CHAPTER 1

A NUMERICAL METHOD OF CALCULATING PLASTIC STRAINS IN STRAIN-HARDENING MATERIALS¹

P. Thoft-Christensen, Technical University of Denmark

Summary
A numerical method is described based on approximating the stress-path, for calculating plastic strains for strain-hardening materials. The method is considered to offer advantages when a number of different stress paths have to be analyzed in statically determined stress states.

In the approximation, an effort is made to transfer the computational advantages of the total type of stress-strain law, to the physically sound stress-strain law of the incremental type. Furthermore, in calculations under special conditions (isotropic strain-hardening, v. Mises’ yield condition and special stress-strain laws) tables have been prepared which simplify the work of calculation considerably.

The method is illustrated by an example in which the correct determination is possible. The example also shows how the method permits the determination of upper and lower values of the plastic strains.

1. INTRODUCTION

During the last few decades, there has been a steady increase in the use of various types of aluminium alloy for constructional purposes. This is also the case for a number of other alloys. In contrast to the usual types of steel, many of these metals exhibit the common feature that they do not in general show any clearly defined tensile yield stress, nor do they behave as perfectly plastic materials when in the inelastic state. In the case of such metals, therefore, stress and strain analysis, which is based on the usual theories for perfectly plastic materials, will be only an approximation.

In consequence, interest has been increasing in the development of theories of the mechanical behaviour of such strain-hardening materials. The theories established for inelastic states are usually based on three physical laws:

¹ Bygningsstatistiske Meddelelser, Vol. 33, 1962, pp. 45-64.
1. An initial yield condition,
2. A strain-hardening law and
3. A stress-strain law.

However, the correct application of these physical laws is extremely complicated, and must in general be rejected in favour of certain methods of approximation. These methods consist either of an idealization of the yield conditions, or of an idealization of the stress-strain law.

While these methods thus represent an idealization of the physical laws, this article will attempt to demonstrate how an idealization of the loading path results in mathematical simplifications in the analysis of statically determined stress states.

2. STRESS-STRAIN LAWS

The only stress-strain laws meeting all reasonable demands made on them (see e.g. Prager [1]) are those of the so-called incremental type. By this term is understood stress-strain laws which uniquely determine the plastic strain-rates for a given stress-rate $\dot{\sigma}_{ij}$ from a given stress state $\sigma_{ij}$. $\epsilon_{ij}$ and $\sigma_{ij}$ designate here the strain tensor and stress tensor respectively, expressed in a rectangular system of cartesian coordinates. Here and in what follows, the dot over the symbols signifies differentiation with respect to time or to a monotonic increasing function of time. Prager [2] gives the following typical example of such a stress-strain law

$$2G_0 \dot{\epsilon}_{ij} = \dot{s}_{ij} + h(\sigma_{ij})\dot{\sigma}_{ij}$$

for $\sigma_{ij} > 0$

$$2G_0 \dot{e}_{ij} = \dot{s}_{ij}$$

for $\sigma_{ij} < 0$ (1)

where $G_0$ is the elastic shear modulus, $\sigma_{ij} = \sigma_{ij}(\sigma_{ij})$ is a specified positive effective stress, $h = h(\sigma_{ij})$ is a function of the effective stress, and the stress deviation tensor $s_{ij}$ is defined by

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$$

(2)

where $\delta_{ij}$ is Kronecker’s delta. The usual summation convention is used in equation (2). In the present article, this means that repetition of the index in a term indicates that summation should be carried out over the index in question, for the values 1, 2 and 3. For example

$$\sigma_{kk} = \sigma_{11} + \sigma_{22} + \sigma_{33}$$

and

$$\sigma_{ij}\sigma_{ik} = \sigma_{11}\sigma_{1k} + \sigma_{22}\sigma_{2k} + \sigma_{33}\sigma_{3k}$$

where $\sigma_{1}$, $\sigma_{2}$, and $\sigma_{3}$ are the principal stresses.

