coding rate achievable over a two-user broadcast channel for the scenario where a common message is transmitted using variable-length stop-feedback codes. Specifically, upon decoding the common message, each decoder sends a stop signal to the encoder, which transmits continuously until it receives both stop signals. For the point-to-point case, Polyanskiy, Poor, and Verdú (2011) recently demonstrated that variable-length coding combined with stop feedback significantly increases the speed at which the maximum coding rate converges to capacity. This speed-up manifests itself in the absence of a square-root penalty in the asymptotic expansion of the maximum coding rate for large blocklengths, a result a.k.a. *zero dispersion*. In this paper, we show that this speed-up does not necessarily occur for the broadcast channel with common message. Specifically, there exist scenarios for which variable-length stop-feedback codes yield a positive dispersion.**

I. Introduction

We consider the setup where an encoder wishes to convey a common message over a broadcast channel with noiseless feedback to two decoders. Similarly to the single-decoder (SD) case, noiseless feedback combined with fixed-blocklength codes does not improve capacity, which is given by [1, p. 126]

$$C = \sup_P \min\{I(P, W_1), I(P, W_2)\}\,. \tag{1}$$

Here, W_1 and W_2 denote the channels to decoder 1 and 2, respectively, and the supremum is over all input distributions P. For the case when there is no feedback, the speed at which C is approached as the blocklength n increases is of the order $1/\sqrt{n}$ [2] (same as in the SD case). The constant factor associated to the $1/\sqrt{n}$ term is commonly referred to as channel *dispersion*.

For the SD case, noiseless feedback combined with variable-length codes improve significantly the speed of convergence to capacity. Specifically, it was shown in [3] that

$$\frac{1}{l}\log \widetilde{M}_{\mathrm{f}}^*(l,\epsilon) = \frac{\widetilde{C}}{1-\epsilon} - \mathcal{O}\left(\frac{\log l}{l}\right) \tag{2}$$

where l stands for the average blocklength (average transmission time), $\widetilde{M}_{\mathrm{f}}^*(l,\epsilon)$ is the maximum number of codewords in the SD case, and $\widetilde{C}$ denotes the corresponding capacity. One sees from (2) that no square-root penalty occurs (zero dispersion), which implies a fast convergence to the asymptotic limit. This fast convergence is demonstrated numerically in [3] by means of nonasymptotic bounds. Variable-length stop-feedback (VLSF) codes, i.e., coding schemes where the feedback is used only to stop transmissions, are sufficient to achieve (2).

The purpose of this paper is to investigate whether a similar result holds for the broadcast channel with common message.

Contribution: We consider the subclass of discrete memoryless broadcast channels for which $I(P, W_1)$ and $I(P, W_2)$ are maximized by the same input distribution P^*, which we assume to be unique. In this case, $C = \min\{I(P^*, W_1), I(P^*, W_2)\}$. Focusing on the case when VLSF codes are used, we obtain nonasymptotic achievability and converse bounds on the maximum number of codewords $M_{\mathrm{sf}}^*(l,\epsilon)$ with average blocklength l that can be transmitted with reliability $1-\epsilon$. Here, the subscript "sf" stands for stop feedback. By analyzing these bounds in the large-l regime, we prove that when the two subchannels are independent and have the same capacity and the same dispersion, and when $\epsilon \leq 0.1968$, the asymptotic expansion of $M_{\mathrm{sf}}^*(l,\epsilon)$ contains a square-root penalty (see (18) and (22) for a precise statement of this result). Hence, the fast convergence to the asymptotic limit experienced in the SD case cannot be expected.

The intuition behind this result is as follows: in the SD case, the stochastic variations of the information density that result in the square-root penalty can be virtually eliminated by using variable-length coding with stop-feedback. Indeed, decoding is stopped after the information density exceeds a certain threshold, which yields only negligible stochastic variations. In the broadcast setup, however, the stochastic variations in the difference between the stopping times at the two decoders make the square-root penalty reappear. Note that our result does not necessarily imply that feedback is useless. It only shows that VLSF codes cannot be used to speed-up convergence to the same level as in the SD case.

Proof techniques: The achievability bound is an extension of [3, Th 3]; the converse bound is based on an optimal stopping problem, where the probability that the stopping time exceeds a given threshold is minimized under a constraint on the "stopped" information density process. The asymptotic analysis of the converse bound relies on Hoeffding's inequality and on the Berry-Esseen central limit theorem, whereas the asymptotic analysis of the achievability bound relies on asymptotic results for random walks [4] and on a Berry-Esseen-type theorem that holds for random summations [5].

Notation: Upper case, lower case, and calligraphic letters denote random variables (RV), deterministic quantities, and sets, respectively. The probability density function of a standard Gaussian RV is denoted by $\phi(x)$. Furthermore, $\Phi(x) \triangleq 1 - Q(x)$ is its cumulative distribution, where $Q(x)$ is the Q-function. We let x^+ and x^- denote $\max(0,x)$ and $\min\{0,x\}$, respectively. Throughout the paper, the index k belongs always to the set $\{1,2\}$, although this is sometimes omitted. Furthermore, $\bar{k} \triangleq 3-k$. We adopt the convention that $\sum_{i=j}^{j-1} a_i = 0$ for all $\{a_i\}$ and all integers j. We use "$\mathbb{c}$" to denote a finite nonnegative constant. Its value may change at each occurrence. Finally, $\mathbb{N}$ denotes the set of positive integers and $\mathbb{Z}_+ = \mathbb{N}\cup\{0\}$.

II. System Model

A common-message discrete memoryless broadcast channel with two decoders is defined by the finite input alphabet $\mathcal{X}$ and the finite output alphabets $\mathcal{Y}_k$, along with the stochastic matrices W_k, where $W_k(y_k|x)$ denotes the probability that $y_k \in \mathcal{Y}_k$ is observed at decoder k given $x \in \mathcal{X}$. We assume that the outputs at each time i are conditionally independent given the input, i.e.,

$$P_{Y_{1,i},Y_{2,i}|X_i}(y_{1,i},y_{2,i}|x_i) \triangleq W_1(y_{1,i}|x_i)W_2(y_{2,i}|x_i). \quad (3)$$

Define the set of probability distributions on $\mathcal{X}$ by $\mathcal{P}(\mathcal{X})$. Let $P \times W_k : (x, y_k) \to P(x)W(y_k|x)$ denote the joint distribution of input and output at decoder k, and let $PW_k : y_k \to \sum_{x\in\mathcal{X}} P(x)W_k(y_k|x)$ denote the marginal distribution on $\mathcal{Y}_k$. For every $P \in \mathcal{P}(\mathcal{X})$, the information density is defined as

$$\imath_{P,W_k}(x^n; y_k^n) \triangleq \sum_{i=1}^{n} \log \frac{W_k(y_{k,i}|x_i)}{PW_k(y_{k,i})}. \quad (4)$$

We let $I(P, W_k) \triangleq \mathbb{E}_{P\times W_k}[\imath_{P,W_k}(X;Y_k)]$ be the mutual information, $V(P,W_k) \triangleq \mathrm{Var}_{P\times W_k}[\imath_{P,W_k}(X;Y_k)]$ be the (unconditional) information variance, and $T(P,W_k) \triangleq \mathbb{E}_{P\times W_k}\left[|\imath_{P,W_k}(X;Y_k) - I(P,W_k)|^3\right]$ be the third absolute moment of the information density. We restrict ourselves to the case, where there exists a unique probability distribution $P^* \in \mathcal{P}(\mathcal{X})$ that maximizes simultaneously both $I(P,W_1)$ and $I(P,W_2)$. In this case, the capacity is given by

$$C \triangleq \min\{C_1, C_2\} \quad (5)$$

where $C_k \triangleq I(P^*, W_k)$. The corresponding (unique) capacity-achieving output distributions are denoted by $P^*_{Y_k}$. Finally, we also define the dispersions $V_k \triangleq V(P^*, W_k)$.

We are now ready to formally define a VLSF code for the broadcast channel with common message.

Definition 1: An (l, M, ϵ)-VLSF code for the broadcast channel with common message consists of:

1) A RV $U \in \mathcal{U}$, with $|\mathcal{U}| \le 3$, which is known by the encoder and by both decoders.
2) A sequence of encoders $f_n : \mathcal{U} \times \mathcal{M} \to \mathcal{X}$, each one mapping the message $J \in \mathcal{M} = \{1,\dots,M\}$, drawn uniformly at random, to the channel input according to $X_n = f_n(U, J)$.
3) Two nonnegative integer-valued RVs τ_1 and τ_2 that are stopping times with respect to the filtrations $\mathcal{F}(U, Y_1^n)$ and $\mathcal{F}(U, Y_2^n)$, respectively, and which satisfy

$$\mathbb{E}[\max\{\tau_1, \tau_2\}] \le l. \quad (6)$$

4) A sequence of decoders $g_{k,n} : \mathcal{U} \times \mathcal{Y}_i^n \to \mathcal{M}$ satisfying

$$\Pr[J \ne g_{k,\tau_k}(U, Y_k^{\tau_k})] \le \epsilon, \qquad k \in \{1,2\}. \quad (7)$$

Remark 1: The RV U serves as common randomness, and enables the use of randomized codes [6]. To establish the cardinality bound on U, we proceed as in [3, Th. 19] to show that $|\mathcal{U}| \le 4$ is sufficient. This bound can be further improved to $|\mathcal{U}| \le 3$ by using the Fenchel-Eggleston theorem [7, p. 35].

Remark 2: VLSF codes require a feedback link from the decoders to the encoder. This feedback consists of a 1-bit stop signal per decoder, which is sent by decoder k at time τ_k. The encoder continuously transmits until both decoders have fed back a stop signal. Hence, the blocklength is $\max\{\tau_1, \tau_2\}$.

