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CHAPTER 2

A CLASS OF YIELD CONDITIONS FOR THE CUBIC CRYSTAL SYSTEM[1]

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Summary
A class of anisotropic yield conditions satisfying the symmetry requirements for the cubic system is presented. An effort is made to obtain yield conditions, which are in some sense simple. The linear yield condition for plane strain used by E. M. Shoemaker is a special case of the yield conditions considered.

1. INTRODUCTION

It is well known that structural materials are generally anisotropic with respect to their mechanical behavior. It is, for example, an experimental fact that metals usually are anisotropic in their plastic state, and it is therefore not surprising that the problem of anisotropy in the plastic state has been the topic for numerous papers although a thorough theoretical investigation has not yet been made - even though some classes of anisotropy have been investigated. The most frequently used class of anisotropic yield conditions was formulated by Hill [19] as a generalization of v. Mises' isotropic yield condition. Berman & Hodge [2] have used a piecewise linear anisotropic yield condition for those cases where the principal directions of stress and strain in each element of the deformed material coincide and remain fixed throughout the loading process.

An interesting application of an anisotropic yield condition is due to Shoemaker [3], who solved the classical punch problem on the basis of an anisotropic piecewise linear yield condition in the state of plane strain. This special yield condition for plane strain was derived from a generalized Tresca yield condition, but the general case was not investigated further.

During recent years the special anisotropic phenomena due to the symmetry properties of the crystal classes has been investigated in detail. These investigations

[1] Bygningsstatiske Meddelelser, Vol. 36, 1965, pp. 25-37.

were initiated for the elastic case by Smith & RIVLIN [4] and later extended to more general cases in the theory of continuum mechanics. For the case of perfect plasticity, Smith [5] has formulated the general anisotropic yield conditions for the two most important crystal classes.

This paper presents a class of anisotropic yield conditions included in the more general class formulated by Smith [5] for the cubic crystal system. It is the hope that such a class of yield conditions not only will have meaning when applied to single cubic crystals but also will be of some value for polycrystalline materials in cases where a material has an equivalent symmetric behavior because of its treatment. The considered class of anisotropic yield conditions includes as a special case v. Mises' isotropic yield condition. Only perfectly plastic behavior is considered, but the theory can easily be extended to the strain hardening case by assuming that the subsequent yield surfaces are "blown up" versions of the initial yield surface as in the theory of Hill [l]. The associated flow rules are formulated, and it is shown that a special yield condition belonging to the considered class of yield conditions is reduced to the simple yield condition used by Shoemaker [3] for the case of plane strain.

2. SYMMETRY PROPERTIES OF MATERIALS

Let us assume for the sake of simplicity that the given material is homogeneous in the mechanical sense; in other words we assume that the mechanical properties are independent of the point considered in the material. If we cut from the material test specimens with different orientations with respect to a fixed coordinate system and if these test specimens show identical mechanical properties then we will call the material *mechanically isotropic;* such a material has no preferred directions. Since we will only consider the mechanical behavior of the material, we will leave out the word mechanical and simply call the material *isotropic.*

However, isotropic behavior will, as already mentioned, be the exception. Often the mechanical properties vary with the orientation of the specimen. The material is then called *anisotropic.* A special class of anisotropic materials is *symmetric* materials where the mechanical behavior is the same for a specimen with a given direction and for specimens with directions determined from the first specimen direction by a group of orthogonal transformations. The so-called *crystal classes* are symmetric materials with three preferred directions, i. e. directions which for a given crystal class are intrinsic associated with the material and are a part of its definition. The preferred directions can be chosen in different *equivalent* ways. If we define one set of these directions by unit vectors $\boldsymbol{h}_i$ $(i = 1, 2, 3)$, we can define the *symmetry properties* by the group of orthogonal transformations which transforms any set of preferred directions into an equivalent set of directions.

In the case of isotropy the directions $\boldsymbol{h}_i$ can be chosen at random. We can thus define an isotropic material as a symmetric material with the full (or proper) orthogonal transformation group.

The group of transformations appropriate to each of the crystal classes has been determined by Smith & Rivlin [4] (see also Green & Adkins [6]) in connection with a derivation of the restrictions imposed on the form of the strain energy function for elastic materials by the symmetry of the material.