Included in (1) are the linear elastic relations for an incompressible material

$$2G_0 \dot{\epsilon}_{ij}^{el} = \dot{s}_{ij}$$

(3)
Assuming that
\[ \dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{el} + \dot{\varepsilon}_{ij}^{pl} \]
the plastic part of the stress-strain law (1) appears as
\[ 2G_0\dot{\varepsilon}_{ij}^{pl} = h(\sigma_\varepsilon)\dot{\sigma}_s \] for \( \dot{\sigma}_e > 0 \)
\[ 2G_0\dot{\varepsilon}_{ij}^{pl} = 0 \] for \( \dot{\sigma}_e < 0 \)

On the basis of the stress-strain law (4), the plastic strain state is determined at any given instant by integration along the stress path. In contrast to the elastic deformations, therefore, the result will depend on the actual stress path, and not on the final stress state alone. As a result, stress-strain laws of the incremental type cannot only be a differentiated form of a suitable total stress-strain law, by which is understood a stress-strain law in which the plastic strains are given direct as functions of the stresses. These conditions have been examined for a relatively general stress-strain law by Handelman & Warner [3].

A radial loading is then considered, i.e. one where all components in the stress deviation tensor \( s_{ij} \) increase proportionally with a time parameter \( t \) or a monotonic increasing function of this, i.e.
\[ s_{ij} = s_{ij}^* t \]
where \( s_{ij}^* \) is a symmetrical tensor, which is independent of \( t \). For such a stress path, Ilyushin [4] has demonstrated that the known stress-strain laws of the incremental type can be integrated and thus brought into total form. In particular, for stress-strain law (4), Prager [2) shows that the following is obtained
\[ 2G_0\varepsilon_{ij}^{pl} = H(\sigma_\varepsilon)s_{ij} \]
where
\[ H(\sigma_\varepsilon) = \frac{1}{\sigma_\varepsilon} \int_0^{\sigma_\varepsilon} h(\sigma)\sigma d\sigma \]

As appears from the above, stress-strain laws of total type are special cases of stress-strain laws of incremental type. In spite of this, stress-strain laws of the total type are used extensively in load paths other than radial (see e.g. Nadai [5], [6] and Sokolovskij [7], although the radial path is the condition necessary to justify the use the total type of law. The widespread use of stress-strain laws of the al type is due to the considerable simplification achieved in mathematical treatment. Using the incremental type of stress-strain laws, it is generally necessary to employ inconvenient methods of numerical integration. As previously mentioned, various efforts have therefore been made to retain the simplifying influence of total theories. For ample the use of piecewise linear yield conditions can be mentioned (see Hodge [8]), and the use of singular yield conditions (Sanders [9]). Finally, it might be mentioned that the presence of singular points or singular regimes in the yield condition in connection with certain almost-radial loading paths also allows the use of stress-strain laws of total type, as shown by Budiansky [10] and Kliushnikov [11].
3. SETTING-UP THE METHOD

We consider once more the stress-strain law (4), set up for stress changes involving \( \sigma_r > 0 \) (loading) or \( \sigma_r < 0 \) (unloading). For the special case, where \( \sigma_r = 0 \) during the changes in stress,

\[
\dot{\varepsilon}_{pl}^{\sigma} = 0
\]  
(8)

is obtained as a limiting value from both expressions in (4). This change in loading is therefore considered to be neutral.

The numerical method, which will now be set up, is based on the following two properties of the most commonly used stress-strain laws of the incremental type:

a. Neutral loading \((\sigma_r \equiv 0)\) involves no changes in the plastic strains.

b. In radial loading, the changes in the plastic strains can be determined by total relations corresponding to stress-strain laws of the total type.

Figure 1 shows parts of two yield surfaces I and II associated with the effective stresses \( \sigma^I \) and \( \sigma^II \) \((\sigma^II > \sigma^I)\) represented in the nine-dimensional stress space \( \sigma_{ij} \).

Further, an interrupted line is drawn showing a non-radial stress path, "cutting" the yield surfaces in A and D, respectively. The stress path between A and D is now approximated by the path ABCD, where AB runs in the stress surface I, CD in the stress surface II, and BC in the radial direction (the extension of the intercept BC passes through the point of origin of the coordinate system, O). On the basis of the above remarks, there will for the approximate stress path ABCD be plastic strain changes only over the stretch BC. Since the stress path here is radial, these plastic strain changes can be determined by the principles of total theory, i.e., as differences between two sets of numbers.