Our aim is to characterize the largest number of codewords $M^*_{\mathrm{sf}}(l, \epsilon)$, whose average length is l, that can be transmitted with reliability $1-\epsilon$ using a VLSF code.

III. Main Results

A. Achievability bound

We first present an achievability bound. Its proof (omitted) follows closely the proof of [3, Th. 3].

Theorem 1: Fix $P \in \mathcal{P}(\mathcal{X})$. Let $\gamma_1, \gamma_2 \ge 0$ and $0 \le q \le 1$ be arbitrary scalars. Let the stopping times τ_k and $\bar{\tau}_k$, $k \in \{1,2\}$, be defined as

$$\tau_k \triangleq \inf\{n \ge 0 : \imath_{P,W_k}(X^n; Y_k^n) \ge \gamma_k\} \quad (8)$$

$$\bar{\tau}_k \triangleq \inf\{n \ge 0 : \imath_{P,W_k}(\bar{X}^n; Y_k^n) \ge \gamma_k\} \quad (9)$$

where $(X^n, \bar{X}^n, Y_1^n, Y_2^n)$ are jointly distributed according to

$$P_{X^n,\bar{X}^n,Y_1^n,Y_2^n}(x^n, \bar{x}^n, y_1^n, y_2^n) = P_{Y_1^n,Y_2^n|X^n}(y_1^n, y_2^n|x^n) \prod_{i=1}^{n} P(x_i)P(\bar{x}_i). \quad (10)$$

For every M, there exists an (l, M, ϵ)-VLSF code such that

$$l \le (1-q)\mathbb{E}[\max\{\tau_1, \tau_2\}] \quad (11)$$

and

$$\epsilon \le q + (1-q)(M-1)\Pr[\tau_k \ge \bar{\tau}_k]. \quad (12)$$

Remark 3: Following the same steps as in [3, Eq. (111)–(118)], ϵ in (12) can be further upper-bounded as

$$\epsilon \le q + (1-q)(M-1)\exp\{-\gamma_k\}. \quad (13)$$

This bound is easier to evaluate and to analyze asymptotically.

B. Converse bound

Let $P_{\mathbf{x}^n} \in \mathcal{P}(\mathcal{X})$ be the type [8, Def. 2.1] of the sequence $\mathbf{x}^n \in \mathcal{X}^n$. We are now ready to state our converse bound.

Theorem 2: For every M, $t \in \mathbb{Z}_+$ and $\delta > 0$, let

$$\lambda_t \triangleq \log M - \log\log M - \delta - (|\mathcal{X}|-1)\log(t+1) \quad (14)$$

and let

$$L_t \triangleq \prod_{k=1}^{2} \max_{\mathbf{x}^t\in\mathcal{X}^t} \left\{\Pr\left[\imath_{P_{\mathbf{x}^t},W_k}(\mathbf{x}^t; Y_k^t) > \lambda_t\right]\right\} + \varepsilon_M \left(1 + \min_k \max_{\mathbf{x}^t\in\mathcal{X}^t} \Pr\left[\imath_{P_{\mathbf{x}^t},W_k}(\mathbf{x}^t; Y_k^t) > \lambda_t\right]\right) \quad (15)$$

where $\varepsilon_M = \epsilon + (\log M)^{-1}$. Then, for every (l, M, ϵ)-VLSF code, we have

$$l \geq \sum_{t=0}^{\infty} (1 - L_t)^+ . \quad (16)$$

Proof: See Section IV. ■

C. Asymptotic expansion

Analyzing (13) and (16) in the limit $l \to \infty$, we obtain the following asymptotic characterization of $M^*_{\text{sf}}(l, \epsilon)$.

Theorem 3: Let $Z_k \sim \mathcal{N}(0, 1)$, $V = \sqrt{V_1 V_2}$, $\varrho_k = (V_k / V_{\bar{k}})^{1/4}$, and let $y = \tilde{Q}^{-1}(x)$ be the solution of

$$\prod_{k=1}^{2} Q(-\varrho_k y) + x \left(1 + \min_k Q(-\varrho_k y)\right) = 1. \quad (17)$$

For every discrete memoryless broadcast channel with $C_1 = C_2$ and every $\epsilon \in (0, 1)$, we have

$$\frac{Cl}{1-\epsilon} - \Xi_{\text{a}}\sqrt{l} - \mathcal{O}\left(l^{1/4+\delta}\right) \leq \log M^*_{\text{sf}}(l, \epsilon) \leq \frac{Cl}{1-\epsilon} - \Xi_{\text{c}}\sqrt{l} + \mathcal{O}(\log l) \quad (18)$$

where $\delta > 0$ is an arbitrarily small constant,

$$\Xi_{\text{a}} \triangleq \sqrt{\frac{V_1 + V_2}{2\pi(1-\epsilon)}} \quad (19)$$

and

$$\Xi_{\text{c}} \triangleq \sqrt{\frac{V}{(1-\epsilon)^3}} \left(\mathbb{E}\left[\min\left\{\tilde{Q}^{-1}(\epsilon), \max_k \varrho_k Z_k\right\}\right] - \epsilon\left(2\tilde{Q}^{-1}(\epsilon) - \min_k \mathbb{E}\left[\min\left\{\tilde{Q}^{-1}(\epsilon), \varrho_k Z_k\right\}\right]\right)\right). \quad (20)$$

Proof: The converse bound in (18) is proved in Section V and the achievability bound is proved in Section VI. ■

Remark 4: When $C_1 \neq C_2$, it can be shown that the square-root penalty on the LHS of (18) vanishes. In this case, the problem reduces to the point-to-point transmission to the weakest decoder, for which the zero-dispersion result in [3] applies.

Remark 5: For the case when $P_{Y_{1,i}, Y_{2,i} | X_i}$ does not satisfy (3), a bound similar to the LHS of (18) can be obtained by replacing Ξ_{a} in (19) with

$$\sqrt{\frac{V_1 + V_2 - 2\text{Cov}(\imath_{P^*, W_1}(X; Y_1), \imath_{P^*, W_2}(X; Y_2))}{2\pi(1-\epsilon)}}. \quad (21)$$

Remark 6: When $\varrho_1 = \varrho_2 = 1$ (and, hence, $V_1 = V_2$), one can simplify the RHS of (18) as follows:

$$\log M^*_{\text{sf}}(l, \epsilon) \leq \frac{Cl}{1-\epsilon} - \sqrt{\frac{Vl}{(1-\epsilon)^3}} \times \left(\frac{1}{\sqrt{\pi}}\left(1 - Q\left(\sqrt{2}Q^{-1}(\epsilon)\right)\right) + (\epsilon - 2)\phi(Q^{-1}(\epsilon))\right) - \mathcal{O}(\log l). \quad (22)$$

The second-order term in (22) is strictly negative for all $\epsilon \leq 0.1968$. This implies that, when $C_1 = C_2$, $V_1 = V_2$, and $\epsilon \leq 0.1968$, the asymptotic expansion of $\log M^*_{\text{sf}}(l, \epsilon)$ contains a square-root penalty.

IV. Proof of Theorem 2

Fix M and ϵ. To establish Theorem 2, we derive a lower bound on l that holds for all VLSF codes having M codewords and probability of error no larger than ϵ. Since,

$$l \geq \mathbb{E}[\max\{\tau_1, \tau_2\}] = \sum_{t=0}^{\infty} (1 - \Pr[\max\{\tau_1, \tau_2\} \leq t]) \quad (23)$$

we can lower-bound l by upper-bounding $\Pr[\max\{\tau_1, \tau_2\} \leq t]$ for every $t \in \mathbb{Z}_+$. The following property (proven in Appendix I-A) turns out to be useful.

Property 1: Fix $t \in \mathbb{Z}_+$ and $\alpha \in [0, 1]$, and suppose there exists an (l, M, ϵ)-VLSF code with $\Pr[\max\{\tau_1, \tau_2\} \leq t] \leq \alpha$. Then there exists an (l', M, ϵ)-VLSF code for some $l' \geq l$, for which $\Pr[\max\{\tau_1, \tau_2\} \leq t] \leq \alpha$ and $\tau_1, \tau_2 \in \{t, t+1, \ldots\}$.

Fix an arbitrary (l, M, ϵ)-VLSF code, defined by the tuple $(f_n, g_{1,n}, g_{2,n}, \tau_1, \tau_2, U)$. By Property 1, it is sufficient to consider codes for which $\tau_1, \tau_2 \in \{t, t+1, \cdots\}$. Let $\epsilon_k^{(u)}$, $u \in \mathcal{U}$, be constants in $[0, 1]$ such that $\sum_{u \in \mathcal{U}} P_U(u) \epsilon_k^{(u)} \leq \epsilon$ and $\Pr[J \neq g_{k,\tau_k}(U, Y_k^{\tau_k}) | U = u] \leq \epsilon_k^{(u)}$.

