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Material representations in mathematical research practice

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Abstract

Mathematicians' use of external representations constitutes an important focal point in current philosophical attempts to understand mathematical practice. In this paper, we add to this understanding by presenting and analyzing how research mathematicians use and interact with external representations. The empirical basis of the article consists of a qualitative interview study we conducted with active research mathematicians. In our analysis of the empirical material, we primarily used the empirically based frameworks provided by distributed cognition and cognitive semantics as well as the broader theory of cognitive integration as an analytical lens. We conclude that research mathematicians engage in generative feedback loops with material representations, that they use representations to facilitate the use of experiences of handling the physical world as a resource in mathematical work, and that their use of representations is socially sanctioned and enabled. These results verify the validity of the cognitive frameworks used as the basis for our analysis, but also show the need for augmentation and revision. Especially, we conclude that the social and cultural context cannot be excluded from cognitive analysis of mathematicians' use of external representations. Rather, representations are socially sanctioned and enabled in an enculturation process.

Keywords

Mathematical practice; mathematical cognition; embodied cognition; distributed cognition; cognitive semantics; enculturation; external representations; diagrams.

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1.0 Cognitive support in mathematical practice

It is well known that most mathematical practices are crucially dependent on representations, material artifacts and other forms of cognitive support. Although humans (and several other species of animal) have an inborn ability to perform certain basic mathematical activities (such as subitizing and estimating the relative size of collections (Feigenson et al. 2004)), this capacity is extremely limited. If we wish to engage in more than trivial tasks, we are compelled to rely on cognitive strategies that extend and augment our innate mathematical abilities (cf. Frank et al. 2008; Núñez 2009). In this paper, we aim to understand how research mathematicians use and interact with cognitive tools in order to scaffold and support their work with mathematics.

Mathematicians' use of representations for cognitive support has been explored from several different perspectives, such as cognitive science, philosophy, semiotics, and mathematics education. In relation to the literature on mathematical practice we will focus on here, the affordances and cognitive functions of various representational systems have been analyzed (e.g. Clark 1989, p.133; De Cruz and De Smedt 2013; Schlimm and Neth 2008; Zhang and Norman 1995), and historical case studies have pointed out that the choice of representational form can influence the theoretical and conceptual development of mathematics (e.g. Epple 2004; Johansen and Misfeldt 2015; Kjeldsen 2009; Steensen and Johansen 2016). The cognitive function of diagrams has also been explored through the investigation of examples and case studies (De Toffoli 2017; De Toffoli and Giardino 2014; Johansen 2014; Larkin and Simon 1987), where it has been shown that diagrams not only support syntactic transformations but also possess a number of qualitatively different cognitive affordances. Diagrams thus offer support for geometric and manipulative imagination and more intuitive forms of reasoning such as perceptual inference and inferences based in common, everyday experiences. In a case study on proofs in the mathematical field "analysis", Carter (2010) demonstrated that although diagrams may not be present in the published proof, they can play a decisive role in concept formation and in discovering and formulating a proof.

The role played by various writing media, especially blackboards, has been explored in two recent papers (Barany and MacKenzie 2014; Greiffenhagen 2014). The first of these followed the weekly seminars of a group of mathematics researchers; based on their observations, the authors concluded that there is a close connection between inscriptions and mathematical thinking, and that "[m]athematical writing and the mathematical thinking that goes with it are markedly dependent on the media available to the mathematician" (Barany and MacKenzie 2014, p.123). In a similar vein, Misfeldt (2011) investigated the mediation of mathematicians' work as a writing process, and unveiled how the different functions that the creation of written representations serves are supported by different media (e.g. paper and digital media). Following a qualitative study focused on the experience of being a mathematician, Leone Burton (2004) suggested that thinking about mathematics can be organized into three distinct styles: visual, analytical, and conceptual. In relation to material representations, the visual thinking style will often be dynamic and pictorial (Burton 2004, p.55), whereas the analytical style is more focused on algebraic representations and rule-based manipulations. The conceptual thinking style often builds on visualizations, but only in order to support the generation of a mental "image" with a

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weaker relation to the material representations than the visual thinker. Often, mathematicians will use two of the thinking styles; however, very few mathematicians will use all three (Burton 2004, p.60). A similar description of the varied styles of reasoning in mathematics can be found in Thurston (1994).

Apart from the use of writing and drawing, the use of conceptual metaphors as a source of cognitive support has also attracted recent attention. Analysis of historical cases and of the language and gestures used among mathematics teachers and students involved in collaborative problem solving indicates that conceptual metaphors linking mathematical ideas and concepts to body-based experiences play an important role in some mathematical activities (Lakoff and Núñez 2000; Marghetis and Núñez 2010; Núñez 2004).

1.1 Aim and scope

The aim of this paper is to include the perspectives of research mathematicians more directly in the understanding of the role played by cognitive support in mathematics. Drawing from a qualitative interview study we conducted with active research mathematicians, we will analyze the processes that take place when mathematicians use the material world, and especially external representations, as a source of cognitive support. In other words, we will investigate the question: What is the qualitative nature of the processes and interactions that arise when research mathematicians use external representations for cognitive support?

As we answered this question, it became clear to us that even though it is in a sense trivial that mathematical cognition cannot be completely separated from the social context it plays out within, the actual nature of the interactions between this context, the mathematical ideas and the material manifestations involved, are not well understood in the literature. The first research question thus led to another, namely: What is the connection between external representations and the cultural practices and processes that govern their use?

Finally, it should be noted that although we take departure in theories concerning human cognition in general our focus is on the cognitive practices we see in mathematics. It will thus be outside the scope of the paper to compare mathematical cognition to other fields and to discuss whether the cognitive processes we see in mathematics can also be found in other scientific disciplines.

