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## CHAPTER 9

### FUNDAMENTALS OF STRUCTURAL RELIABILITY<sup>1</sup>

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#### 1. INTRODUCTION

In the traditional way of designing structures decisive parameters such as dimensions, material strengths and loads are usually characterized by a number of constants, e.g. average values. On the basis of these constants a mathematical model for the behaviour of the structure is used to examine whether the structure is safe or not. To improve the safety of the structural the variables are often replaced with “worst case” values. Such a basis for designing a structure by a “worst case” philosophy is usually too conservative from an economic point of view because of the small risk that all the variables at the same time take on their »worst« values. The procedure described here is called the *deterministic* approach.

However, it is a well-known fact that for example strength varies from structural element to structural element, so that the strength of an element cannot be characterized adequately by a single value. Further it is also in some cases necessary to take into account time variations. The same kind of uncertainty is present in relation to structural dimensions and loading. Especially the so-called natural loads due to waves, currents, winds and earthquakes are difficult to deal with in a deterministic way. It is also important to remember that some uncertainty is involved in the choice of the mathematical models used for the analysis of a structure.

The purpose of using a *probabilistic* approach rather than the simple deterministic approach is to try to take into account the uncertainties mentioned above so that a more realistic analysis of the safety of a structure can be performed.

The history of modern reliability theory has been presented by Borges & Castanheta [1] for the period up to 1970. Only a brief survey will be given here. The first attempt of using statistical concepts in structural reliability seems to be more than 50 years old, more precisely from 1926, and was made by Meyer [2]. But only a limited

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<sup>1</sup> In “Lectures on Structural Reliability”, P. Thoft-Christensen (editor), Aalborg University, Denmark, 1980, pp. 1-28.

number of papers were published in this field before the Second World War. Important progress was however made by people like Prot [3], Weibull [4], [5], [6], Kjellman [7], Wastlund [8] etc.

After 1945 the number of papers has increased steadily. At the University of Columbia, Freudenthal created an Institute for the Study of Fatigue and Reliability, and he produced a great number of papers. The evolution of modern codes was greatly influenced by papers published in 1949 - 1952 by Torkoja and Paez (see e.g. [9]). Johnson [10] suggested in 1953 to apply of extreme value distributions.

Ferry Borges [11] stressed in 1952 the importance of taking into account randomness of dimensions and mechanical properties into randomness of structural behaviour. In a report published by Freudenthal [12] in 1968 it was suggested to represent loads by an extreme value distribution and strength by a logarithmic normal distribution.

In 1968 Benjamin [13] in a paper on seismic force design defends the use of Bayes' probabilistic concepts.

Presentation of load, strength and dimension variables by mean values and variances was proposed by Cornell [14], [15] in 1967 and became the basis for the so-called level II methods. Important contributions in this area are due to Rosenblueth & Esteva [16] in 1972, Ditlevsen [17] in 1973, Hasofer & Lind [18] in 1974 and Veneziano [19] in 1974.

It was in 1968 suggested by Ang & Amin [20] to split the total factor in two parts with the purpose of reducing the sensitivity of the type of distribution. Research by Russian scientists in this field was published by Bolotin [21] in 1969.

Only few papers published since 1970 are mentioned here. This is due to the fact that an enormous amount of papers has appeared in scientific journals or has been presented at conferences in the last ten years. It seems to be too early to try to evaluate in details the great progress achieved in this decade but a number of scientists should be mentioned here bearing in mind that some injustice will certainly be done by such a choice. However, it is reasonable to believe that recent work by the scientists Ang, Amin, Benjamin, Ferry Borges, Cornell, Crandall, Davenport, Ditlevsen, Galambos, Legere, Lind, Marshall, Meyerhof, Moses, Rachwitz, Ravindra, Rosenblueth, Sexsmith, Shah, Siu, Shinozuka, Turkstra, Vanmarcke, Veneziano will be remembered and appreciated in the future.

In this chapter a brief introduction is given to some fundamental concepts and ideas in the probabilistic approach to structural reliability with the purpose of facilitating the reading of the other papers in this book. The presentation is deliberately made very informal so that the main points are not drowned in details. It is intended to make the paper intelligible for a reader without previous knowledge in this field.

To fulfill these intentions the following subjects are briefly treated. In section 2 the most fundamental concepts related to random variables are introduced with special emphasis on distribution functions and the first three moments. As a natural continuation distribution of extreme values are treated in section 3. On this background reliability for single structural members is discussed in section 4. Finally, in section 5 the reliability of some simple structural systems is shortly touched on.

A more extensive treatment can be found in the papers and books referred to in the list of literature at the end of the paper.

## 2. RANDOM VARIABLES

In this section some fundamental concepts concerning random variables will be briefly reviewed. It is outside the scope of the paper to give a detailed presentation of this subject. Let  $X$  be a random variable. Then the *distribution function*  $F_X$  of  $X$  is defined by

$$F_X(x) = P[X \leq x] \quad (1)$$

where  $P[\ ]$  means the probability that the event in  $[ \ ]$  is true.

For a *continuous* random variable  $X$  the *density function*,  $f_X$  is defined by

$$f_X(x) = \frac{dF(x)}{dx} \quad (2)$$

In the case of a *discrete* random variable  $X$  the distribution function is a staircase function with discontinuities at some points  $x_i$  and the density function is defined by

$$f(x) = \sum_i P[X = X_i] \delta(x - x_i) \quad (3)$$

where  $\delta$  is Dirac's Delta function.

In the rest of this section only the continuous case will be treated. It is often useful to describe a random variable by its moments. The *moment*  $m_n$  of order  $n$  is defined

$$m_n = \int_{-\infty}^{\infty} x^n f_X(x) dx \quad (4)$$

The moment  $m_1$  is called the *expected value* or *mean* of  $X$ , and the following two symbols are used

$$\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx \quad (5)$$

Other useful moments are *central moments* defined by

$$m_n^0 = \int_{-\infty}^{\infty} (x - \mu_X)^n f_X(x) dx \quad (6)$$

The moment  $m_2^0$  is called the *variance* of  $X$  and is denoted by  $\sigma_X^2$  or  $\text{Var}[X]$ . The square root  $\sigma_X$  of the variance is called the standard deviation. An important moment of third order is the *skewness coefficient*  $\nu_X$  defined by

$$m_3^0 = \nu_X^3 \sigma_X^3 \quad (7)$$

Note that

$$m_n = E[X^n] \quad \text{and} \quad m_n^0 = E[(X - \mu_X)^n]. \quad (8)$$

Perhaps the most important density function in structural reliability theory is the so-called *normal* distribution, defined by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \quad (9)$$

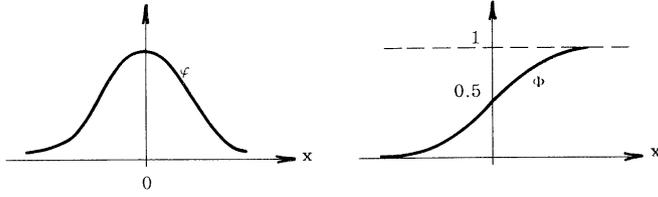


Figure 1.

where  $\mu$  and  $\sigma > 0$  are parameters equal to  $\mu_X$  and  $\sigma_X$ . The *standardized normal density function*  $\phi$  is defined by

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (10)$$

and the corresponding values  $\mu_X$  and  $\sigma_X$  are 0 and 1. The *standardized normal distribution function* is denoted by  $\Phi$  (see figure 1).