Since $\{\tau_k = n\} \in \mathcal{F}(U, Y_k^n)$, we can define a sequence of binary functions $\varphi_k \triangleq \{\varphi_{k,t}, \varphi_{k,t+1}, \cdots\}$ such that $\varphi_{k,n}(u, y_k^n) \triangleq \mathbb{1}\{\tau_k = n\}$. Let $P_{\mathbf{X}}^{(u)}$ be the conditional probability measure on $\mathcal{X}^\infty$ induced by the encoder given $U = u$. Define for $u \in \mathcal{U}$ the set $\bar{\mathcal{Y}}_k^{(u)} \triangleq \{y^n \in \mathcal{Y}_k^n : \varphi_{k,n}(u, y^n) = 1\}$. Note that we must have $Y_k^{\tau_k} \in \bar{\mathcal{Y}}^{(u)}$. Let the length of a sequence of channel outputs $\bar{y} \in \bar{\mathcal{Y}}_k^{(u)}$ be denoted by $|\bar{y}|$. On $\bar{\mathcal{Y}}_k^{(u)}$, define the conditional probability measure $\mathbb{P}_{\bar{Y}|\mathbf{X}}^{(k,u)}$, given $\mathbf{x} \in \mathcal{X}^\infty$ and $u \in \mathcal{U}$, as

$$\mathbb{P}_{\bar{Y}|\mathbf{X}}^{(k,u)}(\bar{y}|\mathbf{x}) \triangleq \prod_{i=1}^{|\bar{y}|} W(\bar{y}_i | \mathbf{x}_i) \quad (24)$$

and the probability measure $\mathbb{P}_{\bar{Y},\mathbf{X}}^{(k,u)}(\bar{y}, \mathbf{x}) \triangleq \mathbb{P}_{\bar{Y}|\mathbf{X}}^{(k,u)}(\bar{y}|\mathbf{x}) P_{\mathbf{X}}^{(u)}(\mathbf{x})$ on $\bar{\mathcal{Y}}^{(u)} \times \mathcal{X}^\infty$. We also need the following auxiliary probability measure $\mathbb{Q}_{\bar{Y}}^{(k,u)}$ on $\bar{\mathcal{Y}}_k^{(u)}$

$$\mathbb{Q}_{\bar{Y}}^{(k,u)}(\bar{y}) \triangleq \sum_{P_{\mathbf{x}^t} \in \mathcal{P}_t(\mathcal{X})} \left(\frac{1}{|\mathcal{P}_t(\mathcal{X})|} \prod_{i=1}^{t} P_{\mathbf{x}^t} W_k(\bar{y}_i) \prod_{i=t+1}^{|\bar{y}|} P^*_{Y_k}(\bar{y}_i) \right) \quad (25)$$

and the probability measure $\mathbb{Q}_{\bar{Y},\mathbf{X}}^{(k,u)}(\bar{y}, \mathbf{x}) = \mathbb{Q}_{\bar{Y}}^{(k)}(\bar{y}) P_{\mathbf{X}}^{(u)}(\mathbf{x})$ on $\bar{\mathcal{Y}}^{(u)} \times \mathcal{X}^\infty$. Here, $\mathcal{P}_t(\mathcal{X}) \subseteq \mathcal{P}(\mathcal{X})$ denotes the set of types formed by length-t sequences.

Using the meta-converse theorem [9, Th. 27], the inequality [9, Eq. (102)], the fact that $\mathbb{Q}_{\bar{Y}_k,\mathbf{X}}^{(k,u)}$ is a convex combination of distributions [10, Lem. 3], and the upper bound $|\mathcal{P}_t(\mathcal{X})| \leq (t+1)^{|\mathcal{X}|-1}$ [11, Lem. 1.1], we conclude that (see Appendix I-B)

$$\mathbb{P}_{\bar{Y},\mathbf{X}}^{(k,u)}\left[\tilde{\imath}_k^{(u)}(\mathbf{X}; \bar{Y}_k) \leq \lambda_t\right] \leq \varepsilon_{k,M}^{(u)} \quad (26)$$

where $\varepsilon_{k,M}^{(u)} \triangleq \epsilon_k^{(u)} + (\log M)^{-1}$ and λ_t is defined in (14). Here,

$$\tilde{\imath}_k^{(u)}(\mathbf{x}; \bar{y}) \triangleq \imath_k(\mathbf{x}^t; y^t) + \sum_{i=t+1}^{|\bar{y}|} \log \frac{W_k(y_i|\mathbf{x}_i)}{P^*_{Y_k}(y_i)} \tag{27}$$

where $\imath_k(\mathbf{x}^t; y^t) \triangleq \imath_{P_{\mathbf{x}^t}, W_k}(\mathbf{x}^t, y^t)$. Next, we minimize $\Pr[\tau_k \leq t | U = u]$ over all stopping times τ_k satisfying (26):

$$\begin{aligned}
&\Pr[\tau_k \leq t | U = u] = \mathbb{P}^{(k,u)}_{\bar{Y}, \mathbf{X}}\left[|\bar{Y}| = t\right] \\
&= \mathbb{P}^{(k,u)}_{\bar{Y}, \mathbf{X}}\left[\tilde{\imath}_k^{(u)}(\mathbf{X}; \bar{Y}_k) > \lambda_t, |\bar{Y}| = t\right] \\
&\quad + \mathbb{P}^{(k,u)}_{\bar{Y}, \mathbf{X}}\left[\tilde{\imath}_k^{(u)}(\mathbf{X}; \bar{Y}_k) \leq \lambda_t, |\bar{Y}| = t\right] && (28) \\
&\leq \min\left\{1, \mathbb{P}^{(k,u)}_{\bar{Y}, \mathbf{X}}\left[\tilde{\imath}_k^{(u)}(\mathbf{X}; \bar{Y}_k) > \lambda_t, |\bar{Y}| = t\right] + \varepsilon_{k,M}^{(u)}\right\} && (29) \\
&\leq \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda_t\right] \\
&\quad + \min\left\{\varepsilon_{k,M}^{(u)}, 1 - \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda_t\right]\right\}. && (30)
\end{aligned}$$

Here, (29) follows from (26). Since the stopping times τ_1 and τ_2 are conditional independent given $U = u$, (30) implies that

$$\Pr[\max\{\tau_1, \tau_2\} \leq t | U = u] = \prod_{k=1}^{2} \mathbb{P}^{(k,u)}_{\bar{Y}, \mathbf{X}}\left[|\bar{Y}_k| = t\right] \tag{31}$$

$$\begin{aligned}
&\leq \prod_{k=1}^{2} \max_{\mathbf{x}^t \in \mathcal{X}^t} \left\{\Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda_t\right]\right\} \\
&\quad + \min_k \left\{\varepsilon_{\bar{k},M}^{(u)} + \varepsilon_{k,M}^{(u)} \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr\left[\imath_k(\mathbf{x}^t; Y_{\bar{k}}^t) > \lambda_t\right]\right\}. && (32)
\end{aligned}$$

Note that (32) holds for all τ_k that satisfies (26). Averaging (32) over $u \in \mathcal{U}$ and using the inequality $\sum_{u \in \mathcal{U}} P_U(u) \varepsilon_{k,M}^{(u)} \leq \epsilon + (\log M)^{-1} = \varepsilon_M$, we obtain (15). The proof is concluded using (23).

V. Asymptotic Analysis: Converse Bound

We analyze L_t in (15) in the limit $l \to \infty$. By (16),

$$l \geq \sum_{t=0}^{\infty} (1 - L_t)^+ \geq \sum_{t=0}^{\lfloor \beta \rfloor} (1 - L_t)^+ \geq \sum_{t=0}^{\lfloor \beta \rfloor} (1 - L_t) \tag{33}$$

where $\beta > 0$ will be specified shortly. Let $\lambda \triangleq \log M - \log\log M - \delta - (|\mathcal{X}| - 1)\log(\beta + 1)$. For all $t \leq \beta$,

$$\max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda_t\right] \leq \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda\right]. \tag{34}$$

The key step is to establish an asymptotic upper bound on $\max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda]$ for every $t \in \mathbb{Z}_+$ as $\lambda \to \infty$.

Let $\alpha \triangleq \frac{\lambda}{C} - \sqrt{\frac{V\lambda}{C^3}} \log \lambda$ and let β be the solution of

$$(\lambda - \beta C)/\sqrt{\beta V} = -\tilde{Q}^{-1}(\epsilon) \tag{35}$$

where C is given in (5), V is defined in Theorem 3, and $\tilde{Q}^{-1}(\epsilon)$ in (17). We divide the asymptotic analysis of $\max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda]$ into three cases: the "large deviations regime" $t \in [0, \alpha)$, where we use Hoeffding's inequality, the "central regime" $t \in [\alpha, \beta)$, where Berry-Esseen central limit theorem is applied, and the case $t \geq \beta$, where the trivial upper bound $\max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda] \leq 1$ suffices.

In the first case, invoking Hoeffding's inequality [12, Th. 2] and using that $I(P_{\mathbf{x}^t}, W_k)$ is upper-bounded by C uniformly, we obtain (see Appendix II-A for details)

$$\sum_{t=0}^{\lfloor \alpha \rfloor} \max_{\mathbf{x}^t \in \mathcal{X}^\infty} \Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda\right] = o(1), \quad \lambda \to \infty \tag{36}$$

and

$$\sum_{t=0}^{\lfloor \alpha \rfloor} \prod_{k=1}^{2} \max_{\mathbf{x}^t \in \mathcal{X}^t} \left\{\Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda\right]\right\} = o(1), \ \lambda \to \infty. \tag{37}$$

In the central regime, we use the Berry-Esseen central limit theorem [13, Th. V.3] to show that

$$\Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda\right] \leq Q\left(\frac{\lambda - tI(P_{\mathbf{x}^t}, W_k)}{\sqrt{tV(P_{\mathbf{x}^t}, W_k)}}\right) + \frac{\mathbb{c}}{\sqrt{t}}. \tag{38}$$