The structure of the rest of the paper will be as follows: After a short description of our theoretical framework (section 1.2) and the empirical method used in the qualitative investigation (section 1.3), we will describe how the mathematicians in our investigation used externalization (section 2.0) and bodily experiences (section 3.0) as sources of cognitive support. Next, we will describe the interaction between the cognitive strategies and the social context (section 4.0) before moving on to a discussion of our main results (section 5.0).

1.2 Theoretical framework

In our analysis, we will take as a point of departure in two well-established theories from current cognitive science: *distributed cognition* and *cognitive semantics*. We have chosen these two approaches because they provide an empirically founded basis for understanding two different and roughly complementary (cf. Johansen 2010) strategies humans use to achieve cognitive support from the material world: We humans distribute cognitive tasks onto external tools and artifacts, and we use basic experiences of interacting with physical reality to structure our abstract thinking.

Although these two strategies are described in slightly different ways and are given different weights by the various positions in the landscape of cognitive theories (sometimes referred to as the 4E: embodied, embedded, enacted, and extended cognition), they are empirically well established and can be seen as fundamental to the contemporary understanding of human cognition. We will furthermore supplement these two theories with elements from *cognitive integration*, which is a later development of distributed cognition and embodied and extended mind theory (Menary 2015). Especially, we will draw on the concepts of *enculturation* and *cognitive niche building*, as explained further below. Hence, our framework builds on distributed cognition, cognitive semantics, and cognitive integration.

The central idea of distributed cognition is the claim that skin and skull are not relevant barriers if we want to understand human cognition. The cognitive processes humans engage in are not always confined to our own brain, but are often distributed over other humans as well as over material objects and artifacts; therefore, if we want to understand human cognition, we need to take its distributed nature into account. In this perspective, material objects and artifacts play a vital role in our cognition, as they often allow cognitive processes to be performed faster, more reliably, or with less effort (Holland et al. 2000; Kirsh and Maglio 1994). Accordingly, the term *cognitive artifacts* has been introduced to designate those physical objects produced to take part in cognitive processes. Mathematical representations can be seen as a prime example of cognitive artifacts, and several of the studies mentioned above are in line with the field of distributed cognition (e.g. Clark 1989; De Cruz and De Smedt 2013; Zhang and Norman 1995).

Cognitive semantics adopts another strategy by focusing on the mapping of inferential structures primarily from material to conceptual domains. The idea here is that human abstract thought is highly dependent on everyday, body-based experiences. According to cognitive semantics, one of the central mechanisms of human cognition consists of the mapping of inferential structures from one domain to another, and we constantly and unconsciously use this ability to map inferential structures from the well-known domain of everyday experience onto more abstract domains (Fauconnier and Turner 2002; Lakoff and Johnson 1980; Lakoff and Núñez 2000). This mechanism is called *conceptual mapping*. Our abstract thought is thus shaped by our bodily presence in the world and our basic interactions with material reality.

Menary's (2015) theory of cognitive integration is a recent development and augmentation of, especially, embedded mind theories (such as distributed cognition). If human cognition is indeed distributed and depends on cognitive artifacts, the skills and practices needed to engage in these distributed processes are of vital importance. These practices are cultural in the sense that they are public and enacted; and using a term from Hutchins (2011), Menary calls the acquisition of such practices *enculturation* (Menary 2015, p.4). He further claims that such cognitive practices "transform our existing biological capacities, allowing us to complete cognitive tasks, in ways that our unenculturated brains and bodies will not allow" (Menary 2015, p. 4). These tools and practices are passed on from one generation to the next, and humans are thus born not only into an ecological niche shaped by our forefathers, but also into a "highly structured cognitive niche that contains not only physical artefact, but also representational systems that embody knowledge (writing systems, number systems, etc.); skills and methods for training and teaching new skills […] and practices for manipulating tools and representations" (Menary 2015, p.6). On the one hand, this niche transforms and sculpts our cognitive capacity (Menary 2015, p.8); on the other, it can be transformed by us through the invention of new tools and practices, although Menary does not explicitly explain how such changes are formed. The cognitive niche is thus used as an analytic tool referring to specific aspects of the environment relevant for understanding a specific mathematical activity from a cognitive perspective.

1.3 Empirical methods

The interview study that forms the empirical foundation of this paper was based on qualitative interviews with 13 active mathematicians of three different nationalities, but all working at Danish universities (see Misfeldt and Johansen (2015) for a thorough description of the empirical investigation). We chose to limit our investigation to researchers of pure mathematics, but we included interviewees from various fields, such as algebra, topology, and analysis. We further selected tenured mathematicians who were mature enough in their careers to have experience with different aspects of mathematical work (e.g. to develop results, publish papers, participate in review and community work and to be part of various forms of research collaborations).

All interviews were between 30 and 60 minutes long, except for one that lasted nearly 100 minutes. The interviews followed a semi-structured approach (Kvale 1996), with prepared questions and clear thematic framing, but were still open-ended enough to allow the actual experiences of the respondents to remain in focus. The interview guide focused on the mathematicians' work and problem-solving processes. More specifically, we asked each respondent the following: *Try to describe how you begin to work on a problem. How do you find a suitable mathematical problem to work on? What are the different stages in your work, and how would you explain the process? What do you imagine, and what do you write down/draw, when you start working on a problem?* We also asked the respondents to provide examples of the notes and sketches they had recently made during their work.

All interviews were recorded, transcribed, and coded in NVivo 10. Photographs or scans of visual representations produced or referred to by the mathematicians in the interviews were also collected.

The data we built our analysis upon were derived from these interviews; more specifically, from interview excerpts coded with "materiality," "inscriptions," and "tools." Our focus on the *cognitive role* means that we used the cognitive frameworks described above as a window for analyzing these data[1].