3. YIELD CONDITIONS FOR CRYSTAL CLASSES

Smith [5] has derived the restrictions imposed on the form of the yield conditions for incompressible perfectly plastic materials by symmetry in two important cases, the cubic system and the hexagonal system.

We will restrict our investigation to the cubic system where the unit vectors $\boldsymbol{h}_i$ are mutually perpendicular. We can thus in what follows take as our reference system a rectangular cartesian coordinate system x_i (i = 1, 2, 3) whose axes are parallel to the vectors $\boldsymbol{h}_i$. The stress components and the deviatoric stress components in the reference system will be denoted by σ_{ij} and s_{ij} respectively. We have by definition

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} \tag{1}$$

where δ_{ij} ½ denotes the Kronecker's delta and where the usual summation convention has been used.

The general yield condition for an incompressible perfectly plastic material

$$F(s_{ij}) = 0 \tag{2}$$

can, as shown by Smith [5], be written for the five classes in the cubic system in the general form

$$F(J_2,..., J_{11}) = 0 \tag{3}$$

for the case of the tetartoidal and the diploidal crystal class and in the form

$$F(J_2,..., J_8) = 0 \tag{4}$$

for the hextetrahedral, gyroidal and hexoctahedral crystal classes. Here

$$\begin{aligned}
J_2 &= s_{11}s_{22} + s_{22}s_{33} + s_{33}s_{11},\\
J_3 &= s_{11}s_{22}s_{33},\\
J_4 &= s_{23}^2 + s_{31}^2 + s_{12}^2,\\
J_5 &= s_{23}^2 s_{31}^2 + s_{31}^2 s_{12}^2 + s_{12}^2 s_{23}^2,\\
J_6 &= s_{23}s_{31}s_{12},\\
J_7 &= s_{11}(s_{31}^2 + s_{12}^2) + s_{22}(s_{12}^2 + s_{23}^2) + s_{33}(s_{23}^2 + s_{31}^2),\\
J_8 &= s_{11}s_{31}^2 s_{12}^2 + s_{22}s_{12}^2 s_{23}^2 + s_{33}s_{23}^2 s_{31}^2,\\
J_9 &= s_{11}(s_{31}^2 - s_{12}^2) + s_{22}(s_{12}^2 - s_{23}^2) + s_{33}(s_{23}^2 - s_{31}^2),\\
J_{10} &= s_{11}s_{22}(s_{11} - s_{22}) + s_{22}s_{33}(s_{22} - s_{33}) + s_{33}s_{11}(s_{33} - s_{11}),\\
J_{11} &= s_{23}^2 s_{31}^2(s_{23}^2 - s_{31}^2) + s_{31}^2 s_{12}^2(s_{31}^2 - s_{12}^2) + s_{12}^2 s_{23}^2(s_{12}^2 - s_{23}^2).
\end{aligned} \tag{5}$$

We note that J_2 and J_4 are of second degree in the stresses while the rest of the quantities in (5) are of third or higher degree in the stresses. Smith has shown that any yield condition of the form (3) or (4) is expressible as a single-valued function of the quantities J_α(α = 2 , . . . , 11 or 2 , . . . , 8 respectively); we are not limited to polynomials in J_α.

We note further that the quantity

$$I_2 = - J_2 + J_4 \tag{6}$$

is the usual deviatoric stress invariant $\frac{1}{2}s_{ij}s_{ij}$.

4. A CLASS OF ANISOTROPIC YIELD CONDITIONS

We now turn to the formulation of a special class of anisotropic yield conditions for the cubic system. We will try to formulate the yield conditions so that the associated theory of the perfectly plastic material satisfies the following requirements:

a. The theory must be simple in mathematical sense not only with respect to the yield condition but also with respect to the flow rule.
b. A special case must be v. Mises' isotropic yield condition I_2 = constant.
c. The linear yield condition for plane strain formulated by Shoemaker [3] must be a special case.
d. The chosen class of yield conditions must satisfy the symmetry requirements for all crystal classes in the cubic system.