It is not possible to mention all density functions of interest in structural reliability theory. However, the distribution of extreme values of random variables will be more detailed treated in the next section and in this connection some frequently used distributions will be presented.

Some new concepts appear when two random variables  $X$  and  $Y$  are considered. It is then easy to extend to more than two variables. The fundamental concept here is the *joint distribution function*  $F_{XY}$  defined by

$$F_{XY}(x, y) = P[X \leq x, Y \leq y] \quad (11)$$

If  $F_{XY}$  has partial derivatives of order up to two the *joint density function*  $f_{XY}$  is given by

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y} \quad (12)$$

Two random variables  $X$  and  $Y$  are called *independent* if for any  $x$  and  $y$

$$P[X \leq x, Y \leq y] = P[X \leq x] P[Y \leq y] \quad (13)$$

It is easy to see for independent random variables  $X$  and  $Y$  that

$$F_{XY}(x, y) = F_X(x) F_Y(y) \quad (14)$$

where  $F_X$  and  $F_Y$  are the marginal distribution functions defined by

$$F_X(x) = F_{XY}(x, \infty) \quad \text{and} \quad F_Y(y) = F_{XY}(\infty, y) \quad (15)$$

Further in this case

$$f_{XY}(x, y) = f_X(x) f_Y(y) \quad (16)$$

where  $f_X$  and  $f_Y$  are the marginal density functions.

The joint density function  $f_{XY}$  given by

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right) \quad (17)$$

is called the *joint normal* density function. The corresponding marginal density functions are normal density functions with  $\mu_X = \mu_1, \sigma_X = \sigma_1$  and  $\mu_Y = \mu_2, \sigma_Y = \sigma_2$ . The remaining parameter  $\rho$  in (17) is called the *correlation coefficient* (see below).

The expected value of a function  $g(X, Y)$  of two random variables  $X$  and  $Y$  is given by

$$E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{XY}(x, y) dx dy \quad (18)$$

where  $f_{XY}$  is the joint density function of  $X$  and  $Y$ .

As an important application of (18) the *covariance*  $\text{Cov}(X, Y)$  of the random variables  $X$  and  $Y$  is defined by

$$\text{Cov}(X, Y) = E[(X - \mu_X)(Y - \mu_Y)] = E[XY] - E[X] E[Y] \quad (19)$$

The ratio

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} \quad (20)$$

is called the *correlation coefficient* of  $X$  and  $Y$ . It can be shown that

$$-1 \leq \rho_{XY} \leq 1 \quad (21)$$

Two random variables  $X$  and  $Y$  are said to be *uncorrelated* if

$$E[XY] = E[X] E[Y] \quad (22)$$

It follows from (19) that  $\text{Cov}(X, Y) = 0$  for uncorrelated random variables  $X$  and  $Y$ .

It is important to note that independent variables are uncorrelated, but uncorrelatedness does not in general imply independency. However, if two jointly normal random variables  $X$  and  $Y$  are uncorrelated they are also independent.

Finally, consider a random variable  $Z$  given by

$$Z = aX + bY \quad (23)$$

where  $a$  and  $b$  are constants. It is then easy to see that

$$\mu_Z = a\mu_X + b\mu_Y \quad (24)$$

and

$$\sigma_Z^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\rho_{XY}\sigma_X\sigma_Y \quad (25)$$

### 3. DISTRIBUTION OF EXTREMES

From a structural reliability point of view it is of great interest to look at distributions of extreme values. In connection with loading it is of course the distribution of maximum values that is important. On the other hand, in relation to strength of materials or structural elements the distribution of minimum values is of interest.

Let  $s_1 < s_2 < \dots < s_n$  be a set of  $n$  measurements of a random load variables  $S$ , with the distribution function  $F_S$  and the density function  $f_S$ . The distribution function  $F_{S, \max}$  for the maximum value  $s_n$  is then easy to determine when the measurements can

be assumed to be independent.  $F_{S,\max}(s)$  is in this case equal to the probability that  $S < s$  in all  $n$  measurements, that is

$$F_{S,\max}(s) = (F_S(s))^n \tag{26}$$

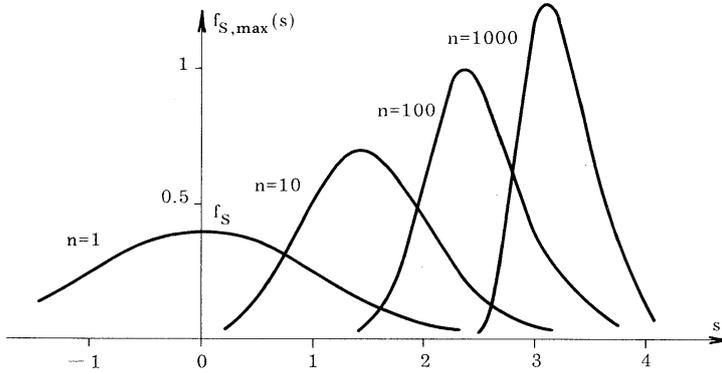


Figure 2.

The corresponding density function  $f_{S,\max}$  is then given by

$$f_{S,\max}(s) = n(F_S(s))^{n-1} f_S(s) \tag{27}$$

and the graph of  $f_{S,\max}$  is as expected shifted to the right compared with the graph of  $f_S$ . Figure 2 shows an example, where  $f_S$  is the standardized normal density function  $\varphi$  defined by (10) and where  $f_{S,\max}$  is calculated for  $n = 10, 100,$  and  $1000$ . Figure 2 is taken from a paper by Fiessler [26], where it is concluded that the density functions shown do not diverge very much from normal functions.