The direct quotations from the interviews included in this paper have been translated where necessary and edited to remove redundancies that result from direct transcription of spoken language (cf. Johansen and Misfeldt (2014: 45); Misfeldt and Johansen (2015: 360); Johansen and Misfeldt (2016)).

Finally, as our analysis is not based on a large-scale quantitative investigation we cannot claim generalizability in the statistical sense of the word. Rather, as we have taken a qualitative and grounded approach, we have based ourselves in the work

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[1] The aspects of the interviews not discussed in the analysis in this paper mainly relate to the selection of mathematical problems and the choice of representation connected to different communicative contexts. These aspects of the investigation are reported in Misfeldt and Johansen (2015) and Johansen and Misfeldt (2016). We refer to the analysis from these papers where appropriate. A general description of the educational implication of the study has been given in Johansen and Misfeldt (2014). In contrast to the papers previously published, the paper at hand focus on cognitive aspects of the use of representations and seeks to understand the relationship between the cognitive and the social aspects of representation use.

of Kathy Charmaz (2006, p 182) and have evaluated our analysis on the central criteria *credibility* and *resonance*. We claim that our analysis is credible because the data was internally compared in the systematic coding and analysis process, and because our interpretation and analysis provide insight into the specific utterances of the mathematicians in a consistent way. Furthermore, the conclusions we have developed seem to resonate with the experience of other mathematicians we have (informally) discussed our work with as well as with the academic field investigating mathematical practices. In that sense and with these cautions in place we do claim that our analysis and conclusions address common aspects of the mathematical work practice.

2.0 Using and interacting with material representations

We begin our presentation of the empirical material by investigating whether the cognitive practices of research mathematicians are distributed in the sense predicted by the theory of distributed cognition. As we shall see, they are: All our respondents considered writing and drawing to be an integral part of their work process. One respondent (R10), for instance, explained that he would always try to have pen and paper at hand because he could not work without these tools: "Of course you could train yourself to do it mentally," he continued, "but this is not how you work. This is not how *I* work. When I work, I write things down on a piece of paper." Asked directly whether he solved problems mentally, another respondent (R11) asserted: "No, no. You write... You write... You write. You calculate a little on different sheets of paper, but usually, you know, they get thrown out or typed into the computer." In a similar vein, R12 explained that doing mathematics is, to a large extent, a matter of "sitting down with a locked door and a pen and a piece of paper" (and not, as one perhaps might expect, sitting down with a locked door and *thinking*)[2].

Other aspects of the distributed cognition framework were also readily confirmed by the practices we encountered; algebraic symbols were used as epistemic artifacts to allow the mathematician to externalize calculations and simply read off the results (De Cruz and De Smedt 2013), and various external representations were used to stabilize complex conceptual structures, thus serving as material anchors in the sense developed by Hutchins (2005)[3].

Having confirmed the existence of distributed elements in mathematicians' cognitive practices, we can move on to describe the nature of this distribution in detail by exploring two examples in greater depth.

One respondent, R12, used a particular drawing as an important element in his research practice. He redrew the drawing for us (Figure 1), and when we asked him why he needed to draw the figure instead of just imagining it, he explained that he needed to do so in order to keep track of all the elements involved in the problem. Pointing out some of these elements in the drawing, he said:

> R12: If I should imagine this in my head. Well? It is just a small step this, and if I had to imagine this. I want to find out how small this angle can get, so it is an advantage

[2] It should be noted that we do not interpret this to suggest that mathematics is always performed in solitude behind closed doors as an individual activity. Rather, the point is that the thinking process (alone or in groups) most often involves writing and sketching.

[3] As pointed out by one of the anonymous reviewers the role of diagrams in mathematics is very diverse, and diagrams may also in some cases be used as epistemic artefact in a way similar to symbols (see e.g. De Toffoli and Giardino 2014).

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> to have written it somewhere, right? So this is alpha, I've found a limit for. So I start calculating how small it can get. [...] And then we have this angle, we could call it Teta [Goes on adding and describing several other features in the drawing]. And then I begin to have several things to keep track of. And if you have to keep this in your head, things start to go wrong.

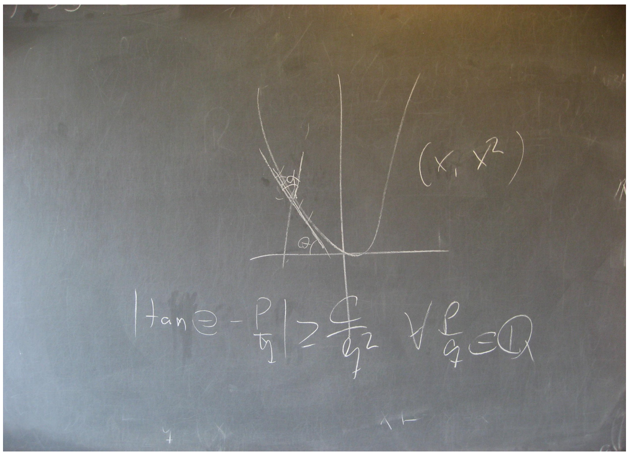

Figure 1: Figure drawn on blackboard by R12.

The drawing, in other words, functions in part as a material anchor used to stabilize an elaborate conceptual structure by capturing ideas and concepts as traces of chalk on a blackboard. This is a clear case of externalizing, and yet, the drawing also played a more active role in the formation of mathematical ideas. R12 went on to describe how the various stages of the representation opened new possible venues for investigation and inference. Some of them were small and incremental, while others were more radical. To exemplify, he reported how he had had a major breakthrough with the problem one night when he started to imagine a particular line segment of the drawing pointing in a different direction. Thus, the drawing—in the form of the geometric properties of an envisioned revision of the diagram on which he had been working—prompted new solution strategies.