In this way, by dividing the complete stress path into parts lying on yield surfaces, and radial parts, the plastic strains can be determined as a sum of differences. By this method, it will be possible to determine the plastic strain state \((\varepsilon_{pl}^{\sigma})_A\) corresponding to a given stress path to a stress state \((\sigma_{ij})_A\), with greater or lesser precision depending on the fineness of the approximation, but for every approximation the calculations will always be in agreement with the principles in the stress-strain laws of the incremental type. Replacing the loading path to the stress point A by the radial loading to A gets the simplest approximation. This approximation corresponds to a direct determination of the plastic strains by a stress-strain law of total type.

4. ISOTROPIC HARDENING BASED ON v. MISES' YIELD CONDITION

We will now examine the practical course of the calculations in the usual case, in which the strain-hardening is isotropic and where the yield conditions are determined by

\[
s_i s_i = \sigma_z^2
\]  
(9)

where \( \sigma_z \geq 0 \) changes during the strain-hardening, and where \( s_i \) are the principal stress deviations. The yield condition (9) is identical with the usual v. Mises' yield condition.
As stress-strain law we assume the equations (4), which take form (6) for radial loading.

As used by Hill [12] among others, the geometric representation of (9) in the usual rectangular coordinate system of principal stresses (σ-system) is right circular cylinders with direction cosines of axis (1/√3, 1/√3, 1/√3). The curves of interception of the cylinders with the so-called Π-plane (σ₁ + σ₂ + σ₃ = 0) are concentric circles. It should also be noted that the stress point (s₁, s₂, s₃) lies in the Π-plane as s₁ + s₂ + s₃ = 0. This Π-plane is shown in fig. 2, and the orthogonal projections of the axes of the coordinate system on the Π-plane are drawn in together with an x'y'z'-coordinate system (z' axis ⊥ to the Π-plane) by means of the transformation formulæ

\[
\begin{bmatrix}
x'

y'

z'
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & 0
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0
\frac{1}{\sqrt{2}} & 0 & 0
\end{bmatrix} \begin{bmatrix}
\sigma_x

\sigma_y

\sigma_z
\end{bmatrix}
\]

The analytic expression for the yield conditions becomes in x'y'z'-coordinates

\[x'^2 + y'^2 = s_j s_j = \sigma^2\]

As all stress states on one and the same generatrix of the yield surface have the same stress-deviation state, the stress-strain law (4) shows that stress changes dσ dt from states of arbitrary stress on the same generatrix give the same plastic strain changes, so long as (s₁, s₂, s₃) is the same for all changes of stress. It may be concluded therefore that all stress paths with the same projection on the Π-plane give the same plastic strain path. In the plastic strain calculation, therefore, it is only necessary to consider the projection of the stress path in the Π-plane, and thus work with a plane problem. All these considerations hold regardless of the form of the yield surface, the type of hardening and the strain-stress law, if only the condition is imposed that a hydrostatic compression or tension has no influence on the plastic deformations (corresponding to the condition that σᵢᵢ and not σᵢⱼ enters directly in the relations).

We now return to the yield conditions (11), and introduce a dimensionless coordinate system with

\[x = x'\sigma_0\]
\[y = y'\sigma_0\]
\[z = z'\sigma_0\]

where σ₀ is a suitably chosen constant stress. The yield conditions (11) then assume the form

\[x^2 + y^2 = L^2\]
In a corresponding manner, the dimensionless principal stress deviations are introduced

\[ u_i = s_i / \sigma_0 \]  

so that the stress-strain law with radial loading (6) assumes the form

\[ \varepsilon_i^{pl} = F(L)u_i \]

where

\[ F(L) = \frac{H(\sigma_r)}{2G_0} \sigma_0 \]

In figure 3, yield curves are drawn for three values of \( L \), namely \( L = L_0, L = L_{n-1} \) and \( L = L_n \), where \( L_n > L_{n-1} \). The yield curve for \( L = L_0 \), which can be chosen arbitrarily, may be designated as the *reference yield curve*.

