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by

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# String cohomology groups of complex projective spaces

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## Abstract

Let  $X$  be a space and write  $LX$  for its free loop space equipped with the action of the circle group  $\mathbb{T}$  given by dilation. We compute the equivariant cohomology  $H^*(LX_{h\mathbb{T}}; \mathbb{Z}/p)$  as a module over  $H^*(B\mathbb{T}; \mathbb{Z}/p)$  when  $X = \mathbb{C}P^r$  for any positive integer  $r$  and any prime number  $p$ . The computation implies that the associated mod  $p$  Serre spectral sequence collapses from the  $E_3$ -page. *MSC*: 55N91; 58E05; 55P35; 18G50

## 1 Introduction

Let  $LX$  be the space of maps from the circle to a space  $X$ . The circle acts on itself by rotation, and this action induces an action of the circle group  $\mathbb{T} = S^1 = SO(2)$  on  $LX$ . (The action extends to an  $O(2)$ -action, but we will not consider the extended action in this paper). The homotopy orbit under the circle action is the space  $E\mathbb{T} \times_{\mathbb{T}} LX$ , which we will also write as  $LX_{h\mathbb{T}}$ . The purpose of this paper is to compute the  $\mathbb{T}$  Borel cohomology with  $\mathbb{F}_p$  coefficients of the free loop space on  $\mathbb{C}P^r$ , that is  $H^*(L\mathbb{C}P^r_{h\mathbb{T}}; \mathbb{F}_p)$ .

There are several motivations for studying this question. In differential topology one studies the spectrum  $TC(M)$ , which is related to the diffeomorphisms of the manifold  $M$ . There is a long exact sequence for the spectrum cohomology of  $TC(M)$ , where the other terms are given by the cohomology and the Borel cohomology of the free loop space on  $M$ . So this result gets us closer to understanding  $TC(\mathbb{C}P^r)$ .

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\*The main part of I. Ottosen's work was done at the University of Aarhus. A smaller part was done at the Institute Mittag-Leffler, Sweden and at the University of Aalborg.

The theory of Chas and Sullivan [CS] constructs algebraic operations on groups related to the free loop space. Part of this structure has been computed in the case  $M = \mathbb{C}P^r$  [CJY]. It would be interesting to write down as much as possible of the Chas-Sullivan structure for the particularly simple example  $M = \mathbb{C}P^r$ . Since one of the groups carrying the structure is the Borel cohomology of the free loops space, a first step is to compute this group.

The usual cohomology of the free loop space has been used to study the existence of closed geodesics [GM], [K1], [SV]. The main problem studied is to decide whether any Riemannian manifold has infinitely many geometrically distinct closed geodesics. This method works better the bigger the cohomology of the free loop space is. For general metrics on  $\mathbb{C}P^r$  one has not succeeded. It is a far out possibility that the knowledge of the equivariant cohomology could possibly be of some help.

We should point out that our computation does not give us much more than the cohomology groups as groups. One can ask for product structure, cohomology operations and higher torsion. We are working on these questions.

The method we use is convoluted, so we should try to give an overview of the computation here.

We know several different spectral sequences that converges to  $H^*(LCP^r_{h\mathbb{T}}; \mathbb{F}_p)$ . The first is the Serre spectral sequence for  $LCP^r \rightarrow LCP^r_{h\mathbb{T}} \rightarrow B\mathbb{T}$ . Since the cohomology of  $LCP^r$  is known, we know the  $E_2$  page of this spectral sequence. The  $d_2$  differential is given if you can compute the action map  $\mathbb{T} \times LCP^r \rightarrow LCP^r$  in the ordinary cohomology of the free loop space. We are able to compute this directly, using methods from [BO2], so we can compute the  $E_3$  page of the Serre spectral sequence. It is a consequence of our final result that there are no further non-trivial differentials, so that the Serre spectral sequence collapses from the  $E_3$  page. One could argue that this is the simplest formulation of our main result. We have no clue as to how to prove this directly using only the Serre spectral sequence.

The second spectral sequence is derived from infinite dimensional Morse theory. For a compact Riemannian manifold  $M$ , the energy function  $E : LM \rightarrow \mathbb{R}$  is a  $\mathbb{T}$ -equivariant map, so we get an equivariant filtration of  $LM$  by energy levels.

For  $M = \mathbb{C}P^r$  with the symmetric space (Fubini–Study) metric, the energy function is a Morse–Bott function. The filtration defines a spectral sequence. We have enough knowledge of the critical manifolds to write down the spectral sequence. This method has previously been used by Klingenberg in [K2] to compute  $H^*(EO(2) \times_{O(2)} LCP^r; \mathbb{F}_2)$ . In this case, Klingenberg proves that the spectral sequence collapses.

In our case, the spectral sequence will eventually collapse, but not from the  $E_1$  page, but only from the  $E_p$ -page. There are definitely non-trivial differentials in the Morse spectral sequence. Not all of these differentials start at the  $\mathbb{T}$  fixed points, that is at the space of constant curves.

The non-triviality of the differentials can be interpreted as a geometrical statement about  $n$ -fold iterated geodesics. It is a consequence of our calculation that on  $\mathbb{C}P^r$  with the standard metric for any  $n$  there is always a curve close to an  $n$ -fold geodesic, such that if the curve moves according to the dynamics given by the gradient of the energy function  $E$ , then it will eventually approximate an  $n - 1$ -fold iterated geodesic.

We study the Morse filtration spectral sequence using localization methods from equivariant homotopy theory. The  $p$ -fold iteration map induces a map

$$H^*(LM_{h\mathbb{T}}; \mathbb{F}_p) \rightarrow H^*(LM_{h\mathbb{T}}^{(p)}; \mathbb{F}_p)$$

where the superscript  $(p)$  means that we have twisted the  $\mathbb{T}$ -action. This map is compatible with the Morse filtration, so it induces a map of spectral sequences. The localization methods prove that under certain technical conditions, after inverting the generator  $u$  of  $H^*(B\mathbb{T}; \mathbb{F}_p)$  this map induces an isomorphism of spectral sequences. It turns out that the Morse spectral sequence for the twisted action collapses, so this isomorphism puts strong conditions on the original spectral sequence.

We apply this to our special case. After some brute calculation in the Morse spectral sequence we obtain the collapse from the  $E_p$  page.

So we have to contend with the first differentials. We know that there are some non-trivial differentials, because the  $E_1$  page of the Morse spectral sequence is strictly bigger than the  $E_3$  term of the Serre spectral sequence. To play these two spectral sequences out against each other, we study the  $\mathbb{T}$ -transfer. This is a degree  $-1$  endomorphism on  $H^*(X)$  defined for any  $\mathbb{T}$ -space  $X$ . What makes it interesting is that we can compute it using homotopy theoretical methods, but it seems hard to calculate the transfer directly in the Morse theory context, primarily because the transfer will not respect the splitting of  $H^*(LCP^r; \mathbb{F}_p)$  into summands corresponding to the quotients of the Morse filtration. So you can translate information about the  $\mathbb{T}$ -transfer into information about how the Morse strata fit together, that is information about the heteroclinic trajectories of the flow on  $LCP^r$  defined by the gradient of the energy function.

A third spectral sequence converging to  $H^*(LCP_{h\mathbb{T}}^r; \mathbb{F}_p)$  is the spectral sequence that we developed in [BO2] and [BO3]. For any space  $X$  the  $E_2$  term of this spectral sequence is algebraically described as non-abelian higher derived functors. If  $X$  is simply connected, it converges towards  $H^*(LX_{h\mathbb{T}}; \mathbb{F}_p)$ . For general  $X$  it seems to be a hard algebraic problem to compute the  $E_2$  page explicitly, but we can handle the algebra for  $\mathbb{C}P^r$ . Preliminary computations show that the size of the  $E_2$  term of this spectral sequence agrees with the size of the Morse spectral sequence, which means that it collapses from the  $E_2$  page.

The individual sections are organized as follows. In sections 2 and 3 we study the space of geodesics on  $\mathbb{C}P^r$ . Except for the constant “geodesics”, this space consists of an infinite number of homeomorphic components. We compute its cohomology by comparing to the cohomology of Grassmann spaces. The  $\mathbb{F}_p$ -cohomology of any  $\mathbb{T}$ -space is a module over  $H^*(B\mathbb{T}; \mathbb{F}_p) \cong \mathbb{F}_p[u]$ , and we determine these module structures. This is very closely related to Klingenberg’s work in [K2]. The main difference is that we are interested in the circle action, while he studies the action of the group  $O(2) \supset \mathbb{T}$ .

In section 4 we compute the equivariant (Borel) cohomology of the spaces of geodesics. The action of  $\mathbb{T}$  is different on the different homeomorphic components. The equivariant cohomology can distinguish at least some of them.

In section 5 we recall some facts about equivariant cohomology theory, in particular the localization theorem, which shows that the relative equivariant cohomology of  $X^{C_p} \subset X$  is annihilated by  $u$ .

In section 6 we study the following abstract construction. If  $X$  is a  $\mathbb{T}$  space, then the fixed points  $X^{C_p}$  is a  $\mathbb{T} \cong \mathbb{T}/C_p$  space. We show that the equivariant cohomology of  $X^{C_p}$  with coefficients in  $\mathbb{F}_p$  does not depend on the action. It is just the tensor product of the ordinary cohomology of  $X$  with  $\mathbb{F}_p[u]$ .

In section 7 we apply the general theory of the two previous section to the free loop space of a manifold. The idea is that the iteration map, which maps a loop to the same loop run through  $p$  times, is really the inclusion of the  $C_p$  fixed points, so that it induces an isomorphism of localized equivariant cohomology and by the results of 6 it is easy to compute this equivariant cohomology.

The localization theorem needs a finite dimensionality condition. This condition is not just technical, but essential. Since the free loop space is infinite dimensional, we have to reduce the situation to a finite dimensional situation. We do this step using Morse theory on the Hilbert manifold version of the free loop space.

In section 8 we analyze how the theory from section 7 influences the Morse spectral sequence. We show that under certain (strong) conditions, the localization of the Morse spectral sequence for equivariant cohomology is a functor in the non-equivariant Morse spectral sequence.

In section 9 we specialize to  $\mathbb{C}P^r$ . We use the results from the sections 3 and 4 to explicitly write down the  $E_1$  page of the Morse spectral sequence, in both the equivariant and the non-equivariant case.

For this space we do know that the non-equivariant spectral sequence collapses, and it follows from section 7 that the localization of the Morse spectral sequence collapses. This does not imply that the un-localized Morse spectral sequence for the equivariant cohomology collapses from the  $E_1$  page, but it is sufficient to prove collapsing from the  $E_p$  page.

Now there is a change of scene to homotopy theoretical methods. In section 11 we compute the  $\mathbb{T}$  transfer map. We do this in the slightly greater generality of spaces whose cohomology are truncated polynomial algebras on one generator. To do the computation, we use the homological methods we developed in [BO2]. The central computation is a computation in non-abelian homological algebra, which we do in section 10.

Even if it is not necessary for the proof of our main result, we study the Serre spectral sequence converging to Borel homology in section 12. In particular we show that our main result implies that the Serre spectral sequence collapses from the  $E_3$  page. We also obtain some information about the product structure in  $H^*(L\mathbb{C}P_{h\mathbb{T}}^r; \mathbb{F}_p)$ , which seems hard to get using only the Morse theory approach.

In section 13 we show how the transfer map forces the existence of non-trivial differentials in the Morse spectral sequence. It seems to be hard to pinpoint these differentials exactly, but we obtain sufficient information to compute the  $E_\infty$  term as a module over  $\mathbb{F}_p[u]$ .

*Notation:* We fix a prime  $p$ . Cohomology groups will always have coefficients in the field  $\mathbb{F}_p$ , unless we explicitly state otherwise. We state otherwise mainly in section 3. The group  $\mathbb{T}$  is the circle group. Spaces are supposed to be homotopy equivalent to CW-complexes, and  $\mathbb{T}$  spaces are supposed homotopy equivalent to  $\mathbb{T}$  CW-complexes.

## 2 Geodesics on $\mathbb{C}\mathbb{P}^r$

Let us fix some notation and recall some basic facts. Consider  $\mathbb{C}^{r+1}$  with the usual Hermitian inner product  $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ , and write  $\langle \cdot, \cdot \rangle$  for its real part. Thus  $\langle \cdot, \cdot \rangle$  is the standard inner product on  $\mathbb{R}^{2r+2} \cong \mathbb{C}^{r+1}$ .

Let  $S^{2r+1}$  be the unit sphere in  $\mathbb{C}^{r+1}$ . Since the left action of  $\mathbb{T}$  on  $S^{2r+1}$  given by  $(z, v) \mapsto zv$  is a free proper action, the Hopf map

$$Q : S^{2r+1} \rightarrow S^{2r+1}/\mathbb{T} = \mathbb{C}\mathbb{P}^r$$

is a smooth submersion. We equip  $\mathbb{C}\mathbb{P}^r$  with the Fubini Study Hermitian metric. This means that for any  $x \in S^{2r+1}$  the map

$$a_x : (\mathbb{C}x)^\perp \subseteq T_x(S^{2r+1}) \xrightarrow{T_x Q} T_{Q(x)}(\mathbb{C}\mathbb{P}^r) \quad (1)$$

is a  $\mathbb{C}$ -linear isometry, where  $(\mathbb{C}x)^\perp = \{v \in \mathbb{C}^{r+1} \mid \langle v, x \rangle_{\mathbb{C}} = 0\}$ . Note that the following identity holds

$$a_{zx}(zv) = a_x(v) \text{ for } z \in S^1. \quad (2)$$

See for example lemma [MT, 14.4] for these facts.

We equip  $S^{2r+1}$  with the Riemannian metric coming from  $\langle \cdot, \cdot \rangle$  and use the real part of the Fubini Study metric as Riemannian metric on  $\mathbb{C}\mathbb{P}^r$ . Let  $H_x$  be the horizontal subspace ie. the orthogonal complement of  $\ker(T_x Q) = \mathbb{R}ix$  in  $T_x(S^{2r+1})$ . Since

$$H_x = \{v \in \mathbb{C}^{r+1} \mid \langle v, x \rangle = \langle v, ix \rangle = 0\} = (\mathbb{C}x)^\perp$$

we see that  $a_x : H_x \rightarrow T_{Q(x)}(\mathbb{C}\mathbb{P}^r)$  is an  $\mathbb{R}$ -linear isometry. So the Hopf map  $Q$  is a Riemannian submersion [GHL, Ch. 2].

We use the Levi-Civita connection corresponding to the standard metrics on  $S^{2r+1}$  and  $\mathbb{C}\mathbb{P}^r$ . Let  $\mathbf{B}_1(\mathbb{C}\mathbb{P}^r)$  denote the set of primitive closed geodesics  $f : [0, 1] \rightarrow \mathbb{C}\mathbb{P}^r$  and let  $\mathbf{B}_q(\mathbb{C}\mathbb{P}^r)$  for integer  $q \geq 1$  be the set of geodesics of the form  $f(qt)$  for  $f \in \mathbf{B}_1(\mathbb{C}\mathbb{P}^r)$ . Note that a geodesic in  $\mathbf{B}_q(\mathbb{C}\mathbb{P}^r)$  has length  $q\pi$  and energy  $q^2\pi^2$ . The free loop space  $L\mathbb{C}\mathbb{P}^r$  defined as the set of closed curves  $g : [0, 1] \rightarrow \mathbb{C}\mathbb{P}^r$  of class  $H^1$  is a Hilbert manifold modeled on  $L(\mathbb{R}^{2r})$  [K3, §1] and by [K2, §1.4] we have that  $\mathbf{B}_q(\mathbb{C}\mathbb{P}^r)$  is a (critical) submanifold of  $L\mathbb{C}\mathbb{P}^r$ .

The left  $\mathbb{T}$  action on  $L\mathbb{C}\mathbb{P}^r$  restricts to an action on  $\mathbf{B}_q(\mathbb{C}\mathbb{P}^r)$ . We view an element in  $\mathbf{B}_q(\mathbb{C}\mathbb{P}^r)$  as a periodic geodesic and then

$$\mathbb{T} \times \mathbf{B}_q(\mathbb{C}\mathbb{P}^r) \rightarrow \mathbf{B}_q(\mathbb{C}\mathbb{P}^r); \quad (e^{2\pi i\theta} * f)(t) = f(t - \theta), \quad \theta \in \mathbb{R}.$$

We will now give an alternative description of  $\mathbf{B}_q(\mathbb{C}\mathbb{P}^r)$ . The description can be found in [K2, §1], but we will write down some explicit maps and add information on the  $\mathbb{T}$ -action. Write  $\mathbf{V}_2(\mathbb{C}^{r+1})$  for the Stiefel manifold of complex *orthonormal* 2-frames in  $\mathbb{C}^{r+1}$ . We can rotate a 2-frame  $(x, v)$  by an angle  $\omega \in \mathbb{R}$  as follows:

$$R(\omega)(x, v) = (\cos(\omega)x + \sin(\omega)v, -\sin(\omega)x + \cos(\omega)v).$$

Write  $\mathbf{P}\mathbf{V}_2(\mathbb{C}^{r+1})$  for the complex projective Stiefel manifold  $\mathbf{V}_2(\mathbb{C}^{r+1})/\text{diag}_2(U(1))$ .

**Definition 2.1.** For integer  $q \geq 1$  we let  $\mathbf{PV}_2^{(q)}(\mathbb{C}^{r+1})$  denote  $\mathbf{PV}_2(\mathbb{C}^{r+1})$  equipped with the well-defined  $\mathbb{T}$ -action

$$\mathbb{T} \times \mathbf{PV}_2(\mathbb{C}^{r+1}) \rightarrow \mathbf{PV}_2(\mathbb{C}^{r+1}); \quad e^{2\pi i\theta} * [x, v] = [R(-q\pi\theta)(x, v)], \quad \theta \in \mathbb{R}.$$

**Proposition 2.2.** *There is a diffeomorphism  $\phi_q : \mathbf{PV}_2^{(q)}(\mathbb{C}^{r+1}) \rightarrow \mathbf{B}_q(\mathbb{CP}^r)$  defined by  $\phi_q([x, v]) = Q \circ c(q, x, v)$  where*

$$c(q, x, v)(t) = \cos(q\pi t)x + \sin(q\pi t)v, \quad 0 \leq t \leq 1.$$

Furthermore,  $\phi_q$  is a  $\mathbb{T}$ -equivariant map.

*Proof.* By [GHL, 2.109 and 2.110] the map  $\phi_q$  is a bijection. A direct computation using the trigonometric addition formulas shows that  $\phi_q$  is  $\mathbb{T}$ -equivariant.  $\square$

Note that the geodesics  $Q \circ c(q, x, v)$  can easily be extended from  $[0, 1]$  to the open interval  $(-\epsilon, 1 + \epsilon)$  for  $\epsilon > 0$ .

**Lemma 2.3.** *There is a  $\mathbb{T}$ -equivariant diffeomorphism*

$$\sigma_q : \mathbf{PV}_2^{(q)}(\mathbb{C}^{r+1}) \rightarrow \widetilde{\mathbf{PV}}_2^{(q)}(\mathbb{C}^{r+1}); \quad [x, v] \mapsto \left[ \frac{x + iv}{\sqrt{2}}, \frac{x - iv}{\sqrt{2}} \right],$$

where  $\widetilde{\mathbf{PV}}_2^{(q)}(\mathbb{C}^{r+1}) = \mathbf{PV}_2(\mathbb{C}^{r+1})$  equipped with the well-defined  $\mathbb{T}$ -action

$$e^{2\pi i\theta} \star [a, b] = [e^{-\pi qi\theta} a, e^{\pi qi\theta} b].$$

*Proof.* We have an automorphism  $\sigma$  of  $\mathbf{V}_2(\mathbb{C}^{r+1})$  as follows:

$$\sigma(x, v) = \frac{1}{\sqrt{2}}(x + iv, x - iv) \quad , \quad \sigma^{-1}(a, b) = \frac{1}{\sqrt{2}}(a + b, -i(a - b)).$$

This automorphism respects the diagonal  $U(1)$  action. Furthermore,

$$\sigma(R(\omega)(x, v)) = \frac{1}{\sqrt{2}}(e^{-i\omega}(x + iv), e^{i\omega}(x - iv)).$$

The result follows.  $\square$

Note that the  $\mathbb{T}$ -action on  $\widetilde{\mathbf{PV}}_2^{(1)}(\mathbb{C}^{r+1})$  is free so by the lemma the  $\mathbb{T}$ -action on  $\mathbf{PV}_2^{(1)}(\mathbb{C}^{r+1})$  is also free.

We have yet another interpretation of the space of primitive geodesics. Let  $\eta : S(\tau(\mathbb{CP}^r)) \rightarrow \mathbb{CP}^r$  denote the sphere bundle of the tangent bundle.

**Proposition 2.4.** *There is a diffeomorphism  $\psi : \mathbf{PV}_2(\mathbb{C}^{r+1}) \rightarrow S(\tau(\mathbb{CP}^r))$  defined by*

$$\psi([x, v]) = (a_x(v))_{Q(x)}.$$

The composition of  $\eta \circ \psi$  is projection on the first factor  $[x, v] \mapsto [x]$ .

*Proof.* The map  $\psi$  is well-defined by (2) and has the inverse  $\psi^{-1}(v_{Q(x)}) = [x, a_x^{-1}(v)]$ . Both  $\psi$  and  $\psi^{-1}$  are smooth maps.  $\square$

We equip  $S(\tau(\mathbb{C}P^r))$  with the  $\mathbb{T}$ -action which makes  $\psi$  an equivariant map. There is a fiber bundle sequence  $\mathbf{P}\mathbf{V}_2^{(1)}(\mathbb{C}^2) \rightarrow \mathbf{P}\mathbf{V}_2^{(1)}(\mathbb{C}^{r+1}) \rightarrow \mathbf{G}_2(\mathbb{C}^{r+1})$  since the Grassmannian is the quotient of the Stiefel manifold by the group  $U(2)$ . Equivalently, we have a fiber bundle

$$S(\tau(\mathbb{C}P^1)) \rightarrow S(\tau(\mathbb{C}P^r)) \rightarrow \mathbf{G}_2(\mathbb{C}^{r+1}).$$

The action of  $\mathbb{T}$  on the total space is free and preserves the fibers. After dividing by it, we get another fiber bundle

$$\Delta_o(\mathbb{C}P^1) \rightarrow \Delta_o(\mathbb{C}P^r) \rightarrow \mathbf{G}_2(\mathbb{C}^{r+1}). \quad (3)$$

Note that we may view a point in  $\Delta_o(\mathbb{C}P^r)$  as the trace of a simple geodesics together with an orientation. Klingenberg [K2] does not consider  $\Delta_o(\mathbb{C}P^r)$ . He divides out by the entire  $O(2)$  action instead and denote the resulting quotient space by  $\Delta(\mathbb{C}P^r)$ .

**Example 2.5.** In the case  $r = 1$  we have that  $\mathbb{C}P^1$  is the standard round sphere  $S^2$  with radius  $\frac{1}{2}$ . A tangent vector  $w \in T_v(S^2)$  is part of a unique ordered basis  $(v, w, v \times w)$ . There is a unique  $A \in SO(3)$  so that  $v = Ae_1, w = Ae_2, v \times w = Ae_3$ . This establishes a diffeomorphism between  $SO(3)$  and  $S(\tau(S^2))$ . The induced action of  $e^{i\theta} \in S^1$  on  $SO(3)$  is given by right multiplication  $A \mapsto A\rho(\theta)$  where  $\rho(\theta)$  is the rotation matrix by angle  $\theta$  around the axis  $e_3$ . The map  $A \mapsto Ae_3$  is invariant under the action. It defines a map  $\Delta_o(\mathbb{C}P^1) \rightarrow S^2$  which is a diffeomorphism.

The example combines with the fibrations to show that we have a diagram of fibration sequences

$$\begin{array}{ccccc} \mathbb{T} & \xlongequal{\quad} & \mathbb{T} & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ SO(3) & \longrightarrow & S(\tau(\mathbb{C}P^r)) & \xrightarrow{q} & \mathbf{G}_2(\mathbb{C}^{r+1}) \\ \downarrow & & \downarrow q' & & \parallel \\ S^2 & \longrightarrow & \Delta_o(\mathbb{C}P^r) & \xrightarrow{\bar{q}} & \mathbf{G}_2(\mathbb{C}^{r+1}) \end{array} \quad (4)$$

Let  $\gamma_2$  be the canonical complex 2-dimensional vector bundle over  $\mathbf{G}_2(\mathbb{C}^{r+1})$  and let  $\mathbb{P}(\gamma_2)$  be the associated projective bundle. A point in the total space of  $\mathbb{P}(\gamma_2)$  is a flag  $V_1 \subset V_2 \subset \mathbb{C}^{r+1}$  where  $V_i$  has complex dimension  $i$ .