The idea that a drawing or a symbolic representation can suggest new moves or prompt ideas you might otherwise not have had forces us to add an important nuance to our understanding of the way mathematical cognition is distributed. The case above cannot be understood as a simple case of cognitive offloading, where an active cognitive agent offloads or distributes a cognitive task onto a passive cognitive tool. Rather, the case marks a clear shift in agency and indicates that the relationship between mathematicians and the cognitive tools they use is dynamic and interactive.

To explore this further, we will investigate another example, in which the respondent (R1) directly addressed this aspect of his use of external representations (more specifically, his use of diagrams):

> R1: I have something in my head, but I need to write it down in order for it to be concrete and correct; that is, sometimes you have a wrong picture in your head.
>
> Interviewer: So it might be that you have a diagram in your head, so to speak, and then...
>
> R1: Yes, not in detail, but... You need to have the structure of the diagram in your head. It doesn't come by itself. What you have in your head is an attempt to structure information. Or the beginning of it. And then you start writing it down, and it might not be exactly what you had expected. You need to change it before it works, or it might not work. That also happens. [...] It is similar to telling it to somebody else.

> When you do that, you also get your thoughts and the information you imagine more structured.

For R1, the diagrams are not externalizations of cut-and-dry ideas and thoughts. Although R1 needs to have an idea before he puts pencil to paper, the externalizing process structures the idea and makes visible the need for elaboration, revision, and change. His use of representations is not a one-way relationship whereby ideas are put into material shape; rather, it is a rich, interactive, and dynamic relationship in which ideas and representations constantly shape each other. R1 does not simply “write things down” or use the external representations as a way to communicate or store information. Rather, the representations constitute an intimate part of his reasoning practice, and his final results are the product of a rich interplay between the mathematician and the representations he has utilized[4]. This dynamic interplay between mathematicians and representations illustrates how firmly the mathematicians are lodged in their cognitive niche. Mathematicians do not use representations in a plug-and-play fashion whereby mathematical ideas and content are developed prior to and independent of representations. Rather, the two are intimately interconnected. The intimacy of the relation between representations and mathematical thinking was seen in all the interviews we conducted, and will be present as an undercurrent in many of the examples we present below. In order to add another layer to our analysis, however, we will move its basis from distributed cognition to cognitive semantics, and explore how mathematicians use everyday experiences as a cognitive tool in their work practice.

3.0 Relating everyday experiences to types of material representations

In connection with the discussion of the representation in Figure 1, R12 also explained that such drawings allowed him to apply everyday experiences of the material world (such as “you cannot take a big thing and put it into a little thing”) to his mathematical work. This suggests that the dynamic relationship emerging from R12’s use of the blackboard not only includes the traces of chalk comprising the representation, but also R12’s previous body-based experiences of handling and manipulating physical objects of the material world.

The use of everyday experience was a common aspect of our respondents’ work practice, and it was expressed in several qualitatively different ways. Some respondents would, for instance, use purely metaphorical expressions. For example, R8 described an object from topology—a spectral sequence—in the following way:

> R8: In topology, you sometimes use these spectral sequences. […] Well, a spectral sequence. You have… You have to imagine a book with pages. And then you have a page 1, which could be the one I have here, and a page 2 […] and so on. Each page is like a bi-graduated object.

So here, we are invited to understand a particular type of mathematical object—spectral sequences—through our preestablished understanding of a class of everyday objects—books.

[4] As one of the anonymous reviewers kindly pointed out such dynamic processes are not confined to mathematical reasoning or reasoning with diagrams, but could also occur in connection to other representations (see e.g. Clark 1998)

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In the case above, the metaphor was established verbally, but in several other cases, the connection between mathematics and everyday experiences was established via a particular external representation. R12's use of everyday experience as mentioned above was, for instance, mediated by and dependent on the particular drawing he used to represent his problem area. In the following, we will describe some of the ways representations are used to mediate the connection between mathematics and everyday experiences.

We will begin with a diagram respondent R5 had been working with (Figure 2).

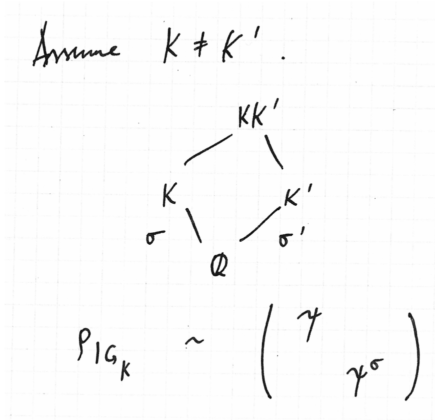

Figure 2: R5's diagrammatic representation of an algebraic number field. The number field is represented by the symbol KK', and its sub-parts by K, K', and Φ.

The diagram represents the structure of a particular type of mathematical set (an algebraic number field). When we asked R5 about the meaning of the diagram, he gave the following description:

> R5: These are very concrete things, right. These are number fields. […]. Up here is something [points to KK'], and it contains this [points to K] and this [points to K'], and both of these contain this [points to Φ].

When we asked whether he actually thought about the situation in the way the diagram describes, he answered:

> R5: I think about it this way. I do. I would imagine these pictures if I thought about it, I would. Because if you don't have these pictures, it becomes very difficult to imagine the situation.