For the radial stress path \( AB \), the plastic strain increments are determined according to (18) by

\[ \Delta \varepsilon_i^{pl} = F(L_n)u_i^n - F(L_{n-1})u_i^{n-1} = \frac{F(L_n) - F(L_{n-1})}{L_0} L_{n-1} L_n u_i^0 \]

where \( u_i^n \) and \( u_i^{n-1} \) designate the dimensionless stress deviations associated with stress states \( B \) and \( A \), while \( u_i^0 \) are the values of the stress deviations for stress state \( C \) on the reference yield curve (see figure 3). In the conversion, the relationships have been used that

\[ u_i^n = \frac{L_n}{L_0} u_i^0 \]

and correspondingly

\[ u_i^{n-1} = \frac{L_{n-1}}{L_0} u_i^0 . \]

If we put

\[ F(L_n) - F(L_{n-1})L_{n-1} = b_n , \]

then (20) can be written

\[ \varepsilon_i^{pl} = \frac{b_n}{L_0} u_i^0 , \]

and if the yield curves are selected for the approximation so that

\[ b_n = b = \text{constant} , \]

the plastic strains in the complete approximation are determined by

\[ \varepsilon_i^{pl} = \frac{b}{L_0} \sum_{n=1}^{N} (u_i^0)_n , \]
where \((u_{i}^{0})_{n}\) are the stress deviations on the reference yield curve corresponding to the radial stress path between the yield curves with and \(L = L_{0}\) and \(L = L_{n-1}\). In figure 4 this stress path is designated by \(n\). \(N\) represents the number of radial parts between subsequent yield surfaces in the approximation.

It should be mentioned that condition (23) has the result that parts between two subsequent yield curves on the same radial direction make contribution to \(u_{i}\). In addition to the constants \(b\) and \(L_{0}\), the expression (24) includes only the dimensionless principal stress deviations \(u_{i}^{0}\) on the reference circle. For the reference circle

\[
x^2 + y^2 = 1
\]

(25)

table 1 gives \(u_{i}\) corresponding to the points of interception, in the 1st and the 4th quadrants, between the circle (25) and the radial lines \(y = \alpha x\),

\[
y = \alpha x
\]

(26)

<table>
<thead>
<tr>
<th>(\alpha n)</th>
<th>1st quadrant</th>
<th>4th quadrant</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(u_{1})</td>
<td>(u_{2})</td>
</tr>
<tr>
<td>0</td>
<td>-0.707</td>
<td>0.707</td>
</tr>
<tr>
<td>1/10</td>
<td>-0.744</td>
<td>0.663</td>
</tr>
<tr>
<td>2/10</td>
<td>-0.773</td>
<td>0.613</td>
</tr>
<tr>
<td>3/10</td>
<td>-0.795</td>
<td>0.560</td>
</tr>
<tr>
<td>4/10</td>
<td>-0.808</td>
<td>0.505</td>
</tr>
<tr>
<td>5/10</td>
<td>-0.815</td>
<td>0.450</td>
</tr>
<tr>
<td>6/10</td>
<td>-0.816</td>
<td>0.396</td>
</tr>
<tr>
<td>7/10</td>
<td>-0.813</td>
<td>0.345</td>
</tr>
<tr>
<td>8/10</td>
<td>-0.807</td>
<td>0.297</td>
</tr>
<tr>
<td>9/10</td>
<td>-0.799</td>
<td>0.253</td>
</tr>
<tr>
<td>1</td>
<td>-0.789</td>
<td>0.211</td>
</tr>
<tr>
<td>1/8</td>
<td>-0.776</td>
<td>0.170</td>
</tr>
<tr>
<td>1/6</td>
<td>-0.761</td>
<td>0.123</td>
</tr>
<tr>
<td>1/7</td>
<td>-0.740</td>
<td>0.071</td>
</tr>
<tr>
<td>1/5</td>
<td>-0.714</td>
<td>0.014</td>
</tr>
<tr>
<td>1/4</td>
<td>-0.681</td>
<td>-0.049</td>
</tr>
<tr>
<td>1/3</td>
<td>-0.642</td>
<td>-0.116</td>
</tr>
<tr>
<td>1/2</td>
<td>-0.594</td>
<td>-0.188</td>
</tr>
<tr>
<td>1/10</td>
<td>-0.539</td>
<td>-0.262</td>
</tr>
<tr>
<td>1</td>
<td>-0.477</td>
<td>-0.336</td>
</tr>
<tr>
<td>(\infty)</td>
<td>-0.408</td>
<td>-0.408</td>
</tr>
</tbody>
</table>

Table 1. The variation in the stress deviations along the yield circle \(L_{0} = 1\).
where $\alpha$ assumes certain selected values. As shown in figures 5-7, which indicate the variation in $u_i$ along the reference circle, the corresponding values of $u_i$ in the 2nd and 3rd quadrants are obtained by reversing the signs of the values in table 1.