**Lemma 2.6.** *The fiber bundle  $S^2 \rightarrow \Delta_o(\mathbb{C}P^r) \rightarrow \mathbf{G}_2(\mathbb{C}^{r+1})$  is isomorphic to the fiber bundle  $\mathbb{C}P^1 \rightarrow \mathbb{P}(\gamma_2) \rightarrow \mathbf{G}_2(\mathbb{C}^{r+1})$ .*

*Proof.* It is enough to show that there is a bundle map  $\widetilde{\mathbf{P}\mathbf{V}}_2^{(1)}(\mathbb{C}^{r+1})/\mathbb{T} \rightarrow \mathbb{P}(\gamma_2)$  which is a fiber wise diffeomorphism. We have a smooth map

$$f : \mathbf{V}_2(\mathbb{C}^{r+1}) \rightarrow \mathbb{P}(\gamma_2) \quad ; \quad (a, b) \mapsto (\mathbb{C}\{a\} \subset \mathbb{C}\{a, b\} \subset \mathbb{C}^{r+1}),$$

which factors through  $\widetilde{\mathbf{P}\mathbf{V}}_2^{(1)}(\mathbb{C}^{r+1})/\mathbb{T}$  since multiplying the generators by units does not change a linear span. It suffices to see that  $\bar{f} : \widetilde{\mathbf{P}\mathbf{V}}_2^{(1)}(\mathbb{C}^{r+1})/\mathbb{T} \rightarrow \mathbb{P}(\gamma_2)$  is a

diffeomorphism when restricted to the fibers over  $\mathbb{C}^2 \subset \mathbb{C}^{r+1}$  since we can then get the result for a general fiber by changing basis. The map of fibers is  $\widetilde{\mathbf{P}\mathbf{V}}_2^{(1)}(\mathbb{C}^2) \rightarrow \mathbb{C}\mathbb{P}^1$  which under the standard identification  $U(2)/(U(1) \times U(1)) \cong \mathbb{C}\mathbb{P}^1$  corresponds to the quotient map

$$\left(\frac{U(2)}{\text{diag}_2(U(1))}\right)/\mathbb{T} \rightarrow \frac{U(2)}{U(1) \times U(1)}.$$

This map is a diffeomorphism because of the following identity of  $2 \times 2$  diagonal matrices:

$$D(e^{2\pi i\alpha}, e^{2\pi i\beta}) = D(e^{\pi i(\alpha+\beta)}, e^{\pi i(\alpha+\beta)})D(e^{\pi i(\alpha-\beta)}, e^{\pi i(\beta-\alpha)}); \quad \alpha, \beta \in \mathbb{R}.$$

□

### 3 The cohomology of spaces of geodesics

The purpose of this section is to compute the cohomology of  $\Delta_o(\mathbb{C}\mathbb{P}^r)$  and  $S(\tau(\mathbb{C}\mathbb{P}^r))$ . It turns out to be convenient to do this computations for cohomology with coefficients in the integers. We first determine the cohomology of the base Grassmann manifold.

**Theorem 3.1.** *Let  $c_1, c_2$  be the two first Chern classes of the canonical bundle  $\gamma_2$  over  $\mathbf{G}_2(\mathbb{C}^{r+1})$ . Then one has*

$$H^*(\mathbf{G}_2(\mathbb{C}^{r+1}); \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2]/(\phi_r, \phi_{r+1}),$$

where  $\phi_i \in \mathbb{Z}[c_1, c_2]$  is the polynomial defined inductively by

$$\phi_0 = 1; \quad \phi_1 = -c_1; \quad \phi_i = -c_1\phi_{i-1} - c_2\phi_{i-2}.$$

*Proof.* According to [B], proposition 31.1, we have an isomorphism

$$H^*(\mathbf{G}_n(\mathbb{C}^{n+m}); \mathbb{Z}) \cong \frac{\mathbb{Z}[x_1, \dots, x_n]^{\Sigma_n} \otimes \mathbb{Z}[x_{n+1}, \dots, x_{n+m}]^{\Sigma_m}}{(\mathbb{Z}[x_1, \dots, x_{n+m}]^{\Sigma_{n+m}})^+} \quad (5)$$

Here the degree of  $x_i$  is 2 for all  $i$  and  $\Sigma_k$  denotes the symmetric group. The plus sign means forming the ideal generated by elements in positive degrees. By [B], 30.2 one sees that this isomorphism comes from the fibration

$$\frac{U(n+m)}{U(n) \times U(m)} \rightarrow \frac{EU(n+m)}{U(n) \times U(m)} \rightarrow \frac{EU(n+m)}{U(n+m)},$$

or equivalently the fibration

$$\mathbf{G}_n(\mathbb{C}^{n+m}) \xrightarrow{j_{n,m}} BU(n) \times BU(m) \xrightarrow{B\rho_{n,m}} BU(n+m),$$

where  $\rho_{n,m} : U(n) \times U(m) \rightarrow U(n+m)$  is the usual block matrix inclusion. The isomorphism above appears as the factorization of  $j_{n,m}^*$  through the positive degree part of the image of  $(B\rho_{n,m})^*$ .

One can describe  $j_{n,m}$  as the composite map

$$j_{n,m} : \mathbf{G}_n(\mathbb{C}^{n+m}) \rightarrow \mathbf{G}_n(\mathbb{C}^{n+m}) \times \mathbf{G}_m(\mathbb{C}^{n+m}) \rightarrow \mathbf{G}_n(\mathbb{C}^\infty) \times \mathbf{G}_m(\mathbb{C}^\infty),$$

where the first map is given by  $V \mapsto (V, V^\perp)$ . Using the fact that the pullback of the canonical bundle  $\gamma_m$  along the map  $\mathbf{G}_n(\mathbb{C}^{n+m}) \rightarrow \mathbf{G}_m(\mathbb{C}^{n+m})$ ,  $V \mapsto V^\perp$  is the orthogonal complement  $\gamma_n^\perp$ , one sees that the Chern classes maps as follows:

$$j_{n,m}^*(c_i(\gamma_n(\mathbb{C}^\infty))) = c_i(\gamma_n) \quad , \quad j_{n,m}^*(c_i(\gamma_m(\mathbb{C}^\infty))) = c_i(\gamma_n^\perp).$$

Put  $c_i = c_i(\gamma_n)$  and  $\bar{c}_j = c_j(\gamma_n^\perp)$ . By the dimension of the bundles we have that  $c_i = 0$  for  $i > n$  and  $\bar{c}_j = 0$  for  $j > m$ . Since  $\gamma_n \oplus \gamma_n^\perp \cong \epsilon^{n+m}$  we also have that  $\sum_{i+j=k} c_i \bar{c}_j = 0$  for  $k > 0$ . This gives us a quotient map into the cohomology of the Grassman manifold, which by (5) is an isomorphism

$$H^*(\mathbf{G}_n(\mathbb{C}^{n+m}); \mathbb{Z}) \cong \mathbb{Z}[c_i, \bar{c}_j | i, j > 0] / (c_i | i > n) + (\bar{c}_j | j > m) + \left( \sum_{i+j=k} c_i \bar{c}_j | k > 0 \right).$$

In the special case  $n = 2$  and  $m = r - 1$  we have that

$$\begin{aligned} & \mathbb{Z}[c_i, \bar{c}_j | i, j > 0] / (c_i | i > 2) + \left( \sum_{i+j=k} c_i \bar{c}_j | k > 0 \right) \cong \\ & \mathbb{Z}[c_1, c_2, \bar{c}_1, \bar{c}_2, \dots] / (\bar{c}_k - \phi_k(c_1, c_2) | k > 0) \end{aligned}$$

Dividing by  $(\bar{c}_j | j > r - 1)$  we see that

$$H^*(\mathbf{G}_2(\mathbb{C}^{r+1}); \mathbb{Z}) \cong \mathbb{Z}[c_1, c_2] / (\phi_i | i \geq r).$$

However, it follows from the inductive definition that the  $\phi_i$  is contained in the ideal generated by  $\phi_r, \phi_{r+1}$  for  $i \geq r$  and this finishes the proof.  $\square$

We now turn to the projective bundle over the Grassmann space.

**Theorem 3.2.** *Let  $\pi : \mathbb{P}(\gamma_2) \rightarrow \mathbf{G}_2(\mathbb{C}^{r+1})$  be the projective bundle of the canonical bundle  $\gamma_2$ . There is an isomorphism*

$$H^*(\mathbb{P}(\gamma_2); \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2] / (Q_r, Q_{r+1}),$$

where  $x_1$  and  $x_2$  have degree 2 and

$$Q_k(x_1, x_2) = \sum_{i=0}^k x_1^i x_2^{k-i} \in \mathbb{Z}[x_1, x_2].$$

We also have that  $\pi^*(c_1(\gamma_2)) = x_1 + x_2$  and  $\pi^*(c_2(\gamma_2)) = x_1 x_2$ .

*Proof.* We use [H], 17.2.5 and 17.2.6. Let  $\lambda \rightarrow \mathbb{P}(\gamma_2)$  be the sub bundle of the pullback  $\pi_*(\gamma_2)$  defined such that a point in the total space of  $\lambda$  is a pair  $(\{V_1 \subset V_2 \subset \mathbb{C}^{r+1}\}, v)$  where the complex dimension of  $V_i$  is  $i$  and  $v \in V_1$ . Let  $\bar{\lambda}$  be the conjugate bundle of  $\lambda$ . Then, we have

$$H^*(\mathbb{P}(\gamma_2); \mathbb{Z}) \cong H^*(\mathbf{G}_2(\mathbb{C}^{r+1}); \mathbb{Z})[c_1(\bar{\lambda})] / (c_1(\bar{\lambda})^2 + c_1(\gamma_2)c_1(\bar{\lambda}) + c_2(\gamma_2)).$$

Combining this with theorem 3.1 we see that  $H^*(\mathbb{P}(\gamma_2); \mathbb{Z})$  is generated by the three classes  $\pi^*(c_1(\gamma_2))$ ,  $\pi^*(c_2(\gamma_2))$  and  $c_1(\lambda)$  with the three relations

$$\begin{aligned}\phi_r(\pi^*(c_1(\gamma_2)), \pi^*(c_2(\gamma_2))) &= 0, \\ \phi_{r+1}(\pi^*(c_1(\gamma_2)), \pi^*(c_2(\gamma_2))) &= 0, \\ \pi^*(c_2(\gamma_2)) &= c_1(\lambda)\pi^*(c_1(\gamma_2)) - c_1(\lambda)^2.\end{aligned}$$

Define  $x_1 = c_1(\lambda)$  and  $x_2 = \pi^*(c_1(\gamma_2)) - c_1(\lambda)$ . Using the last of the above equations to eliminate  $\pi^*(c_2(\gamma_2))$ , we get that  $H^*(\mathbb{P}(\gamma_2); \mathbb{Z})$  is generated by the classes  $x_1$  and  $x_2$  subject to the relations we get by substituting  $\pi^*(c_1(\gamma_2))$  and  $\pi^*(c_2(\gamma_2))$  expressed by  $x_1$  and  $x_2$  into  $\phi_r$  and  $\phi_{r+1}$ . Note that  $\pi^*(c_1(\gamma_2)) = x_1 + x_2$  and that

$$\pi^*c_2(\gamma_2) = c_1(\lambda)\pi^*(c_1(\gamma_2)) - c_1(\lambda)^2 = x_1(x_1 + x_2) - x_1^2 = x_1x_2.$$

So we put  $Q_r(x_1, x_2) = (-1)^r \phi_r(x_1 + x_2, x_1x_2)$ . The two relations are polynomials  $Q_r, Q_{r+1}$  in  $x_1$  and  $x_2$ . The polynomials  $Q_i$  are given inductively by substituting into the inductive definition of  $\phi_i$ . The inductive formula becomes

$$Q_0 = 1, \quad Q_1 = x_1 + x_2, \quad Q_i = (x_1 + x_2)Q_{i-1} - x_1x_2Q_{i-2}.$$

It is easy to check that the polynomials

$$Q_i(x_1, x_2) = \sum_{j=0}^i x_1^j x_2^{i-j} = (x_1^{i+1} - x_2^{i+1}) / (x_1 - x_2)$$

satisfy this inductive definition. □

Because of lemma 2.6 we have the following result:

**Corollary 3.3.** *There is an isomorphism  $H^*(\Delta_o(\mathbb{C}P^r); \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2] / \{Q_r, Q_{r+1}\}$ . The induced of the stabilization map  $\Delta_o(\mathbb{C}P^r) \hookrightarrow \Delta_o(\mathbb{C}P^{r+1})$  maps  $x_1$  and  $x_2$  in  $H^*(\Delta_o(\mathbb{C}P^{r+1}); \mathbb{Z})$  to the classes with the same names in  $H^*(\Delta_o(\mathbb{C}P^r); \mathbb{Z})$ .*

*Proof.* The statement about the stability of the classes  $x_1$  and  $x_2$  follows from the fact that they are defined using Chern classes of the bundles  $\gamma_2$  and  $\lambda$ . These bundles pull back to bundles with the same names. □

We note that it is very easy to check whether a polynomial is in the ideal generated by  $Q_r$  and  $Q_{r+1}$ .

**Lemma 3.4.** *Let  $P = \sum_{i=0}^m p_i x_1^i x_2^{m-i}$  be a homogeneous polynomial of degree  $m$ . Then  $P$  is contained in the ideal  $I = (Q_r, Q_{r+1})$  if and only if  $p_i = p_j$  for all  $i, j$  such that  $m - r \leq i \leq r$  and  $m - r \leq j \leq r$ .*

*Proof.* Since  $x_1^{r+1} = Q_{r+1} - x_1Q_r$ , and similarly for  $x_2^r$ , the monomials  $x_1^i x_2^{m-i}$  are contained in  $I$  if  $0 \leq i \leq m - r - 1$  or  $r + 1 \leq i \leq m$ . It follows that  $P \in I$  if and only if  $\sum_{m-r \leq i \leq r} p_i x_1^i x_2^{m-i} \in I$ . If  $p_{m-r} = p_{m-r+1} = \dots = p_r$ , we see that  $P$  is congruent to  $Q_m$  modulo  $I$ , and  $P \in I$ .

To see that this condition also is necessary, let  $J \subset I$  be the ideal generated by  $x_1^{r+1}$  and  $x_2^{r+1}$ . Then  $x_1Q_k = x_2Q_k = Q_{k+1} \pmod{J}$ , so  $I$  is generated as an abelian group by  $J$  together with  $Q_k$  for  $k \geq r$ . So for any homogeneous polynomial  $P \in I$  of degree  $m$ , there is a  $\lambda \in \mathbb{Z}$  such that  $P - \lambda Q_m \in J$ . This completes the proof. □

Next we consider the space of parametrized geodesics  $S(\tau(\mathbb{C}P^r))$ . We consider two fibration, namely the middle horizontal fibration in (4) and the obvious spherical fibration:

$$SO(3) \rightarrow S(\tau(\mathbb{C}P^r)) \xrightarrow{q} \mathbf{G}_2(\mathbb{C}^{r+1}), \quad S^{2r-1} \rightarrow S(\tau(\mathbb{C}P^r)) \xrightarrow{\eta} \mathbb{C}P^r.$$

Let  $\gamma_1$  be the canonical bundle over  $\mathbb{C}P^r$ , so that

$$H^*(\mathbb{C}P^r; \mathbb{Z}) = \mathbb{Z}[c_1(\gamma_1)]/(c_1(\gamma_1)^{r+1}).$$

We have the following result (compare with the remark in the introduction of [AGMP]):

**Lemma 3.5.** *Put  $t = \eta^*(c_1(\gamma_1))$ . There is a class  $\bar{\sigma} \in H^{2r+1}(S(\tau(\mathbb{C}P^r)); \mathbb{Z})$  such that*

$$H^*(S(\tau(\mathbb{C}P^r)); \mathbb{Z}) = \mathbb{Z}[t, \bar{\sigma}]/(t^{r+1}, \bar{\sigma}^2, (r+1)t^r, t^r \bar{\sigma}).$$

*Furthermore,  $q^*(c_1(\gamma_2)) = 2t$  and  $q^*(c_2(\gamma_2)) = t^2$ .*

*Proof.* We consider the Serre spectral sequence for the spherical fibration  $\eta$ :

$$E_2^{**} = \mathbb{Z}[c_1(\gamma_1), \sigma]/(c_1(\gamma_1)^{r+1}, \sigma^2) \Rightarrow H^*(S(\tau(\mathbb{C}P^r)); \mathbb{Z}),$$

where  $\deg(\sigma) = 2r - 1$ . For dimensional reasons, the only possibly non-trivial differential is  $d_{2r}(\sigma)$  which is given by the Euler characteristic of  $\mathbb{C}P^r$ .  $d_{2r}(\sigma) = e(\eta) = (r+1)c_1(\gamma_1)^r$ . For this, cf [MiSt] corollary 11.12 and theorem 12.2. Let  $\bar{\sigma}$  be a class representing  $c_1(\gamma_1)\sigma$ . There cannot be either further differentials nor extensions for dimensional reasons. This finishes the cohomology computation.

*Claim:* There is a bundle isomorphism  $q^*(\gamma_2) \cong \eta^*(\gamma_1) \oplus \eta^*(\gamma_1)$ .

To prove the claim, we use the diffeomorphism  $\phi$  from lemma 2.4. We have a commutative diagram

$$\begin{array}{ccc} & & \mathbb{C}P^r \\ & \nearrow pr_1 & \uparrow \eta \\ \mathbf{P}V_2(\mathbb{C}^{r+1}) & \xrightarrow[\cong]{\phi} & S(\tau(\mathbb{C}P^r)) \\ & \searrow p & \downarrow q \\ & & \mathbf{G}_2(\mathbb{C}^{r+1}), \end{array}$$

where  $p$  is the canonical projection and  $pr_1$  is given by projection on the first factor. We also have the projection on the second factor  $pr_2$  and  $pr_1^*(\gamma_1) \cong pr_2^*(\gamma_1)$ . So it suffices to show that  $p^*(\gamma_2) \cong pr_1^*(\gamma_1) \oplus pr_2^*(\gamma_1)$ . But this isomorphism is obvious.

We can now compute the total Chern class as follows:

$$1 + c_1(q^*(\gamma_2)) + c_2(q^*(\gamma_2)) = (1 + c_1(\eta^*(\gamma_1)))^2 = 1 + 2c_1(\eta^*(\gamma_1)) + (c_1(\eta^*(\gamma_1)))^2.$$

The final statement of the lemma follows from the naturality of the Chern classes.  $\square$

*Remark 3.6.* Similar to the integral computation we can compute cohomology with coefficients in  $\mathbb{F}_p = \mathbb{Z}/p$  for  $p$  prime. Let  $x$  be the mod  $p$  reduction of  $t$ . If  $p \mid (r+1)$ , there is a class  $\sigma \in H^{2r-1}(S(\tau(\mathbb{C}P^r)); \mathbb{F}_p)$  such that

$$H^*(S(\tau(\mathbb{C}P^r)); \mathbb{F}_p) \cong \mathbb{F}_p[x, \sigma]/(x^{r+1}, \sigma^2).$$

Similarly, if  $p \nmid (r+1)$ , there is a class  $\bar{\sigma} \in H^{2r+1}(S(\tau(\mathbb{C}P^r)); \mathbb{F}_p)$  such that

$$H^*(S(\tau(\mathbb{C}P^r)); \mathbb{F}_p) \cong \mathbb{F}_p[x, \bar{\sigma}]/(x^r, \bar{\sigma}^2).$$

**Corollary 3.7.** *The composition  $S(\tau(\mathbb{C}P^r)) \xrightarrow{q'} \mathbf{\Delta}_o(\mathbb{C}P^r) \xrightarrow{\psi} \mathbb{P}(\gamma_2)$  of the quotient map with the diffeomorphism satisfies*

$$(\psi \circ q')^*(x_1) = (\psi \circ q')^*(x_2) = \eta^*(c_1(\gamma_1)).$$

*The kernel of the map  $(\psi \circ q')^*$  is a copy of the integers, generated by  $x_1 - x_2$ .*

*Proof.* Put  $f = \psi \circ q'$  and consider the composition

$$S(\tau(\mathbb{C}P^r)) \xrightarrow{f} \mathbb{P}(\gamma_2) \xrightarrow{\pi} \mathbf{G}_2(\mathbb{C}^{r+1}),$$

which equals the map  $q$  from (4). By lemma 3.5 we have that

$$(\pi \circ f)^*(c_1(\gamma_2)) = 2t \quad , \quad (\pi \circ f)^*(c_2(\gamma_2)) = t^2,$$

where  $t = \eta^*(c_1(\gamma_1))$ . According to 3.2 we also have that  $\pi^*(c_1(\gamma_2)) = x_1 + x_2$  and  $\pi^*(c_2(\gamma_2)) = x_1 x_2$ . So we conclude that

$$f^*(x_1) + f^*(x_2) = 2t \quad , \quad f^*(x_1)f^*(x_2) = t^2.$$

If  $r \geq 3$  one has  $H^i(S(\tau(\mathbb{C}P^r)); \mathbb{Z}) \cong \mathbb{Z}$  for  $i = 2, 4$  generated by  $t$  and  $t^2$  respectively. So these equations have the unique solution  $f^*(x_1) = f^*(x_2) = t$  which proves the corollary in this case. If  $i \leq 2$ , we use the standard inclusion  $\mathbb{C}^i \subset \mathbb{C}^3$  together with naturality to get the desired result.  $\square$

## 4 Borel cohomology of spaces of geodesics

We consider the space of parametrized geodesics  $S(\tau(\mathbb{C}P^r))$  with the free  $\mathbb{T}$ -action. Let  $S(\tau(\mathbb{C}P^r))^{(n)}$  denote  $S(\tau(\mathbb{C}P^r))$  where we have twisted the  $\mathbb{T}$ -action by composing it with the  $n$ th power map  $\lambda_n : \mathbb{T} \rightarrow \mathbb{T}; z \mapsto z^n$ . Write  $C_n \subseteq \mathbb{T}$  for the cyclic group of order  $n$ . The map  $\lambda_n$  passes to an isomorphism  $\mathbb{T}/C_n \rightarrow \mathbb{T}$  with inverse  $R_n$  given by sending  $z$  to an  $n$ th root of  $z$ . We write  $BC_n$  for  $E\mathbb{T}/C_n$  with  $\mathbb{T}$ -action through  $R_n : \mathbb{T} \rightarrow \mathbb{T}/C_n$ .

The action of  $C_n$  on  $S(\tau(\mathbb{C}P^r))^{(n)}$  is trivial. This makes it possible to consider  $S(\tau(\mathbb{C}P^r))^{(n)}$  as a  $\mathbb{T}/C_n$  space. We get homeomorphisms as follows:

$$E\mathbb{T} \times_{\mathbb{T}} S(\tau(\mathbb{C}P^r))^{(n)} \cong E\mathbb{T}/C_n \times_{\mathbb{T}/C_n} S(\tau(\mathbb{C}P^r))^{(n)} \cong BC_n \times_{\mathbb{T}} S(\tau(\mathbb{C}P^r)).$$

We can write  $\mathbb{T} \rightarrow S(\tau(\mathbb{C}P^r)) \rightarrow \Delta_o(\mathbb{C}P^r)$  as a pullback of the universal  $\mathbb{T}$ -bundle  $\mathbb{T} \rightarrow E\mathbb{T} \rightarrow B\mathbb{T}$  along a map  $f : \Delta_o(\mathbb{C}P^r) \rightarrow B\mathbb{T}$ . By this pullback diagram we get a map of associated fibration sequences:

$$\begin{array}{ccccc} BC_n & \longrightarrow & BC_n \times_{\mathbb{T}} S(\tau(\mathbb{C}P^r)) & \xrightarrow{\rho} & \Delta_o(\mathbb{C}P^r) \\ \parallel & & \downarrow & & \downarrow f \\ BC_n & \longrightarrow & B\mathbb{T} & \longrightarrow & B\mathbb{T}. \end{array} \quad (6)$$

Note that we have used the homotopy equivalence  $pr_1 : BC_n \times_{\mathbb{T}} E\mathbb{T} \xrightarrow{\cong} B\mathbb{T}$  in the middle of the lower part of the diagram. We can now compute the cohomology of the homotopy orbit spaces.

**Theorem 4.1.** *For any prime  $p$  one has*

$$H^*(E\mathbb{T} \times_{\mathbb{T}} S(\tau(\mathbb{C}P^r)))^{(n)} \cong \begin{cases} \mathbb{F}_p[x_1, x_2]/(Q_r, Q_{r+1}), & p \nmid n, \\ \mathbb{F}_p[u, x, \sigma]/(x^{r+1}, \sigma^2), & p \mid n, p \mid (r+1), \\ \mathbb{F}_p[u, x, \bar{\sigma}]/(x^r, \bar{\sigma}^2), & p \mid n, p \nmid (r+1), \end{cases}$$

where  $u, x, x_1, x_2$  have degree 2 and  $\deg(\sigma) = 2r - 1$ ,  $\deg(\bar{\sigma}) = 2r + 1$ .

*Proof.* If  $p$  does not divide  $n$ , then the mod  $p$  cohomology of  $BC_n$  equals that of a point, and by an obvious spectral sequence argument the projection map  $\rho$  induces an isomorphism in  $\mathbb{F}_p$  cohomology. The result follows by corollary 3.3.

Now assume that  $p$  does divide  $n$ . Then one has  $H^*(BC_n) \cong \mathbb{F}_p[v, w]/I_{n,p}$  where  $I_{n,p}$  is the ideal defined by  $I_{n,p} = (v^2 - w)$  if  $p = 2$  and  $4 \nmid n$  and  $I_{n,p} = (v^2)$  otherwise. The degrees are  $\deg(v) = 1$  and  $\deg(w) = 2$ .

The two fibrations of diagram (6) each give a spectral sequences, and the vertical maps induce a map of spectral sequences. Let us denote the spectral sequence derived from the lower row of the diagram by  $E_r(I)$ , the one derived from the upper row by  $E_r(II)$  and the map by  $f^* : E_r(I) \rightarrow E_r(II)$ .

We have that  $E_2(I) = \mathbb{F}_p[u] \otimes \mathbb{F}_p[v, w]/I_{n,p}$  where the only non-trivial differential is  $d_2$ , which is determined by  $d_2w = 0$ ,  $d_2v = u$  and the product structure. This follows since the inclusion of the fiber  $BC_n \rightarrow B\mathbb{T}$  is given by the inclusion  $C_n \subseteq \mathbb{T}$  and the spectral sequence converges to  $H^*(B\mathbb{T})$  such that  $E_\infty(I) = \mathbb{F}_p[w]$ .

We compute the  $E_2$  page of the other spectral sequence:

$$E_2(II) \cong H^*(\Delta_o(\mathbb{C}P^r)) \otimes \mathbb{F}_p[v, w]/I_{n,p} \Rightarrow H^*(BC_n \times_{\mathbb{T}} S(\tau(\mathbb{C}P^r))).$$

The class  $w$  is a permanent cycle, since it is the image of a permanent cycle in the spectral sequence  $E_r(I)$ . The whole spectral sequence is generated by the permanent cycle  $w$  together with the classes in  $E_2^{*,0}(II)$  and  $E_2^{*,1}(II)$ .

But this implies, for formal reasons, that the only possible non-trivial differential is  $d_2$ . Using the product structure we also see that this non-trivial differential is determined by  $d_2(v)$ , which by naturality equals the image  $f^*(d_2(v)) = f^*(u)$ .