R5's description of number fields as "very concrete things" together with his pointing to the diagram and claim that he "think[s] about it this way" support the interpretation that he conceives of the number fields as if they were physical objects: To him number fields have a location in space, they can be decomposed, and the subparts can be arranged in space in the same structured way that a mechanic might place the gears of an engine he is disassembling on the floor of a garage. The lines in the diagram relate to imagined motions needed in order to reassemble the various parts of the object. If this interpretation is correct, then the diagram reveals the existence of a conceptual map that takes physical objects as a source domain and number fields as a target domain, that is: Composite physical objects are mapped onto number fields, and the subparts of physical objects are mapped onto the subparts of number fields. The diagram itself functions as a material anchor for this conceptual map in the sense that

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it resembles a possible configuration of physical objects that could form a source for the conceptual map (cf. Johansen 2014). However, we do not believe that R5 identifies the mathematical objects with the diagram as such. In this case the diagram and the conceptual map are interconnected: The diagram is only meaningful by way of the conceptual map[5].

R5 reassured us that everything in this and similar diagrams can be expressed in a purely symbolic way. He was, in other words, not forced to use the diagram as his representation; rather, it was a purely pragmatic choice:

> R5: If you don't have these pictures, right, it is actually difficult to imagine the situation. [..] And if you have to visualize the situation in any way, it is difficult without the drawing. You could write it up using symbols, but it would be... If you wrote it up using symbols, then it would actually be much harder to read.

The material diagram thus enabled the mathematician to visualize the situation he was working on and to apply his everyday experiences of decomposing and moving physical objects to his mathematical work. In a similar vein, R6 used a circular diagram (see Figure 3) to represent a permutation.[6] When interpreting the diagram, one has to imagine that the numerals connected by a line inside the circle (such as 1 and 3 in the figure) will change place when the permutation works.

Asked why he used the diagram instead of purely formal thinking, R6 gave the following explanation:

> [The diagram] is a good way to depict things. To think. Perhaps because it is nice to think in pictures. To be able to visualize these manipulations. And then you also get a feeling for what you are looking for. […] If you have a visual way of thinking, it can become very obvious. At least we used them [the diagrams] a lot in our work, also as a way of getting ideas for the proofs. At one point, we found methods that could be used to remove such crossings, right.

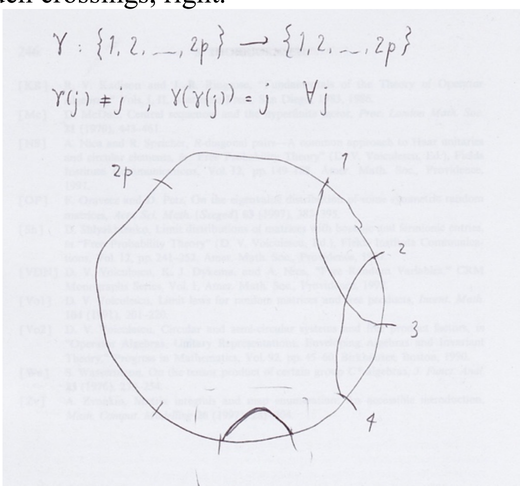

Figure 3: Example of a circular diagram (drawn by R6).

The diagram offers an effective visual organization of information to R6. This can support formal thinking, but there is also a clear connection to physical motions or

[5] Note that by his we do not mean that diagrams cannot support formal thinking. Furthermore, certain diagrams can also be subject to purely formal manipulations.

[6] In mathematics, a permutation is an operation that reorganizes the sequential ordering of the elements of a set.

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gestures. In the quote, R6 is using the diagram as a way to "be able to visualize these manipulations." This indicates that the lines connecting the numerals are to be seen as traces of the movements we would perform if we actually permuted the numeral symbols present in the diagram. So, similar to the example above, the diagram anchors a conceptual map between body-based experiences of manipulating an arrangement of discrete objects and the abstract definition of the mathematical concept "permutation."

In the type of circular diagram used by R6, it is possible that two or more of the lines can cross each other—in Figure 3, for instance, the line connecting 1 and 3 crosses the line connecting 2 and 4. R6 explained that within the theory he was working with, it was common to use the concepts "crossing permutations" and "non-crossing permutations" to distinguish between permutations whose diagrammatic representations have and do not have, respectively, the kind of crossing lines seen in Figure 3. So here we see how particular material features of a diagram—the crossing or non-crossing of lines or movements—can be used to inspire and influence the conceptual development of mathematics; in this case, the distinction between different subclasses of the concept "permutation."

R6 saw the relationship between the formal and diagrammatic definitions of the concept in this way:

> You can define it [the concept] purely formally, right; and this is typically how you do it. But it's just nicer to think or to make such a picture where you can see precisely what it is.

So, although the concepts can be defined formally, they also importantly have a grounded meaning that is connected to particular features of diagrams and the bodily experiences associated with them—with the diagram, "you can see precisely what it is."[7]

The two cases presented sit well within the framework of cognitive semantics: They support the claim that mathematicians actively apply conceptual mapping from body-based experiences to mathematical domains in their research process. The examples furthermore illustrate how the use of this cognitive strategy can be closely tied to the use and development of particular external representations; here, diagrams.

In other examples, the everyday experiences of the mathematicians were not (primarily) exploited by establishing conceptual maps between domains of physical reality and domains of mathematical objects, but rather by the creation of external representations that, in turn, were used and explored as actual physical objects in the sense that purely typographic and topological aspects of the representation tokens on the paper were investigated. Several of our respondents thus used matrices as an important part of their problem-solving and explorative practices. They explained that matrices are useful because mathematically important properties can become visible as patterns and structures in the physical layout of the representations. One respondent (R5), for instance, explained the use of matrices this way:

> You can see ... you can visualize some properties. Let's say for instance that there are only zeroes below the diagonal. That is something you can see in an instant, and you can also form a picture of it in your head, right, but it is also significant mathematically that it looks like this. It's an expression that your map has certain

[7] See also Carter (2010) where the role played by the same type of circular diagrams in finding and constructing a mathematical proof is discussed in depth.

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> properties. You can write them up, but it is much easier to visualize them this way using the matrix form.