The distribution function  $f_{S,\min}$  can be found in a similar way using that the probability  $P(S > s)$  is equal to  $1 - F_S(s)$  for all measurements. The result is therefore

$$F_{S,\min}(s) = 1 - (1 - F_S(s))^n \tag{28}$$

and

$$f_{S,\min}(s) = n(1 - F_S(s))^{n-1} f_S(s) \tag{29}$$

In principle the equations (27) and (29) make it possible to calculate values of the density functions for maximum and minimum values. However, use of (27) and (29) is difficult for large values of  $n$ . It is therefore of great interest to study the so-called asymptotic behaviour for  $n \rightarrow \infty$ . Such a study has been performed by several authors for example by Gumbel [22] and Johnson [10]. Here we will only consider the special case, where the original distribution (parent distribution) of  $S$  is  $N(\mu_S, \sigma_S)$ .

According to Bolotin [21] the following approximate values for the expected values and standard deviations for  $S_{\max}$  and  $S_{\min}$  can in this case be used

$$\mu_{S,\min} \approx \mu_S - \sigma_S \sqrt{\ln n} \tag{30}$$

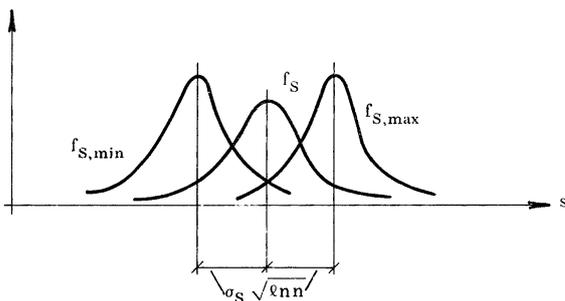


Figure 3.

$$\mu_{S,\max} \approx \mu_S + \sigma_S \sqrt{\ln n} \quad (31)$$

$$\sigma_{S,\min} = \sigma_{S,\max} \approx \frac{\pi}{\sqrt{6 \ln n}} \sigma_S \quad (32)$$

The formulas (30) to (32) are illustrated in figure 3.

The form of the density function  $f_{S,\min}$  and  $f_{S,\max}$  is of course dependent on the original density function  $f_S$  according to (27) and (29). However, it is usually a good approximation to use double exponential laws, for example

$$F_{S,\max}(s) = \exp(-\exp(-\frac{s-a}{b})) \quad (33)$$

and

$$F_{S,\min} = 1 - \exp(-\exp(\frac{s-a}{b})) \quad (34)$$

where  $a$  and  $b > 0$  are parameters. For the maximum distribution  $F_{S,\max}$  it can be shown that

$$\mu_{S,\max} = a + 0.5772b \quad (35)$$

$$\sigma_{S,\max} = \frac{\pi}{\sqrt{6}} b \quad (36)$$

The distributions (33) and (34) are often called extreme value distributions of type I to distinguish them from type II and type III distributions (see Johnson [10]). The type I maximum distribution (33) has been used by Armitt [23] and Dyrbye et al. [24] in connection with wind loading. The maximum value for the wind velocity  $V$  in one year is described by the following distribution function

$$F_V(v) = P[V \leq v] = \exp(-\exp(-\alpha(v-u))) \quad (37)$$

where the two parameters  $\alpha$  and  $u$  are depending on the expected value  $\mu_V$  and the standard deviation by (35) and (36), that is

$$\alpha = \frac{\pi}{6} \frac{1}{\sigma_V} \quad (38)$$

and

$$u = \mu_V - \frac{\sqrt{6}}{\pi} 0.5772 \sigma_V \quad (39)$$

If  $\mu_V = 30$  m/sec and  $\sigma_V = 4$  m/sec then

$$F_V(v) = \exp(-\exp(-0.32(v-28.2))) \quad (40)$$

so that the probability during one year of getting for example a wind velocity greater than 40 m/sec is equal to

$$P[V > 40 \text{ m/sec, 1 year}] = 1 - F_V(40) = 0.023 \quad (41)$$

The reciprocal value of  $1 - F_V(v_M)$  is called the *return period*  $T$  for a wind velocity greater than  $v_M$ . With the figures in (41) one gets a return period of

$$T = \frac{1}{1 - F_V(40)} \approx 44 \text{ years} \quad (42)$$

When  $T = (1 - F_V(v_M))^{-1}$ , the wind velocity  $v_M$  is called the  $T$ -years wind.

In a paper by Schwarz [25] the yield strength  $R$  of mild steel is assumed to be log-normally distributed. The density function  $f_R$  is then given by

$$f_R(r) = \frac{1}{\sqrt{2\pi}\sigma_1 r} \exp\left(-\frac{1}{2}\left(\frac{\ln r - \ln \mu_1}{\sigma_1}\right)^2\right) \quad (43)$$

where  $\sigma_1$  and  $\mu_1$  are parameters defining the distribution. The corresponding extreme value distribution  $f_{R,\min}$  for minimum values is of type III and is called the *Weibull distribution*. It is in its general form a three-parameter distribution, namely

$$f_{R,\min}(r) = \frac{\beta}{k - \zeta} \left(\frac{r - \zeta}{k - \zeta}\right)^{\beta-1} \exp\left(-\left(\frac{r - \zeta}{k - \zeta}\right)^\beta\right) \quad (44)$$

where  $r \geq \zeta$  and  $\beta > 0$ ,  $k > \zeta \geq 0$ . The Weibull-distribution is often used with  $\zeta = 0$ , that is

$$f_{R,\min}(r) = \beta \frac{r^{\beta-1}}{k^\beta} \exp\left(-\left(\frac{r}{k}\right)^\beta\right) \quad (45)$$

According to Schwarz [25] the expected value and standard deviation for  $R_{\min}$  can in this case be calculated by

$$\mu_{R,\min} = m_1 \exp(-\nu\sigma_1) \Gamma\left(1 + \frac{\sigma_1}{\nu}\right) \quad (46)$$

and

$$\sigma_{R,\min} = m_1 \exp(-\nu\sigma_1) \left(\Gamma\left(1 + 2\frac{\sigma_1}{\nu}\right) - \Gamma^2\left(1 + \frac{\sigma_1}{\nu}\right)\right)^{\frac{1}{2}} \quad (47)$$

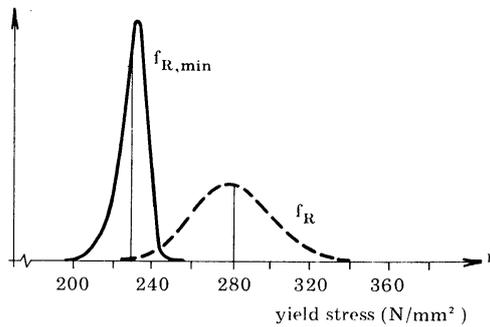


Figure 4

where  $\nu = \left(2 \ln\left(\frac{n}{\sqrt{2\pi}}\right)\right)^{\frac{1}{2}}$  and  $\Gamma$  is the Gamma function. Figure 4 (reproduced from [25]) shows graphs for the original lognormal distribution  $f_R$  with  $\mu_R = 281.6$  N/mm<sup>2</sup>,  $\sigma_R = 19.6$  N/mm<sup>2</sup> and for the extreme Weibull distribution  $f_{R,\min}$  with  $n = 100$ ,  $\mu_{R,\min} = 229.4$  N/mm<sup>2</sup>,  $\sigma_{R,\min} = 7.22$  N/mm<sup>2</sup>.