We next maximize (38) over $\mathbf{x}^t \in \mathcal{X}^t$ following the approach in [10, Prop. 8]. Specifically, we use continuity properties of $I(P, W_k)$ and $V(P, W_k)$ for probability distributions $P \in \mathcal{P}(\mathcal{X})$ close to P^* to show that (see Appendix II-B)

$$\begin{aligned}
&\sum_{t=\lfloor \alpha \rfloor + 1}^{\lfloor \beta \rfloor} \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda\right] \\
&\leq \sqrt{\frac{V\lambda}{C^3}} \Big(\tilde{Q}^{-1}(\epsilon) - \mathbb{E}\Big[\min\Big\{\tilde{Q}^{-1}(\epsilon), \varrho_k Z_k\Big\}\Big]\Big) + \mathcal{O}(\log \lambda)
\end{aligned} \tag{39}$$

where ϱ_k are defined in Theorem 3 and $Z_k \sim \mathcal{N}(0, 1)$. Similarly, we obtain

$$\begin{aligned}
&\sum_{t=\lfloor \alpha \rfloor + 1}^{\lfloor \beta \rfloor} \prod_{k=1}^{2} \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda\right] \\
&\leq \sqrt{\frac{V\lambda}{C^3}} \left(\tilde{Q}^{-1}(\epsilon) - \mathbb{E}\left[\min\left\{\tilde{Q}^{-1}(\epsilon), \max_k \varrho_k Z_k\right\}\right]\right) \\
&\quad + \mathcal{O}(\log \lambda).
\end{aligned} \tag{40}$$

Using (33), (36), (37), (39), and (40), we obtain

$$\begin{aligned}
l &\geq \sum_{t=0}^{\lfloor \beta \rfloor} (1 - L_t) && (41) \\
&\geq \frac{\lambda(1 - \varepsilon_M)}{C} + \sqrt{\frac{V\lambda}{C^3}} \Bigg(\mathbb{E}\left[\min\left\{\tilde{Q}^{-1}(\epsilon), \max_k \varrho_k Z_k\right\}\right] \\
&\quad - \varepsilon_M \left(2\tilde{Q}^{-1}(\epsilon) - \min_k \mathbb{E}\left[\min\left\{\tilde{Q}^{-1}(\epsilon), \varrho_k Z_k\right\}\right]\right)\Bigg) \\
&\quad - \mathcal{O}(\log \lambda) && (42)
\end{aligned}$$

as $\lambda \to \infty$. Finally, we have that

$$\lambda = \log M - \log\log M - \delta - (|\mathcal{X}| - 1)\log(\beta + 1) \quad (43)$$

$$\leq \frac{Cl}{1-\varepsilon_M} - \sqrt{\frac{Vl}{(1-\varepsilon_M)^3}}\left(\mathbb{E}\left[\min\left\{\tilde{Q}^{-1}(\epsilon), \max_k \varrho_k Z_k\right\}\right] - \varepsilon_M\left(2\tilde{Q}^{-1}(\epsilon) - \min_k \mathbb{E}\left[\min\left\{\tilde{Q}^{-1}(\epsilon), \varrho_k Z_k\right\}\right]\right)\right) + \mathcal{O}(\log l) \quad (44)$$

as $l \to \infty$. The final result in (18) is obtained through algebraic manipulations.

VI. Asymptotic Analysis: Achievability Bound

Set $P = P^*$, and fix $r \in \mathbb{N}$, $q = \frac{l'\epsilon - 1}{l' - 1}$, and $l' > 0$, a parameter that will be related to the average blocklength. Let the thresholds be chosen as follows:

$$\gamma \triangleq \gamma_k \triangleq C\left(l' - g(Cl')\right). \quad (45)$$

Here,

$$g(x) \triangleq \sqrt{\frac{V_1 + V_2}{2\pi C^2}}\sqrt{\frac{x}{C}} + b_1 x^{\frac{r+1}{4r+2}} \log x \quad (46)$$

where b_1 will be specified later. If we choose a code with a number of codewords $\tilde{M}$ that satisfies

$$\log \tilde{M} \triangleq C\left(l' - g(Cl')\right) - \log l' \quad (47)$$

we have $(\tilde{M} - 1)\exp\{-\gamma\} \leq 1/l'$. Furthermore, by Remark 3, the average probability of error is upper-bounded by

$$q + (1-q)(\tilde{M} - 1)\exp\{-\gamma_k\} \leq \frac{l'\epsilon - 1}{l' - 1} + \frac{l'(1-\epsilon)}{l' - 1}\frac{1}{l'} = \epsilon. \quad (48)$$

Suppose it can be shown that

$$\mathbb{E}[\max\{\tau_1, \tau_2\}] \leq l' \quad (49)$$

for sufficiently large l'. Then the average blocklength is

$$(1-q)\mathbb{E}[\max\{\tau_1, \tau_2\}] \leq \frac{l'(1-\epsilon)}{l' - 1} l' \triangleq l. \quad (50)$$

Consequently, by Theorem 1, there exists an (l, M, ϵ)-VLSF code with

$$\log M \geq \log \tilde{M} \quad (51)$$

$$= C\left(l' - g(Cl')\right) - \log l' \quad (52)$$

$$= \frac{Cl}{1-\epsilon} - \sqrt{\frac{V_1 + V_2}{2\pi(1-\epsilon)}}\sqrt{l} - \mathcal{O}(l^{\frac{r+1}{4r+2}} \log l) \quad (53)$$

where the last step follows because

$$l = \frac{(l')^2(1-\epsilon)}{l' - 1} = l'(1-\epsilon) + o(1). \quad (54)$$

To establish (49), we proceed as follows. Let $W_n = \imath_{P,W_1}(X_n; Y_{1,n})$ and $Z_n = \imath_{P,W_2}(X_n; Y_{2,n})$. We can then upper-bound $\mathbb{E}[\max\{\tau_1, \tau_2\}]$ using the following lemma, which is proved in Appendix III.

Lemma 1: Let $\{W_n\}$ and $\{Z_n\}$, $n \geq 1$, be i.i.d. discrete RVs with $(W_1, Z_1) \sim P_{W,Z}$, positive mean $\mu_W \triangleq \mathbb{E}[W_1]$ and $\mu_Z \triangleq \mathbb{E}[Z_1]$, respectively, and finite moments of order $r \geq 3$, i.e., $\mathbb{E}[|W_1|^r] < \infty$, and $\mathbb{E}[|Z_1|^r] < \infty$. Define the random walks $U_n \triangleq \sum_{i=1}^n W_i$ and $V_n \triangleq \sum_{i=1}^n Z_i$, and the stopping times $\tau_1 \triangleq \inf\{n \geq 0 : U_n \geq \gamma\}$ and $\tau_2 \triangleq \inf\{n \geq 0 : V_n \geq \gamma\}$ for every $\gamma \in \mathbb{R}$. Then

$$\mathbb{E}[\max\{\tau_1, \tau_2\}] \leq \frac{\gamma}{\min\{\mu_W, \mu_Z\}} + \frac{\sigma}{\sqrt{2\pi}}\sqrt{\frac{\gamma}{\mu_W}}\mathbb{1}\{\mu_W = \mu_Z\} + \mathcal{O}\left(\gamma^{\frac{r+1}{4r+2}} \log \gamma\right) \quad (55)$$

as $\gamma \to \infty$, where $\sigma^2 \triangleq \mathrm{Var}\left[\frac{W_1}{\mu_W} - \frac{Z_1}{\mu_Z}\right]$.

Lemma 1 implies that there exists a constant b_1 such that

$$\mathbb{E}[\max\{\tau_1(\gamma), \tau_2(\gamma)\}] \leq \frac{\gamma}{C} + g(\gamma) \quad (56)$$

for sufficiently large γ. The conditional average blocklength of the VLSF code can be bounded as follows

$$\mathbb{E}[\max\{\tau_1, \tau_2\}] = \mathbb{E}[\max\{\tau_1(\gamma), \tau_2(\gamma)\}] \quad (57)$$

$$\leq \frac{\gamma}{C} + g(\gamma) \quad (58)$$

$$= l' - g(Cl') + g(Cl' - Cg(Cl')) \leq l'. \quad (59)$$

Here, (58) holds by (56), and (59) follows by the definition of γ in (45) and the fact that $g(x)$ is nonnegative and nondecreasing.

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APPENDIX I
STEPS OMITTED IN THE PROOF OF THE CONVERSE BOUND

A. *Proof of Property 1*

Let $(f_n, g_{1,n}, g_{2,n}, \tau_1, \tau_2, U)$ be a tuple defining an (l, M, ϵ)-VLSF code with $\Pr[\max\{\tau_1, \tau_2\} \le t] \le \alpha$. Set

$$\tilde{\tau}_k = \begin{cases} t, & \tau_k \le t \\ \tau_k, & \tau_k > t \end{cases} \tag{60}$$

and

$$\tilde{g}_{k,n}(u, y_k^n) = \begin{cases} g_{k,n}(u, y_k^{\tau_k}), & \tau_k \le n \\ g_{k,n}(u, y_k^n), & \tau_k > n. \end{cases} . \tag{61}$$

Note that $\tilde{\tau}_k$ is also a stopping time with respect to the filtration $\mathcal{F}(U, Y_k^n)$ for $k \in \{1,2\}$. Since τ_k is a function of U and Y_k^n given $\tau_k \le n$, the new decoder $\tilde{g}_{k,n}$ is well-defined. Moreover, the decoders $g_{k,n}$ and $\tilde{g}_{k,n}$ yield the same probability of error. Thus $(f_n, \tilde{g}_{1,n}, \tilde{g}_{2,n}, \tilde{\tau}_1, \tilde{\tau}_2, U)$ defines an (l', M, ϵ)-VLSF code, with $l' \ge l$.