5. CALCULATION OF UPPER AND LOWER VALUES

The value of a numerical method depends to a great extent on the facility for evaluating the computed magnitudes. The present section shows how the numerical method described here permits a convenient determination of both upper and lower values for the plastic strains.
In figure 8, the given stress path $AD$ between yield surfaces I and II with $\sigma_e = \sigma_1$ and $\sigma_\varepsilon = \sigma_\Pi$ respectively, is approximated by $ABD$ and $ACD$. Using the two approximations, the plastic strain increments are calculated according to (6)

### ABD

$$\Delta \varepsilon_{ij}^{pl} = \frac{1}{2G_0} [H(\sigma_\Pi) s_{ij}^B - H(\sigma_1) s_{ij}^A] = \frac{1}{2G_0} [H(\sigma_\Pi) \frac{\sigma_\Pi}{\sigma_1} - H(\sigma_1)] s_{ij}^A = a s_{ij}^A$$  \hspace{1cm} (27)$$

### ACD

$$\Delta \varepsilon_{ij}^{pl} = \frac{1}{2G_0} [H(\sigma_\Pi) s_{ij}^D - H(\sigma_1) s_{ij}^C] = \frac{1}{2G_0} [H(\sigma_\Pi) \frac{\sigma_\Pi}{\sigma_1} - H(\sigma_1)] s_{ij}^C = a s_{ij}^C$$  \hspace{1cm} (28)$$

where $s_{ij}^A$, $s_{ij}^B$, $s_{ij}^C$, and $s_{ij}^D$ are the stress deviations associated with the states $A$, $B$, $C$, and $D$ and

$$a = \frac{1}{2G_0} \left[ H\left(\sigma_\Pi\right) \frac{\sigma_\Pi}{\sigma_1} - H(\sigma_1) \right].$$

All the radial stress paths between the yield surfaces I and II have the same $\alpha$ value. In such approximations, therefore, the magnitude of the plastic strain increments is determined by the stress deviations for the intersection of the radial stress path with yield surface I (or yield surface II). If the variation of $s_{ij}$ along the yield surface is **monotonic**, so that for example

$$s_{ij}^C \geq s_{ij}^A$$  \hspace{1cm} (29)$$

then

$$\Delta \varepsilon_{ij}^{pl} \bigg|_A \geq \Delta \varepsilon_{ij}^{pl} \bigg|_C$$  \hspace{1cm} (30)$$

and, as appears from limits considerations

$$\Delta \varepsilon_{ij}^{pl} \bigg|_A \geq \Delta \varepsilon_{ij}^{pl} \bigg|_A \geq \Delta \varepsilon_{ij}^{pl} \bigg|_C,$$  \hspace{1cm} (31)$$

whereby upper and lower values are determined for the correct value of $\Delta \varepsilon_{ij}^{pl} \bigg|_A$.

If the variation of $s_{ij}$ on the yield surface I (fig. 8) is **not monotonic** along the stretch $AC$, it cannot be concluded that upper and lower values for $\Delta \varepsilon_{ij}^{pl} \bigg|_A$ are determined by the approximations $AB$ and $CD$. However, upper and lower values can in such cases be determined by finding $(s_{ij})_{\text{max}}$ and $(s_{ij})_{\text{min}}$ points (i.e. $(s_{11})_{\text{max}}$, $(s_{11})_{\text{min}}$, $(s_{22})_{\text{max}}$, $(s_{22})_{\text{min}}$, ..., $(s_{23})_{\text{max}}$, $(s_{23})_{\text{min}}$ points) on the stretch $AC$ and letting the approximations run through these points.
For the special case treated in section 4, figures 5-7 show the variation in the dimensionless principal stress deviations $u_i$. It might be mentioned here that there are only two extreme points for each of the stress deviations.