#### 4. SINGLE MEMBER RELIABILITY

As an introduction to structural reliability theory it is useful to consider a very simple idealized example, namely single member reliability. Such an example will be used here to introduce fundamental concepts like probability of failure, probability of safety, reliability index, failure domain, safe domain and failure curve.

For a single structural member it is on some favorable conditions possible to describe the strength by a single random variable  $R$  and the loading by a single random

variable  $S$ . Let  $f_S$  and  $f_R$  be the frequency distributions of  $S$  and  $R$  as illustrated on figure 5.

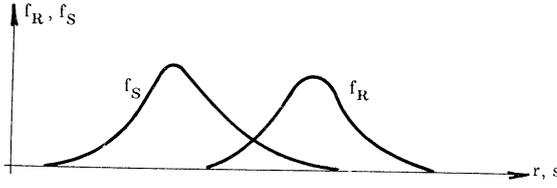


Figure 5.

Failure will occur when

$$R < S \Leftrightarrow R - S < 0 \quad (48)$$

and the *probability of failure*  $P_f$  for such an element is given by

$$P_f = P(R - S < 0) = F_{R-S}(0) \quad (49)$$

where  $F_{R-S}$  is the distribution

function of  $R - S$ .

It is sometimes reasonable to assume statistical independence between the random variables  $R$  and  $S$ . Then it is easy to see that the probability of failure can be expressed by

$$P_f = P(R - S < 0) = \int_{-\infty}^{\infty} F_R(t) f_S(t) dt = \int_{-\infty}^{\infty} (1 - F_S(t)) f_R(t) dt \quad (50)$$

where  $F_R$  and  $F_S$  are the distribution functions of  $R$  and  $S$ .

If independency between  $R$  and  $S$  cannot be assumed the probability of failure is given by

$$P_f = \int_{\omega_f} f_{R,S}(r, s) d\omega \quad (51)$$

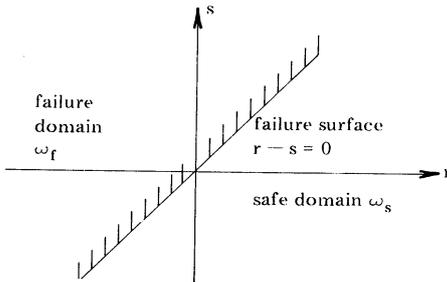


Figure 6.

where  $f_{R,S}$  is the joint density function for  $R$  and  $S$ , and where the area of integration is the *failure domain*  $\omega_f$  (see figure 6) defined by

$$\omega_f = \{(r, s) \in R_2 \mid r - s < 0\} \quad (52)$$

The failure domain  $\omega_f$  and the *safe domain*  $\omega_s$  defined by

$$\omega_s = \{(r, s) \in R_2 \mid r - s \geq 0\} \quad (53)$$

is separated by the *failure surface*  $r - s = 0$ . In this case the failure surface is a straight line.

It is rather obvious that evaluation of  $P_f$  by (50) or (51) is complicated due to mathematical difficulties, but also because the distribution functions for  $R$  and  $S$  are seldom known to a sufficient degree of accuracy. Therefore,  $P_f$  or the *probability of safety*  $= 1 - P_f$  is not a very suitable measures in structural reliability theory. Cornell [14] has instead suggested using the so-called *reliability index*  $\beta$ , defined by

$$\beta = \frac{\mu_{R-S}}{\sigma_{R-S}} \quad (54)$$

where  $\mu_{R-S} = \mu_R - \mu_S$ , and for statistically independent  $R$  and  $S$  (see (25))

$$\sigma_{R-S}^2 = \sigma_R^2 + \sigma_S^2 \quad (55)$$

Therefore, if  $R$  and  $S$  are statistically independent

$$\beta = \frac{\mu_R - \mu_S}{(\sigma_R^2 + \sigma_S^2)^{\frac{1}{2}}} \quad (56)$$

A more comprehensive discussion of the reliability index  $\beta$  is given somewhere else in this book, but it is essential to notice that generally it is not possible to relate  $\beta$  directly to the probability of failure  $P_f$ . However, if the random variables  $R$  and  $S$  are normally distributed it is a simple matter to show that

$$P_f = \Phi(-\beta) \Leftrightarrow \beta = -\Phi^{-1}(P_f) \quad (57)$$

where  $\Phi$  is the standardized normal distribution function.

The fundamental reliability problem formulated above is based on the assumption that only two random variables, namely a load variable  $S$  and a strength variable  $R$  are involved. However, the random variables  $S$  and  $R$  are usually functions of a number of random variables  $S_1, S_2, \dots, S_n$  and  $R_1, R_2, \dots, R_m$ :

$$S = (S_1, S_2, \dots, S_n) \quad (58)$$

$$R = (R_1, R_2, \dots, R_m) \quad (59)$$

By (58) and (59) the distributions of  $S$  and  $R$  are in principle defined by the distributions  $R_1, R_2, \dots, R_m$ , and  $R_1, R_2, \dots, R_m$ , but only  $\mu_S, \mu_R, \sigma_S$ , and  $\sigma_R$  need be determined for calculation of the reliability index  $\beta$  if  $R$  and  $S$  are independent random variables. Exact determination of these values is difficult for general functions  $f$  and  $g$ , but approximate values can be determined in several ways. One possibility is to expand the functions  $f$  and  $g$  about  $(\mu_{S_1}, \dots, \mu_{S_n})$  and  $(\mu_{R_1}, \dots, \mu_{R_m})$  respectively, and retain only the linear terms, e.g.

$$S = f(\mu_{S_1}, \dots, \mu_{S_n}) + \sum_{i=1}^n \frac{\partial f}{\partial S_i} (S_i - \mu_{S_i}) + \dots \quad (60)$$

From (60) approximate values  $\mu_S$  and  $\sigma_S$  are determined by

$$\mu_S \cong f(\mu_{S_1}, \dots, \mu_{S_n}) \quad (61)$$

and

$$\sigma_S^2 \cong \sum_{i=1}^n \left( \frac{\partial f}{\partial S_i} \right)^2 \sigma_{S_i}^2 + \sum_{i=1}^n \sum_{j=1, j \neq i}^n \rho_{ij} \frac{\partial f}{\partial S_i} \frac{\partial f}{\partial S_j} \sigma_{S_i} \sigma_{S_j} \quad (62)$$

where  $\rho_{ij}$  is the coefficient of correlation between  $S_i$  and  $S_j$ . All derivatives in (60) and (62) are evaluated at  $(\mu_{S_1}, \dots, \mu_{S_n})$ .  $\rho_{ij} \equiv 0$  and the last term in (62) disappear when there is no correlation between any pair of variables  $S_i$  and  $S_j$ .