R5 considered the guidance he received from matrix patterns to be "crucial" and concluded by explaining that, in some cases, an essential part of the problem-solving process can be accomplished "just by staring at it [a matrix] and finding a pattern."

To illustrate the strength and diversity of the matrix language, we will investigate another example. Respondent R13 was trying to generalize a theorem from square to rectangular matrices[8] and, as an important part of his working process, he would draw outlines of such matrices. In Figure 4, we see one of these drawings. Here, a rectangular matrix is represented as a large rectangle, and three square matrices are represented as smaller squares. With this representation, parts of the mathematical problem are reduced to a more intuitive problem of fitting small squares into a larger rectangle, and the drawing makes it easier for the mathematician to visualize and handle the different ways this can be done.

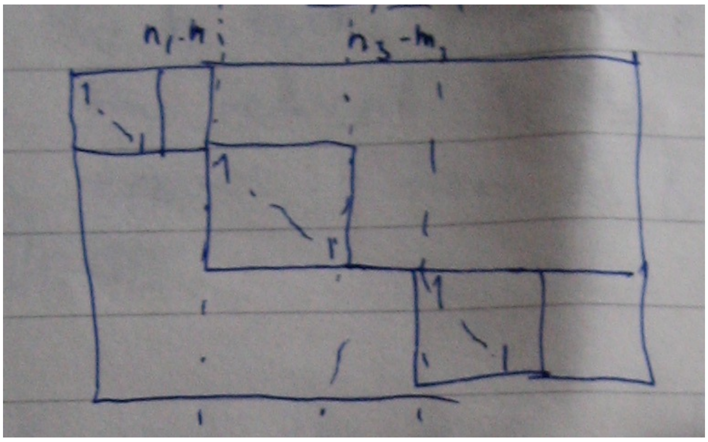

Figure 4: One of R13's drawings. The outer rectangle indicates the outline of a large rectangular matrix. The small squares inside it indicates the outlines and possible positions of three smaller square matrices.

In principle, a matrix can be represented as a list ordered by indices in two dimensions, thus: $\{a_{11}, a_{12}, a_{13}, ..., a_{21}, a_{22}, a_{23}, ...\}$. When we asked R2 whether he used such lists instead of the usual two-dimensional representation, he answered:

> No, no, because part of the problem was to find the right pattern in the matrix we were looking at. So it was important that there were only zeroes in some triangles and that there were some [particular] structures if you looked at the diagonals and so on.

In a similar vein, R5 acknowledged that you could make a list or give an explicit description of the map represented by the matrix:

> But the matrix is a short form [of a list or description]. It is extremely practical because it is something you can keep in your head, right; you can keep this picture in your mind whereas it would be very difficult keeping the explicit description in your mind, right?

---

8 Square matrices have the same number of rows and columns and will consequently typically have a square typographical outline in the usual representational form. In contrast, in rectangular matrices, the number of rows does not coincide with that of the columns.

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So, our respondents were clearly not just focused on creating some external representation that they could point to. Rather, they were interested in the particular visual organization matrices offered to them. There may be several different advantages of using matrices. As R5 explained, the organization offered by matrices is so intuitive that you can easily visualize them and keep them in your mind; but more to the point here, matrices allow mathematicians to tackle mathematical problems using geometric imagination as well as knowledge and skills that are based on the everyday experiences of handling material objects. Due to the matrix representation, the ability to recognize a triangle of zeros in an array of numbers or—in the case of R13 (Fig. 4)—experiences with fitting small square boxes into a larger rectangle can become a valuable resource that the mathematician can exploit in his work practice. Matrices, therefore, are not only tools that allow the mathematician to externally anchor and manipulate mental content. They are also tools used to translate mathematical properties from various fields of mathematics into geometrical features of a representation, and this translation allows the mathematician to activate his or her reservoir of geometric experience obtained by being in, moving through, and handling the geometrically structured physical world.

As these examples illustrate, the two cognitive strategies—externalization and the use of embodied experiences—are interconnected. Representations are used not only to anchor conceptual structures and externalize cognitive tasks, but also to facilitate the use of embodied experiences in mathematical work.

As a final aspect of the complex relationship between representation and experience, the experiences mathematicians had from handling external representations could also be turned into a primary resource of its own, playing a cognitive role similar to embodied physical experiences. Some of our respondents would deliberately aim to build up such experiences. This was particularly clear in the case of R2. He explained how he would often approach a new problem by calculating numerous examples in search for patterns that could be exploited. He was aware that he could have saved time by using the computer to generate examples, but normally he would refrain from doing so; if he had found an interesting pattern using computer calculations, he would even redo the calculations by hand. He was not as much interested in the result of the calculation as in the experience he obtained from handling the formal structure by calculating simple examples. As he explained:

> R2: But for me it is always important to understand things with pen and paper as well [and not only by computer]. In a sense it gets better; I understand it better and remember it better afterwards.

So here we see how an embodied, grounded understanding of a formal structure can be built not only through experiences of a physical nature, but also through direct experiences of manipulating and working with external representations of that structure.

To summarize, mathematicians need material representations in order to work with mathematics. They use these representations not just as a way to store or convey cut-and-dry thoughts and ideas, but also and more importantly as a central aspect of the formation and shaping of their thoughts and ideas. Mathematicians interact with their representations and use them in a dynamic way. Their ideas, thoughts, and concepts both shape and are shaped by the properties of the representations they use. Moreover, external material representations support and facilitate mathematicians' use of embodied experiences in the process of developing mathematical thoughts and ideas. Material representations, in other words, constitute a vital part of the cognitive niche (or niches) of research mathematicians; and for that reason, the choice of representations is of vital importance, not only because different representations support the cognitive process to a lesser or greater extent, but also and more acutely because different representations influence the conceptual development of mathematics in different ways (as we saw in the case of the crossing permutations, Fig. 3) and will allow different aspects of the mathematician's everyday experience to become manifest in mathematical practice (as we, for example, saw in the matrix case, Fig. 4).