For the total path, therefore, upper and lower values for $e_{ij}^{pl}$ are determined from the approximated stress path lying on each side of the correct path, if the correct path only intersects regions where the variations of $s_{ij}$ are monotonic. Otherwise those parts of the approximation where this is not the case, must be inserted by the method described above.

**6. EXAMPLE**

We will now illustrate the numerical method by means of a simple example, in which the correct strain path can be determined analytically. A plane stress state is considered, for which v. Mises' yield conditions (9) take the form

$$
\sigma_\varepsilon^2 = \frac{2}{3}(\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2) ,
$$

or

$$
\gamma_1^2 + \gamma_2^2 - \gamma_1\gamma_2 = \frac{3}{2}(\frac{\sigma_2}{\sigma_0})^2 = \frac{3}{2}L^2 ,
$$

dimensionless stresses being introduced by

$$
\gamma_1 = \sigma_1 / \sigma_0 , \quad \gamma_2 = \sigma_2 / \sigma_0 ,
$$

and $L$ being defined by (16). Further we will assume the stress-strain law

$$
e_{ij}^{pl} = kL u_i = \frac{k}{\sigma_0^2} \sigma_\varepsilon s_{ij} \quad \text{for } \sigma_\varepsilon > 0
$$

$$
e_{ij}^{pl} = 0 \quad \text{for } \sigma_\varepsilon \leq 0
$$

where $k$ is a constant. With radial loading, we get by (6) and (7)

$$
e_{ij}^{pl} = \frac{1}{2} kL u_i ,
$$

as

$$
h(\sigma_\varepsilon) = 2G_0 \frac{k}{\sigma_0^2} ,
$$

and

$$
H(\sigma_\varepsilon) = G_0 \frac{k}{\sigma_0^2} \sigma_\varepsilon .
$$

Finally, we will consider the stress path

$$
\gamma_1 = \frac{1}{5} \gamma_2^2 ,
$$

where $0 \leq \gamma_2 \leq 5$ , and where the initial position is $\gamma_1 = \gamma_2 = 0$ and $e_{ij}^{pl} = 0$. 

16
a. Correct calculation of $e_{pl}^1$.

By (33) and (39) we get

$$L = \frac{\sqrt{6}}{30} \frac{4\gamma_2^2 - 15\gamma_2 + 50}{\sqrt{\gamma_2^2 - 5\gamma_2 + 25}}\gamma_2.$$  \hfill (40)

Further,

$$u_1 = \gamma_1 - \frac{1}{3}(\gamma_1 + \gamma_2) = \frac{1}{15} (2\gamma_2^2 - 5\gamma_2) , \hfill (41)$$

so that

$$e_{pl}^1 = \frac{\sqrt{6k}}{450} \int_0^1 \frac{(2\gamma_2^2 - 5\gamma_2)(4\gamma_2^2 - 15\gamma_2 + 50)}{\sqrt{\gamma_2^2 - 5\gamma_2 + 25}} d\gamma_2 \hfill (42)$$

which gives on integration

$$\frac{e_{pl}^1}{k} = \frac{\sqrt{6}}{3} \frac{25}{32} \ln \left( \frac{2}{5} \gamma_2 - 1 + \frac{2}{5} \sqrt{\gamma_2^2 - 5\gamma_2 + 25} \right)$$

$$+ \frac{\sqrt{\gamma_2^2 - 5\gamma_2 + 25}}{75} \left( \frac{1}{75} \gamma_2^2 - \frac{1}{30} \gamma_2^2 - \frac{1}{8} \gamma_2 - \frac{15}{16} \right) + \frac{75}{16} \hfill (43)$$

In table 2, values of $e_{pl}^1 / k$ for $\gamma_2 = 0, 1, 2, \ldots, 5$ are given, calculated from (43), and the strain variation is shown in fig. 9.

<table>
<thead>
<tr>
<th>$\gamma_2$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_{pl}^1 / k$</td>
<td>0</td>
<td>-0.07</td>
<td>-0.15</td>
<td>-0.11</td>
<td>0.49</td>
<td>1.55</td>
</tr>
</tbody>
</table>

Table 2. Corresponding values of $e_{pl}^1 / k$ and $\gamma_2$.

b. Approximate calculation of $(e_{pl}^1)_A$.