Approximate values for  $\mu_R$  and  $\sigma_R$  can be determined in the same way by expanding the function  $g$ . Note that it is implicitly assumed that the functions  $f$  and  $g$  are differentiable functions.

As a simple example of using this technique consider

$$S = S_1 S_2 \quad (63)$$

With  $\mu_{S_1} = \mu_{S_2} = 1.0$ ,  $\sigma_{S_1} = \sigma_{S_2} = 0.2$ , and  $\rho_{12} = 0.0$ . In this case (61) and (62) give

$$\mu_S \cong \mu_{S_1} \mu_{S_2} = 1.0 \tag{64}$$

$$\sigma_S^2 \cong \mu_{S_2}^2 \sigma_{S_1}^2 + \mu_{S_1}^2 \sigma_{S_2}^2 = 0.50 \tag{65}$$

Rosenblueth [27] has suggested another technique of computing statistical moments of a function of random variables. By his method calculation of derivatives is avoided. The idea behind this technique is to replace the original density function  $f_S$  for a random

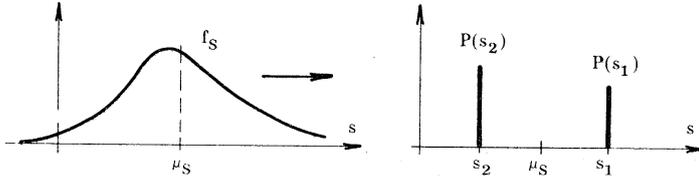


Figure 7.

variable  $S$  with a density function as shown on figure 7 so that the expectation  $\mu_S$ , standard deviation  $\sigma_S$  and the skewness coefficient  $\nu_S$  are the same for the two density functions. These three conditions result in the following values for  $P(s_1)$ ,  $P(s_2)$ ,  $s_1$ , and  $s_2$

$$P(s_1) = \frac{1}{2} \left( 1 \pm \sqrt{1 - \frac{1}{1 + \frac{1}{4} \nu_S^6}} \right); \quad +\text{for } \nu_S < 0, -\text{for } \nu_S > 0 \tag{66}$$

$$P(s_2) = 1 - P(s_1) \tag{67}$$

$$s_1 = \mu_S + \sqrt{\frac{1 - P(s_1)}{P(s_1)}} \sigma_S \tag{68}$$

$$s_2 = \mu_S - \sqrt{\frac{P(s_1)}{1 - P(s_1)}} \sigma_S \tag{69}$$

In the paper [27] values different from those in (66) to (68) are determined. With a symmetrical density function ( $\nu_S = 0$ ) it follows from (66) to (68) that

$$P(s_1) = P(s_2) = 0.5 \tag{70}$$

and

$$s_1 = \mu_S + \sigma_S \tag{71}$$

and

$$s_2 = \mu_S - \sigma_S \tag{72}$$

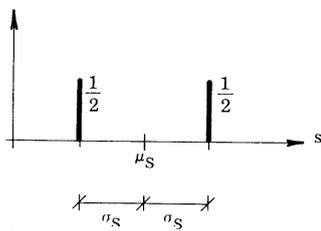


Figure 8.

as shown in figure 8.

Consider the same example as in equation (63) and assume that the density functions for  $S_1$  and  $S_2$  are replaced by density functions as shown on figure 8. In this case it is easy to calculate the density function  $f_S$  for  $S = S_1 S_2$ . The result is shown in figure 9 with  $E[S] = 1$  and  $\sigma_S^2 = 0.5625$ . It is here assumed that the variables  $S_1$  and  $S_2$  are uncorrelated, but in [27] it is also shown how to deal with correlated variables.

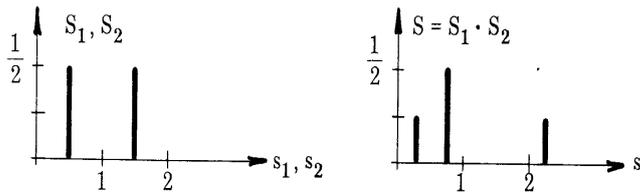


Figure 9.

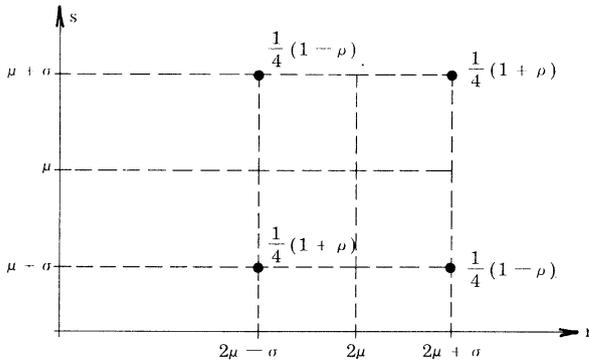


Figure 10.

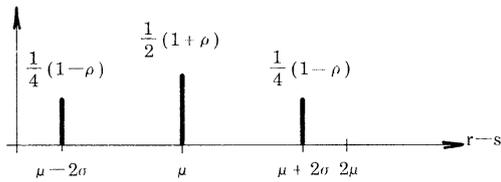


Figure 11.

To illustrate the situation with correlated random variables consider a strength variable  $R$  and a load variable  $S$ , with  $\mu_R = 2\mu, \mu_S = \mu, \sigma_R = \sigma_S = 0$  and the correlation coefficient of  $R$  and  $S$  equal to  $\rho$ .