We claim that  $f^*(u) = \lambda(x_1 - x_2)$  for some  $\lambda \in \mathbb{F}_p \setminus \{0\}$ . To see this, we consider the fibration sequence

$$S(\tau(\mathbb{C}P^r)) \longrightarrow \Delta_o(\mathbb{C}P^r) \xrightarrow{f} B\mathbb{T}.$$

We have already investigated the involved spaces. According to lemma 3.5 and corollary 3.3, the corresponding spectral sequence for integral cohomology has the form

$$E_2 = \mathbb{Z}[u] \otimes \mathbb{Z}[t, \bar{\sigma}] / J \Rightarrow \mathbb{Z}[x_1, x_2] / (Q_r, Q_{r+1}),$$

where  $J = (t^{r+1}, \bar{\sigma}^2, (r+1)t^r, t^r \bar{\sigma})$ . In the notation, we do not distinguish between the classes  $x_1, x_2 \in H^2(\mathbb{P}(\gamma_2); \mathbb{Z})$  and their pull-backs under  $\psi: \Delta_o(\mathbb{C}P^r) \xrightarrow{\psi} \mathbb{P}(\gamma_2)$

Consider the case  $r \geq 2$  first. We have a short exact sequence

$$0 \longrightarrow H^2(B\mathbb{T}; \mathbb{Z}) \xrightarrow{f^*} H^2(\Delta_o(\mathbb{C}P^r); \mathbb{Z}) \xrightarrow{(q')^*} H^2(S(\tau(\mathbb{C}P^r)); \mathbb{Z}) \longrightarrow 0.$$

According to corollary 3.7 the kernel of  $(q')^*$  is the free group generated by  $x_1 - x_2$ . It follows that  $f^*(u) = \pm(x_1 - x_2)$ . Using naturality on the canonical inclusion  $\Delta_o(\mathbb{C}P^1) \subset \Delta_o(\mathbb{C}P^2)$ , which is a  $\mathbb{T}$ -map, we see that this formula is also true for  $r = 1$ . So we have proved the claim.

We return to cohomology with  $\mathbb{F}_p$  coefficients. Let  $K_r$  and  $C_r$  be the kernel and the cokernel of multiplication by the element  $x_1 - x_2$  on  $H^*(\Delta_o(\mathbb{C}P^r))$ . Then

$$E_3(II) = (\mathbb{F}_p[w] \otimes C_r) \oplus (\mathbb{F}_p[w] \otimes vK_r),$$

and the spectral sequence collapses from the  $E_3$ -page.

By theorem 3.2 and the equation  $Q_k(x_1, x_1) = (k+1)x_1^k$  we see that

$$C_r = \begin{cases} \mathbb{F}_p[x_1]/x_1^{r+1}, & p \mid (r+1), \\ \mathbb{F}_p[x_1]/x_1^r, & p \nmid (r+1). \end{cases}$$

The dimension of the cokernel agrees with the dimension of the kernel. So the kernel of multiplication by  $x_1 - x_2$  is a vector space over  $\mathbb{F}_p$  of dimension  $r+1$  if  $p \mid (r+1)$ , and dimension  $r$  if  $p \nmid (r+1)$ .

We need more precise information about the kernel, since we want to compute the multiplicative structure of the spectral sequence  $E_r(II)$ . To determine the kernel, it is enough to exhibit as many linearly independent elements in the kernel as its known dimension. So it suffices to find  $r+1$  non-trivial elements in pairwise different degrees if  $p \mid (r+1)$ , and  $r$  non trivial elements in pairwise different degrees if  $p \nmid (r+1)$ .

Consider the following homogeneous polynomial

$$a_k = x_1^k \sum_{i=0}^{r-1} (i+1)x_1^i x_2^{r-1-i}.$$

A computation shows that  $(x_2 - x_1)a_k = x_1^k(Q_r(x_1, x_2) - (r+1)x_1^r)$ . So if  $p \mid (r+1)$ , the elements  $a_k$  for  $k \geq 0$  are all in the kernel of multiplication by  $x_1 - x_2$ . If  $p \nmid (r+1)$  then  $a_k$  is in the kernel if  $k \geq 1$ , since  $x_1^{r+1} = Q_{r+1} - x_2 Q_r$ .

To show that the kernel is generated by these classes, we have to check that each  $a_k$  is non-trivial in  $\mathbb{F}_p[x_1, x_2]/(Q_r, Q_{r+1})$ , as long as  $k \leq r$ . We use lemma 3.4 to do so. In  $a_k$ , the coefficient of  $x_1^{k-1}x_2^r$  is 0 and the coefficient of  $x_1^k x_2^{r-1}$  is 1. Furthermore, both  $k-1$  and  $k$  lies between  $(k+r-1) - r = k-1$  and  $r$  so by the lemma we conclude that  $a_k \notin (Q_r, Q_{r+1})$  for  $k \leq r$  as desired.

It follows that if  $p \mid (r+1)$  then the  $r+1$  elements  $a_0, a_1, \dots, a_r$  form a basis for the kernel and if  $p \nmid (r+1)$  then the  $r$  elements  $a_1, a_2, \dots, a_r$  form a basis for the kernel.

By the explicit formula for  $a_k$ , we see that the kernel has basis  $a_0, x_1 a_0, \dots, x_1^r a_0$  when  $p \mid (r+1)$  and  $a_1, x_1 a_1, \dots, x_1^{r-1} a_1$  when  $p \nmid (r+1)$ . Let  $\sigma$  represent  $a_0$  when  $p \mid (r+1)$  and let  $\bar{\sigma}$  represent  $a_1$  when  $p \nmid (r+1)$ .

We obtain the  $E_3$  term

$$E_3(II) = \begin{cases} \mathbb{F}_p[w, x_1, \sigma]/(x_1^{r+1}, \sigma^2), & p \mid (r+1), \\ \mathbb{F}_p[w, x_1, \bar{\sigma}]/(x_1^r, \bar{\sigma}^2), & p \nmid (r+1). \end{cases}$$

We already noted that there can be no nontrivial differentials beyond the second one, so  $E_3(II) = E_\infty(II)$ .  $\square$

**Corollary 4.2.** *If  $p$  divides  $n$ , then the mod  $p$  cohomology Serre spectral sequence for the fibration*

$$S(\tau(\mathbb{C}P^r))^{(n)} \xrightarrow{i} E\mathbb{T} \times_{\mathbb{T}} S(\tau(\mathbb{C}P^r))^{(n)} \xrightarrow{pr_1} B\mathbb{T}$$

*collapses at the  $E_2$  page. If  $p$  does not divide  $n$ , the inclusion  $i$  of the fiber induces a surjection in even degrees*

$$i^* : H^{2*}(E\mathbb{T} \times_{\mathbb{T}} S(\tau(\mathbb{C}P^r))^{(n)}) \rightarrow H^{2*}(S(\tau(\mathbb{C}P^r))).$$

*Proof.* Assume that  $p$  divides  $n$ . We can compute the  $E_2$  page by remark 3.6 and we have just computed the cohomology of the total space. It turns out that the  $E_2$  page is isomorphic to the cohomology of the total space. So there is not room for any differentials.

Assume that  $p$  does not divide  $n$ . We only have to check that the classes  $x^j$  are in the image of  $i^*$ . But by the homeomorphisms we used for computing the cohomology of the Borel construction and corollary 3.7 we have that  $i^*(x_1) = x$  and  $i^*(x_2) = x$  so the result follows.  $\square$

## 5 Equivariant vector bundles

In this section we collect results about the homotopy theory of  $\mathbb{T}$ -vector bundles which we will need later. For greater clarity, the level of generality will be greater than strictly necessary.

We are interested in the cohomology of the Borel construction on Thom spaces. Let  $G$  be a compact Lie group (we will only need the case  $G = \mathbb{T}$ ). Assume that  $\xi = (p : E \rightarrow X)$  is a  $G$ -vector-bundle over  $X$  in the sense of [tD, I.9].

**Lemma 5.1.** *The map  $EG \times_G p : EG \times_G E \rightarrow EG \times_G X$  is the projection of a vector bundle which we denote  $\xi_{hG}$ . There is a homeomorphism  $\text{Th}(\xi_{hG}) \cong EG_+ \wedge_G \text{Th}(\xi)$ . Moreover, if  $G$  is connected, then  $\xi_{hG}$  is oriented if and only if  $\xi$  is oriented when viewed as an ordinary vector bundle.*

*Proof.* We have that  $EG \times X$  is a free  $G$ -space. If this  $G$ -space is also strongly locally trivial, then proposition [tD, I.9.4] gives us that  $\xi_{hG}$  is a vector bundle. So we must show that  $EG \times X$  has a tube around all of its point [tD, page 46].

For  $(e, x) \in EG \times X$  we have that  $G/G_{(e,x)} = G$ . The universal principal  $G$ -bundle  $\pi : EG \rightarrow BG$  is locally trivial, so we have a neighborhood  $V$  around  $\pi(e)$  and a trivialization  $\phi : \pi^{-1}(V) \xrightarrow{\cong} V \times G$ . We get a tube around  $(e, x)$  as follows:

$$\pi^{-1}(V) \times X \xrightarrow{pr_1} \pi^{-1}(V) \xrightarrow{\phi} V \times G \xrightarrow{pr_1} G.$$

Thus  $\xi_{hG}$  is a vector bundle as stated.

We may assume that there is a Riemannian metric on  $\xi$  which is invariant under the  $G$  action. We get a Riemannian metric on  $\xi_{hG}$  such that  $D(\xi_{hG}) = EG \times_G D(\xi)$  and  $S(\xi_{hG}) = EG \times_G S(\xi)$ . So we have

$$\begin{aligned} \text{Th}(\xi_{hG}) &= EG \times_G D(\xi) / EG \times_G S(\xi) \cong EG_+ \wedge_G (D(\xi) / S(\xi)) \\ &= EG_+ \wedge_G \text{Th}(\xi). \end{aligned}$$

Regarding orientability, consider the fibration  $G \rightarrow EG \times X \xrightarrow{\pi} EG \times_G X$ . It is the pullback of  $G \rightarrow EG \rightarrow BG$  along  $pr_1 : EG \times_G X \rightarrow BG$ . Furthermore,  $BG$  is 1-connected since  $G$  is connected so we have trivial coefficients in the  $E_2$  pages of the Serre spectral sequences for  $\pi$ . Thus  $H^1(EG \times_G X; \mathbb{F}_2) \rightarrow H^1(X; \mathbb{F}_2)$  is injective. The vector bundle  $\xi_{hG}$  is the pullback of  $\xi$  along  $X \rightarrow EG \times_G X$ . Since orientability is equivalent to the vanishing of the first Stiefel Whitney class the result follows.  $\square$

**Corollary 5.2.** *Let  $\eta$  be an  $n$ -dimensional  $\mathbb{T}$ -vector bundle over the  $\mathbb{T}$ -space  $X$ . Assume that  $\eta$  is oriented for  $H^*(-; \mathbb{Z}/p)$ . Then there is an isomorphism*

$$\tilde{H}^*(\text{Th}(\eta)_{h\mathbb{T}}; \mathbb{F}_p) \cong H^{*-n}(X_{h\mathbb{T}}; \mathbb{F}_p).$$

We are going to need a special case of the localization theorem for Borel cohomology. The set up for the theorem is the following.

**Definition 5.3.** A based space  $X$  homotopy equivalent to a  $\mathbb{T}$  CW complex satisfies the localization finiteness condition if  $X$  contains only finitely many  $\mathbb{T}$  orbit types, and  $X$  is finite dimensional.

Let  $Y$  be any  $\mathbb{T}$  space with a fixed base point. The space  $E\mathbb{T}_+ \wedge Y$  is equipped with a diagonal map  $\tilde{\Delta} : E\mathbb{T}_+ \wedge Y \rightarrow E\mathbb{T}_+ \wedge E\mathbb{T}_+ \wedge Y$  given by  $\tilde{\Delta}(s, y) = (s, s, y)$ . After taking quotients with group actions, we obtain

$$\Delta_Y : E\mathbb{T}_+ \wedge_{\mathbb{T}} Y \rightarrow (B\mathbb{T}_+) \wedge (E\mathbb{T}_+ \wedge_{\mathbb{T}} Y).$$

This map induces a product

$$H^*(B\mathbb{T}) \otimes \tilde{H}^*(E\mathbb{T}_+ \wedge_{\mathbb{T}} Y) \rightarrow \tilde{H}^*(E\mathbb{T}_+ \wedge_{\mathbb{T}} Y),$$

which makes  $\tilde{H}^*(E\mathbb{T}_+ \wedge_{\mathbb{T}} Y)$  into a module over the ring  $H^*(B\mathbb{T})$ . A  $\mathbb{T}$ -equivariant map  $f : Y_1 \rightarrow Y_2$  induces a module map by naturality. This module structure will pervade the whole theory.

One thing we can use it for is to localize. We invert the generator, to form the cohomology localized away from  $u$  as follows:

$$H^*(X_{h\mathbb{T}}) \left[ \frac{1}{u} \right] = H^*(X_{h\mathbb{T}}) \otimes_{\mathbb{F}_p[u]} \mathbb{F}_p[u, u^{-1}].$$

**Theorem 5.4.** *If  $X$  satisfies the localization finiteness condition, the inclusion  $X^{C_p} \subseteq X$  induces an isomorphism of localized cohomology*

$$H^*(X_{h\mathbb{T}}) \left[ \frac{1}{u} \right] \xrightarrow{\cong} H^*((X^{C_p})_{h\mathbb{T}}) \left[ \frac{1}{u} \right].$$

*Proof.* This is a special case of the localization theorem [tD, III.4.2]. The parameters of the localization theorem, as given by tom Dieck, are chosen as follows:  $G = \mathbb{T}$ , the cohomology theory  $H^*(-) = H^*(-; \mathbb{F}_p)$  and the set  $S \subseteq H^*(BG) = H^*(B\mathbb{T}; \mathbb{F}_p)$  is  $S = \{1, u, u^2, \dots\}$ . The subset  $A \subseteq X$  is the empty set.

The statement of the localization theorem is to be interpreted as follows: The family  $\mathcal{F}(S)$  of subgroups of  $\mathbb{T}$  (defined in *ibid* after III.3.1) is the family of subgroups  $C_n \subseteq \mathbb{T}$  with  $n$  not divisible by  $p$ . The set  $X(\mathcal{F}) \subseteq X$  defined in *ibid* I.6.1, is the complement of the set  $X^{C_p}$  in  $X$ . But then  $FX = X^{C_p}$ , so that theorem [tD, III.4.2] indeed specializes to our theorem.  $\square$

The application we want is the following: Let  $X$  be a space, satisfying the localization finiteness condition. Assume that the action is not effective, but that the subgroup  $C_p \subseteq \mathbb{T}$  acts trivially. On the other hand, we assume that all isotropy groups of points in  $X$  are contained in  $C_n \subseteq \mathbb{T}$  for some fixed  $n$ . Let  $\xi$  be a  $\mathbb{T}$ -vector bundle over  $X$ . We do not assume that the action of  $C_p$  on  $\xi$  is trivial. The fixed points of  $C_p$  forms a  $\mathbb{T}$  subbundle  $\xi^{C_p} \subseteq \xi$ .

**Theorem 5.5.** *The following map induces an isomorphism on mod  $p$  cohomology localized away from  $u$ :*

$$E\mathbb{T}_+ \wedge_{\mathbb{T}} \text{Th}(\xi^{C_p}) \rightarrow E\mathbb{T}_+ \wedge_{\mathbb{T}} \text{Th}(\xi).$$

*Proof.* The only possible isotropy groups of  $\text{Th}(\xi)$  are  $\mathbb{T}$  itself (for the base point in the Thom space), and subgroups of  $C_n$ . In particular, there are finitely many orbit types. Since  $\text{Th}(\xi^{C_p}) = \text{Th}(\xi)^{C_p}$ , the result now follows by applying theorem 5.4 on the  $\mathbb{T}$ -space  $\text{Th}(\xi)$ .  $\square$

## 6 The twisted action

Let  $X$  be a  $\mathbb{T}$ -space with action map  $\mu : \mathbb{T} \times X \rightarrow X$ . We can twist this action by the power map  $\lambda_n : \mathbb{T} \rightarrow \mathbb{T}$ ;  $\lambda_n(z) = z^n$  and obtain another  $\mathbb{T}$ -space  $X^{(n)}$ . The underlying spaces of  $X$  and  $X^{(n)}$  are equal, but the action map for  $X^{(n)}$  is  $\mu_n : \mathbb{T} \times X^{(n)} \rightarrow X^{(n)}$ ;  $\mu_n(z, x) = \mu(\lambda_n(z), x)$ .

**Lemma 6.1.** *Let  $X$  be an  $\mathbb{T}$ -space. We have a pullback of fibration sequences which is natural in  $X$  as follows:*

$$\begin{array}{ccccc} BC_n & \longrightarrow & E\mathbb{T} \times_{\mathbb{T}} X^{(n)} & \xrightarrow{g_n} & E\mathbb{T} \times_{\mathbb{T}} X \\ \downarrow & & \downarrow & & \downarrow \\ BC_n & \longrightarrow & B\mathbb{T} & \xrightarrow{B\lambda_n} & B\mathbb{T}. \end{array}$$

*Proof.* It is convenient to consider models of  $E\mathbb{T}$  and  $B\mathbb{T}$  which are realizations of simplicial topological abelian groups. So let  $E\mathbb{T} = |E\mathbb{T}_\bullet|$  and  $B\mathbb{T} = |B\mathbb{T}_\bullet|$ , where  $E\mathbb{T}_q$  is the  $q + 1$ -fold Cartesian product of  $\mathbb{T}$  by itself with

$$d_i(z_0, \dots, z_q) = (z_0, \dots, \widehat{z_i}, \dots, z_q), \quad s_i(z_0, \dots, z_q) = (z_0, \dots, z_i, z_i, \dots, z_q)$$

and  $B\mathbb{T}_q = E\mathbb{T}_q / \text{diag}_q(\mathbb{T})$ . The hat means that the element is left out.

We have simplicial maps  $E(\lambda_n)_\bullet$  and  $B(\lambda_n)_\bullet$  given by rising all elements in a tuple to the  $n$ th power. Since the kernel of  $B(\lambda_n)_q$  is exactly  $(BC_n)_q$ , the bottom row in the diagram is the realization of a short exact sequence of simplicial abelian groups, which means that it is a fibration. Since  $E(\lambda_n)$  is a map over  $B(\lambda_n)$  the right hand square in the diagram commutes when we put  $g_n([e, x]) = [E(\lambda_n)(e), x]$ . Note that this makes  $g_n$  well-defined since  $E(\lambda_n)(ez) = E(\lambda_n)(e)z^n$ . The map of the vertical fibers is the identity so the right hand square is a vertical pullback. It follows that it is also a horizontal pullback.  $\square$

Here is our main result on Borel cohomology of twisted actions.

**Theorem 6.2.** *For any prime  $p$  and  $\mathbb{T}$ -space  $X$  there is an isomorphism, natural in  $X$ :*

$$H^*(X) \otimes \mathbb{F}_p[u] \xrightarrow{\cong} H^*(E\mathbb{T} \times_{\mathbb{T}} X^{(p)}).$$

To prove the theorem, we first show that there exists a natural subgroup  $\mathcal{F}_1^*(X)$  of  $H^*((X^{(p)})_{h\mathbb{T}})$  with the property that the statement is true if we replace  $H^*(X)$  by  $\mathcal{F}_1^*(X)$ . After we have constructed this subgroup, it is easy to identify it with  $H^*(X)$ .

The spectral sequence derived from the fibration of lemma 6.1 defines a filtration

$$\mathcal{F}_0^*(X) \subseteq \mathcal{F}_1^*(X) \subseteq \dots \subseteq H^*(E\mathbb{T} \times_{\mathbb{T}} X^{(p)})$$

such that  $\mathcal{F}_i^*(X) / \mathcal{F}_{i-1}^*(X) \cong \bigoplus_{* \geq 0} E_\infty^{i,*}$ . Each of these filtration subgroups is a functor on the category of  $\mathbb{T}$ -spaces. It is well known how to interpret the first group:

$$\mathcal{F}_0^*(X) = g_p^*(H^*(X_{h\mathbb{T}}))$$

We need to consider the next step in the filtration. This group does not have an equally simple description.

**Definition 6.3.** The group of low elements  $\mathcal{F}_1^*(X) \subseteq H^*(E\mathbb{T} \times_{\mathbb{T}} X^{(p)})$  is the subgroup consisting of those classes which in the Serre spectral sequence represents elements in  $E_\infty^{*,0}$  or  $E_\infty^{*,1}$ .

We define a reduced version of the group in the obvious way. If  $X$  is a pointed  $\mathbb{T}$ -space, we have inclusions of groups, natural on such spaces

$$g_n^*(H^*(E\mathbb{T} \times_{\mathbb{T}} X, B\mathbb{T})) = \mathcal{F}_0^*(X, *) \subseteq \mathcal{F}_1^*(X, *) \subseteq H^*(E\mathbb{T} \times_{\mathbb{T}} X^{(n)}, B\mathbb{T}).$$

Here is a general fact about the low elements corresponding to  $p$ -fold twisting.

**Lemma 6.4.**  *$H^*(X_{h\mathbb{T}}^{(p)})$  is a free module over  $H^*(B\mathbb{T}) = \mathbb{F}_p[u]$ . The low elements  $\mathcal{F}_1^*(X)$  form a basis for it as an  $\mathbb{F}_p[u]$  module.*

*Proof.* The module structure on  $H^*(X_{h\mathbb{T}}^{(p)})$  is given by the map  $pr_1 : X_{h\mathbb{T}}^{(p)} \rightarrow B\mathbb{T}$ . The Serre spectral sequence associated to the upper fibration sequence in lemma 6.1 has the form

$$E_2^{n,m}(X) \cong H^n(X_{h\mathbb{T}}) \otimes H^m(BC_p) \Rightarrow H^{n+m}(X^{(p)}).$$

Remember that  $H^*(BC_p) \cong \mathbb{F}_p[v, w]/I_p$ , where  $I_p$  is a principal ideal, generated by  $v^2 - w$  if  $p = 2$  and by  $v^2$  otherwise.

The Serre spectral sequence of the lower fibration sequence in the lemma has  $E_2^{*,*} = \mathbb{F}_p[u] \otimes \mathbb{F}_p[v, w]/I_p$  and converges towards  $H^*(B\mathbb{T})$  so we must have  $d_2(v) = u$  and  $E_3(*) = E_\infty(*) = \mathbb{F}_p[w]$ . Using this together with naturality in  $X$  of the spectral sequence we get that for any  $X$  the class  $w \in E_2^{0,2}(X)$  is a permanent cycle. Furthermore,

$$E_2^{*,*}(X) \cong \mathbb{F}_p[w] \otimes (E_2^{*,0} \oplus E_2^{*,1})$$

so it follows formally that  $E_3^{*,*}(X) \cong \mathbb{F}_p[w] \otimes (E_3^{*,0} \oplus E_3^{*,1})$ . But then it also follows formally that higher differentials in this spectral sequence vanish, so that

$$E_\infty^{*,*}(X) \cong \mathbb{F}_p[w] \otimes (E_\infty^{*,0} \oplus E_\infty^{*,1}).$$

The low elements are by definition a subspace of  $H^*(X_{h\mathbb{T}}^{(p)})$ . We can extend this inclusion to a unique map of  $\mathbb{F}_p[u]$ -modules

$$f : \mathbb{F}_p[u] \otimes \mathcal{F}_1^*(X) \rightarrow H^*(X_{h\mathbb{T}}^{(p)}).$$

This map sends  $u^i \otimes \mathcal{F}_1^*(X)$  to  $\mathcal{F}_i^*(X)$ , and the map induces an isomorphism on the corresponding graded rings, by the above computation of  $E_\infty^{*,j}$ , and because  $u \in H^2(B\mathbb{T})$  represents  $w \in E_\infty^{0,2}$ . It follows that  $f$  itself is an isomorphism.  $\square$

*proof of theorem 6.2.* We have to exhibit a natural isomorphism  $\mathcal{F}_1^*(X) \rightarrow H^*(X)$ . From the spherical fibration

$$\mathbb{T} \longrightarrow E\mathbb{T} \times X^{(p)} \xrightarrow{i} E\mathbb{T} \times_{\mathbb{T}} X^{(p)}$$

we get a long exact Gysin sequence

$$\longrightarrow H^{*-2}(X_{h\mathbb{T}}^{(p)}) \xrightarrow{\cdot u} H^*(X_{h\mathbb{T}}^{(p)}) \xrightarrow{i^*} H^*(X^{(p)}) \longrightarrow H^{*-1}(X_{h\mathbb{T}}^{(p)}) \longrightarrow$$

(One can use the map  $X \rightarrow *$  to see that the Euler class is  $u$ .) Since  $H^*(X_{h\mathbb{T}}^{(p)})$  is a free  $\mathbb{F}_p[u]$  module, multiplication by  $u$  is injective, and the long exact sequences breaks up into short exact sequences. So via lemma 6.4 we get the short exact sequence

$$0 \longrightarrow \mathbb{F}_p[u] \otimes \mathcal{F}_i^*(X) \xrightarrow{\cdot u} \mathbb{F}_p[u] \otimes \mathcal{F}_i^*(X) \xrightarrow{i^*} H^*(X^{(p)}) \longrightarrow 0.$$

Thus,  $i^*$  factors through  $(\mathbb{F}_p[u] \otimes \mathcal{F}_i^*(X))/\text{im}(\cdot u) = \mathcal{F}_i^*(X)$  and gives the desired isomorphism.  $\square$

## 7 The $n$ -fold iteration map

Let  $M$  be a compact Riemannian manifold. In this chapter we are examining the Hilbert manifold model of the free loop space. We denote this Hilbert manifold by  $LM$ . An element in  $LM$  is a curve  $\gamma : \mathbb{T} \rightarrow M$  of class  $H^1$ . The Hilbert manifold model is homotopy equivalent to the usual continuous mapping space model. The energy integral

$$E : LM \rightarrow \mathbb{R}; \quad E(\gamma) = \int_{\mathbb{T}} |\gamma'(z)|^2 dz$$

is a smooth function on  $LM$ . Its critical points are the closed geodesics on  $M$ . We review the main results of Morse theory in this setting. For details we refer to [K1], especially to chapter 1.

