On the adiabatic behaviour of a bound state when diving into the continuous spectrum
joint work with H. D. Cornean, A. Jensen, and Gh. Nenciu

YRS 2018, Montreal
20–21 July 2018

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The problem

An external potential is tuned adiabatically in time such that the system has some isolated bound state for some time. Then the following question arises:
What is the survival probability of the bound state when the corresponding eigenvalue dives into the continuous spectrum for a while during the adiabatic tuning of the external potential?

We answer this question for a rather simple model, however we are confident that the methods applied in the proof can be generalised to more realistic setting.
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This question is closely related to ‘adiabatic pair creation’ [Nenciu 1987; Pickl, Dürr 2008] and ‘memory effects in mesoscopic transport’ [Cornean, Jensen, Nenciu 2014].
The model

We consider a two-channel model of an atom given by a quantum dot coupled to an open scattering channel. On the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3) \oplus \mathbb{C}$ we define the Hamiltonian

$$H_\tau(E) := \begin{pmatrix} -\Delta & 0 \\ 0 & E \end{pmatrix} + \tau \begin{pmatrix} 0 & \langle \varphi \rangle \\ \langle \varphi | & 0 \end{pmatrix}$$

where

- $E \in \mathbb{R}$ is the dot energy,
- $\tau \in \mathbb{R}$ is the coupling constant and
- the normalised coupling function $\varphi \in L^2(\mathbb{R}^3)$ has to fulfill that $\int_{\mathbb{R}^3} (1 + x^2)^w |\varphi(x)|^2 d^3x < \infty$ for all $w > 0$ and $|k|^{-\nu} \hat{\varphi}$ is continuous at $k = 0$ for some $\nu \geq 1$. 
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Observe that $\sigma_{ac}(H_\tau(E)) = [0, \infty]$ and $H_\tau(E)$ has at most one eigenvalue $\lambda(E)$. 
Tuning of the dot energy

In order to model an adiabatic tuning process the dot energy is varied:
We assume $E(\cdot) \in C^2([-1, 0])$ with $E(-1) = E(0) < 0$ and that there is $s_m \in ]-1, 0[$ such that $E(s_m) > 0$, $E(\cdot)$ is strictly increasing for $s < s_m$ and strictly decreasing for $s > s_m$. 

![Graph showing the behavior of E(\cdot) with a maximum at s_m]
Time evolution and adiabatic limit

The microscopic time is \( t := s / \eta \) with \( \eta > 0 \) and we set \( H(t) := H_\tau(E(\eta t)) \). Then \( \eta \downarrow 0 \) is called adiabatic limit.

The corresponding Heisenberg evolution \( U(t, t_0) \) is the solution of the Schrödinger equation

\[
i \frac{\partial}{\partial t} U(t, t_0) = H(t)U(t, t_0) , \quad U(t_0, t_0) = 1.
\]
Theorem

For any $\tau$ small enough the following holds:

(I) There is $0 < E_c < E(s_m)$ such that $\lambda(E_c) = 0$ is an embedded simple eigenvalue with eigenfunction $\Psi_c \in \mathcal{H}$ and projection $P_c$. Moreover, $\lambda(E) < 0$ is a simple discrete eigenvalue for $E < E_c$ with smooth eigenprojection $P(E)$ and eigenfunction $\Psi(E)$. 
Criticality and different regimes of the dot energy

Note that there are macroscopic times $-1 < s_c < s_m < s'_c < 0$ with $E(s_c) = E(s'_c) = E_c$. So $E(\cdot)$ may look as follows:
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(II) There is a class of coupling functions $\varphi$ for which $H_\tau(E)$ has purely a.c. spectrum for $E_c < E \leq E(s_m)$ and

$$\lim_{\eta \to 0} |\langle \Psi(E(0)), U(0, -1/\eta) \Psi(E(-1)) \rangle|^2 = 0.$$
Summary

- Main ingredients in the proof:
  - Feshbach formula
  - Asymptotic expansions of the resolvent of the Laplacian:
    \[
    r_0(-\kappa^2) = \sum_{j=0}^{n} \kappa^j G_j + O(\kappa^{n+1}) \quad \text{as } \kappa \to 0 \text{ with } \Re \kappa \geq 0
    \]
    [Jensen, Kato 1979]
  - An improved propagation estimate:
    \[
    |\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix} | e^{-itH_a} | \begin{pmatrix} 0 \\ 1 \end{pmatrix} \rangle| \leq C (1 + |t|)^{-5/2} \text{ for some } H_a = H_{\tau}(E_a),
    \]
    \[
    E_a \in ]E_c, E(s_m)\]

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  - Crucial: Existence of an embedded eigenvalue at the threshold
Outlook

- What about more realistic Dirac and Schrödinger operators?

- What about the case when there is no eigenvalue at the threshold (but a resonance)?
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→ There are some more or less rigorous results, generalisation of our method is work in progress

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- What about the case when there is no eigenvalue at the threshold (but a resonance)?
  → Open!