According to [27] the joint probability density is assumed concentrated at points as shown in figure 10. In the same figure the magnitude of the density concentrations is shown. To evaluate the reliability index  $\beta$  the random variable  $R-S$  is considered. The density function  $f_{R-S}$  is shown on figure 11, and

$$\mu_{R-S} = \mu \quad (73)$$

and

$$\sigma_{R-S}^2 = 2\frac{1}{4}(1-\rho)(2\sigma)^2 = 2(1-\rho)\sigma^2 \quad (74)$$

so that

$$\beta = \frac{\mu}{\sqrt{2(1-\rho)}\sigma} = (1-\rho)^{-\frac{1}{2}}\beta^* \quad (75)$$

where  $\beta^*$  corresponds to no correlation between  $R$  and  $S$  ( $\rho = 0$ ). In this simple linear case the results (73) - (75) are of course exact.

In a paper by Mazumdar, Marshall & Chay [28] approximate methods for estimating the probability that a random variable  $S$  exceeds a specified design limit  $S_0$  are evaluated. The random variable  $S$  is as in eq. (58) assumed to be a function  $f$  of random variables  $S_1, \dots, S_n$  with known distribution functions. In [28] four approximate methods are considered, namely

- (1) Linearization of  $f$
- (2) Method of moments
- (3) Simulation
- (4) Numerical integration.

The first of these methods is treated above and is based on a Taylor series expansion (60). In [28] it is concluded that it is the authors' experience that simulation estimates of  $P[S > S_0]$  are more accurate than those obtained by the other methods. A very simple form of simulation is to simulate values for all the random variables  $S_1, \dots, S_n$  on a computer a number of times and in this way get a set of values for  $S = f(S_1, \dots, S_n)$ . From these values  $E[S]$  and  $\sigma_S$  can be estimated. This simple method called the direct Monte Carlo method is not suitable in estimation of small probabilities  $P[S >$

$S_0$ ] due to the great number of simulations needed. But a variant called “importance sampling” seems to be very efficient according to paper [28].

## 5. RELIABILITY OF STRUCTURAL SYSTEMS

In the traditional way of designing structures the reliability of a structural system is connected with the safety of some critical members or cross-sections. For a single element (structural member, cross-section, etc.) evaluation of safety can be performed as shown in chapter 4. However, a real structure is much more complex to analyse. Usually it will consist of a great number of elements with multiple load condition and with more than one failure mode. Further the loading and behaviour of the structure in its entire lifetime must be considered. The idealized goal is to determine the probability of failure of the overall structure, taking into consideration factors like correlation between strength of single elements, statically indeterminateness, multiple failure, etc.

Many authors have treated reliability of structural systems. It is outside the scope of this paper to give a detailed presentation of the “state of the art” in this field. For a more deep discussion papers by Moses [29], [30], Moses & Stevenson [31], Benjamin [32], Marshall & Bea [33], Rackwitz [34], Krzykacs [35] and Grigorium & Turkstra [36] can be referred to.

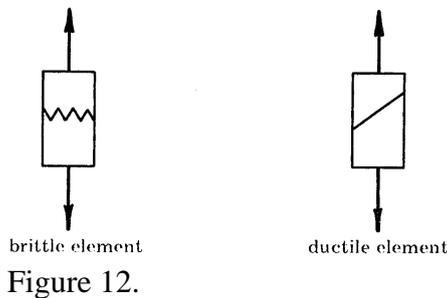


Figure 12.

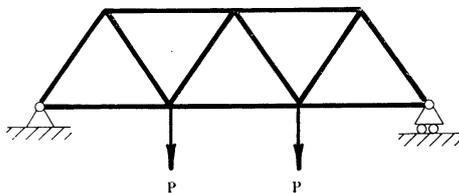


Figure 13.

“weakest link” fails. The structure in figure 13 has 11 elements.

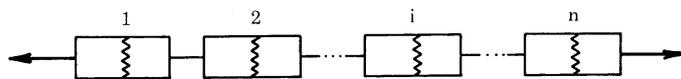


Figure 14.

elements are supposed to be perfectly brittle. If the distribution function for the strength of element  $i$  is  $F_i$  then the probability of survival for element  $i$  loaded with the stress  $x_i$  is  $1 - F_i(x_i)$ . Therefore, the distribution function  $F_R$  for the strength  $R$  of this series system assuming independent strength of the elements is given by

It is of great importance for a structural system whether its elements can be considered *perfectly brittle* or *perfectly ductile*. A structural element is called perfectly brittle if it loses its loading capacity completely after failure, so that it becomes ineffective. A perfectly ductile element maintains its load level after failure. To distinguish these two types of element behaviour symbols as shown in figure 12 can be used.

The two simplest structural systems are the so-called *series system* and *parallel system*. A series system is defined as a system, which is in a state of failure as soon as one element fails. A typical example is a statically determinate structure as shown in figure 13. The structure fails when its

In general a series system with  $n$  elements is symbolized as shown in figure 14. It is important to note that the  $n$  elements have generally different loads. In figure 14 all

$$F_R(x) = 1 - (1 - F_1(x_1)) \dots (1 - F_n(x_n)) = 1 - \prod_{i=1}^n (1 - F_i(x_i)) \quad (76)$$

where  $x$  is the external load on the series system resulting in the stress  $x_i$  in element  $i$ .

If all  $n$  elements in a series system as shown in figure 14 have the same distribution function  $F$  and if the loading is equal to a constant  $x$  then (76) can be reduced to

$$F_R(x) = 1 - (1 - F(x))^n \quad (77)$$

Note the similarity between equations (28) and (77) so that the approximate values for the expected value and standard deviation for the strength of the system can be found in chapter 3. Also note that  $F_R$  in this case will approach an asymptotic extreme-value distribution for increasing  $n$ .

If the strengths  $R_i$  of the elements in the series system on figure 14 are assumed independent random variables it is not so difficult to estimate the expected value  $E[R]$  and  $\text{Var}[R]$  for the strength  $R$  of the series system. However, if some correlation exists, this assumption of independence increases the risk of failure of the system by increasing the probability of failure for the single element. This fact is demonstrated by Grigoriu & Turkstra [36] in several examples. In [36] the reliability index  $\beta_s$  for a series system with  $n$  elements is compared with the reliability index  $\beta_e$  for the single elements on the following assumptions:

- Loading is deterministic and constant in time.
- The strength of the elements is  $N(0, 1)$ .
- The correlation coefficient between any two elements is  $\rho$ .
- The reliability coefficient for any single element is  $\beta_e$ .

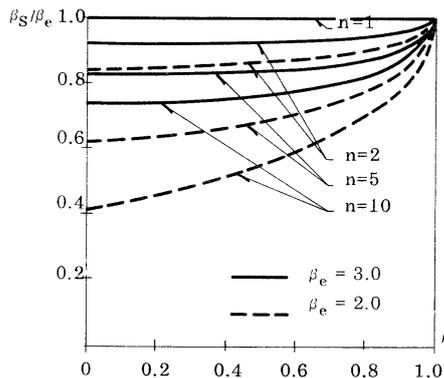


Figure 15.