The power map  $\lambda_n : \mathbb{T} \rightarrow \mathbb{T}; \lambda_n(z) = z^n$  gives us an  $n$ -fold iteration map

$$P_n : LM \rightarrow LM; \quad P_n(\gamma) = \gamma \circ \lambda_n.$$

Just like other versions of the iteration map, this map is not equivariant with respect to the  $\mathbb{T}$ -action, but becomes equivariant if we twist the action. That is, the formula defines an equivariant map  $P_n : (LM)^{(n)} \rightarrow LM$ .

The map preserves energy up to a constant factor,  $E(P_n(\gamma)) = n^2 E(\gamma)$ . That is, it induces an equivariant map of energy filtration  $\mathcal{F}(a) = E^{-1}(-\infty, a)$  as follows:

$$P_n : (\mathcal{F}(a))^{(n)} \rightarrow \mathcal{F}(n^2 a)$$

We also get an equivariant map of quotients

$$P_n : (\mathcal{F}(a)/\mathcal{F}(a - \epsilon))^{(n)} \rightarrow \mathcal{F}(n^2 a)/\mathcal{F}(n^2 a - n^2 \epsilon).$$

Let  $T_\gamma LM$  denote the tangent space of  $LM$  at  $\gamma$ . It consists of the vector fields along  $\gamma$  of class  $H^1$ , and is a real Hilbert space with inner product

$$\langle \xi, \eta \rangle_1 = \int_{\mathbb{T}} (\langle \xi(z), \eta(z) \rangle + \langle \nabla \xi(z), \nabla \eta(z) \rangle) dz.$$

Here  $\nabla$  denotes the covariant differentiation along the curve defined by the Levi-Civita connection of the Riemannian manifold  $M$ . The inner product defines a Riemannian metric on the Hilbert manifold  $LM$ . One technical problem is that the iteration map is not an (injective) isometry with respect to this metric. To study the properties of  $P_n$  it will be convenient to consider a modified inner product on  $T_\gamma LM$ . Let

$$\langle \xi, \eta \rangle_{c,1} = \int_{\mathbb{T}} (\langle \xi(z), \eta(z) \rangle + c \langle \nabla \xi(z), \nabla \eta(z) \rangle) dz.$$

The differential of the iteration map is given by

$$D_\gamma(P_n) : T_\gamma LM \rightarrow T_{P_n(\gamma)} LM; \quad D_\gamma(P_n)(\xi) = \xi \circ \lambda_n.$$

So we have  $\nabla(D_\gamma(P_n)\xi) = n\nabla(\xi) \circ \lambda_n = nD_\gamma(P_n)(\nabla\xi)$  and

$$\langle D_\gamma(P_n)\xi, D_\gamma(P_n)\eta \rangle_1 = \int_{\mathbb{T}} \langle \xi(z^n), \eta(z^n) \rangle + n^2 \langle \nabla(\xi)(z^n), \nabla(\eta)(z^n) \rangle dz = \langle \xi, \eta \rangle_{n^2,1}.$$

This means that even if the iteration map does not preserve the inner product, it becomes an isometry if we replace the inner product on the target by a suitably modified inner product.

The metric on  $M$  determines an exponential map  $\exp_p: T_pM \rightarrow M$  for each  $p \in M$ . This induces an exponential map

$$\exp_{\tilde{\gamma}}: T_{\gamma}LM \rightarrow LM; \quad \exp_{\tilde{\gamma}}(\xi)(t) = \exp_{\gamma(t)}(\xi(t)).$$

This is not the exponential map derived from the Hilbert metric on  $LM$ , but it has the advantage that it is compatible with the iteration map in the sense that the following diagram commutes:

$$\begin{array}{ccc} T_{\gamma}LM & \xrightarrow{D_{\gamma}(P_n)} & T_{P_n(\gamma)}LM \\ \downarrow \exp_{\tilde{\gamma}} & & \downarrow \exp_{\tilde{P}_n(\gamma)} \\ LM & \xrightarrow{P_n} & LM. \end{array}$$

The exponential map is a diffeomorphism from a neighborhood of 0 in  $T_{\gamma}LM$  to a neighborhood of  $\gamma$  in  $LM$ .

Let  $\gamma$  be a closed geodesic on  $M$ . This is a critical point for the energy function  $E$ . The Hessian of the energy function defines a quadratic form  $D^2E(\gamma)$  on  $T_{\gamma}LM$ . This form determines a self adjoint operator  $A_{\gamma}: T_{\gamma}LM \rightarrow T_{\gamma}LM$  by the equation  $D^2E(\gamma)(\xi, \eta) = \langle A_{\gamma}\xi, \eta \rangle_1$ . The operator  $A_{\gamma}$  is the sum of the identity with a compact operator, so there are at most a finite number of negative eigenvalues, each corresponding to a finite dimensional vector space of eigenvectors of  $A_{\gamma}$ .

Obviously the operator  $A_{\gamma}$  depends not just on the Hessian of the energy integral, but also on the metric on  $LM$ . Since we are considering modifications of the inner product, we also have to consider the corresponding modifications of the self adjoint operator. To emphasize this dependence, for a real vector space  $V$  with inner product  $\langle \cdot, \cdot \rangle_{\alpha}$  and with a quadratic form  $Q$ , we define the operator  $A(\langle \cdot, \cdot \rangle_{\alpha}, Q)$  by the property that  $Q(\xi, \eta) = \langle A(\langle \cdot, \cdot \rangle_{\alpha}, Q)\xi, \eta \rangle_{\alpha}$ .

Now let  $K_a$  be the space of critical points of  $E$  with energy level  $a$ . The negative bundle  $\mu^{-}$  over  $K_a$  is the vector bundle whose fiber at  $\gamma$  is the vector space spanned by the eigenvectors belonging to negative eigenvalues of  $A_{\gamma}$ . Similarly,  $\mu^0$  and  $\mu^{+}$  are the vector bundles with fibers spanned by the eigenvectors corresponding to the eigenvalue 0 and the positive eigenvalues respectively.

Using the modified inner product  $\langle \cdot, \cdot \rangle_{c,1}$  we obtain a modified negative bundle  $\mu_c^{-}$  etc. This bundle is not the same as  $\mu^{-}$ . But on the other hand, the difference between these two bundles is not so dramatic, since the orthogonal projection from  $\mu_c^{-}$  to  $\mu^{-}$  with respect to the standard inner product defines an isomorphism of bundles.

**Lemma 7.1.** *The iteration map induces a bijection  $P_n: K_a \rightarrow (K_{an^2})^{C_n}$ . The group  $C_n$  acts on the fiber  $(\mu^{-})_{P_n\gamma}$ , and for any  $\gamma \in K_a$  the differential of the iteration map induces an isomorphism of vector spaces*

$$D_{\gamma}(P_n): (\mu_{n^2}^{-})_{\gamma} \xrightarrow{\cong} ((\mu^{-})_{P_n\gamma})^{C_n}.$$

*Proof.* The property of being a geodesic is a local property of a curve, so  $P_n\gamma$  is a geodesic if and only if  $\gamma$  is a geodesic. It follows that  $P_n(K_a) \subseteq (K_{n^2a})^{C_n}$ . On the other hand, if  $\theta \in (K_{n^2a})^{C_n}$  we can write  $\theta = P_n\gamma$  for some  $\gamma \in LM$ . But  $\gamma$  has to be a geodesic, since  $\theta$  is, and  $E(\gamma) = E(P_n\gamma)/n^2 = a$ , which means that  $\gamma \in K_a$ . Thus,  $P_n$  induces a bijection as stated.

Now we look at the tangent map. We first claim that this map induces an isomorphism

$$D_\gamma(P_n) : T_\gamma LM \xrightarrow{\cong} (T_{P_n\gamma} LM)^{C_n}.$$

This is clear, since  $P_n\gamma$  is periodic of period  $n$ , and any  $n$ -periodic vector field along  $P_n\gamma$  is the image under the differential of the iteration map of a vector field on  $\gamma$ .

We must show that this isomorphism restricts to an isomorphism as stated. The groups  $C_n$  acts on  $V = T_{P_n\gamma} LM$ , let us say that a generator  $\sigma \in C_n$  acts by

$$(\sigma\xi)(z) = \xi(z\zeta_n) \in T_{P_n\gamma(z\zeta_n)} M = T_{P_n\gamma(z)} M; \quad \zeta_n = e^{2\pi i/n}.$$

Let  $N \in \mathbb{C}[C_n]$  be the sum  $\frac{1}{n} \sum_{k=0}^{n-1} \sigma^k$ . This  $N$  acts as an idempotent on  $V$ , and so does  $\mathbf{Id} - N$ . The inner product is manifestly invariant under the action on  $C_n$ , so a simple calculation shows that  $N$  is self adjoint such that  $N$  and  $\mathbf{Id} - N$  are orthogonal idempotents. Moreover, the idempotents commute with the action of the group  $C_n$ . So  $V$  splits as a  $C_n$  representation into an orthogonal sum  $NV \oplus (\mathbf{Id} - N)V$ .

The energy function is also invariant under the group action, so by the same argument as we just used for the inner product, the quadratic form  $(V, D_{P_n\gamma}^2(E))$  splits as a direct sum  $(NV, D_{P_n\gamma}^2(E)) \oplus ((\mathbf{Id} - N)V, D_{P_n\gamma}^2(E))$ .

Let  $A = A(\langle \cdot, \cdot \rangle_1, D_{P_n\gamma}^2(E))$ . Since both the inner product and the quadratic form split as direct sums, the linear endomorphism  $A : V \rightarrow V$  is a direct sum of two self adjoint endomorphisms

$$A|_{NV} : NV \rightarrow NV, \quad A|_{(\mathbf{Id}-N)V} : (\mathbf{Id} - N)V \rightarrow (\mathbf{Id} - N)V.$$

It follows that the subspace of  $V$  generated by negative eigenvectors equals the direct sum of the negative eigenvector spaces of  $A|_{NV}$  and  $A|_{(\mathbf{Id}-N)V}$ .

We claim that  $NV = V^{C_n}$  and  $((\mathbf{Id} - N)V)^{C_n} = 0$ . This follows since  $\sigma N = N$  so that  $NV \subseteq V^{C_n}$  and if  $\sigma(\mathbf{Id} - N)\xi = (\mathbf{Id} - N)\xi$  then  $\sigma\xi = \xi$  so that  $(\mathbf{Id} - N)\xi = 0$ .

To finish the proof, we have to show that  $D_\gamma(P_n)(\mu_{n^2}^-)_\gamma$  equals the negative eigenvector space of  $A|_{NV}$ .

But  $D_\gamma(P_n)$  is an isometry (with respect to the modified inner product), and it preserves  $D^2E$  up to multiplication by  $n^2$ . So have a commutative diagram

$$\begin{array}{ccc} T_\gamma LM & \xrightarrow{D_\gamma P_n} & T_{P_n\gamma} LM \\ n^2 A(\langle \cdot, \cdot \rangle_{n^2, 1}, D_\gamma^2 E) \downarrow & & \downarrow A(\langle \cdot, \cdot \rangle_1, D_{P_n\gamma}^2 E) \\ T_\gamma LM & \xrightarrow{D_\gamma P_n} & T_{P_n\gamma} LM \end{array}$$

by which we get the desired result.  $\square$

Assume that  $N_a \subseteq LM$  is a nondegenerate critical manifold of energy  $a$ . Also assume that  $P_p(N_a) \subseteq N_{p^2a}$  is a non-degenerate critical manifold, and that there are no other critical points at energy levels in the intervals  $(a - \epsilon, a)$  and  $(p^2(a - \epsilon), p^2a)$ .

The Whitney sum  $\mu = \mu^- \oplus \mu^+$  is the normal bundle of the submanifold  $N_a \subseteq LM$  because we are assuming that  $N_a$  satisfies Bott's non-degeneracy condition.

**Theorem 7.2.** *The  $p$ -fold iteration map induces an isomorphism in cohomology localized away from  $u$  as follows:*

$$\tilde{H}^*((\mathcal{F}(a)/\mathcal{F}(a-\epsilon))_{h\mathbb{T}}) \left[ \frac{1}{u} \right] \cong \tilde{H}^*((\mathcal{F}(p^2a)/\mathcal{F}(p^2(a-\epsilon)))_{h\mathbb{T}}^{(p)}) \left[ \frac{1}{u} \right].$$

*Proof.* We have a commutative diagram (possibly after decreasing  $\epsilon$ )

$$\begin{array}{ccc} (D(\mu_{p^2}^-(N_a)), S(\mu_{p^2}^-(N_a))) & \xrightarrow{DP_p} & (D(\mu^-(N_{p^2a}), S(\mu^-(N_{p^2a}))) \\ \downarrow \text{exp}^\sim & & \downarrow \text{exp}^\sim \\ (\mathcal{F}(a), \mathcal{F}(a-\epsilon)) & \xrightarrow{P_p} & (\mathcal{F}(p^2a), \mathcal{F}(p^2(a-\epsilon))). \end{array}$$

According to theorem 5.4 the upper horizontal map induces an isomorphism in mod  $p$  cohomology localized away from  $u$ .

The exponential map in the right column induces a  $\mathbb{T}$  homotopy equivalence on quotients by theorem [K1, 2.4.10]. This is the Hilbert manifold version of the fundamental theorem of Morse theory. So to prove the theorem, we need to see that the left vertical map induces a  $\mathbb{T}$  equivariant homotopy equivalence on quotients.

But this is true for the same reason. The proof of theorem [K1, 2.4.10] does not use the explicit form of the metric on the Hilbert manifold  $LM$ , but only the nondegeneracy of  $D^2E$ . The statement of the variation of the theorem using  $LM$  with the modified metric  $\langle \cdot, \cdot \rangle_{1/p^2, 1}$  is exactly the statement that the left vertical map is a homotopy equivalence.  $\square$

*Remark 7.3.* The theorem and its proof are also valid if  $a$  is not a critical value. In this case the interpretation of the statement is that if  $\lambda$  is a critical value of the energy function, such that  $\lambda/p^2$  is not a critical value, then for  $\epsilon > 0$  sufficiently small,

$$\tilde{H}^*((\mathcal{F}(\lambda)/\mathcal{F}(\lambda-\epsilon))_{h\mathbb{T}}^{(p)}) \left[ \frac{1}{u} \right] = 0.$$

*Remark 7.4.* Since  $P_p$  is a homeomorphism whose image is the  $C_p$  fixed points, we could try to use theorem 5.4 directly on the inclusion  $\mathcal{F}(p^2a)^{C_p} \subseteq \mathcal{F}(p^2a)$ , cleverly bypassing this entire section. The problem with this approach is that you need a finite dimensionality condition for the localization statement 5.4 to be true. The only role of Morse theory in the proof of theorem 7.2 is to reduce the infinite dimensional situation to a finite dimensional one.

It seems likely that this reduction can be done in greater generality, and that the non-degeneracy condition in 7.2 is far stronger than necessary.

## 8 The Morse spectral sequences

Let  $M$  be a compact Riemannian manifold as before. But in this section we are going to assume that the critical points of the energy function on  $LM$  are collected

on compact critical manifolds. We also assume that each of these critical manifolds satisfy the Bott non-degeneracy condition. This is a strong assumption on the metric of  $M$ , but for instance the symmetric spaces satisfy this, according to [Z].

We are considering the filtration induced by the levels of the energy function. Let the critical values of the energy function be  $0 = \lambda_0 < \lambda_1 < \dots$ . There is a filtration of  $LM$  by  $\mathcal{F}(\lambda_i) = E^{-1}(-\infty, \lambda_i)$ . This filtration is equivariant with respect to the action of the circle, and induces spectral sequences of various forms of cohomology. We call these spectral sequences Morse spectral sequences. The condition we impose on the metric of  $M$  means that for each  $\lambda$  we have a finite dimensional critical manifold  $N(\lambda)$ . The tangent bundle of  $LM$  restricted to  $N(\lambda)$  splits  $\mathbb{T}$ -equivariantly into a sum of three bundles.

$$TLM|_{N(\lambda)} \cong \mu^-(\lambda) \oplus \mu^0(\lambda) \oplus \mu^+(\lambda).$$

The standard Morse theory argument can be carried through equivariantly on the Hilbert manifold  $LM$ . This was done by Klingenberg. For an account of this work see section [K1, 2.4], especially theorem 2.4.10. The statement of this theorem implies that we have an equivariant homotopy equivalence

$$\mathcal{F}(\lambda_n)/\mathcal{F}(\lambda_{n-1}) \simeq \text{Th}(\mu^-(\lambda_n)).$$

We will consider cohomology of the homotopy orbits  $LM_{h\mathbb{T}}$ . We are also going to consider the twisted action. This is not because we have a particular interest in  $H^*(LM_{h\mathbb{T}}^{(p)})$  in itself. As we will see, these groups are easy to compute anyhow. The purpose of considering them is rather to be able to use a comparison argument to obtain information about the Morse spectral sequence of  $H^*(LM_{h\mathbb{T}})$ . The abstract situation is as follows:

**Theorem 8.1.** *There are three spectral sequences*

$$\begin{aligned} \{E_r^{n,m}(\mathcal{M})(LM)\} &\Rightarrow H^{n+m}(LM), \\ \{E_r^{n,m}(\mathcal{M})(LM_{h\mathbb{T}})\} &\Rightarrow H^{n+m}(LM_{h\mathbb{T}}), \\ \{E_r^{n,m}(\mathcal{M})(LM_{h\mathbb{T}}^{(p)})\} &\Rightarrow H^{n+m}(LM_{h\mathbb{T}}^{(p)}). \end{aligned}$$

The  $E_1$  pages are given by

$$\tilde{H}^m(\text{Th}(\mu^-(\lambda_n))), \quad \tilde{H}^m(\text{Th}(\mu^-(\lambda_n)_{h\mathbb{T}})), \quad \tilde{H}^m(\text{Th}(\mu^-(\lambda_n)_{h\mathbb{T}}^{(p)}))$$

respectively. We have a natural isomorphism of spectral sequences

$$E_*(\mathcal{M})(LM_{h\mathbb{T}}^{(p)}) \cong \mathbb{F}_p[u] \otimes_{\mathbb{F}_p} E_*(\mathcal{M})(LM)$$

as a module over  $H^*(B\mathbb{T}) \cong \mathbb{F}_p[u]$ .

*Proof.* Consider the energy filtration  $\mathcal{F}(\lambda_0) \subseteq \mathcal{F}(\lambda_1) \subseteq \dots$  with union  $LM$ . Because the filtration is equivariant, it induces filtrations of the spaces  $LM$ ,  $LM_{h\mathbb{T}}$  and  $LM_{h\mathbb{T}}^{(p)}$ . These filtrations give rise to three exact couples which gives the three spectral sequences. The spectral sequences all converge (strongly). One can use [Sp, Ch. 9,1.6] or [Bo, theorem 12.6] and the remark following [Bo, theorem 7.1] to see this.

The functorial isomorphism of  $H(B\mathbb{T}; \mathbb{F}_p)$ -modules from theorem 6.2 imply the natural isomorphism of spectral sequences.  $\square$

We next have to discuss the  $p$ -fold iteration map. If there exists a periodic geodesic of energy  $\lambda$ , its  $p$ -fold iterate is a geodesic of energy  $p^2\lambda$ . So the sequence  $p^2\lambda_0 < p^2\lambda_1 < p^2\lambda_2 < \dots$  is a subsequence of the sequence  $\lambda_0 < \lambda_1 < \lambda_2 < \dots$ . The  $p$ -fold iteration map induces an equivariant map of filtrations:

$$\begin{array}{ccccccc} \mathcal{F}(\lambda_0)^{(p)} & \longrightarrow & \mathcal{F}(\lambda_1)^{(p)} & \longrightarrow & \cdots & \longrightarrow & LM^{(p)} \\ \downarrow & & \downarrow & & & & \downarrow \\ \mathcal{F}(p^2\lambda_0) & \longrightarrow & \mathcal{F}(p^2\lambda_1) & \longrightarrow & \cdots & \longrightarrow & LM. \end{array}$$

**Lemma 8.2.** *Let  $p^2\lambda_{i-1} = \lambda_j < \lambda_{j+1} < \cdots < \lambda_k = p^2\lambda_i$  be the critical values of the energy function in the interval  $[p^2\lambda_{i-1}, p^2\lambda_i]$ . Then*

$$H^*(\mathcal{F}(\lambda_\ell)_{h\mathbb{T}}, \mathcal{F}(\lambda_{\ell-1})_{h\mathbb{T}}; \mathbb{F}_p) \left[ \frac{1}{u} \right] = \begin{cases} 0 & , \ell \neq k, \\ H^*(\mathcal{F}(\lambda_i), \mathcal{F}(\lambda_{i-1}); \mathbb{F}_p) \otimes \mathbb{F}_p[u, u^{-1}] & , \ell = k. \end{cases}$$

The isomorphism is induced by the iteration map followed by the isomorphism of theorem 6.2.

*Proof.* This follows from theorem 7.2 and remark 7.3.  $\square$

**Theorem 8.3.** *Up to a re-indexing of the columns in the spectral sequence the  $p$ -fold iteration map induces a natural isomorphisms of spectral sequences*

$$E_*(\mathcal{M})(LM_{h\mathbb{T}}) \left[ \frac{1}{u} \right] \cong E_*(\mathcal{M})(LM) \otimes \mathbb{F}_p[u, u^{-1}].$$

*Proof.* We have to explain the re-indexing. Let  $j(i)$  be the non-negative number such that  $p^2\lambda_i = \lambda_{j(i)}$ , and  $k(i)$  the largest number  $s \leq i$  such that  $s = j(t)$  for some  $t$ . So by definition,  $k(i) \leq i$ .

Besides the filtration  $\mathcal{F}_i := \mathcal{F}(\lambda_i)$  we consider two derived filtrations of  $LM$ ;

$$\mathcal{F}'_i := \mathcal{F}_{j(i)} \quad , \quad \mathcal{F}''_i := \mathcal{F}_{k(i)}.$$

The filtrations define spectral sequences

$$E_*(\mathcal{M})'(LM_{h\mathbb{T}}) \Rightarrow H^*(LM_{h\mathbb{T}}) \quad , \quad E_*(\mathcal{M})''(LM_{h\mathbb{T}}) \Rightarrow H^*(LM_{h\mathbb{T}}).$$

The  $p$  fold iteration map  $P := P_p$  induces a map of filtrations  $P : \mathcal{F}_i \rightarrow \mathcal{F}_{j(i)}$ , so we have a ladder

$$\begin{array}{ccccccc} (\mathcal{F}_0)^{(p)}_{h\mathbb{T}} & \longrightarrow & (\mathcal{F}_1)^{(p)}_{h\mathbb{T}} & \longrightarrow & \cdots & \longrightarrow & LM^{(p)}_{h\mathbb{T}} \\ \downarrow P & & \downarrow P & & & & \downarrow P \\ (\mathcal{F}'_0)_{h\mathbb{T}} & \longrightarrow & (\mathcal{F}'_1)_{h\mathbb{T}} & \longrightarrow & \cdots & \longrightarrow & LM_{h\mathbb{T}}. \end{array}$$

According to theorem 7.2 we get an isomorphism of spectral sequences

$$P^* : E_*(\mathcal{M})'(LM_{h\mathbb{T}}) \left[ \frac{1}{u} \right] \xrightarrow{\cong} E_*(\mathcal{M})(LM_{h\mathbb{T}}^{(p)}) \left[ \frac{1}{u} \right].$$

Note that the spectral sequence on the right hand side can be rewritten by the isomorphism in theorem 8.1.

The filtration  $\mathcal{F}_i''$  is just a trivial reindexing of the filtration  $\mathcal{F}_i'$ . We have that

$$\mathcal{F}'_0 = \mathcal{F}''_0 = \mathcal{F}''_1 = \cdots = \mathcal{F}''_{j(1)-1} \subseteq \mathcal{F}'_1 = \mathcal{F}''_{j(1)} = \mathcal{F}''_{j(1)+1} = \cdots$$

So the two spectral sequences

$$E_*(\mathcal{M})''(LM_{h\mathbb{T}}) \left[ \frac{1}{u} \right] , \quad E_*(\mathcal{M})'(LM_{h\mathbb{T}}) \left[ \frac{1}{u} \right]$$

are the same up to a re-indexing of the columns.

Finally, since  $\mathcal{F}_i'' = \mathcal{F}_{k(i)} \subseteq \mathcal{F}_i$  we have an inclusion of filtrations, which induces a map of spectral sequences

$$E_*(\mathcal{M})''(LM_{h\mathbb{T}}) \left[ \frac{1}{u} \right] \longrightarrow E_*(\mathcal{M})(LM_{h\mathbb{T}}) \left[ \frac{1}{u} \right] .$$

According to lemma 8.2 this map is an isomorphism of  $E_1$  pages, and thus of spectral sequences. This proves the theorem.  $\square$

## 9 The Morse spectral sequences for $\mathbb{C}P^r$

We now turn to the special case  $M = \mathbb{C}P^r$  with the usual symmetric space (Fubini–Study) metric. We know that this space satisfies the conditions of section 8, in particular theorem 8.3 is valid. We are going to study the Morse spectral sequence for  $H^*(L\mathbb{C}P^r_{h\mathbb{T}})$ . Using theorem 8.3 we can pin down the possible structure of the differentials, but we cannot completely determine them. In particular, using only differential geometry, we can not prove that the differentials are non-trivial. But we will make an effort to obtain as much information as possible about the Morse spectral sequence with the information we already have available at this point.

It will turn out in section 13 that there are non-trivial differentials in the Morse spectral sequence. In that section we are going to compare the Morse spectral sequence to a purely homotopy theoretical spectral sequence. This will not quite solve the differentials, but it will at least determine the dimension of  $H^*(L\mathbb{C}P^r_{h\mathbb{T}})$  as a vector space over  $\mathbb{F}_p$ .

