On the adiabatic behaviour of a bound state when diving into the continuous spectrum

joint work with H. D. Cornean, A. Jensen, and Gh. Nenciu

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The problem

An external potential is tuned adiabatically in time such that the system has some isolated bound state for some time. Then the following question arises:
What is the survival probability of the bound state when the corresponding eigenvalue dives into the continuous spectrum for a while during the adiabatic tuning of the external potential?

We answer this question for a rather simple model, however we are confident that the methods applied in the proof can be generalised to more realistic setting.

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This question is closely related to ‘adiabatic pair creation’ [Nenciu 1987; Pickl, Dürr 2008] and ‘memory effects in mesoscopic transport’ [Cornean, Jensen, Nenciu 2014].

The model

We consider a two-channel model of an atom given by a quantum dot coupled to an open scattering channel. On the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3) \oplus \mathbb{C}$ we define the Hamiltonian

$$H_\tau(E) := \begin{pmatrix} -\Delta & 0 \\ 0 & E \end{pmatrix} + \tau \begin{pmatrix} 0 & |\varphi\rangle \\ \langle\varphi| & 0 \end{pmatrix}$$

where

- $E \in \mathbb{R}$ is the dot energy,
- $\tau \in \mathbb{R}$ is the coupling constant and
- the normalised coupling function $\varphi \in L^2(\mathbb{R}^3)$ has to fulfill that $\int_{\mathbb{R}^3} (1 + x^2)^w |\varphi(x)|^2 \mathrm{d}^3x < \infty$ for all $w > 0$ and $|k|^{-\nu}\, \widehat{\varphi}$ is continuous at $k = 0$ for some $\nu \geq 1$

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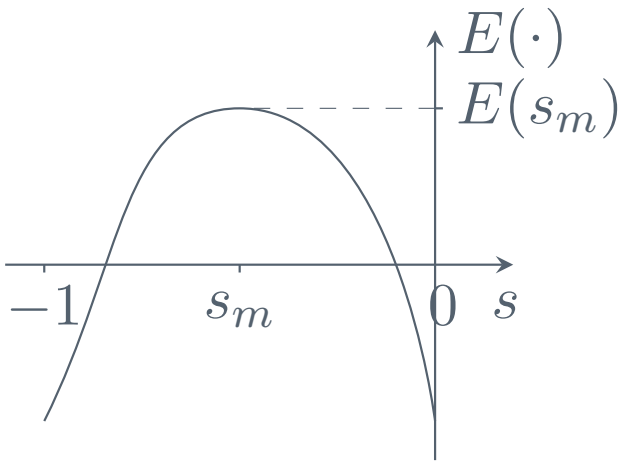
Observe that $\sigma_{ac}(H_\tau(E)) = [0, \infty[$ and $H_\tau(E)$ has at most one eigenvalue $\lambda(E)$.

Tuning of the dot energy

In order to model an adiabatic tuning process the dot energy is varied:
We assume $E(\cdot) \in C^2([-1,0])$ with $E(-1) = E(0) < 0$ and that there is $s_m \in]-1,0[$ such that $E(s_m) > 0$, $E(\cdot)$ is strictly increasing for $s < s_m$ and strictly decreasing for $s > s_m$.



Time evolution and adiabatic limit

The microscopic time is $t := s/\eta$ with $\eta > 0$ and we set $H(t) := H_\tau(E(\eta t))$. Then $\eta \searrow 0$ is called adiabatic limit.

The corresponding Heisenberg evolution $U(t, t_0)$ is the solution of the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}U(t, t_0) = H(t)U(t, t_0)\,, \quad U(t_0, t_0) = \mathbb{1}\,.$$