In figure 15 one of the interesting results from [36] is shown, namely the dependence of  $\beta_s / \beta_e$  on the correlation coefficient  $\rho$  for  $\beta_e = 3.0$  and  $\beta_e = 2.0$  and for several values of the number of elements  $n$ .

Next consider a statically indeterminate structure like the one shown in figure 16. For such a structure it might be too conservative to use a series system because the reserve survival probability after the first element has failed is neglected. A detailed analysis of such a structure is complicated because of the great

number of failure modes. If for an indeterminate structure two or more elements behave in such a way that a redistribution of load to remaining elements following failure in a single element is possible, these elements can sometimes be considered a parallel sub-system of the structure.

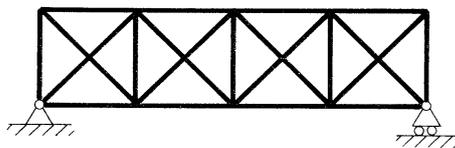


Figure 16.

In a parallel system the structure will only fail when all elements in the system fail. The behaviour of such a system is clearly dependent on whether or not the elements are

perfectly ductile or perfectly brittle.

In figure 17 a parallel system with  $n$  perfectly ductile elements is shown. If the strength variable for element  $i$  is the random variable  $R_i$  the strength  $R$  of the parallel system is

$$R = \sum_{i=1}^n R_i \quad (78)$$

because the total strength of such a system is the sum of the strengths of the single elements due to the assumption of ductile behaviour. A parallel system is especially

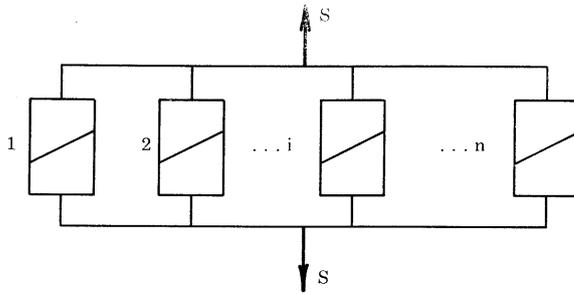


Figure 17.

easy to analyse when the single elements are normally distributed, because  $R$  will then also be normally distributed due to the linearity of (78). But even if this is not the case the Central Limit Theorem makes it a good approximation to assume  $R$  to be normally distributed when the number of elements,  $n$ , is not too low.

If the correlation coefficient between any two elements in the parallel system in figure 17 is equal to a constant  $\rho$  we have from (78)

$$E[R] = \sum_{i=1}^n E[R_i] \quad (79)$$

and

$$\text{Var}[R] = \sum_{i=1}^n \text{Var}[R_i] + \rho \sum_{i,j=1, i \neq j}^n (\text{Var}[R_i] \text{Var}[R_j])^{1/2} \quad (80)$$

For constant  $E[R_i]$  and  $\text{Var}[R_i]$  and common element reliability index  $\beta_e$  for all elements Grigoriu & Turkstra [36] have derived the following relation between the reliability index  $\beta_s$  for the parallel system and the reliability index  $\beta_e$  for the single elements

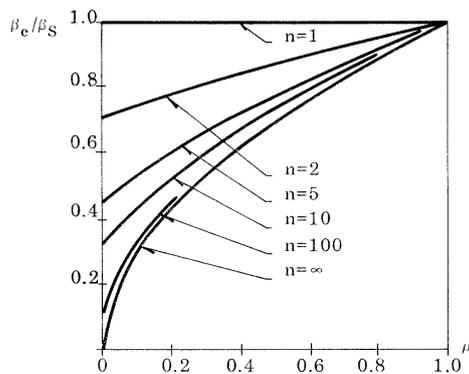


Figure 18.

$$\beta_s = \beta_e \sqrt{\frac{n}{1 + \rho(n-1)}} \quad (81)$$

where  $n$  is the number of elements and  $\rho$  is the correlation coefficient. The dependence between the ratio  $\beta_e / \beta_s$  and  $\rho$  is illustrated for some values of  $n$  in figure 18 (from [36]). As expected  $\beta_s / \beta_e$  is increased when  $\rho$  is decreased or  $n$  is increased. Therefore, the probability of failure for such a parallel system is underestimated in contrast to a series system if independence is assumed, when some correlation exists.

Finally consider a parallel system with perfectly brittle elements as shown in figure 19. Let the  $n$  elements have identically distributed strength  $R_e$  and let  $r_1, r_2, \dots, r_n$ , where  $r_1 < r_2 < \dots < r_n$  be randomly

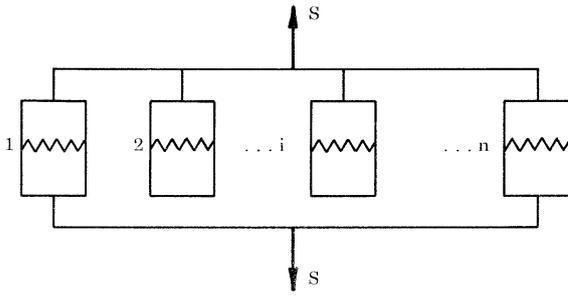


Figure 19.

sampled values of the strength of the single elements. The strength  $r_s$  of the parallel system is then the maximum of  $nr_1, (n-1)r_2, \dots, 2r_{n-1}$  or  $r_n$  that is

$$r_s = \max(nr_1, (n-1)r_2, \dots, 2r_{n-1}, r_n) \quad (82)$$

It has been shown by Daniels [37] (see also Rackwitz [34]) that under certain conditions the distribution of  $r_s$  approaches a normal distribution  $N(\mu_R, \sigma_R)$  for large  $n$ , where

$$\mu_R = nr_0(1 - F_{R_e}(r_0)) \quad (83)$$

$$\sigma_R^2 = nr_0^2 F_{R_e}(r_0)(1 - F_{R_e}(r_0)) \quad (84)$$

and where the maximum value of  $x(1 - F_{R_e}(r))$  occurs for  $x = r_0$ .