Point d is easily satisfied by using functions of only the quantities $J_2, \ldots, J_8$. Point a is more difficult to satisfy because “simple in mathematical sense" is not a very objective statement; however it seems reasonable to try to use yield conditions involving only the quantities J_2 and J_4, which are of second degree in the stresses. We consider thus yield conditions of the form

$$F(J_2, J_4) = 0. \tag{7}$$

We note that because of (6) we still have the possibility of satisfying point b. Finally we will make a further simplification by only including one parameter in the class of yield conditions.

It is easy to see from (5) that $J_2 \leq 0$ and $J_4 \geq 0$, so we can for the sake of simplicity use the following two dimensionless quantities

$$\begin{aligned} i_2 &= \frac{1}{\sigma_F}\sqrt{-3J_2}, \\ i_4 &= \frac{1}{\sigma_F}\sqrt{3J_4}, \end{aligned} \tag{8}$$

where σ_F denotes the yield stress in pure tension in the preferred directions $\boldsymbol{h}_i$. V. Mises' yield condition can then be written

$$i_2^2 + i_4^2 = 1 \tag{9}$$

As we shall see, the yield condition

$$i_2 + i_4 = 1 \tag{10}$$

satisfies point c above. It is therefore reasonable to try to use the following one-parameter class of yield conditions

$$i_2^\alpha + i_4^\alpha = 1 \tag{11}$$

where $1 \leq \alpha \leq 2$. Formulated in terms of J_2 and J_4, (11) takes the

$$(-J_2)^{\alpha/2} + J_4^{\alpha/2} = (\tfrac{\sqrt{3}}{3}\sigma_F)^\alpha \tag{12}$$

The special case (10) becomes

$$\sqrt{-3J_2} + \sqrt{3J_4} = \sigma_F \tag{13}$$

Smith [5] gives no flow rule related to the yield conditions (3) and (4), but it is a simple matter to formulate the associated flow rule for smooth parts of the yield conditions. The yield conditions (12) will generally have corners where the plastic strain increments are determined in the usual way. It must be noted that the symmetry of the deviatoric stress components s_{ij} has been used in the formulation of the quantities (5) and it is helpful in formulating the flow rule to know the fact that

$$\frac{\partial s_{kl}}{\partial \sigma_{ij}} = \delta_{ik}\delta_{jl} - \frac{1}{3}\delta_{ij}\delta_{kl} \tag{14}$$

The flow rule associated with the yield condition (7) has the simple form

$$\dot{\varepsilon}_{ij}^{pl} = \begin{cases} -\lambda \dfrac{\partial F}{\partial J_2} s_{ij} & \text{for } i = j \\ \lambda \dfrac{\partial F}{\partial J_4} s_{ij} & \text{for } i \neq j \end{cases} \tag{15}$$

where $\dot{\varepsilon}_{ij}^{pl}$ are the components of the plastic strain rate tensor and λ an undetermined multiplier.

The flow rules associated with the class of yield conditions under consideration (12) are

$$\dot{\varepsilon}_{ij}^{pl} = \begin{cases} \lambda(-J_2)^{\frac{\alpha}{2}-1} s_{ij} & \text{for } i = j \\ \lambda J_4^{\frac{\alpha}{2}-1} s_{ij} & \text{for } i \neq j \end{cases} \tag{16}$$

In the special case of v. Mises' yield condition, α = 2 and

$$\dot{\varepsilon}_{ij}^{pl} = \lambda s_{ij}. \tag{17}$$

It must be noted that the yield conditions have the form (12) only when the reference system x_i is parallel with the preferred directions $\boldsymbol{h}_i$. If the yield conditions in another reference system are wanted then the new form can be found by transforming the stress components.

For those special stress states where the principal stress directions are parallel to the preferred directions $\boldsymbol{h}_i$, $J_4 = 0$, and the theory will take the same form as the v. Mises' theory.

We will in Sec. 5 consider the special case where α = 1, and we will in Sec. 6 investigate the variation of the yield stress in pure tension with orientation.