Theorem

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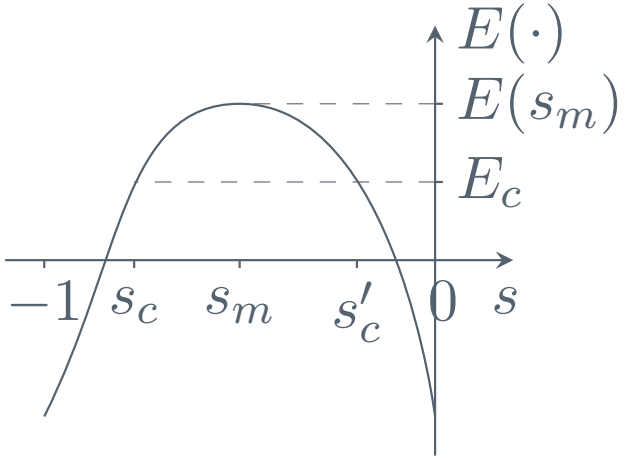
For any τ small enough the following holds:

(I) *There is $0 < E_c < E(s_m)$ such that $\lambda(E_c) = 0$ is an embedded simple eigenvalue with eigenfunction $\Psi_c \in \mathcal{H}$ and projection P_c. Moreover, $\lambda(E) < 0$ is a simple discrete eigenvalue for $E < E_c$ with smooth eigenprojection $P(E)$ and eigenfunction $\Psi(E)$.*

Criticality and different regimes of the dot energy

Note that there are macroscopic times $-1 < s_c < s_m < s_c' < 0$ with $E(s_c) = E(s_c') = E_c$. So $E(\cdot)$ may look as follows:



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(II) *There is a class of coupling functions φ for which $H_\tau(E)$ has purely a.c. spectrum for $E_c < E \leq E(s_m)$ and*

$$\lim_{\eta \searrow 0} |\langle \Psi(E(0)), U(0, -1/\eta)\Psi(E(-1))\rangle|^2 = 0\,.$$

Summary

- Main ingredients in the proof:
  - Feshbach formula
  - asymptotic expansions of the resolvent of the Laplacian: $r_0(-\kappa^2) = \sum_{j=0}^{n} \kappa^j G_j + \mathcal{O}(\kappa^{n+1})$ as $\kappa \to 0$ with $\operatorname{Re}\kappa \geq 0$ [Jensen, Kato 1979]
  - an improved propagation estimate: $|\langle \binom{0}{1} | \mathrm{e}^{-\mathrm{i}tH_a} \binom{0}{1} \rangle| \leq C\,(1+|t|)^{-5/2}$ for some $H_a = H_\tau(E_a)$, $E_a \in]E_c, E(s_m)]$

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- Crucial: Existence of an embedded eigenvalue at the threshold

Outlook

- What about more realistic Dirac and Schrödinger operators?
- What about the case when there is no eigenvalue at the threshold (but a resonance)?

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  $\rightarrow$ Open!

References

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- Cornean, Jensen, Nenciu, Ann. Henri Poincaré 15, 1919–1943 (2014).
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Thank you for your attention!

The Feshbach formula

Let $z \in \mathbb{C}$ with $\operatorname{Im} z \neq 0$. The full resolvent is $R(z) = (H_\tau(E) - z)^{-1}$. With

$$F(z, E) = E - z - \tau^2 \langle \varphi | r_0(z) \varphi \rangle$$

and $r_0(z) = (-\Delta - z)^{-1}$ we can rewrite the resolvent as

$$R(z) = \begin{pmatrix} r_0(z) & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{F(z, E)} \begin{pmatrix} \tau^2 r_0(z)|\varphi\rangle\langle\varphi|r_0(z) & -\tau r_0(z)|\varphi\rangle \\ -\tau\langle\varphi|r_0(z) & 1 \end{pmatrix}.$$

In particular, $\langle \zeta | R(z) \zeta \rangle = 1/F(z, E)$ with $\zeta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Then $\lambda = \lambda(E) \in \mathbb{R}$ is the only eigenvalue of $H_\tau(E)$ iff $F(\lambda, E) = 0$.

An asymptotic expansion of the free resolvent

Denote $z = -\kappa^2$ with $\operatorname{Im} z \geq 0$.

Lemma (Asymptotic expansion [Jensen, Kato 1979])
As $\kappa \to 0$ with $\operatorname{Re} \kappa \geq 0$,

$$r_0(-\kappa^2) = \sum_{j=0}^{n} \kappa^j G_j + \mathcal{O}(\kappa^{n+1})$$

where G_j are bounded operators (in an appropriate topology) with integral kernel $G_j(x,y) = \frac{(-1)^j}{4\pi j!}|x-y|^{j-1}$.