B. *Proof of* (26)

For each decoder k, the average probability of error is no larger than $\epsilon_k^{(u)}$ under $\mathbb{P}_{\bar{Y}_k,\mathbf{X}}^{(k,u)}$ and it is no larger than $1-1/M$ under $\mathbb{Q}_{\bar{Y},\mathbf{X}}^{(k,u)}$. Hence, using the meta-converse theorem [9, Th. 27] and the inequality [9, Eq. (102)], we conclude that

$$\log M \le \log \tilde{\gamma}_k^{(u)} - \log\left(\mathbb{P}_{\bar{Y}_k,\mathbf{X}}^{(k,u)}\left[\imath_k^{(u)}(\mathbf{X};\bar{Y}_k) \le \log \tilde{\gamma}_k^{(u)}\right] - \epsilon_k^{(u)}\right) \tag{62}$$

for all $\tilde{\gamma}_k^{(u)}$ such that $\mathbb{P}_{\bar{Y}_k,\mathbf{X}}^{(k,u)}\left[\imath_k(\mathbf{X};\bar{Y}_k) \le \log \tilde{\gamma}_k^{(u)}\right] > \epsilon_k^{(u)}$. Here,

$$\imath_k^{(u)}(\mathbf{x};\bar{y}_k) \triangleq \log \frac{\mathbb{P}_{\bar{Y}_k,\mathbf{X}}^{(k,u)}(\bar{y},\mathbf{x})}{\mathbb{Q}_{\bar{Y}_k,\mathbf{X}}^{(k,u)}(\bar{y},\mathbf{x})} = \log \frac{\mathbb{P}_{\bar{Y}|\mathbf{X}}^{(k,u)}(\bar{y}_k|\mathbf{x})}{\mathbb{Q}_{\bar{Y}}^{(k,u)}(\bar{y}_k)} \tag{63}$$

for all $\mathbf{x} \in \mathcal{X}^\infty$ and all $\bar{y}_k \in \mathcal{Y}_k^{(u)}$. Let now $\varepsilon_{k,M}^{(u)} = \epsilon_k^{(u)} + (\log M)^{-1}$ and set $\tilde{\gamma}^{(u)} = \gamma_k^{(u)}$ where

$$\gamma_k^{(u)} \triangleq \sup\left\{\nu \in \mathbb{R} : \mathbb{P}_{\bar{Y},\mathbf{X}}^{(k,u)}\left[\imath_k^{(u)}(\mathbf{X};\bar{Y}_k) \le \log \nu\right] \le \varepsilon_{k,M}^{(u)}\right\}. \tag{64}$$

Note that there exists an arbitrary small positive constant δ, which is independent of $\log M$, such that

$$\mathbb{P}_{\bar{Y},\mathbf{X}}^{(k,u)}\left[\imath_k^{(u)}(\mathbf{X};\bar{Y}_k) \le \log \gamma_k^{(u)} - \delta\right] \le \varepsilon_{k,M}^{(u)} \le \mathbb{P}_{\bar{Y},\mathbf{X}}^{(k,u)}\left[\imath_k^{(u)}(\mathbf{X};\bar{Y}_k) \le \log \gamma_k^{(u)}\right]. \tag{65}$$

Using (64) in (62), we obtain

$$\log M \le \log \gamma_k^{(u)} - \log\left(\mathbb{P}_{\bar{Y},\mathbf{X}}^{(k,u)}\left[\imath_k^{(u)}(\mathbf{X};\bar{Y}_k) \le \log \gamma_k^{(u)}\right] - \epsilon_k^{(u)}\right) \tag{66}$$

$$\le \log \gamma_k^{(u)} + \log\log M. \tag{67}$$

Finally, by (65) and (67), we have

$$\mathbb{P}_{\bar{Y},\mathbf{X}}^{(k,u)}\left[\imath_k^{(u)}(\mathbf{X};\bar{Y}_k) \le \log M - \log\log M - \delta\right] \le \mathbb{P}_{\bar{Y},\mathbf{X}}^{(k,u)}\left[\imath_k^{(u)}(\mathbf{X};\bar{Y}_k) \le \log \gamma_k^{(u)} - \delta\right] \tag{68}$$

$$\le \varepsilon_{k,M}^{(u)}. \tag{69}$$

Using [10, Lem. 3] and the fact that $\mathbb{Q}_{\bar{Y}}^{(k,u)}$ is a convex combination of distributions, we obtain the following relation between $\imath_k^{(u)}(\mathbf{x};\bar{y})$ and $\tilde{\imath}_k^{(u)}(\mathbf{x};\bar{y})$

$$\imath_k^{(u)}(\mathbf{x};\bar{y}) \le \tilde{\imath}_k^{(u)}(\mathbf{x};\bar{y}) - \log \frac{1}{|P_t(\mathcal{X})|}. \tag{70}$$

The inequality in (69) can then be rewritten using (70), as follows:

$$\varepsilon_{k,M}^{(u)} \ge \mathbb{P}_{\bar{Y},\mathbf{X}}^{(k,u)}\left[\tilde{\imath}_k^{(u)}(\mathbf{X};\bar{Y}_k) \le \log M - \log\log M - \delta - \log|\mathcal{P}_t(\mathcal{X})|\right] \tag{71}$$

$$\ge \mathbb{P}_{\bar{Y},\mathbf{X}}^{(k,u)}\left[\tilde{\imath}_k^{(u)}(\mathbf{X};\bar{Y}_k) \le \lambda_t\right]. \tag{72}$$

Here, (72) follows by the definition of λ_t in (14), and because the number of types $|\mathcal{P}_t(\mathcal{X})|$ is upper bounded by $(t+1)^{|\mathcal{X}-1|}$ [11, Lem. 1.1].

APPENDIX II
STEPS OMITTED IN THE ASYMPTOTIC ANALYSIS OF THE CONVERSE BOUND

We will need the following property, whose proof follows from standard algebraic manipulations.

Property 2: Fix arbitrary $x \in \mathbb{R}$, $a > 0, b > 0$, and $\lambda > 0$. Suppose that $\xi > 0$ is the unique solution to the equation

$$\frac{\lambda - \xi a}{\sqrt{b\xi}} = x. \tag{73}$$

Then

$$0 \le \xi - \left(\frac{\lambda}{a} - x\sqrt{\frac{b\lambda}{a^3}}\right) \le \frac{b}{a^2}x^2. \tag{74}$$

For notational convenience, we will denote the mean, variance and third absolute moment of $\imath_k(\mathbf{x}^t; Y_k^t)$ by

$$I_k(P_{\mathbf{x}^t}) \triangleq I(P_{\mathbf{x}^t}, W_k) \tag{75}$$

$$V_k(P_{\mathbf{x}^t}) \triangleq V(P_{\mathbf{x}^t}, W_k) \tag{76}$$

$$T_k(P_{\mathbf{x}^t}) \triangleq T(P_{\mathbf{x}^t}, W_k). \tag{77}$$

According to (74) and since β satisfies (35), we have

$$0 \le \beta - \left(\frac{\lambda}{C} + \tilde{Q}^{-1}(\epsilon)\sqrt{\frac{V\lambda}{C^3}}\right) \le \mathbb{c}. \tag{78}$$

A. *Proof of* (36) *and* (37)

For the case $t < [0, \alpha)$, we use the following large-deviation bound

$$\max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda\right]$$

$$\leq \max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr\left[\frac{\imath_k(\mathbf{x}^t; Y_k^t)}{t} - I_k(P_{\mathbf{x}^t}) \geq \frac{\lambda}{t} - I_k(P_{\mathbf{x}^t})\right] \quad (79)$$

$$\leq \max_{\mathbf{x}^t \in \mathcal{X}^t} \exp\left(-\mathbb{c}\left(\frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{t}}\right)^2\right) \quad (80)$$

$$\leq \exp\left(-\mathbb{c}\log^2 \lambda\right) \quad (81)$$

$$\leq \left(\frac{1}{\lambda}\right)^{\mathbb{c}\log\lambda} \quad (82)$$

where (80) follows from Hoeffding's inequality [12, Th. 2] and (81) follows because $t < \alpha$ and because $I_k(P_{\mathbf{x}^t})$ is uniformly upper bounded by C. It follows from (82) that

$$\sum_{t=0}^{\lfloor \alpha \rfloor} \max_{\mathbf{x}^t \in \mathcal{X}^\infty} \Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda\right]$$

$$\leq (\alpha + 1)\left(\frac{1}{\lambda}\right)^{\mathbb{c}\log\lambda} \quad (83)$$

$$\leq \mathbb{c}\left(\frac{1}{\lambda}\right)^{\mathbb{c}\log\lambda - 1} = o(1). \quad (84)$$

Using similar argument, one establishes (37).

B. *Proof of* (39) *and* (40)

For the case when $t \in [\alpha, \beta)$, we need tighter bounds on $I_k(P_{\mathbf{x}^t})$ and $V_k(P_{\mathbf{x}^t})$. Let Π_μ be the set of probability distributions that are at distance no larger than μ from P^*:

$$\Pi_\mu \triangleq \left\{P \in \mathcal{P}(\mathcal{X}) : ||P - P^*||_2 \leq \mu\right\}. \quad (85)$$

Here, $||P - P^*||_2^2 \triangleq \sum_{x \in \mathcal{X}} (P(x) - P^*(x))^2$. Bounds on $I_k(P_{\mathbf{x}^t})$ and $V_k(P_{\mathbf{x}^t})$ are then supplied by [10, Lem. 7], which yields positive constants ς, μ and ρ for which

$$I_k(P_{\mathbf{x}^t}) \leq C - \varsigma\,||P_{\mathbf{x}^t} - P^*||_2^2 \quad (86)$$

$$V_k(P_{\mathbf{x}^t}) \geq \frac{V_k}{2} \quad (87)$$

and

$$\left|\sqrt{V_k(P_{\mathbf{x}^t})} - \sqrt{V_k}\right| \leq \rho\,||P_{\mathbf{x}^t} - P^*||_2 \quad (88)$$

for all $P_{\mathbf{x}^t} \in \Pi_\mu$.