**Theorem 9.1.** *The Morse spectral sequence  $E_{*,*}(\mathcal{M})(L\mathbb{C}P^r_{h\mathbb{T}})$  is a spectral sequence of  $H(B\mathbb{T}) = \mathbb{F}_p[u]$  modules converging towards  $H^*(L\mathbb{C}P^r_{h\mathbb{T}})$ . In case  $p \mid (r+1)$  the  $E_1$  page is given by*

$$\begin{aligned} E_1^{0,*} &= \mathbb{F}_p[u, x]/(x^{r+1}), \\ E_1^{pm+k,*} &= \alpha_{pm+k} \mathbb{F}_p[x_1, x_2]/(Q_r, Q_{r+1}), \quad 0 \leq m, \quad 1 \leq k \leq p-1, \\ E_1^{pm,*} &= \alpha_{pm} \mathbb{F}_p[x, u]/(x^{r+1}) \oplus \zeta_{pm} \mathbb{F}_p[x, u]/(x^{r+1}), \quad 1 \leq m. \end{aligned}$$

In case  $p \nmid (r+1)$  the  $E_1$  page is given by

$$\begin{aligned} E_1^{0,*} &= \mathbb{F}_p[u, x]/(x^{r+1}), \\ E_1^{pm+k,*} &= \alpha_{pm+k} \mathbb{F}_p[x_1, x_2]/(Q_r, Q_{r+1}), \quad 0 \leq m, \quad 1 \leq k \leq p-1, \\ E_1^{pm,*} &= \alpha_{pm} \mathbb{F}_p[x, u]/(x^r) \oplus \bar{\zeta}_{pm} \mathbb{F}_p[x, u]/(x^r), \quad 1 \leq m. \end{aligned}$$

The total degree of the element  $\alpha_{pm+k}x_1^i x_2^j$  is  $2r(pm+k-1) + 2i + 2j + 1$  and the filtration degree is  $pm+k$ , where  $1 \leq k \leq p-1$ . The generators which are free  $\mathbb{F}_p[u]$ -module generators have total degree and filtration degree as follows:

class	case	total degree	filtration
$\alpha_{pm}x^i$	all $r, 0 \leq i \leq r-1$	$2r(pm-1) + 2i + 1$	$pm$
$\alpha_{pm}x^r$	$p \mid (r+1)$	$2r(pm-1) + 2r + 1$	$pm$
$\zeta_{pm}x^i$	$p \mid (r+1)$ and $0 \leq i \leq r$	$2rpm + 2i$	$pm$
$\zeta_{pm}x^i$	$p \nmid (r+1)$ and $0 \leq i \leq r-1$	$2rpm + 2 + 2i$	$pm$

*Proof.* The sequence of critical values for the energy function is given by  $\lambda_n = n^2$ . So in order to determine the  $E_1$  page we have to compute  $\tilde{H}^*(\text{Th}(\mu^-(n^2))_{h\mathbb{T}})$  for each  $n$ .

The case  $n = 0$  is special. The critical manifold  $N(0)$  consists of the constant curves, so it is diffeomorphic to  $\mathbb{C}\mathbb{P}^r$  itself, with trivial action of  $\mathbb{T}$ . Since the constant curves are the absolute minima of the energy function, the negative bundle is the trivial bundle. We find

$$E_1^{0,*}(\mathcal{M})(L\mathbb{C}\mathbb{P}_{h\mathbb{T}}^r) = H^*(B\mathbb{T} \times \mathbb{C}\mathbb{P}^r) = H^*(B\mathbb{T}) \otimes H^*(\mathbb{C}\mathbb{P}^r).$$

This is what the theorem states for  $E_1^{0,*}$ .

Now consider  $n > 0$ . The negative bundle  $\mu^-(n^2)$  is a  $\mathbb{T}$ -vector bundle over  $N(n^2) \cong S(\tau(\mathbb{C}\mathbb{P}^r))^{(n)}$ . We know from theorem 8.1 that

$$E_1^{n,*}(\mathcal{M})(L\mathbb{C}\mathbb{P}_{h\mathbb{T}}^r) \cong \tilde{H}^*(\text{Th}(\mu^-(n^2))_{h\mathbb{T}}).$$

Recall from [BO4] that if we forget the  $\mathbb{T}$ -action, then  $\mu^-(n^2)$  is the vector bundle  $(n-1)\eta^*(\tau(\mathbb{C}\mathbb{P}^r)) \oplus \epsilon$  over  $S(\tau(\mathbb{C}\mathbb{P}^r))$ . In particular, this means that  $\mu^-(n^2)$  is orientable. So by lemma 5.1, the vector bundle  $\mu^-(n^2)_{h\mathbb{T}}$  is orientable and  $\text{Th}(\mu^-(n^2))_{h\mathbb{T}} \cong \text{Th}(\mu^-(n^2))_{h\mathbb{T}}$ . The vector bundle  $\mu^-(n^2)_{h\mathbb{T}}$  is of the same dimension as  $\mu^-(n^2)$ , that is of dimension  $2r(n-1) + 1$ . So by the Thom isomorphism we find that

$$E_1^{n,*}(L\mathbb{C}\mathbb{P}_{h\mathbb{T}}^r) \cong \tilde{H}^*(\text{Th}(\mu^-(n^2))_{h\mathbb{T}}) \cong H^{*-(2r(n-1)+1)}(N(n^2)_{h\mathbb{T}}).$$

The theorem now follows from the computation of the Borel cohomology of the space of geodesics in theorem 4.1.  $\square$

*Remark 9.2.* The symbol  $\alpha_n x^i$  refers to the cup product in a critical manifold. The precise meaning is that it is the Thom isomorphism of the bundle  $\mu_n^-$  applied to the class  $x^i$ . In particular, the class  $\alpha_n$  is the Thom class itself. The product is not defined on the cohomology of the Thom space of the negative bundle. It is also not defined in the spectral sequence. So strictly speaking, the product notation is improper. Products with  $u$  on the other hand are genuine products, which are defined in the spectral sequence.

We also consider the non-equivariant case.

**Lemma 9.3.** *Consider the Morse spectral sequence*

$$\{E_r^{*,*}(\mathcal{M})(LCP^r)\} \Rightarrow H^*(LCP^r).$$

We have the following formulas for it's  $E_1$  page:

$$E_1^{n,*}(\mathcal{M})(LCP^r) = \begin{cases} \mathbb{F}_p[x]/(x^{r+1}) & , n = 0, \\ \alpha_n \mathbb{F}_p[x, \sigma]/(x^{r+1}, \sigma^2) & , n \geq 1, \quad p \mid (r+1), \\ \alpha_n \mathbb{F}_p[x, \bar{\sigma}]/(x^r, \bar{\sigma}^2) & , n \geq 1, \quad p \nmid (r+1). \end{cases}$$

The total degree of  $\alpha_n$  is  $2r(n-1) + 1$ , the total degree of  $\sigma$  is  $2r - 1$  and the total degree of  $\bar{\sigma}$  is  $2r + 1$ . If  $i \geq rn + 1$  and  $p \mid (r+1)$  or if  $i \geq rn$  and  $p \nmid (r+1)$  then  $E_1^{n, 2i+1-n}(\mathcal{M})(LCP^r) = 0$  for all  $n$ . Finally, the canonical map

$$E_1^{n, 2j+1-n}(\mathcal{M})(LCP_{h\mathbb{T}}^r) \rightarrow E_1^{n, 2j+1-n}(\mathcal{M})(LCP^r)$$

is surjective for all  $n$  and all  $j$ .

*Proof.* Most of this follows directly from remark 3.6. The class of highest even degree in  $\mathbb{F}_p[x, \sigma]/(x^{r+1}, \sigma^2)$  is  $x^r$  of degree  $2r$  and in  $\mathbb{F}_p[x, \bar{\sigma}]/(x^r, \bar{\sigma}^2)$  it is  $x^{r-1}$  of degree  $2r - 2$ . The stated vanishing result follows from this observation. The final surjectivity statement follows from the surjectivity result in corollary 4.2.  $\square$

A basic fact, which we will use again and again, is the following:

**Theorem 9.4.** *The Morse spectral sequence for non-equivariant cohomology collapses.*

This theorem follows for instance from the stable splitting of  $LCP^r$  that we constructed in [BO4]. But the theorem definitely does not belong to us. To prove the basic fact we only need a splitting on homology level, and such a splitting was known to Ziller [Z].

**Corollary 9.5.** *Let  $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq LCP^r$  be the energy filtration and let  $m \geq 0$ . If  $p \mid (r+1)$ , the odd degree cohomology  $H^{\text{odd}}(\mathcal{F}_m)$  is a vector space of dimension  $(r+1)m$ . If  $p \nmid (r+1)$ , its dimension is  $rm$ .*

*Proof.* According to theorem 9.4 the spectral sequence  $\{E_r^{*,*}\} = \{E_r^{*,*}(\mathcal{M})(LCP^r)\}$  collapses, so that

$$H^{\text{odd}}(\mathcal{F}_m) \cong \bigoplus_{i,n} E_\infty^{n, 2i+1-n} \cong \bigoplus_{i,n} E_1^{n, 2i+1-n},$$

where the direct sums are taken over pairs  $n, i$  such that  $0 \leq n \leq m$  and  $n \leq 2i + 1$ . By lemma 9.3 we have that  $E_1^{0, 2i+1} = 0$ , and for a fixed  $n$  with  $n \geq 1$  we have that

$$\sum_{n \leq 2i+1} \dim(E_1^{n, 2i+1-n}) = \begin{cases} r+1 & \text{if } p \mid (r+1), \\ r & \text{if } p \nmid (r+1), \end{cases}$$

independently of  $n$ . It follows that

$$\dim H^{\text{odd}}(\mathcal{F}_m) = \sum_{1 \leq n \leq m} \sum_{n \leq 2i+1} \dim(E_1^{n, 2i+1-n}) = \begin{cases} (r+1)m & \text{if } p \mid (r+1), \\ rm & \text{if } p \nmid (r+1). \end{cases}$$

$\square$

Later it will be convenient to be able to estimate the sheer size of the Morse spectral sequence using a coarse dimension counting. So let's get that over with.

**Lemma 9.6.** *In case  $p \mid (r + 1)$ , the Poincaré series of  $E_1^{*,*}(\mathcal{M})(L\mathbb{C}P_{h\mathbb{T}}^r)$  is*

$$\frac{1 - t^{2r+2}}{(1-t)(1-t^2)(1-t^{2pr})}.$$

*In case  $p \nmid (r + 1)$ , the Poincaré series of  $E_1^{*,*}(\mathcal{M})(L\mathbb{C}P_{h\mathbb{T}}^r)$  is*

$$\frac{1 - t^{2r+2} - t^{2pr} + t^{2pr+2}}{(1-t)(1-t^2)(1-t^{2pr})}.$$

*Proof.* In theorem 9.1 the  $E_1$  term is written as a direct sum of three terms. We compute the Poincaré series of each term, and add. First note that the Poincaré series of the algebra  $\mathbb{F}_p[x_1, x_2]/(Q_r, Q_{r+1})$  is

$$\frac{1 - t^{2r}}{1 - t^2} \frac{1 - t^{2r+2}}{1 - t^2}$$

as one sees by writing down a basis of monomials.

Here is the case  $p \mid (r + 1)$ :

$$\begin{aligned} & \frac{1}{1-t^2} \frac{1-t^{2r+2}}{1-t^2} + \frac{t}{1-t^{2pr}} \frac{1-t^{2r(p-1)}}{1-t^{2r}} \frac{1-t^{2r}}{1-t^2} \frac{1-t^{2r+2}}{1-t^2} + \\ & \frac{1}{1-t^2} \frac{1-t^{2r+2}}{1-t^2} \frac{t^{2r(p-1)+1} + t^{2rp}}{1-t^{2pr}}, \end{aligned}$$

which equals

$$\begin{aligned} & \frac{1 - t^{2r+2}}{(1-t^2)^2(1-t^{2pr})} [(1-t^{2pr}) + (t-t^{2r(p-1)+1}) + (t^{2pr} + t^{2r(p-1)+1})] = \\ & \frac{(1-t^{2r+2})(1+t)}{(1-t^2)^2(1-t^{2pr})} = \frac{1-t^{2r+2}}{(1-t)(1-t^2)(1-t^{2pr})}. \end{aligned}$$

In case  $p \nmid (r + 1)$ , the last term in the direct sum is changed and thus the Poincaré series equals the sum

$$\begin{aligned} & \frac{1}{1-t^2} \frac{1-t^{2r+2}}{1-t^2} + \frac{t}{1-t^{2pr}} \frac{1-t^{2r(p-1)}}{1-t^{2r}} \frac{1-t^{2r}}{1-t^2} \frac{1-t^{2r+2}}{1-t^2} + \\ & \frac{1}{1-t^2} \frac{1-t^{2r}}{1-t^2} \frac{t^{2r(p-1)+1} + t^{2pr+2}}{1-t^{2pr}}, \end{aligned}$$

which equals

$$\begin{aligned} & \frac{1}{(1-t^2)^2(1-t^{2pr})} [(1-t^{2r+2} - t^{2pr} + t^{2pr+2r+2}) \\ & + (t-t^{2(p-1)r+1} - t^{2r+3} + t^{2pr+3}) + (t^{2(p-1)r+1} + t^{2pr+2} - t^{2pr+1} - t^{2pr+2r+2})] = \\ & \frac{1 - t^{2r+2} - t^{2pr} + t^{2pr+2}}{(1-t)(1-t^2)(1-t^{2pr})}. \end{aligned}$$

□

Now that we have completed the census, we will investigate what we can say about the algebraic properties of the Morse spectral sequence for  $\mathbb{C}P^r$ .

**Lemma 9.7.** *The classes  $\zeta_{pm}x^i$ ,  $\bar{\zeta}_{pm}x^i$ ,  $\alpha_{pm}x^i$  and  $\alpha_nx_1^i$  are not in the image of any differential.*

*Proof.* The inclusion map  $i : LCP^r \rightarrow LCP_{h\mathbb{T}}^r$  induces a map of filtrations, and a map of cohomology Morse spectral sequences. We know from theorem 9.4 that the target spectral sequence collapses.

But on the  $E^1$  level, the classes  $\zeta_{pm}x^i$  and  $\bar{\zeta}_{pm}x^i$  survive to the classes represented by the Thom isomorphism applied to  $\sigma x^i$  respectively  $\bar{\sigma}x^i$ , according to corollary 4.2. These classes are not in the image of a differential (since all differentials vanish). So by naturality, the classes  $\zeta_{pm}x^i$  and  $\bar{\zeta}_{pm}x^i$  cannot be in the image of a differential either.

Similarly, the classes  $\alpha_{pm}x^i$  and  $\alpha_nx_1^i$  are mapped non-trivially according to corollary 4.2 and the result follows.  $\square$

**Lemma 9.8.** *In the Morse spectral sequence  $E_*^{*,*}(\mathcal{M})(LCP_{h\mathbb{T}}^r)$  every non-trivial differential starts in an even total degree.*

*Proof.* We have to do some counting, but the strategy of the proof is simple. First we note that it is enough to consider the generators of  $E_1$ , secondly we see that except for a very few special cases these generators cannot map non-trivially for dimensional reasons. Thirdly we use lemma 9.7 to dispose of the remaining cases.

The elements of odd degree in the spectral sequence are of the form  $\alpha_{pm}x^i u^j$  or  $\alpha_nx_1^i x_2^j$ . Because the spectral sequence is a spectral sequence of  $\mathbb{F}_p[u]$  modules, to prove the lemma it suffices to show the special case that all differentials vanish on the generators  $\alpha_{pm}x^i$  and  $\alpha_nx_1^i$ .

We consider a class  $d_s(\alpha_{pm}x^i)$ . We wish to prove that it is trivial. This class has filtration  $pm + s$  and total degree  $2r(pm - 1) + 2i + 2$ . We ask, when does there exist a non trivial class of this filtration and total degree? To figure this out, we inspect the table of theorem 9.1, and realize the following facts.

- There exist non-trivial classes in  $E_1(\mathcal{M})(LCP_{h\mathbb{T}}^r; \mathbb{F}_p)$  of total even degree, and of filtration  $n$  if and only if  $n$  is divisible by  $p$ .
- If  $n$  is divisible by  $p$  there is a unique such class of filtration  $n$  with lowest possible even dimension. In case  $p$  divides  $r + 1$  it is  $\zeta_n$  of dimension  $2rn$ , if  $p$  does not divide  $r + 1$  it is  $\bar{\zeta}_n$  of dimension  $2rn + 2$ .

The class  $d_s(\alpha_{pm}x^i)$  has total even degree, so if it were non-trivial it would to have degree at least equal to the lowest possible degree of such a class. That is

$$2r(pm - 1) + 2i + 2 \geq \begin{cases} 2r(pm + s), & p \mid (r + 1), \\ 2r(pm + s) + 2, & p \nmid (r + 1). \end{cases}$$

If  $p$  does not divide  $r + 1$ , we have the two inequalities  $i \geq r(s + 1)$  and  $r - 1 \geq i$ . This system has no solution with  $s \geq 1$ . If  $p$  divides  $r + 1$ , we have the two inequalities  $i + 1 \geq r(s + 1)$  and  $r \geq i$ . Now there is the unique solution  $s = r = i = 1$ . That

is, we would have that  $d_1(\alpha_{pm-1}x) = \lambda\zeta_{pm}$  for a scalar  $\lambda \neq 0$ . But this contradicts lemma 9.7.

A similar argument shows that  $d_s(\alpha_n x_1^i) = 0$ .  $\square$

At this point, to a certain extent we can make a shortcut in the theory. Using the elementary method as in the proof of lemma 9.8 we can prove the following statement (theorem 9.9). The term “elementary” should be read as “without using theorem 8.3”. Together with the purely homotopy theoretical analysis of the Serre spectral sequence (which we discuss in detail in section 12), theorem 9.9 is sufficient to compute the cohomology  $H^*(LCP_{h\mathbb{T}}^r)$  in the special case  $p \mid (r+1)$ . It does not seem possible to cover the case  $p \nmid (r+1)$  using a similar method. It is precisely because of this that we need theorem 8.3.

**Theorem 9.9.** *If  $p$  divides  $r+1$  then there is a class  $\hat{\zeta}_{pm} \in H^{2pmr}(LCP_{h\mathbb{T}}^r)$  such that the restriction  $i^*(\hat{\zeta}_{pm}) \in H^{2pmr}(LCP^r)$  is non-trivial.*

*Proof.* The argument is very similar to the proof of lemma 9.8. We show that the class  $\zeta_{pm} \in E_1^{pm, (2r-1)pm}(\mathcal{M})(LCP_{h\mathbb{T}}^r)$  of theorem 9.1 survives to  $E_\infty$ .

Suppose that  $d_s(\zeta_{pm}) = \alpha_{pm+s}y$  for some nontrivial scalar  $y$ . Equality of total degrees gives the relation

$$2rpm + 1 = \deg(\zeta_{pm}) + 1 = \deg(\alpha_{pm+s}y) = 2r(pm + s - 1) + \deg(y) + 1 \geq 2rpm + 1.$$

So we conclude that  $s = 1$ ,  $\deg(y) = 0$ , and  $d_1(\zeta_{pm}) = \lambda\alpha_{pm+1}$  for a non-trivial scalar  $\lambda$ . But this contradicts lemma 9.7. That is,  $\zeta_{pm}$  is a permanent cycle. Furthermore,  $\zeta_{pm}$  is not in the image of any differential by lemma 9.7, so it survives to  $E_\infty$ .

We claim that any class  $\hat{\zeta}_{pm} \in H^{2pmr}(LCP_{h\mathbb{T}}^r; \mathbb{F}_p)$  representing  $\zeta_{pm}$  will satisfy the property of the theorem. By naturality,  $\zeta_{pm}$  maps nontrivially by  $i^*$  to the  $E_1$  page of the Morse spectral sequence converging to  $H^*(LCP^r; \mathbb{F}_p)$ . The theorem follows, again since this spectral sequence collapses.  $\square$

In order to prove this sections final result we need a lemma on non-negatively graded modules over the graded ring  $\mathbb{F}_p[u]$  (graded by  $\deg(u) = 2$ ). We say that the graded module  $M$  is trivial in degrees strictly greater than  $n$  if  $M^i = 0$  for all  $i > n$ .

Let us also say that a graded  $\mathbb{F}_p[u]$ -module is generated in degrees less or equal to  $m$  if there is a set of generators which have degrees less or equal to  $m$ . Because  $M$  is bounded from below, a set of elements  $\{x_\alpha\}$  generate  $M$  exactly if the reductions  $[x_\alpha]$  span the  $\mathbb{F}_p$ -vector space  $M/uM$ .

**Lemma 9.10.** *Let  $f : M \rightarrow N$  be a degree preserving map of non-negatively graded  $\mathbb{F}_p[u]$  modules. Assume that  $M$  is generated in degrees less or equal to  $m$ , and that  $N$  is trivial in degrees strictly greater than  $n$ . Then the kernel of  $f$  is generated in degrees less or equal to  $\max(m, n)$ .*

*Proof.* Assume without loss of generality that  $f$  is surjective. Multiplication by  $u$  defines a map of the short exact sequence  $0 \rightarrow \ker(f) \rightarrow M \rightarrow N \rightarrow 0$  to itself. The snake lemma produces an exact sequence

$$\ker(u : N \rightarrow N) \longrightarrow \ker(f)/u\ker(f) \longrightarrow M/uM \longrightarrow N/uN \longrightarrow 0.$$

That  $M$  is generated in degrees less or equal to  $m$  implies that the graded vector space  $M/uM$  is zero in degrees strictly greater than  $m$ . The exact sequence proves that  $\ker(f)/u\ker(f)$  is trivial in degrees greater than  $\max(m, n)$ . Pick a set of spanning classes  $\{[x_\alpha]\}$  in  $\ker(f)/u\ker(f)$ , and lift them arbitrarily to classes  $\{x_\alpha\}$  in  $\ker(f)$  of degree less or equal to  $\max(m, n)$ . These classes generate  $\ker(f)$ .  $\square$

We are now ready to apply our general theorem on the localized Morse spectral sequence to our particular specimen.

**Theorem 9.11.** *The spectral sequence  $E_*(\mathcal{M})(LCP_{h\mathbb{T}}^r)$  collapses from the  $E_p$  page.*

*Proof.* Recall from lemma 9.8 that all non-trivial differentials are defined on classes in even total degree. It is easy to see from theorem 9.1 that the only classes of even total degree sit in filtrations divisible by  $p$ . Let  $X_s^{pm,*} \subseteq E_s^{pm,*}(\mathcal{M})(LCP_{h\mathbb{T}}^r)$  be the subgroup of elements of even total degree. This is a module over the ring  $\mathbb{F}_p[u]$ , and for all  $s$  one has

$$X_{s+1}^{pm,*} = \ker(d_s : X_s^{pm,*} \rightarrow E_s^{pm+s,*}).$$

*Claim.* If  $1 \leq s \leq p-1$ , then the  $\mathbb{F}_p[u]$ -module  $X_{s+1}^{pm,*}$  is generated by elements in degrees less than  $2r(pm+s+1) - 2$ .

The claim follows from lemma 9.10 and induction over  $s$ . By theorem 9.1 we see that  $X_1^{pm,*}$  in all cases is generated in degrees less or equal to  $2pmr+2r$ .  $E_s^{pm+s,*}$  is a quotient of the  $\mathbb{F}_p[u]$ -module  $E_1^{pm+s,*}$ . Using theorem 9.1 we see that for  $1 \leq s \leq p-1$  this last module is trivial in dimensions strictly greater than  $2r(pm+s) + 2r - 1$ . Now we use lemma 9.10 on the differential

$$d_1 : X_1^{pm,*} \rightarrow E_1^{pm+1,*}.$$

The differential raises total degree by 1. Taking this into account, lemma 9.10 shows that  $X_1^{pm}$  is generated in degree less than  $\max(2pmr+2r, 2r(pm+1)+2r-2) = 2r(pm+2) - 2$ , which proves the claim for  $s = 1$ . Using lemma 9.10 inductively on  $d_s$  for  $2 \leq s \leq p-1$  proves the claim.

In particular,  $X_p^{mp,*}$  is generated in degrees less or equal to  $2r(pm+p) - 2$ . The basic fact 9.4 says that the Morse spectral sequence  $E_*(\mathcal{M})(LCP^r)$  collapses from the  $E_1$  page. Because of theorem 8.3 this implies that the localized spectral sequence  $E_*(\mathcal{M})(LCP_{h\mathbb{T}}^r)[1/u]$  collapses from the  $E_1$  page.

From our computation in theorem 9.1 we can read off that the localization map

$$E_1^{pm,*}(\mathcal{M})(LCP_{h\mathbb{T}}^r) \longrightarrow E_1^{pm,*}(\mathcal{M})(LCP_{h\mathbb{T}}^r) \left[ \frac{1}{u} \right]$$

is injective. By naturality, this means that no non-trivial differentials are arriving at  $E_1^{pm,*}(\mathcal{M})(LCP_{h\mathbb{T}}^r)$ . So in particular, the  $d_p$  differential on  $X_p^{mp,*}$  is trivial, and  $X_{p+1}^{mp,*} = X_p^{mp,*}$  is generated in degrees less than  $2r(pm+p) - 2$ .

We claim that all higher differentials  $d_s$  vanish on  $X_{p+1}^{pm,*}$ . Assume to the contrary that  $s$  is the smallest  $s \geq p+1$  which does not vanish on  $X_s^{pm,*}$ . Since  $X_s^{pm,*} = X_{p+1}^{pm,*}$ , we know that  $X_s^{pm,*}$  is generated in degrees less or equal to  $2r(pm+p) - 2$ .

It is enough to prove that the differential  $d_s$  vanishes on the generators. A generator of  $X_{p+1}^{pm,*}$  will be mapped by  $d_s$  to a class in some  $E_{p+1}^{pm+s,*}$  of total degree

less or equal to  $2r(pm + p) - 1$ . We have to show that there is no such non-trivial class.

But the non-zero class of lowest degree in  $E_{p+1}^{pm+s,*}$  is  $\alpha_{pm+s}$  which has degree  $2r(pm + s - 1) + 1$ . Since we are assuming that  $s \geq p + 1$ , this degree is larger than or equal to  $2r(pm + p) + 1$ . This finishes the proof.  $\square$

## 10 Derived functors at odd primes

In this section and in the following two sections we are studying the cohomology of the free loop space by homotopy theoretical methods. The main result we are aiming for is a computation of the action of the circle on the cohomology. It seems hard to obtain this using Morse theory methods, since the action will cause non-trivial interaction between the layers in the Morse filtration. In section 13 we will show that this computation has consequences for the Morse spectral sequence  $E_*(\mathcal{M})(LCP_{h\mathbb{T}}^r)$ , and essentially solves its differentials.

In this section we make a preliminary algebraic computation, which we will need for the homotopy theory calculation.