4.0 Enculturation: The material and the social

The question of representation choice was not a part of our original research focus (which was: problem solving, the support delivered by artifacts and the mathematical writing process; Misfeldt and Johansen 2015), but the topic emerged in our interviews as an important category in the mathematicians' work practice. Our data show that mathematicians are very aware of their choice of representation, and the analysis presented above suggests at least one reason: Material representations constitute a central cognitive tool to the mathematician; consequently, the cognitive possibilities of the mathematician are closely tied to the affordances of the representation he or she is using. In our interviews, it was, however, apparent that the mathematicians' choice of a particular representation was founded not just in cognitive efficiency, but also in concerns of a more social nature. Therefore, it makes sense to investigate this aspect of their practice using the concept of enculturation.

When our respondents reported the results of their work in mathematical journals, they clearly related to social norms and genre conventions. This relation was partly a subordination in the sense that the respondents needed to balance the cognitive affordances of figural representations (diagrams, figures, graphs, etc.) with journal genres and community norms that promote a more formal approach to mathematics; as indicated above, drawings and diagrams played a significant role in our respondents' everyday practice, yet they were often downplayed in the published work. The respondents agreed that formalizable proofs are necessary, but they regretted the loss of figural representations in the formalization process (cf. Johansen and Misfeldt 2016). For instance, relating to the matrix outline in Figure 4, respondent R13 noted that the figure would not be in the final paper: "But really, it should be included because if I give a talk or try to explain it to anybody, I would draw the figure." Along the same line, another respondent (R10) explained that the pictorial reasoning he had explained to us would not be present in the published papers, "Although," he added, "you actually understand things better, I would say, [with pictures]" (Johansen and Misfeldt 2016, p.265). As these quotes reflect, the mathematicians clearly submitted to cultural norms governing the dissemination of research in mathematical journals, even though they were aware of the potential cognitive disadvantages these norms might have. Hence, mathematicians may feel in conflict with the norms and values governing the use of the cognitive tools present in their particular cognitive niche.

It is perhaps not so surprising that mathematicians submit themselves to genres and norms when they write papers for publication, but we were more surprised to find that mathematicians also felt pressure to subordinate themselves to cultural norms and practices when they were working alone behind the closed doors of their offices. Accordingly, all the mathematicians we spoke with tended to express themselves in preexisting representational languages instead of using idiosyncratic visualizations and representations, even in cases where the preexisting language did not represent the mathematical objects exactly as the mathematicians visualized them and where they could envision ways of representing the objects that came closer to their own conceptual image.

This is seen in the following quote by R5, who continued his comments on the diagram reproduced in Figure 3 with the following observation:

> R5: This is the way *I* think about it. If you take this precise situation in my field, everybody in the whole world would draw it like this. It is simply because a culture has emerged concerning how... It has emerged because a particular mathematician at a particular point in time began drawing it like this.
>
> […]
>
> R5: But I would add, that the pictures you have in your head do not only consist of diagrams and little symbols. There are also other things, right, that I do not draw because I cannot draw them.
>
> Interviewer: So the standard representation, it doesn't capture everything that you imagine?
>
> R5: No no, it doesn't. I would say that it's only the bottom level of it, right.

So, R5 chose to express himself in a preexisting representational language, although it did not fully capture what he imagined. There might be several explanations for this choice. One is cognitive economy. It takes time and effort to create a suitable representation—it might even not be possible—and consequently it is easier to speak using a preexisting one. From this point of view, a particular representational form can be seen as a culturally created resource that is available as part of certain cognitive niches (as explained by Menary 2015). This also means that the mathematicians' decision to use a particular representation not only depends on the cognitive needs of the here and now, but is also historically situated and reflects the outcome of developments and negotiations of the past. To illustrate this, we can briefly return to the drawing made by R12 (Fig. 1). In the figure, we see a parabola drawn in a coordinate system, and although R12 was clearly interested in the geometric properties of the drawing (such as angles between lines, etc.), the problem he was tackling in fact belonged to number theory. The parabola in the drawing is thus a representation of the set of pairs of rational numbers, where the second number is the square of the first ($\{(x, x^2) | x \in \mathbb{Q}\}$). It is not obvious how you get from a set of numbers to geometry, and the representation of the set above as a parabola is only possible with the aid of a sophisticated artifact (the Cartesian coordinate system) and a highly developed theoretical construct (analytic geometry). So, when R12 used a parabola to represent his number theoretical problem, he depended on the cognitive tools left for him to use by past mathematicians, on his and his predecessors' development and appropriation of the cultural practices surrounding these tools, and on the negotiations that rendered these practices and tools acceptable parts of the cognitive niche inhabited by number theorists like himself (cf. Pycior 1997).

In an in-depth analysis of the respondents' choice of problem to work on (published in Misfeldt and Johansen 2015), we further observed that the possibility to resonate with trends in the mathematical community is one of the central criteria for choosing problems and research directions as a mathematician. For example, one researcher (R4) described how his supervisor always told him to think about what other mathematicians are interested in. It takes time and effort to read the work of other mathematicians, and if no one is willing to invest the necessary time in reading your research, "you might as well not have done it" (Misfeldt and Johansen 2015, p. 367).

Hence, mathematicians have a clear interest in getting their work recognized. For both emotional and strategic reasons, they need to get the attention of and resonate with other practitioners in their field. If you use your own idiosyncratic representations, it will be difficult for you to share ideas and collaborate with other mathematicians, not only because they might not understand the conventions of your representational language, but also because they might not understand you at a more fundamental cognitive level. This observation does not rule out the possibility that some mathematicians may succeed in introducing new notation. It only serves to explain why it is the exception rather than the norm.