Let $P_{\mathbf{x}^t} \in \Pi_\mu$. The Berry-Esseen central limit theorem yields the following estimate

$$\Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda\right]$$

$$\leq Q\left(\frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{tV_k(P_{\mathbf{x}^t})}}\right) + \frac{6tT_k(P_{\mathbf{x}^t})}{(tV_k(P_{\mathbf{x}^t}))^{3/2}} \quad (89)$$

$$\leq Q\left(\frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{tV_k(P_{\mathbf{x}^t})}}\right) + \frac{\mathbb{c}}{\sqrt{t}} \quad (90)$$

where the last inequality follows from (87) and because $T_k(P_{\mathbf{x}^t}) < \mathbb{c}$ uniformly in Π_μ.

This also implies that for all $P_{\mathbf{x}^t} \in \Pi_\mu$,

$$\max_{\mathbf{x}^t \in \mathcal{X}^t} \left\{\Pr\left[\imath_1(\mathbf{x}^t; Y_1^t) > \lambda\right]\right\} \max_{\mathbf{x}^t \in \mathcal{X}^t} \left\{\Pr\left[\imath_2(\mathbf{x}^t; Y_2^t) > \lambda\right]\right\}$$

$$\leq \prod_{k=1}^{2} \max_{\mathbf{x}^t \in \mathcal{X}^t} Q\left(\frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{tV_k(P_{\mathbf{x}^t})}}\right) + \frac{\mathbb{c}}{\sqrt{t}}. \quad (91)$$

For the case when $P_{\mathbf{x}^t} \notin \Pi_\mu$, we use Chebyshev's inequality to obtain the estimate

$$\Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda\right] \leq \frac{tV_k(P_{\mathbf{x}^t})}{(\lambda - tI_k(P_{\mathbf{x}^t}))^2} \quad (92)$$

for all $\lambda > tI_k(P_{\mathbf{x}^t})$. Since $P_{\mathbf{x}^t} \notin \Pi_\mu$, there exists a constant C' such that $I_k(P_{\mathbf{x}^t}) \leq C' < C$. Hence, for sufficiently large λ, the condition $t \leq \beta$ implies that $\lambda > tI_k(P_{\mathbf{x}^t})$. Therefore, by (92), we have that

$$\max_{P_{\mathbf{x}^t} \notin \Pi_\mu} \Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda\right]$$

$$\leq \max_{P_{\mathbf{x}^t} \notin \Pi_\mu} \frac{tV_k(P_{\mathbf{x}^t})}{(\lambda - tI_k(P_{\mathbf{x}^t}))^2} \quad (93)$$

$$\leq \frac{\mathbb{c}t}{(\lambda - tC')^2} \quad (94)$$

$$\leq \frac{\mathbb{c}\lambda}{(\lambda - \lambda C'/C - \mathbb{c}\sqrt{\lambda} - \mathbb{c})^2} \quad (95)$$

$$\leq \frac{\mathbb{c}}{\lambda} \quad (96)$$

where we have used that $t \leq \beta \leq 2t$ for sufficiently large λ and that $V_k(P_{\mathbf{x}^t})$ is uniformly upper-bounded [10, pp. 7048]. We see that $\max_{P_{\mathbf{x}^t} \notin \Pi_\mu} \Pr[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda]$ can be driven arbitrarily close to zero by having λ sufficiently large. This implies that we only need to consider the input vectors $\mathbf{x}^t$ for which $P_{\mathbf{x}^t} \in \Pi_\mu$, i.e.,

$$\max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda\right]$$

$$\leq \max_{P_{\mathbf{x}^t} \in \Pi_\mu} \Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda\right] + \frac{\mathbb{c}}{\lambda}. \quad (97)$$

Using (90) and (97), we obtain

$$\max_{\mathbf{x}^t \in \mathcal{X}^t} \Pr\left[\imath_k(\mathbf{x}^t; Y_k^t) > \lambda\right]$$

$$\leq \max_{P_{\mathbf{x}^t} \in \Pi_\mu} Q\left(\frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{tV_k(P_{\mathbf{x}^t})}}\right) + \frac{\mathbb{c}}{\sqrt{t}} + \frac{\mathbb{c}}{\lambda} \quad (98)$$

$$\leq Q\left(\min_{P_{\mathbf{x}^t} \in \Pi_\mu} \frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{tV_k(P_{\mathbf{x}^t})}}\right) + \frac{\mathbb{c}}{\sqrt{t}} + \frac{\mathbb{c}}{t} \quad (99)$$

$$= \int_{-\infty}^{\infty} \phi(x)\mathbb{1}\left\{\min_{P_{\mathbf{x}^t} \in \Pi_\mu} \frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{tV_k(P_{\mathbf{x}^t})}} \leq z\right\} \mathrm{d}z + \frac{\mathbb{c}}{\sqrt{t}} \quad (100)$$

for all sufficiently large λ. The indicator function in (100) can be upper bounded as

$$\mathbb{1}\left\{\min_{P_{\mathbf{x}^t}\in\Pi_\mu}\left\{\frac{\lambda - tI_k(P_{\mathbf{x}^t})}{\sqrt{tV_k(P_{\mathbf{x}^t})}} - z\right\} \leq 0\right\}$$
$$= \mathbb{1}\left\{\max_{P_{\mathbf{x}^t}\in\Pi_\mu}\left\{tI_k(P_{\mathbf{x}^t}) + z\sqrt{tV_k(P_{\mathbf{x}^t})} - \lambda\right\} \geq 0\right\} \quad (101)$$
$$\leq \mathbb{1}\left\{tC - t\varsigma\xi^2 + z\sqrt{tV_k} + |z|\sqrt{t}\rho\xi - \lambda \geq 0\right\} \quad (102)$$
$$\leq \mathbb{1}\left\{tC + z\sqrt{tV_k} + \frac{|z|\rho}{2\varsigma} - \lambda \geq 0\right\} \quad (103)$$
$$\leq \mathbb{1}\left\{\frac{\lambda - \frac{|z|\rho}{2\varsigma} - tC}{\sqrt{tV_k}} \leq z\right\} \quad (104)$$
$$\leq \mathbb{1}\left\{\frac{\lambda}{C} - z\sqrt{\frac{\lambda V_k}{C^3}} - \frac{|z|\rho}{2C\varsigma} \leq t\right\} \quad (105)$$

where (101) follows since $\sqrt{tV_k(\mathbf{x}^t)} > 0$ for $P_{\mathbf{x}^t} \in \Pi_\mu$ by (87), (102) follows by (86) and (88) with $\xi \triangleq ||P_{\mathbf{x}^t} - P^*||_2$, (103) follows because $-\varsigma\xi^2 t + |z|\rho\xi\sqrt{t}$ is a quadratic expression in $\xi\sqrt{t}$ with maximum $\frac{|z|\rho}{2\varsigma}$ and (105) follows from (74). The steps (101)–(103) essentially follow from [10, Prop. 8]. Substituting (105) into (100) and summing from $(\lfloor\alpha\rfloor+1)$ to $\lfloor\beta\rfloor$, we obtain

$$\sum_{t=\lfloor\alpha\rfloor+1}^{\lfloor\beta\rfloor}\max_{\mathbf{x}^t\in\mathcal{X}^t}\Pr\left[\imath_k(\mathbf{x}^t;Y_k^t) > \lambda\right]$$
$$\leq \sum_{t=0}^{\lfloor\beta\rfloor}\int_{-\infty}^{\infty}\phi(z)\mathbb{1}\left\{\frac{\lambda}{C} - z\sqrt{\frac{V_k\lambda}{C^3}} - \frac{|z|\rho}{2C\varsigma} \leq t\right\}\mathrm{d}z + \mathcal{O}(\log\lambda) \quad (106)$$
$$\leq \int_0^{\beta}\int_{-\infty}^{\infty}\phi(z)\mathbb{1}\left\{\frac{\lambda}{C} - z\sqrt{\frac{V_k\lambda}{C^3}} - \frac{|z|\rho}{2C\varsigma} \leq t\right\}\mathrm{d}z\,\mathrm{d}t + \mathcal{O}(\log\lambda) \quad (107)$$
$$\leq \int_{-\infty}^{\infty}\phi(z)\int_0^{\beta}\mathbb{1}\left\{\frac{\lambda}{C} - z\sqrt{\frac{V_k\lambda}{C^3}} - \frac{|z|\rho}{2C\varsigma} \leq t\right\}\mathrm{d}t\,\mathrm{d}z + \mathcal{O}(\log\lambda) \quad (108)$$
$$\leq \beta - \mathbb{E}\left[\min\left\{\beta, \left(\frac{\lambda}{C} - Z_k\sqrt{\frac{V_k\lambda}{C^3}}\right)\right\}\right] + \mathcal{O}(\log\lambda) \quad (109)$$
$$\leq \sqrt{\frac{V\lambda}{C^3}}\left(\tilde{Q}^{-1}(\epsilon) - \mathbb{E}\left[\min\left\{\tilde{Q}^{-1}(\epsilon)\,, \varrho_k Z_k\right\}\right]\right) + \mathcal{O}(\log\lambda) \quad (110)$$

where ϱ_k are defined in Theorem 3 and $Z_k \sim \mathcal{N}(0,1)$. Here, (107) follows because the indicator function is nondecreasing in t, in (108) the order of the integrals is interchangeable by Tonelli's theorem, and in (109) we have used (74).

By following the same approach, we obtain (40).