Let as before  $p$  be a prime number. Write  $T_r(x)$  for the truncated polynomial algebra  $\mathbb{F}_p[x]/(x^{r+1})$ , where  $r \geq 1$  and the degree of the generator  $|x|$  is a positive even number. In [BO3] we defined the derived functors  $H_*(T_r(x); \overline{\Omega})$ , and in the case  $p = 2$  we computed them. In this section we extend this calculation to compute these derived functors for odd  $p$ . We will use definitions and notation from [BO3].

Assume that  $p$  is an odd prime. Recall the functor  $\overline{\Omega} : \mathcal{F} \rightarrow \mathcal{Alg}$  from [BO2]. The cohomology with  $\mathbb{F}_p$  coefficients of a space defines an object in  $\mathcal{F}$ , and  $\mathcal{Alg}$  is the category of non-negatively graded algebras  $A$  over  $\mathbb{F}_p$  such that  $a^p = a$  for  $a \in A^0$ . We view  $T_n(x)$  as an object in  $\mathcal{F}$  with  $\lambda = 0$  and  $\beta = 0$ . Note that by definition of  $\mathcal{F}$ , any polynomial algebra  $\mathbb{F}_p[z_i | i \in I]$  on even dimensional generators is a free object in  $\mathcal{F}$ . This special type of free object has trivial  $\lambda$  and  $\beta$ . By a similar argument as in [BO3] Theorem 2.1 we find the following result:

**Theorem 10.1.** *For odd primes  $p$ , there is an almost free simplicial resolution  $R_\bullet \in s\mathcal{F}$  of the object  $T_r(x) \in \mathcal{F}$  as follows:  $R_q = \mathbb{F}_p[x, y_1, \dots, y_q]$  for  $q \geq 0$  where  $|y_i| = (r + 1)|x|$ . The face and degeneracy maps are given by  $s_i(x) = x$ ,  $d_i(x) = x$ ,*

$$s_i(y_j) = \begin{cases} y_j & , i \geq j, \\ y_{j+1} & , i < j, \end{cases}$$

$$d_i(y_j) = \begin{cases} x^{r+1} & , i = 0, j = 1, \\ y_{j-1} & , i < j, j > 1, \\ y_j & , i \geq j, j < q, \\ 0 & , i = q, j = q. \end{cases}$$

Using this resolution, the derived functors can be computed as the homotopy groups  $H_i(T_r(x); \overline{\Omega}) = \pi_i \overline{\Omega}(R_\bullet)$ . It is convenient to use the normalized chain complex  $N_*(\overline{\Omega}R_\bullet)$  with  $N_i(\overline{\Omega}R_\bullet) = \cap_{j=1}^i \ker(d_j)$  and differential  $d_0$  for this purpose.

The de Rham differential on  $\overline{\Omega}R_\bullet$  is not the only simplicial derivation. There is another one which turns out to be useful.

**Lemma 10.2.** *There is a well defined derivation  $\theta : \overline{\Omega}R_q \rightarrow \overline{\Omega}R_q$  of degree 1 for each  $q \geq 0$  which satisfies  $\theta(ab) = \theta(a)b + (-1)^{|a|}a\theta(b)$  for all  $a, b \in \overline{\Omega}R_q$  and is defined by the following for  $1 \leq j \leq q$ :*

$$\theta(x) = 0, \quad \theta(y_j) = 0, \quad \theta(\mathbf{d}x) = x, \quad \theta(\mathbf{d}y_j) = (r+1)y_j.$$

*One has that  $\theta \circ \theta = 0$ . Furthermore,  $\theta$  commutes with the simplicial face and degeneracy maps and hence defines a simplicial derivation  $\theta : \overline{\Omega}R_\bullet \rightarrow \overline{\Omega}R_\bullet$ .*

*Proof.* We have that  $\overline{\Omega}R_q = \mathbb{F}_p[x, y_1, \dots, y_q] \otimes \Lambda(\mathbf{d}x, \mathbf{d}y_1, \dots, \mathbf{d}y_q)$ . By the derivation property we see that  $\theta((\mathbf{d}x)^2) = 0$  and  $\theta((\mathbf{d}y_j)^2) = 0$  so  $\theta$  is well defined. The derivation property for  $\theta$  also implies that  $\theta \circ \theta$  is a derivation of degree two. So  $\theta \circ \theta$  is zero since it maps the algebra generators to zero.

One checks that  $\theta(s_i z) = s_i \theta(z)$  and  $\theta(d_i z) = d_i \theta(z)$  for each algebra generator  $z$  by direct computations. The most interesting case goes as follows:

$$\begin{aligned} \theta(d_0(\mathbf{d}y_1)) &= \theta(\mathbf{d}(x^{r+1})) = \theta((r+1)x^r \mathbf{d}x) = (r+1)x^{r+1} = d_0((r+1)y_1) \\ &= d_0 \theta(\mathbf{d}y_1). \end{aligned}$$

□

Here is a complete computation of the derived functors.

**Theorem 10.3.** *If  $p$  is a prime such that  $p \mid (r+1)$ , Then there is an isomorphism of bigraded  $\mathbb{F}_p$ -algebras*

$$H_*(T_r(x); \overline{\Omega}) \cong T_r(x) \otimes \Lambda(\mathbf{d}x) \otimes \Gamma[\omega],$$

where  $\|x\| = (0, |x|)$ ,  $\|\mathbf{d}x\| = (0, |x| - 1)$ ,  $\|\gamma_i(\omega)\| = (i, i((r+1)|x| - 1))$ . The algebra generators are represented by cycles in the normalized chain complex  $N_*(\overline{\Omega}R_\bullet)$  as follows:  $x = [x]$ ,  $\mathbf{d}x = [\mathbf{d}x]$ ,  $\gamma_i(\omega) = [\mathbf{d}y_1 \dots \mathbf{d}y_i]$ . The de Rham differential induces the map

$$\mathbf{d}_* : H_*(T_r(x); \overline{\Omega}) \rightarrow H_*(T_r(x); \overline{\Omega}); \quad x \mapsto \mathbf{d}x, \quad \mathbf{d}x \mapsto 0, \quad \gamma_i(\omega) \mapsto 0.$$

*If  $p$  is a prime such that  $p \nmid (r+1)$ , then there is an isomorphism of bigraded  $\mathbb{F}_p$ -algebras*

$$H_*(T_r(x); \overline{\Omega}) \cong \mathbb{F}_p[a_i, b_i | i \geq 0] / I_r,$$

where  $I_r$  is the ideal generated by the following elements for  $i, j \geq 0$ :

$$a_i a_j, \quad b_i b_j - \binom{i+j}{i} b_0 b_{i+j}, \quad a_i b_j - \binom{i+j}{i} b_0 a_{i+j}, \quad b_0^r b_i, \quad b_0^r a_i.$$

Here  $\|a_i\| = (i, i((r+1)|x| - 1) + |x| - 1)$  and  $\|b_i\| = \|a_i\| + (0, 1)$ . The algebra generators are represented by cycles in the normalized chain complex  $N_*(\overline{\Omega}R_\bullet)$  as follows:  $a_i = [\mathbf{d}x \mathbf{d}y_1 \dots \mathbf{d}y_i]$ ,  $b_i = [\theta(\mathbf{d}x \mathbf{d}y_1 \dots \mathbf{d}y_i)]$ . The de Rham differential induces the map

$$\mathbf{d}_* : H_*(T_r(x); \overline{\Omega}) \rightarrow H_*(T_r(x); \overline{\Omega}); \quad a_i \mapsto 0, \quad b_i \mapsto (1 + (r+1)i)a_i.$$

*Remark 10.4.* Explicitly, the cycle that represents  $b_i$  is

$$\theta(\mathbf{d}x\mathbf{d}y_1 \dots \mathbf{d}y_i) = x\mathbf{d}y_1 \dots \mathbf{d}y_i + (r+1)\mathbf{d}x \sum_{k=1}^i (-1)^k y_k \mathbf{d}y_1 \dots \widehat{\mathbf{d}y_k} \dots \mathbf{d}y_i.$$

*Proof.* For  $p = 2$  these results were proved in [BO3] so assume that  $p$  is an odd prime. We first compute the derived functors as  $\mathbb{F}_p$ -vector spaces.

We have that  $T_r(x)$  is the pushout of the diagram  $\mathbb{F}_p \leftarrow \mathbb{F}_p[y] \rightarrow \mathbb{F}_p[x]$  in  $\mathcal{F}$  where  $y \mapsto x^{r+1}$ . Since  $\overline{\Omega}$  commutes with colimits (appendix of [O]) and  $\mathbb{F}_p[x]$  is a free module over  $\mathbb{F}_p[y]$ , proposition 6.3 of [BO2] gives us a Quillen spectral sequence

$$E_{i,j}^2 = \mathrm{Tor}_i^{H_*(\mathbb{F}_p[y]; \overline{\Omega})}(\mathbb{F}_p, H_*(\mathbb{F}_p[x]; \overline{\Omega}))_j \Rightarrow H_{i+j}(T_r(x); \overline{\Omega}).$$

Since polynomial algebras on even dimensional generators are free objects in  $\mathcal{F}$ , we see that  $E_{i,j}^2 = 0$  for  $j > 0$  and that

$$H_i(T_r(x); \overline{\Omega}) \cong E_{i,0}^2 \cong \mathrm{Tor}_i^{\overline{\Omega}(\mathbb{F}_p[y])}(\mathbb{F}_p, \overline{\Omega}(\mathbb{F}_p[x])).$$

Note that  $\overline{\Omega}(\mathbb{F}_p[y]) = \mathbb{F}_p[y] \otimes \Lambda(\mathbf{d}y)$  is the free graded commutative algebra on  $\{y, \mathbf{d}y\}$  so we have the Koszul resolution  $(K_*, \partial)$  of  $\mathbb{F}_p$  by free  $\overline{\Omega}(\mathbb{F}_p[y])$ -modules:

$$K_* = \Lambda(v) \otimes \Gamma[w] \otimes \overline{\Omega}(\mathbb{F}_p[y]); \quad \partial v = y, \quad \partial \gamma_i(w) = \gamma_{i-1}(w) \mathbf{d}y,$$

where  $v \in K_1$  and  $\gamma_i(w) \in K_i$ . In order to compute the group  $\mathrm{Tor}_i^{\overline{\Omega}(\mathbb{F}_p[y])}(\mathbb{F}_p, \overline{\Omega}(\mathbb{F}_p[x]))$  we tensor this Koszul resolution with  $\overline{\Omega}(\mathbb{F}_p[x])$  over  $\overline{\Omega}(\mathbb{F}_p[y])$  and get a chain complex  $(C_*, \partial)$  with

$$C_* = \Lambda(v) \otimes \Gamma[w] \otimes \overline{\Omega}(\mathbb{F}_p[x]), \quad \partial v = x^{r+1}, \quad \partial \gamma_i(w) = (r+1)\gamma_{i-1}(w)x^r \mathbf{d}x.$$

Computing the homology of this chain complex, we find that if  $p \mid (r+1)$  then

$$H_*(T_r(x); \overline{\Omega}) \cong T_r(x) \otimes \Lambda(\mathbf{d}x) \otimes \Gamma[w].$$

If  $p \nmid (r+1)$  we compute that  $H_0(T_r(x); \overline{\Omega}) \cong T_r(x) \otimes \Lambda(\mathbf{d}x)/(x^r \mathbf{d}x)$ . For  $i > 0$ ,  $H_i(T_r(x); \overline{\Omega})$  is a sum of two copies of  $T_r(x)/(x^r)$ , the generators being  $\mathbf{d}x\gamma_i(w)$  and  $x\gamma_i(w) - (r+1)\mathbf{d}xv\gamma_{i-1}(w)$ . This completes the computation of the derived functors as  $\mathbb{F}_p$ -vector spaces.

We now check that the listed representatives are indeed cycles in the chain complex  $N_*(\overline{\Omega}R_\bullet)$ . Define elements in  $\overline{\Omega}R_i$  using the derivation  $\theta$  as follows:

$$\omega_i = \mathbf{d}y_1 \dots \mathbf{d}y_i, \quad \alpha_i = \mathbf{d}x\omega_i, \quad \beta_i = \theta(\alpha_i).$$

Here by convention  $\omega_0 = 1$  such that  $\alpha_0 = \mathbf{d}x$  and  $\beta_0 = x$ . We have that  $d_j\omega_i = 0$  for  $0 < j \leq i$  since  $(\mathbf{d}y_j)^2 = 0$  and  $d_i y_i = 0$ . Furthermore,  $d_0\omega_i = (r+1)x^r \mathbf{d}x\omega_{i-1}$ . Thus  $\omega_i$  is a cycle when  $p \mid (r+1)$  and  $\alpha_i, \beta_i$  are cycles when  $p \nmid (r+1)$  as stated.

By a similar argument as the one presented in [BO3] theorem 2.5 one checks that

$$[x^j(\mathbf{d}x)^\epsilon \omega_i], \quad 0 \leq j \leq n, \quad \epsilon \in \{0, 1\}$$

are linearly independent in  $H_i(N_*\overline{\Omega}R_\bullet)$  when  $p \mid (r+1)$  and that

$$[\beta_0^j \alpha_i], [\beta_0^j \beta_i], \quad 0 \leq j \leq r-1$$

are linearly independent in  $H_i(N_*\overline{\Omega}R_\bullet)$  when  $p \nmid (r+1)$ . By a dimension count, these two sets are then vector spaces bases.

We now prove that the algebra structure of  $H_*(T_r(x); \overline{\Omega})$  is as stated. The algebra structure comes from the chain map

$$\rho : C_*(\overline{\Omega}R) \otimes C_*(\overline{\Omega}R) \xrightarrow{g} C_*(\overline{\Omega}R \otimes \overline{\Omega}R) \xrightarrow{C_*(m)} C_*(\overline{\Omega}R)$$

where  $g$  is the Eilenberg–MacLane shuffle map [ML] and  $m$  the multiplication map on  $\overline{\Omega}R_\bullet$ . By  $C_*(V)$  for a simplicial  $\mathbb{F}_p$ -vector space  $V$ , we mean the chain complex with  $C_i(V) = V_i$  and differential  $\sum_{j=0}^i (-1)^j d_j$ . So we have the formula

$$\rho(v_i \otimes w_j) = \sum_{(\mu, \nu)} (-1)^{\epsilon(\mu)} s_\nu(v_i) s_\mu(w_j)$$

where the sum is over all  $(i, j)$  shuffles  $(\mu, \nu)$  of the set  $\{0, 1, \dots, i+j-1\}$  and  $\epsilon(\mu) = \sum_{k=1}^i (\mu_k - (k-1))$ .

Lemma 2.4 of [BO3] still holds for  $\mathbb{F}_p$ -coefficients. (There is a small misprint in the lemma: There should be a hat over  $\nu_j$  in the index set of the last formula.) So we find that

$$\rho(\omega_i \otimes \omega_j) = \binom{i+j}{i} \omega_{i+j}.$$

By this formula and remark 10.4 it follows directly that

$$\rho(\alpha_i \otimes \alpha_j) = 0 \quad , \quad \rho(\alpha_i \otimes \beta_j) = \binom{i+j}{i} \beta_0 \alpha_{i+j}.$$

Since the degeneracy maps commutes with  $\theta$ , there is a commutative diagram as follows where  $A_\bullet = \overline{\Omega}R_\bullet$  :

$$\begin{array}{ccccc} A_i \otimes A_j & \xrightarrow{g} & A_{i+j} \otimes A_{i+j} & \xrightarrow{m} & A_{i+j} \\ \theta \otimes 1 + (-1)^{i+j} 1 \otimes \theta \downarrow & & \theta \otimes 1 + (-1)^{i+j} 1 \otimes \theta \downarrow & & \downarrow \theta \\ A_i \otimes A_j & \xrightarrow{g} & A_{i+j} \otimes A_{i+j} & \xrightarrow{m} & A_{i+j} \end{array}$$

By mapping  $\alpha_i \otimes \theta(\alpha_j)$  both ways around one finds that

$$\rho(\beta_i \otimes \beta_j) = \binom{i+j}{i} \beta_0 \beta_{i+j}.$$

Thus the algebra structure is as stated. It follows directly from the formulas for the representing cycles, that the de Rham map is as stated.  $\square$

## 11 Cohomology of the free loop space

**Theorem 11.1.** *Let  $p$  be a prime. Assume that  $X$  is a 1-connected space with mod  $p$  homology of finite type. Assume also that*

$$H^*(X) = \mathbb{F}_p[x]/(x^{r+1}),$$

where the degree  $|x|$  of  $x$  is even and  $r \geq 1$ . Put  $\alpha = |x|$  and  $\rho = (r+1)\alpha - 2$ .

1) If  $p \mid (r+1)$  then there is an algebra isomorphism

$$H^*(LX) \cong \mathbb{F}_p[x]/(x^{r+1}) \otimes \Lambda(\mathbf{d}x) \otimes \Gamma[\omega],$$

where  $|x| = \alpha$ ,  $|\mathbf{d}x| = \alpha - 1$ ,  $|\gamma_i(\omega)| = \rho i$ . The action differential is given by

$$d : H^*(LX) \rightarrow H^*(LX); \quad d(x) = \mathbf{d}x, \quad d(\mathbf{d}x) = 0, \quad d(\gamma_i(\omega)) = 0.$$

2) If  $p \nmid (r+1)$  then there is an algebra isomorphism

$$H^*(LX) \cong \mathbb{F}_p[a_i, b_i | i \geq 0]/I,$$

where  $I$  is the ideal generated by the following elements for  $i, j \geq 0$ :

$$a_i a_j, \quad b_i b_j - \binom{i+j}{i} b_0 b_{i+j}, \quad b_i a_j - \binom{i+j}{i} b_0 a_{i+j}, \quad b_0^r b_i, \quad b_0^r a_i.$$

The degrees of the generators are  $|a_i| = \rho i + \alpha - 1$  and  $|b_i| = \rho i + \alpha$ . In particular, the dimensions of  $H^{2k}(LX)$  and  $H^{2k-1}(LX)$  are the same. The action differential is given by

$$d : H^*(LX) \rightarrow H^*(LX); \quad d(a_i) = \kappa_i b_0^{r-1} b_i, \quad d(b_i) = ((r+1)i + 1)a_i.$$

where  $\kappa_i = 0$  unless  $\alpha = 2$ ,  $p = 2$ ,  $r$  is even and  $i$  is odd.

*Remark 11.2.* When  $p \nmid (r+1)$  we have the following formula for  $0 \leq i$ ,  $1 \leq j \leq r$ :

$$d(b_0^{j-1} b_i) = ((r+1)i + j) b_0^{j-1} a_i.$$

We will show later (in corollary 13.8) that  $\kappa_i = 0$ . So we have  $d(b_0^{j-1} a_i) = 0$ .

*Proof.* According to [BO2] there is a strongly convergent second quadrant spectral sequence of cohomology type

$$E_2^{-i,j} = H_i(H^*(X); \overline{\Omega})^j \Rightarrow H^*(LX),$$

where the  $E_2$  page is the derived functors which we computed in theorem 10.3. The spectral sequence is a spectral sequence of algebras, and the induced of the de Rham differential  $\mathbf{d}_*$  on  $E_2$  corresponds to the action differential  $d$  on  $H^*(LX)$ .

There are two distinct cases:  $p \mid (r+1)$  and  $p \nmid (r+1)$ . In both cases, the  $E^2$  page does not admit any differentials because of its distribution of zeros, so  $E_\infty \cong E_2$  as a vector space over  $\mathbb{F}_p$ . The theorem can be paraphrased as that the  $E_2$  page is isomorphic to  $H^*(LX)$  as an algebra, and that also the action differential agrees.

We have to look close to exclude the possibility of multiplicative extensions as well as extension problems concerning the action differential. Write  $|\cdot|$  for the total degree in the spectral sequence.

1) Assume that  $p \mid (r+1)$ . Since  $x$  and  $\mathbf{d}x$  lie in  $H_0(T_r(x); \overline{\Omega})$  they have unique representatives in  $H^*(LX)$  which satisfy  $x^{r+1} = 0$ ,  $(\mathbf{d}x)^2 = 0$  and  $d(x) = \mathbf{d}x$ . So the possible extension questions are if the relations  $\gamma_i^p = 0$  and  $d\gamma_i = 0$  are valid. Let us denote a class in  $H^*(LX)$  representing  $\gamma_i \in E_\infty$  by the symbol  $\overline{\gamma}_i$ .

We look at  $\overline{\gamma}_i^p$ . We know that  $\overline{\gamma}_i^p = 0$  up to classes of the same total degree and strictly higher filtration. The class  $\gamma_{ip}$  has the same total degree, but also the same filtration degree as  $\gamma_i^p$ . Using that  $|\gamma_{j+1}| = |x^r \gamma_j| + \alpha - 2$ , we see that if  $\alpha \geq 4$  this is the only class of the same total degree as  $\gamma_i^p$ , so that independently of the choice of  $\overline{\gamma}_i$ , we indeed have that  $\overline{\gamma}_i^p = 0$ .

Similarly  $d\gamma_i = 0$  up to classes of the same total degree and of strictly higher filtration. But if  $\alpha \geq 4$ , using that  $|d\gamma_{j+1}| = |x^{r-1} \mathbf{d}x \gamma_j| + \alpha - 2$  we see that there is no such class. So  $d\overline{\gamma}_i = 0$ .

Now we consider the slightly more complicated case  $\alpha = 2$ . In this case the class  $x^r \gamma_j$  has the same total degree as  $\gamma_{j+1}$  and strictly higher filtration. It is the only such class. Similarly,  $x^{r-1} \mathbf{d}x \gamma_j$  is the only class of same total degree as  $d\gamma_{j+1}$  and of strictly higher filtration.

The filtration of  $H^* = H^*(LX)$  has the form

$$H^* \supseteq \dots \supseteq F^{-i}H^* \supseteq F^{-i+1}H^* \supseteq \dots \supseteq F^0H^* \supseteq F^1H^* = 0.$$

We now show that we can choose  $\overline{\gamma}_i \in F^{-i}H^*$  such that  $[\overline{\gamma}_i] = \gamma_i \in E_\infty^{-i,*}$  and  $d\overline{\gamma}_i = 0$ . For  $i = 0$  the unit  $\overline{\gamma}_0 = 1$  has these properties. Assume that  $\overline{\gamma}_j$  has been chosen with these properties for  $1 \leq j < i$ . Choose  $\gamma'_i \in F^{-i}H^*$  such that  $[\gamma'_i] = \gamma_i \in E_\infty^{-i,*}$ . We have  $[d\gamma'_i] = 0$  so  $d\gamma'_i = kx^{r-1} \mathbf{d}x \overline{\gamma}_{i-1}$  for some  $k \in \mathbb{F}_p$ . Put  $\overline{\gamma}_i = \gamma'_i - \frac{k}{r} x^r \overline{\gamma}_{i-1}$ . Then,  $[\overline{\gamma}_i] = \gamma_i \in E_\infty^{-i,*}$  and  $d\overline{\gamma}_i = d\gamma'_i - kx^{r-1} \mathbf{d}x \overline{\gamma}_{i-1} = 0$ .

We claim that these representatives satisfy  $(\overline{\gamma}_{p^i})^p = 0$  for  $i \geq 0$ . From the spectral sequence we see that  $(\overline{\gamma}_{p^i})^p = cx^r \overline{\gamma}_{p^{i+1}-1}$  for some  $c \in \mathbb{F}_p$ . We apply the action differential on this equation and find

$$0 = d((\overline{\gamma}_{p^i})^p) = cx^{r-1} \mathbf{d}x \overline{\gamma}_{p^{i+1}-1},$$

and since  $r \equiv -1 \pmod{p}$  it follows that  $c = 0$ .

So there is a well defined algebra map

$$\phi : \mathbb{F}_p[x]/(x^{r+1}) \otimes \Lambda(\mathbf{d}x) \otimes \Gamma[\omega] \rightarrow H^*(LX),$$

which maps  $\gamma_i(\omega)$  to  $\overline{\gamma}_i$ . We can filter the domain such that  $\phi$  becomes a map of filtered algebras. Since  $\phi$  is an isomorphism on associated graded objects, it is itself an isomorphism.

2) Assume that  $p \nmid (r+1)$ . The classes and multiplicative relations in  $E_2$  are as follows:

class or relation	type	total degree/parity	filtration
$b_0^i a_j, \quad 0 \leq i \leq r-1$	class	$\rho j + \alpha(i+1) - 1$ , odd	$-j$
$b_0^i b_j, \quad 0 \leq i \leq r-1$	class	$\rho j + \alpha(i+1)$ , even	$-j$
$b_i a_j - \binom{i+j}{i} b_0 a_{i+j}$	relation	$\rho(i+j) + 2\alpha - 1$ , odd	$-i - j$
$b_0^r a_j$	relation	$\rho j + \alpha(r+1) - 1$ , odd	$-j$
$a_i a_j$	relation	$\rho(i+j) + 2\alpha - 2$ , even	$-i - j$
$b_i b_j - \binom{i+j}{i} b_0 b_{i+j}$	relation	$\rho(i+j) + 2\alpha$ , even	$-i - j$
$b_0^r b_j$	relation	$\rho j + \alpha(r+1)$ , even	$-j$

We note that if  $0 \leq i \leq r-1$ , then

$$0 < \alpha \leq \alpha(i+1) \leq \alpha r \leq \rho.$$

From this and the list of classes above it follows easily that there is at most one class in each total degree. A differential in the spectral sequence raises total degree by one, and it strictly increases the filtration. The unique class of total degree one higher than  $a_j$  is  $b_j$ , which has the same filtration as  $a_j$ , so that  $a_j$  is a permanent cycle. If  $\alpha \geq 4$ , there is no non-trivial class of degree one higher than  $b_j$ . If  $\alpha = 2$ , the unique class of degree one higher than  $b_j$  is  $b_0 a_j$  which again has the same filtration so  $b_j$  is also a permanent cycle.

Each of the relations given above is true for any lifting of the generators in  $E_\infty$  to generators of  $H^*(LX)$  up to classes of the same total degree and strictly higher filtration. We claim that in each case, there are no non-trivial such classes.

If  $\alpha \geq 4$ , there are no nontrivial classes of the same dimension as  $b_0^r a_j$  or  $b_0^r b_j$ . In case  $\alpha = 2$ , there are the unique classes  $a_{j+1}$  respectively  $b_{j+1}$ . But these have lower filtration than the relation, so they cannot contribute to extensions.