In (36), the stress-strain law is given in total form (18), as

$$F(L) = \frac{1}{2}kL.$$  \hfill (44)

In order to be able to use the formulae established in chapter 4, the yield curves must be so drawn in that

$$b = \frac{1}{2}k(L_n^2 - L_{n-1}^2) = \text{constant}. \hfill (45)$$

By (24), we then obtain

$$\frac{e_{pl}^1}{k} = \frac{1}{2} \frac{b^*}{L_0} \sum_{n=1}^{N} (u_n^0). \hfill (46)$$
where

\[ b^* = L_n^2 - L_{n-1}^2 = \text{constant}. \]  

(47)

The orthogonal projection of the stress path (39) on the \( \Pi \)-plane is determined by writing the stress path in the parameter form

\[ (\gamma'_1, \gamma'_2, \gamma'_3) = (\frac{1}{3} t^2, t, 0) \]  

(48)

where \( 0 \leq t \leq 5 \), and then use (10)

\[ x = \frac{\sqrt{2}}{10} (5t - t^2), \quad y = -\frac{\sqrt{6}}{30} (5t + t^2). \]  

(49)

In figure 10, the projection of the stress path is drawn in on the \( \Pi \)-plane (marked \( c_1 \)), and the radial lines corresponding to table 1 are drawn in for use in the approximation. We wish to determine the first plastic principal strain \( (\varepsilon_{1}^{pl})_A \) corresponding to the stress state \( A \) (figure 10), where \( t = 5 \) and \( (x, y) = (0, -5\sqrt{6}/3) \). We choose to insert five yield curves \( (N = 5) \) and let the fifth curve \( (L = L_5; \) run through \( A \), i.e. by (15)

\[ L_3 = \frac{5}{3} \sqrt{6}, \]  

(50)

and according to (47)

\[ 5b^* = L_3^2, \quad b^* = 10/3 \]  

(51)

In this manner all the yield curves are determined and can be drawn in on figure 10. The approximated path is likewise drawn in on figure 10 (marked \( c_2 \)). \( (\varepsilon_{1}^{pl})_A \) can then be calculated by means of (46), where \( L_0 = 1 \), as table 1 is used

\[ \frac{\varepsilon_{1}^{pl}}{k} = \frac{1}{2} \sum_{n=1}^{5} (u_{1}^n)_n = \frac{5}{3} (-0.123 + 0.049 + 0.262 + 0.336 + 0.408) = 1.55, \]  

(52)

which is in fact the actual value determined by (43).

c. Calculation of upper value for \( (\varepsilon_{1}^{pl})_A \).
As a consequence of the monotonic variation of $u_1$ along the yield circle in the 4th quadrant (see figure 5), an upper value for $(\varepsilon_{1pl}^p)_A$ is determined by the stress path marked $c_3$ in figure 10,

$$(\frac{\varepsilon_{1pl}^p}{k})_A = \frac{5}{3}(-0.114+0.262+0.336+0.408+0.408)=2.33$$  \quad (53)

### d. Calculation of lower value for $(\varepsilon_{1pl}^p)_A$.

In a similar manner, a lower value for $(\varepsilon_{1pl}^p)_A$ is determined by the stress path marked $c_4$.

$$(\frac{\varepsilon_{1pl}^p}{k})_A = \frac{5}{3}(-0.450-0.071+0.188+0.262+0.336) = 0.44$$  \quad (54)

### e. Direct calculation of $(\varepsilon_{1pl}^p)_A$ by total expression.

The direct approximated determination by the total relation (36) is analogous to using the radial stress path $OA$ (marked $c_5$),

$$(\frac{\varepsilon_{1pl}^p}{k})_A = \frac{1}{2}L_5(u_1)_A = 3.40$$  \quad (55)

<table>
<thead>
<tr>
<th>Stress path</th>
<th>$(\varepsilon_{1pl}^p/k)_A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correct calculation .</td>
<td>$\varepsilon_1$</td>
</tr>
<tr>
<td>Approx. calculation .</td>
<td>$\varepsilon_2$</td>
</tr>
<tr>
<td>Upper value calculation .</td>
<td>$\varepsilon_3$</td>
</tr>
<tr>
<td>Lower value calculation .</td>
<td>$\varepsilon_4$</td>
</tr>
<tr>
<td>Total theory calculation .</td>
<td>$\varepsilon_5$</td>
</tr>
</tbody>
</table>

Table 3. Calculated values of $(\varepsilon_{1pl}^p)_A$.