Appendix III
Proof of Lemma 1

Fix $\gamma \in \mathbb{R}$. We define the following two random walks, which are equivalent to U_n and V_n, but more convenient to analyze:

$$A_n \triangleq U_n/\mu_W + V_n/\mu_Z \quad (111)$$
$$B_n \triangleq U_n/\mu_W - V_n/\mu_Z. \quad (112)$$

We also define the additional stopping time

$$\tau_{12} \triangleq \inf\left\{n \geq 0 : A_n \geq \gamma\frac{\mu_W + \mu_Z}{\mu_W\mu_Z}\right\}. \quad (113)$$

We shall next show that

$$\mathbb{E}[\max\{\tau_1,\tau_2\}] \leq \mathbb{E}[\tau_{12} + \tau_1'(\gamma - U_{\tau_{12}}) + \tau_2'(\gamma - V_{\tau_{12}})] \quad (114)$$

where $\tau_1'(\cdot)$ and $\tau_2'(\cdot)$ are defined as

$$\tau_1'(\tilde{\gamma}) = \inf\left\{n \geq 0 : \sum_{i=1}^{n}\tilde{W}_i \geq \tilde{\gamma}\right\} \quad (115)$$
$$\tau_2'(\tilde{\gamma}) = \inf\left\{n \geq 0 : \sum_{i=1}^{n}\tilde{Z}_i \geq \tilde{\gamma}\right\} \quad (116)$$

and where $\{\tilde{W}_k, \tilde{Z}_k\}$ are i.i.d. and $(\tilde{W}_1, \tilde{Z}_1) \sim P_{W,Z}$ but independent of W_j, Z_j for all $j \in \mathbb{N}$. Note that τ_1' and τ_2' are independent of $U_{\tau_{12}}$ and $V_{\tau_{12}}$.

To prove (114), we use the following argument. At time τ_{12}, we have that $U_{\tau_{12}}/\mu_W + V_{\tau_{12}}/\mu_Z \geq \gamma\frac{\mu_W+\mu_Z}{\mu_W\mu_Z}$. This implies that either $\tau_1 \leq \tau_{12}$ or $\tau_2 \leq \tau_{12}$ (or both) are satisfied. Consider the case $\tau_1 \leq \tau_{12}$ and $\tau_2 > \tau_{12}$. To bound $\mathbb{E}[\max\{\tau_1,\tau_2\}]$, we need to characterize the remaining time until the random walk V_n hits the threshold γ. This time is given by $\min\{n \geq 0 : V_{\tau_{12}+n} \geq \gamma\}$, which has the same distribution as (116) computed at $\gamma - V_{\tau_{12}}$. Note also that $\tau_k'(\tilde{\gamma}) = 0$ for every $\tilde{\gamma} \leq 0$ since we use the convention $\sum_{i=1}^{0}(\cdot) = 0$. The inequality in (114) follows because there exist events for which $\max\{\tau_1,\tau_2\} < \tau_{12}$. The case $\tau_2 \leq \tau_{12}$ and $\tau_1 > \tau_{12}$ can be analyzed similarly.

By [4, Th. 3.9.4] (or by Wald's equality when W_1 and Z_1 have bounded support [3, Eq. (106)–(107)]), we have

$$\frac{\tilde{\gamma}}{\mu_W} \leq \mathbb{E}[\tau_1'(\tilde{\gamma})] \leq \frac{\tilde{\gamma}}{\mu_W} + \mathbb{c} \quad (117)$$
$$\frac{\tilde{\gamma}}{\mu_W} \leq \mathbb{E}[\tau_2'(\tilde{\gamma})] \leq \frac{\tilde{\gamma}}{\mu_Z} + \mathbb{c} \quad (118)$$
$$\gamma\frac{\mu_W + \mu_Z}{2\mu_W\mu_Z} \leq \mathbb{E}[\tau_{12}] \leq \gamma\frac{\mu_W + \mu_Z}{2\mu_W\mu_Z} + \mathbb{c}. \quad (119)$$

Using (114), the linearity of expectation, (117)–(119), and the fact that

$$\mathbb{E}[\tau_1'(\gamma - U_{\tau_{12}})] = \mathbb{E}[\mathbb{E}[\tau_1'(\gamma - U_{\tau_{12}})|U_{\tau_{12}}]] \leq \frac{1}{\mu_W}\mathbb{E}\left[(\gamma - U_{\tau_{12}})^+\right] + \mathbb{c} \quad (120)$$

we conclude that

$$
\begin{aligned}
&\mathbb{E}[\max\{\tau_1,\tau_2\}] - \gamma\frac{\mu_W+\mu_Z}{2\mu_W\mu_Z} \\
&\le \frac{1}{\mu_W}\mathbb{E}\Big[(\gamma - U_{\tau_{12}})^+\Big] + \frac{1}{\mu_Z}\mathbb{E}\Big[(\gamma - V_{\tau_{12}})^+\Big] + \mathbb{c} \quad (121)\\
&= \frac{1}{\mu_W}\mathbb{E}\left[\left(\gamma - \frac{1}{2}\mu_W(A_{\tau_{12}} + B_{\tau_{12}})\right)^+\right] \\
&\quad + \frac{1}{\mu_Z}\mathbb{E}\left[\left(\gamma - \frac{1}{2}\mu_Z\,(A_{\tau_{12}} - B_{\tau_{12}})\right)^+\right] + \mathbb{c} \quad (122)\\
&\le \mathbb{E}\left[\left(\frac{\gamma}{\mu_W} - \frac{1}{2}\left(\gamma\frac{\mu_W+\mu_Z}{\mu_W\mu_Z} + B_{\tau_{12}}\right)\right)^+\right] \\
&\quad + \mathbb{E}\left[\left(\frac{\gamma}{\mu_Z} - \frac{1}{2}\left(\gamma\frac{\mu_W+\mu_Z}{\mu_W\mu_Z} - B_{\tau_{12}}\right)\right)^+\right] + \mathbb{c} \quad (123)\\
&= \frac{1}{2}\mathbb{E}\left[\left|\gamma\frac{\mu_Z-\mu_W}{\mu_W\mu_Z} - B_{\tau_{12}}\right|\right] + \mathbb{c} \quad (124)
\end{aligned}
$$

where (123) follows from the definition of τ_{12} (see (113)) which implies that $A_{\tau_{12}} \ge \gamma\frac{\mu_W+\mu_Z}{\mu_W\mu_Z}$.

We next show that the RHS of (124) is upper-bounded by the RHS of (55) by the following two steps. First, we shall approximate $B_{\tau_{12}}$ by a Gaussian RV using a variation of the Berry-Esseen theorem that holds when the number of terms in the summation is a RV (see Lemma 2 below). Then, we shall establish (55) using standard properties of Gaussian RVs.

Lemma 2: ([5, Th. 1]) Let $\{\xi_n, n\ge 1\}$ be i.i.d. RVs with zero mean, positive variance σ^2, and finite third absolute moment. Let $\{N_n, n\in\mathbb{N}\}$ be a sequence of positive integer-valued RVs and assume that

$$\Pr\left[\left|\frac{N_n}{n\nu} - 1\right| > \zeta_n\right] = \mathcal{O}\left(\sqrt{\zeta_n}\right) \quad (125)$$

for some constant ν and a sequence $\{\zeta_n\}$ that vanishes as $n\to\infty$ and that satisfies $\frac{1}{n}\le\zeta_n$ for all n. Then

$$\sup_{\lambda\in\mathbb{R}}\left|\Pr\left[\sum_{i=1}^{N_n}\xi_i \le \sigma\sqrt{n\nu}\lambda\right] - \Phi(\lambda)\right| = \mathcal{O}\left(\sqrt{\zeta_n}\right). \quad (126)$$

The RV $B_{\tau_{12}}$ and its variance satisfies [4, Th. 4.2.4 (ii')]

$$\mathrm{Var}[B_{\tau_{12}}] = \sigma^2\gamma\frac{\mu_W+\mu_Z}{2\mu_W\mu_Z} + \mathcal{O}(1) \quad (127)$$

as $\gamma\to\infty$. For some constant $\nu>0$, let $\gamma_n \triangleq \frac{2\nu\mu_W\mu_Z n}{\mu_W+\mu_Z}$, $N_n \triangleq \tau_{12}(\gamma_n)$, $\xi_n \triangleq \frac{W_n}{\mu_W} - \frac{Z_n}{\mu_Z}$ and $\zeta_n \triangleq n^{-\frac{r}{2r+1}}$ for $n\in\mathbb{N}$. Note that by (119), we have

$$
\begin{aligned}
\mathbb{E}[N_n] &= \mathbb{E}[\tau_{12}(\gamma_n)] \quad (128)\\
&= \gamma_n\frac{\mu_W+\mu_Z}{2\mu_W\mu_Z} + \mathcal{O}(1) \quad (129)\\
&= \nu n + \mathcal{O}(1), \qquad n\to\infty. \quad (130)
\end{aligned}
$$

We next show that condition (125) in Lemma 2 is satisfied. Indeed,

$$
\begin{aligned}
&\Pr\left[\left|\frac{N_n}{\nu n} - 1\right| \ge \zeta_n\right] \\
&= \Pr\left[\left|\frac{N_n - \nu n}{\sqrt{\nu n}}\right|^r \ge \left(\sqrt{\nu n}\zeta_n\right)^r\right] \quad (131)\\
&\le \frac{\mathbb{E}\left[\left|\frac{N_n-\nu n}{\sqrt{\nu}}\right|^r\right]}{(\sqrt{\nu n}\zeta_n)^r} \quad (132)\\
&= \frac{\mathbb{c}}{(\sqrt{\nu n}\zeta_n)^r} \quad (133)\\
&= \frac{\mathbb{c}}{n^{r/2}\left(n^{-\frac{r}{2r+1}}\right)^r} = \frac{\mathbb{c}}{n^{\frac{r}{4r+2}}} = \mathcal{O}(\sqrt{\zeta_n}) \quad (134)
\end{aligned}
$$

as $n\to\infty$. Here, (132) follows from Markov's inequality and (133) follows from [4, Th. 3.8.4(i)].