5. A SPECIAL ANISOTROPIC YIELD CONDITION

We will now in more detail consider the special anisotropic yield condition (13)

$$F(J_2, J_4) = \sqrt{-3J_2} + \sqrt{3J_4} - \sigma_F = 0 \tag{18}$$

with the associated flow rule

$$\dot{\varepsilon}_{ij}^{pl} = \begin{cases} \lambda(-J_2)^{\frac{\alpha}{2}-1} s_{ij} & \text{for } i = j \\ \lambda J_4^{\frac{\alpha}{2}-1} s_{ij} & \text{for } i \neq j \end{cases} \tag{19}$$

For *plane stress* where $\sigma_{33} \equiv \sigma_{23} \equiv \sigma_{13} \equiv 0$ we have

$$s_{ij} = \frac{1}{3}\begin{bmatrix} 2\sigma_{11} - \sigma_{22} & 3\sigma_{12} & 0 \\ 3\sigma_{12} & 2\sigma_{22} - \sigma_{11} & 0 \\ 0 & 0 & -(\sigma_{11} + \sigma_{22}) \end{bmatrix} \tag{20}$$

and

$$\begin{aligned} J_2 &= -\frac{1}{3}(\sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22}), \\ J_4 &= \sigma_{12}^2. \end{aligned} \tag{21}$$

Thus the yield condition becomes

$$\sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22} = (\sqrt{3}\sigma_{12} \pm \sigma_F)^2. \tag{22}$$

The special case $\sigma_{22} \equiv 0$ gives the linear yield condition

$$\text{Max}[\pm\sigma_{11} \pm \sqrt{3}\sigma_{12}] = \sigma_F, \tag{23}$$

where all four combinations of signs are applicable. Another special case of (22) $\sigma_{12} \equiv 0$ gives, of course, v. Mises' ellipse

$$\sigma_{11}^2 + \sigma_{22}^2 - \sigma_{11}\sigma_{22} = \sigma_F^2. \tag{24}$$

The yield condition (18) (and therefore (22)) is formulated in a reference system x_i parallel to the preferred directions. If the yield condition in another reference system is wanted, the stress components must, as mentioned above, be transformed according to the transformation rule for second order tensors.

We consider now the case of *plane strain* where $\dot{\varepsilon}_{23}^{pl} \equiv \dot{\varepsilon}_{13}^{pl} \equiv \dot{\varepsilon}_{33}^{pl} \equiv 0$. In this case we get from (19) that $s_{33} \equiv s_{23} \equiv s_{13} \equiv 0$. We then have

$$s_{ij} = \begin{bmatrix} \frac{1}{2}(\sigma_{11} - \sigma_{22}) & \sigma_{12} & 0 \\ \sigma_{11} & -\frac{1}{2}(\sigma_{11} - \sigma_{22}) & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{25}$$

and

$$\begin{aligned} J_2 &= -\tfrac{1}{4}(\sigma_{11} - \sigma_{22})^2, \\ J_4 &= \sigma_{12}^2. \end{aligned} \tag{26}$$

Thus the yield condition becomes

$$\text{Max}[\pm(\sigma_{11} - \sigma_{22}) \pm 2\sigma_{12}] = \frac{2}{3}\sqrt{3}\sigma_F, \tag{27}$$

where all four combinations of signs are applicable. The yield condition (27) is the same linear yield condition as proposed by Shoemaker [3]. Shoemaker has used (27) to solve the classical punch problem. The flow rule associated with (27) is

$$\dot{\varepsilon}_{ij}^{pl} = \pm\lambda \begin{bmatrix} 1 & \pm 1 & 0 \\ \pm 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{28}$$

where the plus and minus signs under the matrix symbol must be chosen so that the matrix is always symmetric. There are thus four possible combinations, one for each of the four plane parts of the yield surface (27)

We will in the next chapter see how the yield stress in pure tension is changed with orientation for the yield condition (18).

6. VARIATION OF THE YIELD STRESS WITH ORIENTATION

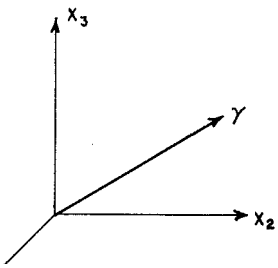

Figure 1. The reference system.