Above only structural systems that can be modelled as series systems or parallel systems are treated. Real structures, of course, must be modelled by a much more complicated system. However, it is often possible to subdivide a structure in a number of substructures and model these substructures by series systems or parallel systems. By combining the subsystems again in series or parallel systems a rather general system can be modelled. In evaluation of the probability of failure of a real structure on the basis of an idealized model it is important to take into account the uncertainty connected with using such a mode

## REFERENCES

- [1] Borges, J. Ferry & M. Castanheta. *Structural Safety*. Laboratório Nacional de Engenharia Civil, Lissabon, 1971.
- [2] Meyer, M. *Die Sicherheit der Bauwerke*. Springer Verlag, Berlin, 1926.
- [3] Prot, M. *Note sur la notion de coefficient de securite*. *Annales des Ponts et Chaussées*, no: 27, Paris, 1936.
- [4] Weibull, W. *Investigations into Strength Properties of Brittle Materials*. *Ingeniör Vetenskaps Akademin Proceedings*, 149, Stockholm, 1938.
- [5] Weibull, W. *A Statistical Theory of Strength of Materials*. *Ingeniör Vetenskaps Akademin Proceedings*, 151, Stockholm, 1939.
- [6] Weibull, W. *The Phenomenon of Rupture in Solids*. *Ingeniör Vetenskaps Akademin Proceedings*, 153, Stockholm, 1939.
- [7] Kjellman, W. *Säkerhetsproblemet ur principiell och teoretisk synpunkt*. *Ingeniör Vetenskaps Akademin Proceedings*, 156, Stockholm, 1940.
- [8] Wastlund, G. *Säkerhetsproblemet ur praktisk-konstruktiv synpunkt*. *Ingeniör Vetenskaps Akademin Proceedings*, 156, Stockholm, 1940.
- [9] Torroja, E. & A. Paez. *Calcul du coefficient de securite*. Publication preliminaire du 4 'erne Congres de l' Association Internationale des Ponts et Charpentes, Cambridge et London, 1952.

- [10] Johnson, A. I. *Strength, Safety and Economical Dimensions of Structures*. Kgl. Tekniska Hogskolan, Institutionen for Byggnadsstatik, Medd. Nr: 12, Stockholm, 1953.
- [11] Borges, J. Ferry. *Critical View of the Methods for the Probabilistic Calculation of Safety*. Final report, 4' Congress, Int. Ass. Bridge and Struct. Eng., Cambridge, 1952.
- [12] Freudenthal, A. M. *Critical Appraisal of Safety Criteria and their Basic Concepts*. Preliminary publication, 8' Congress Int. Ass. Bridge and Struct. Eng., N. Y., 1968.
- [13] Benjamin, J. R. *Probabilistic Models for Seismic Force Design*. Proc. Amer. Soc. Civil Eng., N. Y., 94 (ST5), May 1968.
- [14] Cornell, C. A. *Bounds on the Reliability of Structural Systems*. Proc. Amer. Soc. Civil Eng., N. Y., 93(ST1), February 1967.
- [15] Cornell, C. A. *A Probability-based Structural Code*, ACI Journal, Vol. 66, 1969, pp. 974-985.
- [16] Rosenblueth, E. & L. Esteva. *Reliability Basis for some Mexican Codes*. ACI Publication SP-31, 1972, pp. 1-41.
- [17] Ditlevsen, O. *Structural Reliability and the Invariance Problem*. Res. Rep. No. 22, Solid Mech. Div., Univ. of Waterloo, Canada, 1973.
- [18] Hasofer, A. M. & N. C. Lind. *An Exact and Invariant First-Order Reliability Format*. Proc. ASCE, J. Eng. Mech. Div., Vol. 100, 1974, pp. 111-121.
- [19] Veneziano, D. *Contributions to Second Moment Reliability Theory*. Res. Rep. R 74-33, Dept. Civil Eng., M.I.T., Cambridge, Mass., 1974.
- [20] Ang, A. H. & M. Amin. *Reliability of Structures and Structural Systems*. Proc. Amer. Soc. Civ. Eng., N. Y. (EM2), April, 1968.
- [21] Bolotin, V. V. *Statistical Methods in Structural Mechanics*. Holden-Day, San Francisco, 1969.
- [22] Gumbel, E. J. *Statistics of Extremes*. Columbia University Press, N. Y., 1958.
- [23] Armit, J.: *Wind Structures*. VOJ1 Karman Institute for Fluid Dynamics, Lecture Series 89, Wind Effects on Buildings and Structures, Bruxelles, 1976.
- [24] Dyrbye, C., S. Gravesen, S. Krenk, N. C. Lind & H. O. Madsen. *Konstruktioners sikkerhed (in Danish)*. Den private Ingeniørfond ved Danmarks tekniske Højskole, Copenhagen 1979.
- [25] Schwarz, R. F. *On the Reliability of Statically Indeterminate Structures with Particular Reference to Load Combinations*. ICOSSAR '77, Munich, Germany, 1977, 63-80.
- [26] Fiessler, B.: *Kombination von zufälligen Lastfolgen*. Berichte zur Zuverlässigkeitstheorie der Bauwerke. LKI, Technische Universität München, Heft 25, 1978, pp. 67-82.
- [27] Rosenblueth, E.: *Point Estimates for Probability Moments*. Proc. Nat. Acad. Sci., U.S.A., Vol. 72, 1975, pp. 3812-3814.
- [28] Mazumdar, M., J. A. Marshall & S. C. Chay. *Propagation of Uncertainties in Problems of Structural Reliability*. Nuclear Engineering and Design, Vol. 50, 1978, pp. 163-167.
- [29] Moses, F.: *Reliability of Structural Systems*. BOSS '76, Trondheim, Norway, 1976, pp. 912-923.
- [30] Moses, F.: *Structural System Reliability and Optimization*. Computers & Structures, Vol. 7, 1977, pp. 283-290.

- [31] Moses, F. & J. D. Stevenson: *Reliability-based structural design*. J. Struct. Div., Feb. 1970, pp. 221-244.
- [32] Benjamin, J. R.: *Reliability Studies in Reinforced Concrete Design*. Structural reliability and codified design (editor N. C. Lind). University of Waterloo, Canada, 1970.
- [33] Marshall, P. W. & R. G. Bea: *Failure Modes of Offshore Platforms*. BOSS '76, Trondheim, Norway, 1976, pp. 579-635.
- [34] Rackwitz, R.: *First Order Reliability Theories and Stochastic Models*. ICOSSAR '77, Munich, 1977, pp. 199-220.
- [35] Krzykaes, B.: *Zur Zuverliissigkeit von Systemen*. Berichte zur Zuverlassigkeitstheorie der Bauwerke, SFB 96, TH München, Heft. 25/1978, pp. 83-104.
- [36] Grigorium, M. & C. Turkstra: *Safety of structural systems with correlated resistances*. Appl. Math. Modelling, Vol. 3, 1979, pp. 130-136.
- [37] Daniels, H. E.: *The Statistical Theory of the Strength of Bundles of Threads*. Royal Stat. Soc., Series A, Vol. 183, 1945, p. 405.