If  $\alpha \geq 4$  there is no class which has the same total degree as the relation  $a_i a_j$ . In case  $\alpha = 2$ , there is the unique possibility  $b_{i+j}$ , which does not matter anyway since its filtration is too low.

Also, the relations  $b_i a_j - \binom{i+j}{i} b_0 a_{i+j}$  and  $b_i b_j - \binom{i+j}{i} b_0 b_{i+j}$  have the same total degree as  $b_0 a_{i+j}$  respectively  $b_0 b_{i+j}$ . By filtration check, there can be no extension problems.

Finally we have to consider the action differential. The filtration argument says that  $db_i$  is as stated. We have to argue that  $da_i = 0$ . If  $\alpha \geq 4$ , there is no non-trivial possibility for  $da_i$ . If  $\alpha = 2$ , the class  $da_i$  has the same total degree as  $b_{i-1} b_0^{r-1}$ . Because  $d$  is a differential,  $d(a_i) = 0$  unless

$$0 = db_i = (1 + (r+1)i)a_i,$$

and

$$0 = d(b_{i-1} b_0^{r-1}) = (1+(r+1)(i-1))b_0^{r-1} a_{i-1} + (r-1)b_{i-1} b_0^{r-2} a_0 = ((r+1)i-1)b_0^{r-1} a_{i-1}.$$

Solving this, we get that  $2 = 0$  (that is  $p = 2$ ), that  $r$  is even and that  $i$  is odd.  $\square$

We will later want to know how big the cokernel of the action differential is, so we do the counting now. We write  $\mathbb{N} = \{1, 2, 3, \dots\}$  for the set of natural numbers.

**Definition 11.3.** Let  $p$  be a prime and let  $r, \alpha \in \mathbb{N}$  with  $\alpha$  even. Put  $\rho = (r+1)\alpha - 2$  and let

$$\chi_p(s) = \begin{cases} 0 & \text{if } p \mid s, \\ 1 & \text{if } p \nmid s. \end{cases}$$

We define two subsets of  $\mathbb{N}$  as follows:

$$\begin{aligned} \mathcal{IF}(r, p, \alpha) &= \{\rho i + \alpha j \mid \chi_p(r+1) \leq j \leq r, 0 \leq i \text{ and } p \mid ((r+1)i + j)\} \setminus \{0\}, \\ \mathcal{IT}(r, p, \alpha) &= \{\rho i + \alpha j \mid \chi_p(r+1) \leq j \leq r, 0 \leq i \text{ and } p \nmid ((r+1)i + j)\}. \end{aligned}$$

The notation  $\mathcal{IF}$  refers to an *index* set for *free* generators and  $\mathcal{IT}$  refers to an *index* set for *transfer* generators. This choice of notation will make sense later on.

**Lemma 11.4.** *Whether a natural number  $k$  is contained in  $\mathcal{IF}(r, p, \alpha)$  respectively in  $\mathcal{IT}(r, p, \alpha)$  only depends on the congruence class of  $k$  modulo  $\rho p$ . Furthermore,*

$$\begin{aligned} \mathcal{IF}(r, p, \alpha) \cap \mathcal{IT}(r, p, \alpha) &= \begin{cases} 2r\mathbb{N} & \text{if } p \mid (r+1) \text{ and } \alpha = 2, \\ \emptyset & \text{otherwise,} \end{cases} \\ \mathcal{IF}(r, p, 2) \cup \mathcal{IT}(r, p, 2) &= 2\mathbb{N}. \end{aligned}$$

*If a number is in  $\mathcal{IF}(r, p, \alpha)$  or in  $\mathcal{IT}(r, p, \alpha)$ , there is a unique choice of numbers  $i, j$  that displays it as such. If  $p \mid (r+1)$ , the set  $\{0 < 2k \leq \rho p m \mid 2k \in \mathcal{IF}(p, r, \alpha)\}$  has  $m(r+1)$  elements. If  $p \nmid (r+1)$ , the set has  $mr$  elements.*

*Proof.* Note that  $0 \notin \mathbb{N}$ . We warm up by first ignoring the congruence conditions. Assume that  $i, i' \in \mathbb{Z}$  and that  $\chi_p(r+1) \leq j, j' \leq r$ . We make two claims about this situation.

1. If  $i' > i$  and  $\rho i + \alpha j = \rho i' + \alpha j'$ , then  $\alpha = 2$ ,  $p \mid (r+1)$ ,  $i' = i + 1$ ,  $j' = 0$  and  $j = r$ .
2. If  $\rho i + \alpha j > 0$ , then  $i \geq 0$ .

To prove the first claim, note that  $\alpha(j - j') = \rho(i' - i) \geq \rho$ . So

$$j - j' \geq \frac{\rho}{\alpha} = r + \frac{\alpha - 2}{\alpha}.$$

Since  $0 \leq j, j' \leq r$  and  $\alpha \geq 2$ , this is only possible if  $j = r$ ,  $j' = 0$ ,  $\alpha = 2$ . But this implies that  $i' = i + 1$  and by assumption, if  $j' = 0$ , then  $p \mid (r+1)$ .

To prove the second claim, note that

$$i > -\frac{\alpha j}{\rho} \geq -\frac{\alpha r}{\rho} = -\frac{\alpha r}{\alpha(r+1) - 2} \geq -\frac{\alpha r}{\alpha r} = -1.$$

We now prove the lemma. First assume that  $k$  and  $k + m\rho p$  are natural numbers. Assume also that  $k \in \mathcal{IF}(r, p, \alpha)$ . We can write  $k = \rho i + \alpha j$ , where  $i, j$  satisfy the appropriate conditions. Then  $k + m\rho p = \rho(i + mp) + \alpha j$ , and the pair  $(i + mp, j)$  satisfies the same congruence conditions, and conditions on  $j$ . This proves that

$k + m\rho p \in \mathcal{IF}(r, p, \alpha)$ , if we know that  $i + mp \geq 0$ . But this follows from claim 2. together with our assumption that  $k + m\rho p$  is a natural number.

The same argument shows that  $\mathcal{IT}(r, p, \alpha)$  is also a union of congruence classes of natural numbers.

If  $x \in \mathcal{IF}(r, p, \alpha) \cap \mathcal{IT}(r, p, \alpha)$ , it must be possible to write  $x$  in two different ways in the form  $x = \rho i + \alpha j$ .

By the first claim, we get that the only possible way this can happen is that  $\alpha = 2$ ,  $p \mid (r+1)$  and  $x = 2ri + 2r = 2r(i+1) + 0$ . This proves that  $\mathcal{IF}(r, p, \alpha) \cap \mathcal{IT}(r, p, \alpha)$  is empty unless  $p \mid (r+1)$  and  $\alpha = 2$ . It also shows that  $\mathcal{IF}(r, p, 2) \cap \mathcal{IT}(r, p, 2) \subseteq 2r\mathbb{N}$ . We have to show that if  $p \mid (r+1)$  then  $2r\mathbb{N} \subseteq \mathcal{IF}(r, p, 2) \cap \mathcal{IT}(r, p, 2)$ . In this case we write  $2rm = \rho i + \alpha j = \rho i' + \alpha j'$  for  $(i, j) = (m, 0)$  and  $(i', j') = (m-1, r)$ . This proves the claim, since  $p \mid j$  but  $p \nmid j'$ .

We have  $\mathcal{IF}(r, p, 2) \cup \mathcal{IT}(r, p, 2) = \{2(ri + j) \mid \chi_p(r+1) \leq j \leq r, 0 \leq i\}$ , which equals  $2\mathbb{N}$  as stated. The uniqueness statement on  $i, j$  follows directly from claim 1.

To prove the final statement about the number of elements, it is enough to show that the number of congruence classes in  $\mathcal{IF}(r, p, \alpha)$  modulo  $\rho p$  is  $r+1$  respectively  $r$ . In case  $p \mid (r+1)$  the congruence classes of  $2k$  are the classes of form  $\rho i + \alpha p j'$  for  $0 \leq i < p$  and  $0 \leq j' < (r+1)/p$ , and there are clearly  $p((r+1)/p) = r+1$  of those. In case  $p \nmid (r+1)$ , each  $j$  uniquely determines a congruence class  $i(j)$  modulo  $p$  such that  $(r+1)i(j) + j \equiv 0 \pmod{p}$ . That is, each  $j$  with  $1 \leq j \leq r$  uniquely determines an  $i$ ,  $0 \leq i < p$  such that  $(r+1)i + j \equiv 0 \pmod{p}$ . So there are exactly  $r$  pairs  $(i, j)$  qualifying, and there are  $r$  congruence classes in  $\mathcal{IF}(r, p, \alpha)$ .  $\square$

**Example 11.5.** Let  $\alpha = 2$ ,  $r = 2$ . Then  $\rho = 4$  and

$$\begin{aligned}\mathcal{IF}(2, 2, 2) &= \{4 + 8m, 6 + 8m \mid m \geq 0\}, \\ \mathcal{IF}(2, 3, 2) &= 4\mathbb{N}, \\ \mathcal{IF}(2, 5, 2) &= \{8 + 20m, 14 + 20m \mid m \geq 0\}, \\ \mathcal{IF}(2, 7, 2) &= \{10 + 28m, 20 + 28m \mid m \geq 0\}.\end{aligned}$$

**Lemma 11.6.** Let  $X$  be as in theorem 11.1,  $k \in \mathbb{N}$  and put  $H^* = H^*(LX)$ .

1. The kernel of the action differential  $d : H^{2k} \rightarrow H^{2k-1}$  is either a trivial or a one dimensional vector space. It is non-trivial if and only if  $2k \in \mathcal{IF}(r, p, \alpha)$ .
2. The image of  $d : H^{2k} \rightarrow H^{2k-1}$  is either a trivial or a one dimensional vector space. It is non-trivial if and only if  $2k \in \mathcal{IT}(r, p, \alpha)$ .
3. The cokernel of  $d : H^{2k} \rightarrow H^{2k-1}$  is either a trivial or a one dimensional vector space. In case  $p \nmid (r+1)$ , it is non-trivial if and only if  $2k \in \mathcal{IF}(r, p, \alpha)$ . In case  $p \mid (r+1)$  it is non-trivial if and only if either  $2k \in \mathcal{IF}(r, p, \alpha)$  and  $\rho \nmid 2k$  or if  $k > 1$  and  $2k \equiv 2 \pmod{\rho}$ .
4. The kernel of the map  $d : \bigoplus_{2 \leq 2k \leq \rho pm} H^{2k} \rightarrow \bigoplus_{1 \leq 2k-1 \leq \rho pm-1} H^{2k-1}$  is a vector space of dimension  $mr$  if  $p \nmid (r+1)$ , and of dimension  $m(r+1)$  if  $p \mid (r+1)$ .
5. The cokernel of the map  $d : \bigoplus_{2 \leq 2k \leq \rho pm} H^{2k} \rightarrow \bigoplus_{1 \leq 2k-1 \leq \rho pm-1} H^{2k-1}$  is a vector space of dimension  $rm$  when  $p \nmid (r+1)$ .

6. The cokernel of the map  $d : \bigoplus_{2 \leq 2k \leq \rho pm + 2} H^{2k} \rightarrow \bigoplus_{1 \leq 2k-1 \leq \rho pm + 1} H^{2k-1}$  is a vector space of dimension  $(r+1)m$  when  $p \mid (r+1)$ .

*Proof.* For the action differential  $d : H^{2k} \rightarrow H^{2k-1}$  we have the equation

$$\begin{aligned} \dim \operatorname{coker}(d) &= \dim H^{2k-1} - \dim \operatorname{im}(d) \\ &= \dim H^{2k-1} - (\dim H^{2k} - \dim \operatorname{ker}(d)) \\ &= -(\dim H^{2k} - \dim H^{2k-1}) + \dim \operatorname{ker}(d). \end{aligned} \quad (7)$$

We first consider the case  $p \nmid (r+1)$ . The even part  $H^{\text{even}}$  has basis  $\{b_0^{j-1}b_i\}$  and the odd part  $H^{\text{odd}}$  has basis  $\{b_0^{j-1}a_i\}$  where  $0 \leq i, 1 \leq j \leq r$ . The basis elements sit in degrees  $\rho i + \alpha j$  and  $\rho i + \alpha j - 1$  respectively. The kernel of the action differential  $d : H^{\text{even}} \rightarrow H^{\text{odd}}$  is generated by those  $b_0^{j-1}b_i$  for which  $p \mid ((r+1)i + j)$  and its image of those  $b_0^{j-1}a_i$  for which  $p \nmid ((r+1)i + j)$ . Combining this with the uniqueness statement for  $i, j$  in lemma 11.4 we see that 1. and 2. are valid. Furthermore we see that  $\dim H^{2k} = \dim H^{2k-1}$  which combined with (7) gives us 3.

Next we consider the case  $p \mid (r+1)$ . Here  $H^{\text{even}}$  has basis  $\{x^j \gamma_i(\omega)\}$  and  $H^{\text{odd}}$  has basis  $\{x^j dx \gamma_i(\omega)\}$  where  $0 \leq i, 0 \leq j \leq r$ . The basis elements sit in degrees  $\rho i + \alpha j$  and  $\rho i + \alpha j + \alpha - 1$  respectively. The kernel for the action differential  $d : H^{\text{even}} \rightarrow H^{\text{odd}}$  is generated by those  $x^j \gamma_i(\omega)$  for which  $p \mid j$  and the image is generated by those  $x^{j-1} dx \gamma_i(\omega)$  for which  $p \nmid j$ . As above lemma 11.4 give us that 1. and 2. are valid.

We now prove 3. One checks it directly for  $(r, p, \alpha) = (1, 2, 2)$ . Assume that  $(r, p, \alpha) \neq (1, 2, 2)$  which implies  $\rho > 2$ . By a counting argument we find the following:

$$\dim H^{2k} - \dim H^{2k-1} = \begin{cases} 1 & \text{if } \rho \mid 2k, \\ -1 & \text{if } \rho \mid (2k-2) \text{ and } k > 1, \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

We combine (8) and (7) in order to prove statement 3.

If  $\rho \mid 2k$  we have  $2k \in \mathcal{IF}(r, p, \alpha)$  so the dimension of the cokernel becomes 0. Assume that  $\rho \mid (2k-2)$  and  $k > 1$ . We claim that this implies that  $2k \notin \mathcal{IF}(r, p, \alpha)$ . For if  $2k \in \mathcal{IF}(r, p, \alpha)$  we have that  $2k = \rho i + \alpha j$  where  $0 \leq i, 0 \leq j \leq r$  and  $p \mid j$ . It follows that  $2 \equiv 2k \equiv \alpha j \pmod{\rho}$ . Since  $\rho > 2$ , we cannot have  $j = 0$ . We conclude that  $1 \leq p \leq j$ . On the other hand,  $0 \equiv 2k - 2 = \rho i + \alpha j - 2 \equiv \alpha j - 2 \pmod{\rho}$ , so  $\rho \leq \alpha j - 2$ . Now we have our contradiction, finishing the proof of 3. since  $\alpha j - 2 \leq \alpha r - 2 < \rho$ .

We have reduced the last three statements of the lemma to statements about the sets  $\mathcal{IT}(r, p, \alpha)$  and  $\mathcal{IF}(r, p, \alpha)$ . We see that 4. is equivalent to the statement that the set  $\{0 < 2k \leq \rho pm \mid 2k \in \mathcal{IF}(r, p, \alpha)\}$  has  $rm$  elements if  $p \nmid (r+1)$  and  $(r+1)m$  elements if  $p \mid (r+1)$ . But this is exactly the content of the last statement of lemma 11.4. Statement 5. follows from statement 4. and (7).

Finally, we prove statement 6. One verifies it directly for  $(r, p, \alpha) = (1, 2, 2)$ . Assume that  $(r, p, \alpha) \neq (1, 2, 2)$ . Formula (8) gives us that

$$\sum_{2 \leq 2k \leq \rho pm + 2} (\dim H^{2k} - \dim H^{2k-1}) = 0.$$

So by (7), 4. and 1. it suffices to check that  $\rho pm + 2 \notin \mathcal{IF}(r, p, \alpha)$ . This follows from lemma 11.4 since  $\rho pm + 2 \equiv 2 \pmod{\rho p}$ .  $\square$

We define two formal power series in  $\mathbb{Z}[[t]]$  by

$$P_{\mathcal{IT}(r,p,\alpha)}(t) = \sum_{n \in \mathcal{IT}(r,p,\alpha)} t^n, \quad P_{\mathcal{IF}(r,p,\alpha)}(t) = \sum_{n \in \mathcal{IF}(r,p,\alpha)} t^n.$$

Note that the constant terms are zero in both series. Furthermore, if the numbers  $\kappa_i$  in theorem 11.1 are zero for all  $i$ , then  $P_{\mathcal{IT}(r,p,\alpha)}(t) = tP_{\text{im}(d)}(t)$  where  $P_{\text{im}(d)}(t)$  is the Poincaré series for the image of the action differential. By lemma 11.4 we find the following result:

**Lemma 11.7.** *When  $\alpha = 2$  such that  $\rho = 2r$  we have that*

$$P_{\mathcal{IT}(r,p,2)}(t) + P_{\mathcal{IF}(r,p,2)}(t) = \frac{t^2}{1-t^2} + (1 - \chi_p(r+1)) \frac{t^{2r}}{1-t^{2r}}.$$

## 12 The $E_3$ -term of the Serre spectral sequence

Let  $Y$  be a  $\mathbb{T}$ -space with  $H_*(Y)$  of finite type. Write  $q : E\mathbb{T} \times Y \rightarrow E\mathbb{T} \times_{\mathbb{T}} Y$  for the quotient map. As described in appendix 14 there is a  $\mathbb{T}$ -transfer map  $\tau$  such that the composite  $q^* \circ \tau$  equals the action differential:

$$d : H^{*+1}(Y) \xrightarrow{\tau} H^*(Y_{h\mathbb{T}}) \xrightarrow{q^*} H^*(Y).$$

Since  $B\mathbb{T}$  is 1-connected and  $H^*(B\mathbb{T}) = \mathbb{F}_p[u]$  where  $|u| = 2$ , the Serre spectral sequence for the fibration  $Y \rightarrow Y_{h\mathbb{T}} \rightarrow B\mathbb{T}$  has the following form:

$$E_2^{*,*} = \mathbb{F}_p[u] \otimes H^*(Y) \Rightarrow H^*(Y_{h\mathbb{T}}).$$

The  $d_2$  differential is determined by the action differential since  $d_2 y = udy$  for all  $y \in H^*(Y)$ . Thus, the  $E_3$  term has the following form:

$$E_3^{*,*} = \text{im}(d) \oplus (\mathbb{F}_p[u] \otimes H(d)),$$

where  $\text{im}(d)$  and  $H(d)$  denotes the image and the homology of the action differential respectively.

**Proposition 12.1.** *The subspace  $\text{im}(d) \subseteq E_3^{*,*}$  survives to  $E_\infty^{*,*}$ . For any  $a \in H^*(Y)$  one has that  $\tau(a) \in H^*(Y_{h\mathbb{T}})$  represents  $da \in E_\infty^{0,*}$  and that  $pr_1^*(u)\tau(a) = 0$  in  $H^*(Y_{h\mathbb{T}})$ , where  $pr_1 : Y_{h\mathbb{T}} \rightarrow B\mathbb{T}$  denotes the projection on the first factor.*

*Proof.* There are two commutative diagrams

$$\begin{array}{ccc} H^*(Y) & \xrightarrow{\tau} & H^*(Y_{h\mathbb{T}}) \longrightarrow E_\infty^{0,*} \\ & \searrow d & \downarrow q^* \\ & & H^*(Y) = E_2^{0,*} \end{array} \quad \begin{array}{ccc} H^*(Y_{h\mathbb{T}}) & \xrightarrow{q^*} & H^*(E\mathbb{T} \times Y) \\ \uparrow pr_1^* & & \uparrow pr_1^* \\ H^*(B\mathbb{T}) & \longrightarrow & H^*(E\mathbb{T}) \end{array}$$

Assume that  $a \in H^*(Y)$  has  $da \neq 0$ . The diagram to the left shows that  $\tau(a) \neq 0$  and that  $da$  survives to  $E_\infty$  and is represented by  $\tau(a) \in H^*(Y_{h\mathbb{T}})$ . The diagram to the right shows that  $q^* \circ pr_1^*(u) = 0$  since  $E\mathbb{T}$  is contractible. By Frobenius reciprocity  $pr_1^*(u)\tau(b) = \tau(q^*(pr_1^*(u))b) = 0$ .  $\square$

We now take  $Y = LX$ . By the result in the previous section we can compute the  $E_3$  term of the Serre spectral sequence.

**Proposition 12.2.** *Let  $p$  be a prime. Assume that  $X$  is a 1-connected space with mod  $p$  homology of finite type. Assume also that*

$$H^*(X; \mathbb{F}_p) = \mathbb{F}_p[x]/(x^{r+1}),$$

where  $\alpha = |x|$  is even and  $r \geq 1$ . Put  $\rho = (r+1)\alpha - 2$ .

1) If  $p \mid (r+1)$  then

$$E_3^{*,*} \cong (\mathbb{F}_p[u, \phi, q, \delta_0, \delta_1, \dots, \delta_{p-2}]/I) \otimes \Gamma[\omega],$$

where  $I$  is the ideal

$$I = (\phi^{(r+1)/p}, q^2, \delta_j u, \delta_j q, \delta_j \delta_k \mid 0 \leq j \leq p-2, 0 \leq k \leq p-2).$$

The bidegrees are  $\|u\| = (2, 0)$ ,  $\|\phi\| = (0, p\alpha)$ ,  $\|q\| = (0, p\alpha-1)$ ,  $\|\delta_j\| = (0, j\alpha+\alpha-1)$  and  $\|\gamma_i(\omega)\| = (0, \rho i)$ . The generators are represented by elements in the  $E_2$  term as follows:

$$u = [u], \quad \phi = [x^p], \quad q = [x^{p-1}dx], \quad \delta_j = [x^j dx], \quad \gamma_i(\omega) = [\gamma_i(\omega)].$$

2) If  $p \nmid (r+1)$  and the numbers  $\kappa_t$  from theorem 11.1 are zero for all  $t$ , then

$$E_3^{*,*} \cong \mathbb{F}_p[v_i^{(k)}, w_i^{(k)}, \tau_i^{(h)}, u \mid 1 \leq k \leq r, p \mid ((r+1)i+k), \\ 1 \leq h \leq r, p \nmid ((r+1)i+h), 0 \leq i \leq r],$$

where  $I$  is the ideal generated by the elements

$$\begin{aligned} & \tau_i^{(h)} u, \quad \tau_i^{(h)} w_j^{(\ell)}, \quad \tau_i^{(h)} \tau_j^{(m)}, \quad w_i^{(k)} w_j^{(\ell)}, \\ & v_i^{(k)} v_j^{(\ell)} - \epsilon_r(k+\ell) \binom{i+j}{i} v_{i+j}^{(k+\ell)}, \\ & v_i^{(k)} w_j^{(\ell)} - \epsilon_r(k+\ell) \binom{i+j}{i} w_{i+j}^{(k+\ell)}, \\ & v_i^{(k)} \tau_j^{(h)} - \epsilon_r(k+h) \binom{i+j}{i} \tau_{i+j}^{(k+h)}. \end{aligned}$$

Here the number  $\epsilon_r(s)$  equals 1 if  $1 \leq s \leq r$  and 0 otherwise. The bidegrees of the generators are

$$\begin{aligned} \|v_i^{(k)}\| &= (0, \rho i + \alpha k), & \|w_i^{(k)}\| &= (0, \rho i + \alpha k - 1), \\ \|\tau_i^{(h)}\| &= (0, \rho i + \alpha h - 1), & \|u\| &= (2, 0), \end{aligned}$$

and the generators are represented by elements in the  $E_2$  term as follows:

$$v_i^{(k)} = [b_0^{k-1} b_i], \quad w_i^{(k)} = [b_0^{k-1} a_i], \quad \tau_i^{(h)} = [b_0^{h-1} a_i], \quad u = [u].$$

*Proof.* 1) Assume that  $p \mid (r + 1)$  such that  $r + 1 = mp$  for some  $m \geq 1$ . By the Künneth formula we have that

$$H(d) = H(\mathbb{F}_p[x]/(x^{mp}) \otimes \Lambda(dx); d) \otimes \Gamma[\omega].$$

Since  $d(x^j) = jx^{j-1}dx$  and  $d(x^j dx) = 0$ , the kernel and image of the differential on  $\mathbb{F}_p[x]/(x^{mp}) \otimes \Lambda(dx)$  has the following  $\mathbb{F}_p$ -bases:

$$\begin{aligned} &\{x^{kp}, x^j dx \mid 0 \leq k \leq m - 1, 0 \leq j \leq mp - 1\}, \\ &\{x^{j-1} dx \mid 1 \leq j \leq mp - 1, p \nmid j\}. \end{aligned}$$

It follows that  $H(d) = \mathbb{F}_p[\phi]/(\phi^m) \otimes \Lambda(q) \otimes \Gamma[\omega]$ , where  $\phi = x^p$  and  $q = x^{p-1}dx$ . We can get the basis elements in  $\text{im}(d)$  by multiplying the elements in the set  $\{x^k dx \mid 0 \leq k \leq p - 2\}$  by the elements  $\gamma_i(\omega)$  and powers of  $\phi$ . The result follows.