It is a characteristic of anisotropic materials that the yield stress in pure tension varies with the orientation of the test specimen. Experimentally this variation is perhaps the simplest anisotropic test to perform. It is, therefore, of major interest to see how any anisotropic yield condition predicts the yield stress variation.

The determination of the yield stress in pure tension in a direction y given by its direction-cosines γ_i (i = 1, 2, 3) with respect to a fixed reference system is a simple matter when the yield condition in the reference system is known (see Figure 1). We note first that

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1 \tag{29}$$

Assuming that the stress state is a pure tension σ in the γ-direction, we can obtain the stress tensor in the reference system from the transformation law

$$\sigma_{ij} = \gamma_i \gamma_j \sigma \tag{30}$$

and

$$s_{ij} = (\gamma_i \gamma_j - \frac{1}{3}\delta_{ij})\sigma. \tag{31}$$

From (5) we get

$$\left.\begin{aligned} J_2 &= \sigma^2\left(a - \frac{1}{3}\right) \\ J_4 &= \sigma^2 a \end{aligned}\right\} \tag{32}$$

where $a = \gamma_1^2\gamma_2^2 + \gamma_2^2\gamma_3^2 + \gamma_3^2\gamma_1^2$. The yield condition (12) then gives the yield stress ratio

$$\frac{\sigma}{\sigma_F} = \frac{\sqrt{3}}{3}\left[(\frac{1}{3} - a)^{\frac{\alpha}{2}} + a^{\frac{\alpha}{2}}\right]^{\frac{1}{\alpha}} \tag{33}$$

where σ_F is the yield stress in pure tension in the x_i-directions. It is easy to see from (33) that the directions with minimum σ/σ_F are determined by a = 1/6 and are independent of α. The minimum values are determined by

$$\left(\frac{\sigma}{\sigma_F}\right)_{\min} = 2^{\frac{1}{2}-\frac{1}{\alpha}}. \tag{34}$$

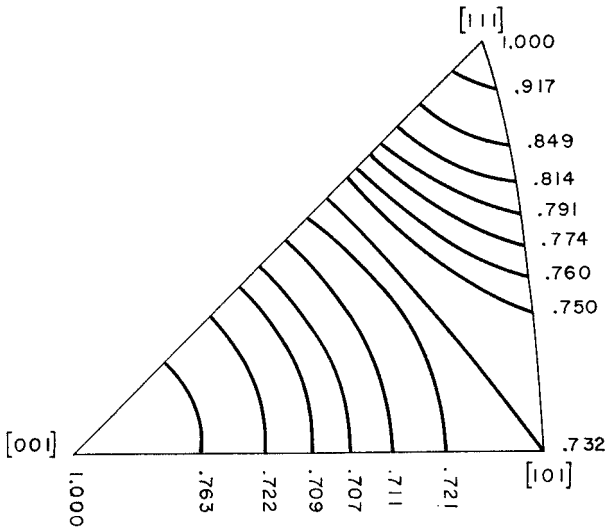

Figure 2. The variation of σ/σ_F for α = 1.00.

Because of the symmetry of the material the variation of the yield stress is completely determined by its variation in one sixth of an octant. In Figure 2 the variation for α = 1.00 is shown in the usual stereographic projection as an illustration of (33). The Miller-indices are used referred to the reference system Figure 1. However, the pattern shown in Figure 2 is the same for all values of α because it is determined alone by the variation of a. The forms of the curves with constant σ/σ_F are invariant with respect to α but the values of σ/σ_F associated with the curves are dependent on α. We will discuss Figure 2 in more detail. First we notice that the absolute minimum value of σ/σ_F is $\sqrt{2}/2$ = 0.707 and that this minimum is not very distinct. Next we notice that the directions with absolute maximum values of σ/σ_F are isolated directions and that the direction [111] is a direction with maximum value of σ/σ_F. Finally the variation of σ/σ_F is more marked in the neighbourhood of the maximum directions than in the neighbourhood of the minimum directions.