Thank you for your attention!
Let $z \in \mathbb{C}$ with $\text{Im } z \neq 0$. The full resolvent is $R(z) = (H_\tau(E) - z)^{-1}$. With

$$F(z, E) = E - z - \tau^2 \langle \varphi | r_0(z) \varphi \rangle$$

and $r_0(z) = (-\Delta - z)^{-1}$ we can rewrite the resolvent as

$$R(z) = \begin{pmatrix} r_0(z) & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{F(z, E)} \begin{pmatrix} \tau^2 r_0(z) | \varphi \rangle \langle \varphi | r_0(z) & -\tau r_0(z) | \varphi \rangle \\ -\tau \langle \varphi | r_0(z) & 1 \end{pmatrix}.$$ 

In particular, $\langle \zeta | R(z) \zeta \rangle = 1/F(z, E)$ with $\zeta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then $\lambda = \lambda(E) \in \mathbb{R}$ is the only eigenvalue of $H_\tau(E)$ iff $F(\lambda, E) = 0$. 

The Feshbach formula
An asymptotic expansion of the free resolvent

Denote $z = -\kappa^2$ with $\text{Im} \ z \geq 0$.

Lemma (Asymptotic expansion [Jensen, Kato 1979])

As $\kappa \to 0$ with $\text{Re} \ \kappa \geq 0$,

$$r_0(-\kappa^2) = \sum_{j=0}^{n} \kappa^j G_j + \mathcal{O}(\kappa^{n+1})$$

where $G_j$ are bounded operators (in an appropriate topology) with integral kernel $G_j(x, y) = \frac{(-1)^j}{4\pi j!} |x-y|^{j-1}$.

We have:

$|k|^{-\nu} \hat{\varphi}$ continuous at $k = 0 \Rightarrow \hat{\varphi}(0) = 0 \Rightarrow G_1 \varphi = 0$,

$G_0 \varphi \in L^2$ and $\langle \varphi | G_2 \varphi \rangle_w = -\langle G_0 \varphi | G_0 \varphi \rangle \leq 0$
Eigenvalue and projection

- For $E < E_c = \tau^2 \langle \varphi | G_0 \varphi \rangle_w$, $\lambda(E) < 0$ and the corresponding orthogonal projection onto the eigenspace is obtained from the Feshbach formula for the resolvent by Cauchy’s residue theorem. An eigenvector is

$$
\Psi(E) = \frac{1}{\sqrt{1 + \tau^2 \langle \varphi | r_0^2(\lambda(E)) \varphi \rangle}} \begin{pmatrix}
-\tau r_0(\lambda(E)) | \varphi \rangle \\
1
\end{pmatrix}.
$$

- For $E = E_c > 0$ we have $\lambda(E_c) = 0$ and

$$
\Psi_c = \frac{1}{\sqrt{1 - \tau^2 \langle \varphi | G_2 \varphi \rangle}} \begin{pmatrix}
-\tau G_0 | \varphi \rangle \\
1
\end{pmatrix}.
$$

- For $E > E_c$ there is no eigenvalue for (II) and one eigenvalue embedded in the continuous spectrum for (III).
A fundamental identity

We want to study the overlap $|\langle \Psi_c | U(t'_c, t_c) \Psi_c \rangle|$ for $\eta \downarrow 0$.

Assume that there is $E_a \in (E_c, E_m]$ such that $\inf_{r \in \mathbb{R}} |F(r^2 + i0_+, E_a)| \geq c > 0$. Then there is $t_a \in (t_c, t'_c)$ with $E(\eta t_a) = E_a$. We set $H_a = H_\tau(E_a)$ and $\varepsilon(t) = E_a - E(-\eta t)$. With Dyson’s equation we obtain for the Heisenberg evolution:

$$U(t'_c, t_c) = U(t'_c, t_a) U(t_a, t_c)$$

$$= e^{-i(t'_c - t_c)H_a} + i \int_{t_c}^{t_a} \varepsilon(u)e^{-i(t'_c - u)H_a} |\zeta\rangle \langle \zeta| U(u, t_c) \, du$$

$$+ i \int_{t_a}^{t'_c} \varepsilon(v) U(t'_c, v) |\zeta\rangle \langle \zeta| e^{-i(v - t_c)H_a} \, dv$$

$$- \int_{t_a}^{t'_c} \int_{t_c}^{t_a} \varepsilon(u) \varepsilon(v) U(t'_c, v) |\zeta\rangle \langle \zeta| e^{-i(v - u)H_a} \zeta \rangle \langle \zeta| U(u, t_c) \, dudv$$
A propagation estimate

We find that all terms, but the last one vanish with $\eta$ by standard arguments. For the last term, we need an estimate for $|\langle \zeta | e^{-i(v-u)H_a} \zeta \rangle|$ which is provided by the following

**Proposition (Propagation estimate)**

Assume $w > 9/2$. There is a constant $C > 0$ such that for all $t \in \mathbb{R}$,

$$|\langle \zeta | e^{-itH_a} \zeta \rangle| \leq C (1 + |t|)^{-5/2}.$$ 

Therefore $\lim_{\eta \downarrow 0} |\langle \Psi_c | U(t'_c, t_c) \Psi_c \rangle| = 0$ and the survival probability is zero.