Let $F(\lambda) \triangleq \Pr[B_{N_n} \le \sigma\sqrt{\upsilon n}\lambda]$. We can now use Lemma 2, which for sufficiently large n implies that

$$\sup_{\lambda\in\mathbb{R}}|F(\lambda) - \Phi(\lambda)| \le \mathbb{c}n^{-\frac{r}{4r+2}}. \quad (135)$$

We next refine our estimate in (135) using Lemma 3 below.

Lemma 3: ([13, Th. 9]) Let $F(x)$ be the cumulative distribution function of a RV that has finite moment of order p. Suppose that $0 < \Delta \triangleq \sup_x|F(x) - \Phi(x)| \le 1/\sqrt{e}$. Then there exists a constant C_p, that depends only on p, such that

$$|F(x) - \Phi(x)| \le \frac{C_p\Delta\left(\log\frac{1}{\Delta}\right)^{p/2} + \rho_p}{1+|x|^p} \quad (136)$$

for all x. Here

$$\rho_p = \left|\int_{-\infty}^{\infty}|x|^p\mathrm{d}F(x) - \int_{-\infty}^{\infty}|x|^p\mathrm{d}\Phi(x)\right|. \quad (137)$$

Using Lemma 3 and (135), we have that

$$|F(\lambda) - \Phi(\lambda)| \le \frac{\mathbb{c}n^{-\frac{r}{4r+2}}\log n + \rho_2(n)}{1+\lambda^2}. \quad (138)$$

for $\lambda\in\mathbb{R}$ and sufficiently large n. Here,

$$\rho_2(n) = \left|\frac{\mathrm{Var}[B_{N_n}]}{\sigma^2 n\nu} - 1\right| = \left|\frac{n+\mathcal{O}(1)}{n} - 1\right| \le \frac{\mathbb{c}}{n}. \quad (139)$$

Fix an arbitrary $a\in\mathbb{R}$. Using (138), we obtain the following upper bound

$$
\begin{aligned}
&\mathbb{E}[|a - B_{N_n}|] \\
&= \sigma\sqrt{\nu n}\int_0^\infty 1 + F\left(\frac{a}{\sigma\sqrt{\nu n}} - x\right) - F\left(\frac{a}{\sigma\sqrt{\nu n}} + x\right)\mathrm{d}x \quad (140)\\
&\le \sigma\sqrt{\nu n}\int_0^\infty\left[\Phi\left(\frac{a}{\sigma\sqrt{\nu n}} - x\right) + \left(1 - \Phi\left(\frac{a}{\sigma\sqrt{\nu n}} + x\right)\right)\right.\\
&\quad\left. + \frac{\mathbb{c}n^{-\frac{r}{4r+2}}\log n + \mathbb{c}/n}{1+(\frac{a}{\sigma\sqrt{\nu n}}+x)^2} + \frac{\mathbb{c}n^{-\frac{r}{4r+2}}\log n + \mathbb{c}/n}{1+(\frac{a}{\sigma\sqrt{\nu n}}-x)^2}\right]\mathrm{d}x \quad (141)\\
&= \sigma\sqrt{\nu n}\mathbb{E}\left[\left|\frac{a}{\sigma\sqrt{\nu n}} - Z\right|\right]
\end{aligned}
$$

$$+ \pi\sigma\sqrt{\nu}\left(\mathbb{c}n^{\frac{1}{2}-\frac{r}{4r+2}}\log n + \mathbb{c}/\sqrt{n}\right) \tag{142}$$

$$= \sqrt{\frac{2}{\pi}}\sigma\sqrt{\nu n}\psi\left(\frac{a}{\sigma\sqrt{\nu n}}\right) + |a| + \mathcal{O}(n^{\frac{r+1}{4r+2}}\log n) \tag{143}$$

as $n \to \infty$, where $Z \sim \mathcal{N}(0,1)$ and

$$\psi(x) \triangleq \sqrt{\frac{\pi}{2}}(\mathbb{E}[|x-Z|] - |x|) \tag{144}$$

$$= \exp\left(-\frac{x^2}{2}\right) + x\sqrt{\frac{\pi}{2}}\left(\text{erf}\left(\frac{x}{\sqrt{2}}\right) - \text{sgn}(x)\right). \tag{145}$$

The positive function $\psi(x)$ is unimodal with maximum 1 attained at $x = 0$ and decays exponentially to 0 as $|x| \to \infty$.

Substituting $a = \gamma_n\frac{\mu_Z-\mu_W}{\mu_W\mu_Z} = \frac{2\nu n(\mu_Z-\mu_W)}{\mu_W+\mu_Z}$ into (143), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\left|\gamma_n\frac{\mu_Z-\mu_W}{\mu_W\mu_Z} - B_{N_n}\right|\right] \\
&\leq \sqrt{\frac{2}{\pi}}\sigma\sqrt{\nu n}\psi\left(\frac{2\sqrt{\nu n}(\mu_Z-\mu_W)}{\sigma(\mu_W+\mu_Z)}\right) \\
&\quad + \gamma_n\left|\frac{\mu_Z-\mu_W}{\mu_W\mu_Z}\right| + \mathcal{O}(n^{\frac{r+1}{4r+2}}\log n). \qquad (146)
\end{aligned}$$

Note that for the case $\mu_Z \neq \mu_W$, we have that $\sqrt{n}\psi\left(\frac{2\sqrt{\nu n}(\mu_Z-\mu_W)}{\sigma(\mu_W+\mu_Z)}\right) = o(1)$ as $n \to \infty$. Substituting (146) into (124), we obtain

$$\begin{aligned}
&\mathbb{E}[\max\{\tau_1(\gamma_n), \tau_2(\gamma_n)\}] \\
&\leq \gamma_n\frac{\mu_W+\mu_Z}{2\mu_W\mu_Z} \\
&\quad + \frac{1}{2}\mathbb{E}\left[\left|\gamma_n\frac{\mu_Z-\mu_W}{\mu_W\mu_Z} - B_{\tau_{12}(\gamma_n)}\right|\right] + \mathcal{O}(1) \qquad (147) \\
&= \gamma_n\frac{\mu_W+\mu_Z}{2\mu_W\mu_Z} + \gamma_n\left|\frac{\mu_Z-\mu_W}{2\mu_W\mu_Z}\right| \\
&\quad + \frac{\sigma}{\sqrt{2\pi}}\sqrt{\nu n}\mathbb{1}\{\mu_W = \mu_Z\} + \mathcal{O}(n^{\frac{r+1}{4r+2}}\log n) \qquad (148) \\
&= \frac{\gamma_n}{\min\{\mu_W, \mu_Z\}} + \frac{\sigma}{\sqrt{2\pi}}\sqrt{\nu n}\mathbb{1}\{\mu_W = \mu_Z\} \\
&\quad + \mathcal{O}(n^{\frac{r+1}{4r+2}}\log n), \qquad n \to \infty \qquad (149)
\end{aligned}$$

where (149) follows from the identity $a + b + |a-b| = 2\max\{a, b\}$.

To complete the proof, let $n_1 \triangleq \lceil\frac{\gamma}{\min(\mu_W,\mu_Z)}\rceil$, $\Psi(x) \triangleq x + \frac{\sigma}{\sqrt{2\pi}}\sqrt{\nu x}\mathbb{1}\{\mu_W = \mu_Z\} + b_1 x^{\frac{r+1}{4r+2}}\log x$, and set $\nu \triangleq \frac{\mu_W+\mu_Z}{2\max\{\mu_W,\mu_Z\}}$, i.e.

$$\gamma_n = \min\{\mu_W, \mu_Z\}\, n. \tag{150}$$

Note that $\Psi(x)$ is nondecreasing, concave and differentiable in $x \in [1, \infty]$. Then there exists a constant $b_1 > 0$ such that

$$\begin{aligned}
&\mathbb{E}[\max\{\tau_1(\gamma), \tau_2(\gamma)\}] \\
&\leq \mathbb{E}[\max\{\tau_1(\gamma_{n_1}), \tau_2(\gamma_{n_1})\}] \qquad (151) \\
&\leq n_1 + \frac{\sigma}{\sqrt{2\pi}}\sqrt{\nu n_1}\mathbb{1}\{\mu_W = \mu_Z\} + b_1 n_1^{\frac{r+1}{4r+2}}\log n_1 \qquad (152) \\
&= \Psi\left(\left\lceil\frac{\gamma}{\min(\mu_W,\mu_Z)}\right\rceil\right) \qquad (153) \\
&\leq \Psi\left(\frac{\gamma}{\min\{\mu_W,\mu_Z\}} + 1\right) \qquad (154) \\
&\leq \Psi\left(\frac{\gamma}{\min(\mu_W,\mu_Z)}\right) + \mathbb{c} \qquad (155) \\
&= \frac{\gamma}{\min\{\mu_W,\mu_Z\}} + \frac{\sigma}{2\sqrt{\pi}}\sqrt{\frac{\gamma(\mu_W+\mu_Z)}{\mu_W\mu_Z}}\mathbb{1}\{\mu_W = \mu_Z\} \\
&\quad + \mathcal{O}(\gamma^{\frac{r+1}{4r+2}}\log\gamma) \qquad (156) \\
&= \frac{\gamma}{\min\{\mu_W,\mu_Z\}} + \frac{\sigma}{\sqrt{2\pi}}\sqrt{\frac{\gamma}{\mu_W}}\mathbb{1}\{\mu_W = \mu_Z\} \\
&\quad + \mathcal{O}(\gamma^{\frac{r+1}{4r+2}}\log\gamma). \qquad (157)
\end{aligned}$$

Here, (151) follows because $\mathbb{E}[\max\{\tau_1(\gamma), \tau_2(\gamma)\}]$ is nondecreasing in γ.