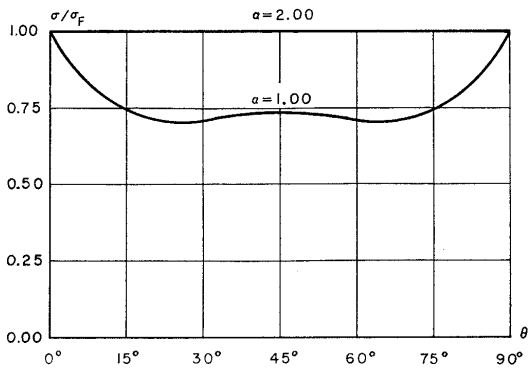

Figure 3. The variation of σ/σ_F for a = 1.00 and 2.00 for directions in the x_1x_2-plane.

A better illustration of the variation is perhaps given on Figure 3 where the variation of σ/σ_F for directions in a plane determined by two of the preferred directions; say x_1 and x_2 is shown. The angle θ is the angle between the direction considered and x_1 (or x_2). This variation has some special interest because of its meaning for a rolled sheet. For a rolled sheet it is reasonable to expect that one of the preferred directions is perpendicular to the plane of the sheet. If the anisotropy of the sheet in such a case can be described by the theory presented here, then Figure 3 shows the variation of the yield stress in pure tension for specimens parallel to the plane of the sheet. We notice that the minimum value is not in the 45° direction.

For the more general case where the yield condition has the form (7), it is interesting to notice that the variation-pattern shown on Figure 2 is still valid because J_2

and J_4 are determined by σ and α alone. Thus it is possible to change only the absolute values of σ / σ_F, the essential properties of the anisotropy remaining unchanged.

7. DISCUSSION

There can be at least two reasons for using an anisotropic yield condition. The first reason would be if a more precise description of the behavior of an anisotropic material than an isotropic theory can give is wanted. In such a case the appropriate yield condition must be chosen so that it approximates an experimentally determined yield condition as well as possible; on the other hand it is often necessary to choose a simple analytical expression to be able to use the theory in solving problems.

The second reason for using an anisotropic yield condition might be the desire to determine upper or lower bounds for the limit load in a given problem where the material actually is isotropic, but where the isotropic solution is more difficult to find than the anisotropic solution. In such a case the isotropic yield condition is approximated by a suitable simple yield condition, e.g. a linear yield condition as in the case investigated by Shoemaker [3].

In this paper is presented a class of anisotropic yield conditions satisfying certain requirements. It has been the aim to make the associated theory simple, but it is clear that the proposed class can be extended to any desired degree by using more than one parameter. In one sense the anisotropic yield conditions used by Hill [1] are more general because we here must satisfy the symmetry requirements, but on the other hand, we are not here limited to using yield conditions which are quadratic in the stress components. It is therefore not possible to compare the two types of yield conditions directly.

It is the author's opinion that the general form of anisotropic yield conditions derived by Smith [5] is the first rational approach to the problem of anisotropy in the plastic state. The complete general form is, of course, in itself not very applicable but it gives the foundation for simplified theories, the value of which must be checked by experiments. It would seem to be an advantage to have available a certain number of anisotropic yield conditions since it is not reasonable to expect a single class to be sufficient for all anisotropic materials under any circumstances. It is in the light of these considerations that the anisotropic yield conditions formulated here are presented.

8. REFERENCES

[1] Hill, R. *The Mathematical Theory* of *Plasticity.* Oxford 1950.

[2] Berman, I. & P. G. Hodge. *A general theory of piecewise linear plasticity for initially anisotropic materials.* Arch. Mech. Stos., Vol. 11, 1959, pp. 513-540.

[3] Shoemaker, E. M. *A theory of linear plasticity for plane strain.* Arch. Rat. Mech. Anal., Vol. 14, 1963, pp.283-300.

[4] Smith, G. F. & R. S. Rivlin. *The strain energy functions for anisotropic materials.* Trans. Amer. Math. Soc., Vol. 88, 1958, pp. 175-193.

[5] Smith, G. F. *On the yield condition for anisotropic materials.* Q. Appl. Math., Vol. 20, 1962, pp.241-247.

[6] Green, A. E. & J. E. Adkins. *Large Elastic Deformations.* Oxford 1960.