In a very literal sense, representations can create shared meaning for the inhabitants of a cognitive niche, even in cases where the representations are not used in published work. R6, for instance, did not use the circular diagrams (Fig. 3) in the publication of his results, but because basic concepts, such as crossing permutations, derive their meaning from the diagram, it would be difficult (or at least more difficult) for the mathematicians R6 was collaborating with to understand and resonate with his reasoning if they were using radically different diagrams or no diagrams at all. Exactly because material representations form a deeply integrated part of the mathematical reasoning process, and are not just a tool mathematicians use to convey ideas and thoughts, collaboration in mathematics is facilitated better if the mathematicians express themselves in shared representational languages.

The need to share information with others, to focus on problems that resonate with peers, and to use commonly accepted representations is therefore not only a strategic and career-related consideration. It is also a way to participate in the shared and public cultural practices of a cognitive niche that transcend the psychological and strategic needs of the individual mathematicians, and is thus necessary in order to do mathematics as it is most commonly practiced, i.e., as collaborative work (cf. Burton 2004). The idea that mathematicians enter a cognitive niche and are shaped by the cognitive tools and cultural practices existing in the niche is a very clear example of enculturation in the sense of the concept as used by Menary (2015).

Although the strategic and emotional need for collaboration was strongly present in our interviews, it should be noted that such considerations may not be the only motivating factor in representational choice. Epistemic considerations may be of importance, but sociological factors such as the political and social outlook of the mathematicians may also play a role (e.g. Bloor 1981). It is however beyond the scope of this paper to pursue this perspective.

5.0 An integrated understanding of cognitive mathematical practice

In sum, our analysis of the empirical material has shown that (1) mathematicians interact with their material representations, and during this interactive work process, both their representations and ideas are changed and shaped. (2) Mathematicians use experiences of handling the physical world as a resource in their mathematical work. External, material representations can be used to facilitate the activation of such experiences, and they can constitute new domains for embodied experiences. (3) Mathematical work is done in cognitive niches constituted in part by the tools and cultural practices displayed with external representations. Individual mathematicians are enculturated in the sense that they take over or surrender to the practices that dominate the niche they are working within.

Our analysis furthermore showed that these three aspects of the mathematical practice are interconnected and cannot be understood in isolation; the use of external representations as cognitive tools is interconnected with the use of embodied experiences and both of these strategies play out in a cognitive niche that is governed by social and cultural practices. The analysis thus shows the strength of using the basic concepts and ideas from the theory of cultural integration in the analysis of mathematical cognition. Cultural integration makes it possible to analyze and integrate different aspects of mathematical cognition without falling into eclecticism, and–of special importance for the paper at hand–the theory furthermore offers a view of the role played by mathematical representations that neither considers social and cultural factors to be superfluous nor regards mathematical representations to be a stable resource that is unaffected by culture.

Still, our analysis also augments the theory of cognitive integration as it is presented by Menary (2015). As a first such augmentation Menary seems to focus on the development of cognitive artifacts and the related cognitive practices. As a central result of our analysis we saw that in mathematical cognition the use of external representations in distributed cognitive processes has a direct connection to the use of bodily experiences. The cognitive tools and practices that are found in a cognitive niche are closely tied to the embodied experiences of the cognitive agents exploiting the niche. As we have seen in section 3, this relationship can play out in several different ways; some representations may either by design or by accident facilitate the use of embodied experiences in a mathematical task (e.g. the matrix outline, Fig. 4), while on the other hand embodied experiences may add a crucial dimension of shared meaning to a particular representational form (e.g. the circular diagrams, Fig. 3). These phenomena seem crucial in the data we have presented here, and they should not be subordinated to, but rather integrated in the fruitful analysis of the connection between culture and cognitive tools that has been undertaken in cognitive integration.

As a second augmentation cognitive niches are not fixed and stable. We may inherit a niche and use the tools it offers us, but the niche can also be shaped by us if we introduce new tools or manage to change current practices. Menary mentions symbolic novelty as one way a niche can be expanded, but he is aware that he does not have a good answer to the general question of novelty in mathematics (Menary 2015: 15-16). Clearly, we are also not able to answer the question in full generality, but from our data we can point to more of the mechanisms that are at play in expansion and mathematical novelty. First, novelty in modern mathematics is not confined to new ways of using symbols. As we have seen in both section 2 and 3, diagrams and other forms of pictorial representations can have a generative power in the mathematical practice. Representations can not only suggest new proof strategies (as the drawing in Fig. 1 did to R12), they can also play a generative role in the development of mathematical concepts and in forming new connections to embodied experiences. These roles of representations can have consequences both on the level of individual cognition and on a more general cultural level as they may change mathematics e.g. in terms of central concepts and research agendas (see e.g. Johansen and Misfeldt 2015). Therefore, it is important to understand the processes that develop and change the cognitive niche of mathematics.

So in sum, in this paper we have investigated the question of cognitive support in mathematics by analyzing the roles material representations play in the practice of research mathematicians. Our focus has been on the practices connected to the creation of new mathematical knowledge as experienced and described by research mathematicians. As our main results, we saw that mathematicians depend heavily on external, material representations when they acquire new knowledge, that these representations play a generative role in the mathematical practice, and that the use of material representations is tightly connected to the use of embodied knowledge taking mathematicians' everyday experiences as source domain. The cognitive practice of mathematicians is furthermore clearly situated in and influenced by its social and cultural context, and finally, we have exemplified how central concepts and ideas from the theory of cognitive integration can be used to tie together these diverse aspects of mathematical cognition into a combined whole.

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