We have:
$|k|^{-\nu}\,\widehat{\varphi}$ continuous at $k = 0 \Rightarrow \widehat{\varphi}(0) = 0 \Rightarrow G_1\varphi = 0$,
$G_0\varphi \in L^2$ and $\langle \varphi | G_2 \varphi \rangle_w = -\langle G_0\varphi | G_0\varphi \rangle \leq 0$

Eigenvalue and projection

- For $E < E_c = \tau^2 \langle \varphi | G_0 \varphi \rangle_w$, $\lambda(E) < 0$ and the corresponding orthogonal projection onto the eigenspace is obtained from the Feshbach formula for the resolvent by Cauchy's residue theorem. An eigenvector is

$$\Psi(E) = \frac{1}{\sqrt{1 + \tau^2 \langle \varphi | r_0^2(\lambda(E)) \varphi \rangle}} \begin{pmatrix} -\tau r_0(\lambda(E)) | \varphi \rangle \\ 1 \end{pmatrix}.$$

- For $E = E_c > 0$ we have $\lambda(E_c) = 0$ and $\Psi_c = \frac{1}{\sqrt{1 - \tau^2 \langle \varphi | G_2 \varphi \rangle}} \begin{pmatrix} -\tau G_0 | \varphi \rangle \\ 1 \end{pmatrix}$.
- For $E > E_c$ there is no eigenvalue for (II) and one eigenvalue embedded in the continuous spectrum for (III).

A fundamental identity

We want to study the overlap $|\langle\Psi_c|U(t_c',t_c)\Psi_c\rangle|$ for $\eta\downarrow 0$.

Assume that there is $E_a\in(E_c,E_m]$ such that $\inf_{r\in\mathbb{R}}|F(r^2+i0_+,E_a)|\geq c>0$. Then there is $t_a\in(t_c,t_c')$ with $E(\eta t_a)=E_a$. We set $H_a=H_\tau(E_a)$ and $\varepsilon(t)=E_a-E(-\eta t)$. With Dyson's equation we obtain for the Heisenberg evolution:

$$
\begin{aligned}
U(t_c',t_c) &= U(t_c',t_a)U(t_a,t_c)\\
&= \mathrm{e}^{-\mathrm{i}(t_c'-t_c)H_a}+\mathrm{i}\int_{t_c}^{t_a}\varepsilon(u)\mathrm{e}^{-\mathrm{i}(t_c'-u)H_a}|\zeta\rangle\langle\zeta|U(u,t_c)\,\mathrm{d}u\\
&\quad+\mathrm{i}\int_{t_a}^{t_c'}\varepsilon(v)U(t_c',v)|\zeta\rangle\langle\zeta|\mathrm{e}^{-\mathrm{i}(v-t_c)H_a}\,\mathrm{d}v\\
&\quad-\int_{t_a}^{t_c'}\int_{t_c}^{t_a}\varepsilon(u)\varepsilon(v)U(t_c',v)|\zeta\rangle\langle\zeta|\mathrm{e}^{-\mathrm{i}(v-u)H_a}\zeta\rangle\langle\zeta|U(u,t_c)\,\mathrm{d}u\mathrm{d}v
\end{aligned}
$$

A propagation estimate

We find that all terms, but the last one vanish with η by standard arguments. For the last term, we need an estimate for $|\langle\zeta|\mathrm{e}^{-\mathrm{i}(v-u)H_a}\zeta\rangle|$ which is provided by the following

Proposition (Propagation estimate)

Assume $w > 9/2$. There is a constant $C > 0$ such that for all $t \in \mathbb{R}$,

$$|\langle\zeta|\mathrm{e}^{-\mathrm{i}tH_a}\zeta\rangle| \leq C\,(1+|t|)^{-5/2}\,.$$

Therefore $\lim_{\eta\downarrow 0}|\langle\Psi_c|U(t_c',t_c)\Psi_c\rangle| = 0$ and the survival probability is zero.