Thus, with the stress paths $c_1-c_5$, figure 10, the values for $(\varepsilon_{1pl}^p)_A$ shown in table 3, have been determined. All determinations for the stress paths shown have been in full agreement with the physical laws, but considered as determination of $(\varepsilon_{1pl}^p/k)_A$ associated with the stress path $c_1$, the values from $c_1$-$c_5$ are only approximations.

### 7. DISCUSSION

An attempt will be made in this section to evaluate the advantages and disadvantages of the numerical method described, in comparison with the usual methods of numerical integration. However, it does not seem possible to come to a general decision on this question, and individual cases must be evaluated separately. Various main features may however be discussed.

One of the advantages of the method appears to be that it permits a large part of the work of computation to be done once and for all. Thus, for materials where v. Mises' yield condition is a good approximation during strain hardening, numerical values corresponding to table 1 can be calculated independent of the stress-strain law of the materials in question (although these stress-strain laws will normally vary). If the stress-strain laws can be brought into form (4), the yield curves necessary for the approximation can be inserted according to the principle described, whereby the work of computation will be further facilitated. If this principle cannot be used, the yield curves can be inserted arbitrarily without any great increase in the work of
computation. If the yield surfaces are inserted, and the numerical values mentioned calculated, arbitrary statically determined stress paths, involving both loading and unloading, can be treated in a simple manner. In particular, the influence of the stress path on the plastic strain state corresponding to a given stress state will be easy to investigate. With the usual numerical methods of integration, where the integration interval is divided into sub-intervals, and the integrants in each of these sub-intervals are replaced by a polynomial, the same amount of work, will be necessary for all the different stress paths, as the integrants are dependent on the stress paths and must therefore be approached differently in the individual cases. It seems possible to conclude, therefore, that the method will be advantageous, when several stress paths have to be studied with materials with the same mechanical properties.

With regard to the error in the values calculated by the approximation, it seems that this can be held at a reasonable level even with the use of quite few yield surfaces and radial paths in the approximation. The case treated in the example, together with other worked out examples, gives the expected result, that in general the best approximation is got when the approximated path runs "equally over and under" the actual stress path. An exception to this occurs, when the stress path intercepts regions where the stress deviations do not have monotonic variation along the yield surfaces (stress-strain law (4) assumed). Some evaluation of the numerical determination is possible by calculating the upper and lower values by the method indicated. The determination of these upper and lower values is very easy, but in order that the gap should not be too great, a very considerable sub division in the approximation is required (cf. the example worked out). The principle established in chapter 4 for the insertion of yield curve in isotropic hardening for special, often-employed stress-strain laws must be assumed to increase the precision of the determinations, as the result achieved by this principle is that the individual contributions to the summation are of the same order of magnitude. On the other hand if the yield curves are inserted at equidistant intervals (in the example $L_n - L_{n-1} = \text{constant}$), the contributions to the plastic strains will be relatively greater for the radial paths between the "outside" yield curves.

The simple determination of upper (or lower) values can be useful where some degree of orientation is desired as to the magnitude of the plastic strains, e.g. when the strains have to be kept below a certain absolute magnitude.

8. CONCLUSIONS

With the limitation inherent in the material presented, an attempt is made to establish certain tentative conclusions:

a. The numerical method is considerably simpler than direct determination by integration, which is in any case not always possible.

b. The fineness of the approximations need not normally be especially great in order to give a good approximation.

c. In most cases the approximated path gives the best approach, when it runs "equally above and below" the true stress path.

d. In isotropic hardening, formulae and tables can be set up which allow of a saving in computation. In certain types of stress-strain laws the approximation can be so arranged that the result will probably be determined with good accuracy in relation to the number of yield curves chosen in the approximation.
e. In analyzing a single stress path only, it is doubtful whether the method is more advantageous than other numerical methods (as for example Simpson’s method) unless the tables mentioned are already available.

9. REFERENCES