2) Assume that  $p \nmid (r + 1)$ . By remark 11.2 we see that  $\ker(d)$ ,  $\text{im}(d)$ ,  $H(d)$  has respective  $\mathbb{F}_p$ -bases as follows:

$$\begin{aligned} &b_0^{j-1} a_i, b_0^{k-1} b_i && \text{for } p \mid ((r + 1)i + k), \\ &b_0^{h-1} a_i && \text{for } p \nmid ((r + 1)i + h), \\ &[b_0^{k-1} b_i], [b_0^{k-1} a_i] && \text{for } p \mid ((r + 1)i + k), \end{aligned}$$

where  $0 \leq i$  and  $1 \leq h, k, j \leq r$ . The result follows.  $\square$

**Corollary 12.3.** *If  $X$  satisfies the hypothesis of proposition 12.2, then the Poincaré series for the  $E_3$  term of the Serre spectral sequence is given by the following when  $p \mid (r + 1)$ :*

$$\frac{1 - t^{(r+1)\alpha}}{(1 - t^{p\alpha})(1 - t^{(r+1)\alpha-2})} \cdot \left( \frac{1 + t^{p\alpha-1}}{1 - t^2} + \frac{t^{\alpha-1} - t^{p\alpha-1}}{1 - t^\alpha} \right),$$

and by the following when  $p \nmid (r + 1)$ :

$$\frac{1}{1 - t^2} \left( 1 + P_{IF(r,p,\alpha)}(t) + \frac{1}{t} P_{IF(r,p,\alpha)}(t) \right) + \frac{1}{t} P_{IT(r,p,\alpha)}(t).$$

By proposition 12.1 and proposition 12.2 we have

**Corollary 12.4.** *If  $X$  satisfies the hypothesis of proposition 12.2 then,*

1. *If  $p \mid (r + 1)$  then the element  $\gamma_i(\omega)\phi^j \delta_k \in E_3$  survives to  $E_\infty$  and is represented by  $\tau(x^{pj+k+1}\gamma_i(\omega))$  for  $0 \leq i$ ,  $0 \leq j \leq (r + 1)/p$  and  $0 \leq k \leq p - 2$ .*
2. *If  $p \nmid (r + 1)$  then the generator  $\tau_i^{(h)} \in E_3$  survives to  $E_\infty$  and is represented by  $\tau(b_0^{h-1} b_i)$  for  $1 \leq h \leq r$  with  $p \nmid ((r + 1)i + h)$  and  $0 \leq i$ .*

## 13 Comparing the two spectral sequences.

In this section we will complete the investigation of the Morse spectral sequence  $E_*(\mathcal{M})(LCP_{h\mathbb{T}}^r)$ . We write  $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \subseteq LCP^r$  for the energy filtration. So far, the main structural facts which we proved in section 9 are the following:

SF(1) The classes of even total degree are concentrated in  $\bigoplus_m E_*^{pm,*}(\mathcal{M})(LCP_{h\mathbb{T}}^r)$  (theorem 9.1).

SF(2)  $E_*^{pm,*}(\mathcal{M})(LCP_{h\mathbb{T}}^r)$  is a free  $\mathbb{F}_p[u]$  module. If  $p \nmid n$ , then  $E_*^{n,*}(\mathcal{M})(LCP_{h\mathbb{T}}^r)$  is a finite dimensional vector space (theorem 9.1).

SF(3) Every non-trivial differential goes from even total degree to odd total degree (lemma 9.8).

SF(4) The spectral sequence collapses from the  $E_p$  page (theorem 9.11).

*Remark 13.1.* It follows from SF(3) that the inclusion  $j : (\mathcal{F}_n)_{h\mathbb{T}} \hookrightarrow LCP_{h\mathbb{T}}^r$  induces a surjective map  $j^* : H^{\text{odd}}(LCP_{h\mathbb{T}}^r) \rightarrow H^{\text{odd}}((\mathcal{F}_n)_{h\mathbb{T}})$  for all  $n \geq 0$ .

By SF(1) and SF(4) we see that the only possibly non-trivial differentials in the spectral sequence are

$$E_s^{pm,*}(\mathcal{M})(LCP_{h\mathbb{T}}^r) \xrightarrow{d_s} E_s^{pm+s,*}(\mathcal{M})(LCP_{h\mathbb{T}}^r) \cong E_1^{pm+s,*}(\mathcal{M})(LCP_{h\mathbb{T}}^r)$$

for  $1 \leq s \leq p-1$ .

We are also going to use the non-equivariant spectral sequence  $E_*(\mathcal{M})(LCP^r)$ . From theorem 9.3 we have the following structural facts:

SF(5)  $E_1^{n,2i+1-n}(\mathcal{M})(LCP^r) = 0$  if  $p \mid (r+1)$  and  $i \geq rn+1$  or if  $p \nmid (r+1)$  and  $i \geq rn$ .

SF(6) The map  $E_1^{\text{odd}}(\mathcal{M})(LCP_{h\mathbb{T}}^r) \rightarrow E_1^{\text{odd}}(\mathcal{M})(LCP^r)$  is surjective.

*Remark 13.2.* By SF(6) and SF(3) the map  $E_\infty^{\text{odd}}(\mathcal{M})(LCP_{h\mathbb{T}}^r) \rightarrow E_\infty^{\text{odd}}(\mathcal{M})(LCP^r)$  is a surjection. A filtration argument shows that if a map in some degree induces a surjective map on the  $E_\infty$  pages, then it also induces a surjective map on the cohomology of the targets of the spectral sequences. So  $H^{\text{odd}}(LCP_{h\mathbb{T}}^r) \rightarrow H^{\text{odd}}(LCP^r)$  is also surjective.

Our plan for computing  $H^*(LCP_{h\mathbb{T}}^r)$  goes as follows. First, we concentrate on the odd part of  $H^*(LCP_{h\mathbb{T}}^r)$ . The sum of the odd dimensional cohomology groups  $H^{\text{odd}}(LCP_{h\mathbb{T}}^r)$  is a submodule of  $H^*(LCP_{h\mathbb{T}}^r)$  over  $H^*(B\mathbb{T})$ , since this ring is concentrated in even degrees. We will list a set of elements in this module, and use the above properties of the spectral sequences to show that these elements are generators. We also give the relations satisfied by these elements, thus computing the odd part of the Borel cohomology.

To determine the odd part of the cohomology is not quite the same as to determine the differentials in the Morse spectral sequence, but in our situation it is close enough. We can use the knowledge of the odd cohomology together with the spectral sequences to determine the even dimensional cohomology.

The first step in this program is to find elements in  $H^{\text{odd}}(LCP_{h\mathbb{T}}^r)$  which can serve as generators. We use the results on the transfer map to find the right elements.

Consider the  $\mathbb{T}$  transfer map  $\tau$  in the context of the Morse filtration of  $LCP^r$ . Let  $i : LCP^r \rightarrow LCP_{h\mathbb{T}}^r$  be the inclusion. It follows from theorem 14.1 that the composite

$$H^{*+1}(LCP^r) \xrightarrow{\tau} H^*(LCP_{h\mathbb{T}}^r) \xrightarrow{i^*} H^*(LCP^r)$$

equals the action differential  $d$ .

We can now chose one half of our generators. This bunch of generators come with the relation that the generators are annihilated by multiplication by  $u$ .

**Lemma 13.3.** *There is a graded subgroup  $\mathcal{T}^* \subseteq H^{\text{odd}}(\text{LCP}_{h\mathbb{T}}^r)$  such that*

1.  $u\mathcal{T}^* = 0$ .
2. The map  $i^* : H^*(\text{LCP}_{h\mathbb{T}}^r) \rightarrow H^*(\text{LCP}^r)$  restricts to an injective map on  $\mathcal{T}^*$ .
3. The subgroup  $i^*(\mathcal{T}^*) \subseteq H^*(\text{LCP}^r)$  agrees with the image of the composite map  $i^* \circ \tau : H^{*+1}(\text{LCP}^r) \rightarrow H^*(\text{LCP}^r)$ .

*Proof.* We chose a graded subgroup  $\overline{\mathcal{T}}^{*+1} \subseteq \tilde{H}^{*+1}(\text{LCP}^r)$  which maps isomorphically to the image of  $d$ . That is, we chose (arbitrarily) a splitting of the surjective map  $d : H^{*+1}(\text{LCP}^r) \rightarrow \text{im}(d)^*$ . Then  $\mathcal{T}^* = \tau(\overline{\mathcal{T}}^{*+1}) \subseteq H^*(\text{LCP}_{h\mathbb{T}}^r)$  is a subgroup which by its definition satisfies the second and third property. It also satisfies the first property, since  $u\tau = 0$  by theorem 14.1.  $\square$

Our second bunch of generators is not involved with any relations.

**Lemma 13.4.** *There is a graded subgroup  $\mathcal{U}^* \subseteq H^{\text{odd}}(\text{LCP}_{h\mathbb{T}}^r)$  such that the composite*

$$\mathcal{T}^* \oplus \mathcal{U}^* \longrightarrow H^{\text{odd}}(\text{LCP}_{h\mathbb{T}}^r) \xrightarrow{i^*} H^{\text{odd}}(\text{LCP}^r)$$

*is an isomorphism. In addition to this, the restriction*

$$\mathcal{U}^{2i+1} \longrightarrow H^{2i+1}(\text{LCP}_{h\mathbb{T}}^r) \xrightarrow{j^*} H^{2i+1}((\mathcal{F}_{pm})_{h\mathbb{T}})$$

*is trivial if either  $p \nmid (r+1)$  and  $i \geq rpm$ , or  $p \mid (r+1)$  and  $i \geq rpm + 1$ .*

*Proof.* To construct  $\mathcal{U}^*$ , we make a choice  $\overline{\mathcal{U}}^* \subseteq H^{\text{odd}}(\text{LCP}^r)$  of a complementary subgroup of  $i^*(\mathcal{T})$ , so that we have a direct sum decomposition of vector spaces  $H^{\text{odd}}(\text{LCP}^r) \cong i^*(\mathcal{T}^*) \oplus \overline{\mathcal{U}}^*$ . We intend to find  $\mathcal{U}^* \subseteq H^{\text{odd}}(\text{LCP}_{h\mathbb{T}}^r)$  such that  $i^*$  maps this subgroup isomorphically to  $\overline{\mathcal{U}}^*$  and such that the statement for the restriction map holds.

According to the long exact sequence of theorem 14.1, the following diagram has exact rows. Because of remark 13.1 the left and middle vertical maps are surjections, because of remark 13.2 the upper right horizontal map is a surjection.

$$\begin{array}{ccccc} H^{2i-1}(\text{LCP}_{h\mathbb{T}}^r) & \xrightarrow{u} & H^{2i+1}(\text{LCP}_{h\mathbb{T}}^r) & \xrightarrow{i^*} & H^{2i+1}(\text{LCP}^r) \\ \downarrow & & \downarrow & & \downarrow \\ H^{2i-1}((\mathcal{F}_{pm})_{h\mathbb{T}}) & \xrightarrow{u} & H^{2i+1}((\mathcal{F}_{pm})_{h\mathbb{T}}) & \longrightarrow & H^{2i+1}(\mathcal{F}_{pm}). \end{array}$$

Assume that  $p \nmid (r+1)$  and  $i \geq rpm$ , or  $p \mid (r+1)$  and  $i \geq rpm + 1$ . By SF(5) this implies that  $H^{2i+1}(\mathcal{F}_n, \mathcal{F}_{n-1}) = 0$  for  $0 \leq n \leq pm$  such that  $H^{2i+1}(\mathcal{F}_{pm}) = 0$ . Thus  $\overline{\mathcal{U}}^{2i+1}$  is contained in the kernel of the right vertical map.

The rest is a diagram chase. By the surjectivity of the upper right map  $\bar{\mathcal{U}}^*$  is the isomorphic image of a subgroup of  $H^*(LCP_{h\mathbb{T}}^r)$ . The degree  $2i + 1$  part of this subgroup might not itself be in the kernel of the middle vertical map, but using the surjectivity of the left vertical map, we can replace it with a subgroup  $\mathcal{U}^{2i+1} \subseteq H^{2i+1}(LCP_{h\mathbb{T}}^r)$  which also maps isomorphically to  $\bar{\mathcal{U}}^*$ , and such that  $\mathcal{U}^{2i+1}$  is in the kernel of the middle vertical map.  $\square$

*Remark 13.5.* It follows by 5. and 6. of lemma 11.6, that if  $p \nmid (r + 1)$ , the dimension of the group  $\bigoplus_{1 \leq 2k-1 \leq 2rpm-1} \mathcal{U}^{2k-1}$  is  $rm$ . If  $p \mid (r + 1)$  the dimension of the group  $\bigoplus_{1 \leq 2k-1 \leq 2rpm+1} \mathcal{U}^{2k-1}$  is  $(r + 1)m$ .

**Theorem 13.6.** *There is a map of  $\mathbb{F}_p[u]$ -modules*

$$h_1 \oplus h_2 : (\mathbb{F}_p[u] \otimes \mathcal{U}^*) \oplus \mathcal{T}^* \rightarrow H^{\text{odd}}(LCP_{h\mathbb{T}}^r)$$

which is an isomorphism of  $\mathbb{F}_p[u]$ -modules.

*Proof.* We can extend the inclusion of  $\mathcal{U}^*$  in a unique way to an  $\mathbb{F}_p[u]$ -linear map  $h_1 : \mathbb{F}_p[u] \otimes \mathcal{U}^* \rightarrow H^{\text{odd}}(LCP_{h\mathbb{T}}^r)$ . Because of lemma 13.3 the inclusion of  $\mathcal{T}^*$  is already an  $\mathbb{F}_p[u]$ -linear map  $h_2 : \mathcal{T}^* \hookrightarrow H^{\text{odd}}(LCP_{h\mathbb{T}}^r)$ .

Lemma 13.4 and the exact sequence

$$H^{2i-1}(LCP_{h\mathbb{T}}^r) \xrightarrow{u} H^{2i+1}(LCP_{h\mathbb{T}}^r) \xrightarrow{i^*} H^{2i+1}(LCP^r)$$

shows that  $h_1 \oplus h_2$  is surjective on indecomposables. But then it is a surjective  $\mathbb{F}_p[u]$ -linear map, and it suffices to show that it is also injective.

The main step is to prove injectivity of the localized map  $(h_1 \oplus h_2)[1/u]$ . Because of the vanishing statement of lemma 13.4 we have a commutative diagram as follows, where  $\chi_p(r + 1)$  is the number defined in 11.3:

$$\begin{array}{ccc} \mathbb{F}_p[u] \otimes \bigoplus_{0 \leq i} \mathcal{U}^{2i+1} & \xrightarrow{h_1} & H^{\text{odd}}(LCP_{h\mathbb{T}}^r) \\ \downarrow \text{id} \otimes pr & & \downarrow j^* \\ \mathbb{F}_p[u] \otimes \bigoplus_{0 \leq i \leq rpm - \chi_p(r+1)} \mathcal{U}^{2i+1} & \xrightarrow{\bar{h}_1} & H^{\text{odd}}((\mathcal{F}_{pm})_{h\mathbb{T}}). \end{array}$$

The map  $j^*$  is surjective by remark 13.1.

We localize by inverting  $u$ . Because all modules are of finite type, this localization agrees with tensoring with  $\mathbb{F}_p[u, u^{-1}]$  over  $\mathbb{F}_p[u]$ . Since  $h_1 \oplus h_2$  is surjective, and the localization of  $h_2$  vanishes, we know that

$$h_1\left[\frac{1}{u}\right] : \mathbb{F}_p[u, u^{-1}] \otimes \mathcal{U}^* \longrightarrow H^{\text{odd}}(LCP_{h\mathbb{T}}^r)\left[\frac{1}{u}\right]$$

is surjective. It follows by the diagram that the map

$$\bar{h}_1\left[\frac{1}{u}\right] : \mathbb{F}_p[u, u^{-1}] \otimes \bigoplus_{0 \leq i \leq rpm - \chi_p(r+1)} \mathcal{U}^{2i+1} \longrightarrow H^{\text{odd}}((\mathcal{F}_{pm})_{h\mathbb{T}})\left[\frac{1}{u}\right]$$

is also surjective. We know from remark 13.5 that as abstract module, the domain space is given by

$$\mathbb{F}_p[u, u^{-1}] \otimes \bigoplus_{0 \leq i \leq rpm - \chi_p(r+1)} \mathcal{U}^{2i+1} \cong \begin{cases} \mathbb{F}_p[u, u^{-1}]^{\oplus (r+1)m} & \text{if } p \mid (r + 1) \\ \mathbb{F}_p[u, u^{-1}]^{\oplus rm} & \text{if } p \nmid (r + 1). \end{cases}$$

The target space is determined by theorem 8.3 and corollary 9.5. By the proof of theorem 8.3 the re-indexing is given by multiplying the filtration degree by  $p$ . We find that

$$H^{\text{odd}}((\mathcal{F}_{pm})_{h\mathbb{T}}) \left[ \frac{1}{u} \right] \cong \begin{cases} \mathbb{F}_p[u, u^{-1}]^{\oplus(r+1)m} & \text{if } p \mid (r+1) \\ \mathbb{F}_p[u, u^{-1}]^{\oplus rm} & \text{if } p \nmid (r+1). \end{cases}$$

The punch line is that  $h_1[1/u]$  is a surjective map between two finitely generated, free  $\mathbb{F}_p[u, u^{-1}]$ -modules of the same rank. But then the map has to be an isomorphism.

This proves the injectivity of the localization  $h_1$ . To get the injectivity of  $h_1 \oplus h_2$ , we consider an element  $(c, t) \in \ker(h_1 \oplus h_2)$ . We have to prove that this element is trivial. Since the localization of  $t$  vanishes, the localization of  $c$  is in the kernel of the localization of  $h_1$ . This localized map is injective, so the localization of  $c$  vanishes. But the canonical map  $\mathbb{F}_p[u] \otimes \mathcal{U}^* \rightarrow \mathbb{F}_p[u, u^{-1}] \otimes \mathcal{U}^*$  is injective, so  $c$  itself vanishes, and  $t$  is in the kernel of  $h_2$ . But  $h_2$  is injective on  $\mathcal{T}$ , which proves that  $t = 0$ .  $\square$

We can finally complete the main calculation of the paper. Recall the index sets  $\mathcal{IT}(r, p, 2)$  and  $\mathcal{IF}(r, p, 2)$  from definition 11.3. We need a small perturbation as follows:

$$\mathcal{IF}'(r, p, 2) = \begin{cases} (\mathcal{IF}(r, p, 2) \setminus 2r\mathbb{N}) \cup (2 + 2r\mathbb{N}) & \text{if } p \mid (r+1), \\ \mathcal{IF}(r, p, 2) & \text{if } p \nmid (r+1). \end{cases}$$

This makes sense, since  $2r\mathbb{N} \subseteq \mathcal{IF}(r, p, 2)$  by lemma 11.4 when  $p \mid (r+1)$ .

**Theorem 13.7.** *As a graded  $\mathbb{F}_p[u]$ -module,  $H^*(LCP_{h\mathbb{T}}^r; \mathbb{F}_p)$  is isomorphic to the direct sum*

$$\mathbb{F}_p[u] \oplus \bigoplus_{2k \in \mathcal{IF}(r, p, 2)} \mathbb{F}_p[u] f_{2k} \oplus \bigoplus_{2k \in \mathcal{IF}'(r, p, 2)} \mathbb{F}_p[u] f_{2k-1} \oplus \bigoplus_{2k \in \mathcal{IT}(r, p, 2)} (\mathbb{F}_p[u]/(u)) t_{2k-1}.$$

Here the lower index denotes the degree of the generator.

*Proof.* In theorem 13.6 we proved the formula for the group of odd degree elements. We have to show that  $H^{\text{even}}(LCP_{h\mathbb{T}}^r)$  is a free  $\mathbb{F}_p[u]$ -module, with generators in the stated degrees. Put  $E_r^{**} = E_r^{**}(\mathcal{M})(LCP_{h\mathbb{T}}^r)$ . From SF(1) and SF(2) we see that  $E_1^{\text{even}}$  is a free  $\mathbb{F}_p[u]$ -module. The degrees of the generators can be read off theorem 9.1. We see that

$$E_1^{(0,*)(\text{even})} \cong \bigoplus_{0 \leq 2i \leq 2r} \mathbb{F}_p[u] x_{2i},$$

$$E_1^{(pm,*)(\text{even})} \cong \begin{cases} \bigoplus_{2rpm \leq 2i \leq 2r(pm+1)} \mathbb{F}_p[u] x_{2i} & \text{if } p \mid (r+1), \\ \bigoplus_{2rpm+2 \leq 2i \leq 2r(pm+1)} \mathbb{F}_p[u] x_{2i} & \text{if } p \nmid (r+1). \end{cases}$$

Because of SF(4) together with SF(2) we see that  $E_{\infty}^{(pm,*)(\text{even})}$  is a submodule of finite index in  $E_1^{(pm,*)(\text{even})}$ . By abstract structure theory of  $\mathbb{F}_p[u]$ -modules, it follows that  $E_{\infty}^{(pm,*)(\text{even})}$  is a free module on certain generators. If we can filter a graded module with quotient which are free modules, the original module was also a free

module. The generators are in the same degrees as the generators for the direct sum of the quotients.

What is left is to figure out the degrees of the generators of the free module  $E_\infty^{(pm,*)^{(\text{even})}}$ . It would be best if we could compute the differentials. Unfortunately, we cannot do this. What we can do, is to compute the dimension of  $E_\infty^{\text{even}}$  in each degree, and recover the degrees of the generators from this.

To do this, we consider Poincaré series. Let  $P(t)$ ,  $P^{\text{even}}(t)$  and  $P^{\text{odd}}(t)$  be the Poincaré series of  $H^*(LCP_{h\mathbb{T}}^r)$ ,  $H^{\text{even}}(LCP_{h\mathbb{T}}^r)$  and  $H^{\text{odd}}(LCP_{h\mathbb{T}}^r)$  respectively. Similarly, let  $P_r(t)$ ,  $P_r^{\text{even}}(t)$  and  $P_r^{\text{odd}}(t)$  be the Poincaré series of  $E_r$ , the even total degree part of  $E_r$  and the odd total degree part  $E_r$  respectively.

The series  $P_1^{\text{even}}(t)$  and  $P_1^{\text{odd}}(t)$  can be recovered from  $P_1(t)$  as the sum of the even respectively the sum of the odd monomials occurring in it. From the computation in Lemma 9.6, we recall that if  $p \mid (r+1)$ , then

$$P_1(t) = \frac{1}{1-t} \cdot \frac{1-t^{2r+2}}{(1-t^2)(1-t^{2pr})}$$

It follows that

$$P_1^{\text{even}}(t) = \frac{1}{1-t^2} \cdot \frac{1-t^{2r+2}}{(1-t^2)(1-t^{2pr})}, \quad P_1^{\text{odd}}(t) = \frac{t}{1-t^2} \cdot \frac{1-t^{2r+2}}{(1-t^2)(1-t^{2pr})}.$$

The important thing here is that  $P_1^{\text{odd}}(t) = tP_1^{\text{even}}(t)$ . This means that the dimensions of the group of classes of total degree  $2i$  in  $E_1$  agrees with the dimension of the classes of total degree  $2i+1$ . Now recall that by SF(3), all differentials in the spectral sequence go from even total degree to odd total degree. It follows that for any  $r$ , including  $r = \infty$ , we have that  $P_r^{\text{odd}}(t) = tP_r^{\text{even}}(t)$ , and also that  $P^{\text{odd}}(t) = tP^{\text{even}}(t)$ .

In case  $p \nmid (r+1)$  the formula for  $P_1(t)$  in lemma 9.6 is different, but the argument above still applies, so that  $P^{\text{odd}}(t) = tP^{\text{even}}(t)$  in both cases.

We can use theorem 13.6 to give an expression for  $P^{\text{odd}}(t) = P_\infty^{\text{odd}}(t)$ . Using the notation of 11.3, we see that

$$tP^{\text{odd}}(t) = P_{IT(r,p,2)}(t) + \frac{1}{1-t^2} P_{IF'(r,p,2)}(t).$$

By definition of  $IF'(r,p,2)$ , we have

$$P_{IF'(r,p,2)}(t) = P_{IF(r,p,2)}(t) - (1 - \chi_p(r+1)) \frac{t^{2r}(1-t^2)}{1-t^{2r}}, \quad (9)$$

which we insert above and find

$$tP^{\text{odd}}(t) = P_{IT(r,p,2)}(t) + \frac{1}{1-t^2} P_{IF(r,p,2)}(t) - (1 - \chi_p(r+1)) \frac{t^{2r}}{1-t^{2r}}.$$

Rewriting the first two terms, and using the result  $P^{\text{odd}}(t) = tP^{\text{even}}(t)$ , we see that

$$\begin{aligned} t^2 P^{\text{even}}(t) &= tP^{\text{odd}}(t) \\ &= P_{IT(r,p,2)}(t) + P_{IF(r,p,2)}(t) + \frac{t^2}{1-t^2} P_{IF(r,p,2)}(t) - (1 - \chi_p(r+1)) \frac{t^{2r}}{1-t^{2r}}. \end{aligned}$$

We rewrite the sum of the first two terms using lemma 11.7 and obtain

$$t^2 P^{\text{even}}(t) = \frac{t^2}{1-t^2} + \frac{t^2}{1-t^2} P_{IF(r,p,2)}(t),$$

which completes the proof.  $\square$

**Corollary 13.8.** *The numbers  $\kappa_i$  in theorem 11.1 are always zero.*

*Proof.* In theorem 11.1 this is proved for  $\alpha \geq 4$ , and in some cases also when  $\alpha = 2$ . So assume that  $\alpha = 2$ . Then an obstruction argument shows that  $X$  is homotopy equivalent to  $\mathbb{C}P^r$ . So we can without loss of generality assume that  $X = \mathbb{C}P^r$ . Since the action differential factors over the transfer map, it sufficient to show that the transfer map

$$\tau : H^{\text{odd}}(L\mathbb{C}P^r) \rightarrow H^{\text{even}}(L\mathbb{C}P_{h\mathbb{T}}^r)$$

is the zero map. The image of the transfer map is annihilated by multiplication by  $u$  (theorem 14.1). According to theorem 13.7 we also have that  $H^{\text{even}}(L\mathbb{C}P_{h\mathbb{T}}^r)$  is a free module over  $\mathbb{F}_p[u]$ , so multiplication by  $u$  is injective. The result follows.  $\square$

We can now give an other version of our main result. Recall that

$$IF(r, p, 2) = \{2(ri + j) \mid 0 \leq i, \chi_p(r+1) \leq j \leq r, p \mid ((r+1)i + j)\},$$

where  $\chi_p(s)$  equals 0 when  $p$  divides  $s$  and 1 when  $p$  does not divide  $s$ .

**Theorem 13.9.** *Let  $\{E_*\}$  be the mod  $p$  Serre spectral sequence for the fibration sequence  $L\mathbb{C}P^r \rightarrow (L\mathbb{C}P^r)_{h\mathbb{T}} \rightarrow B\mathbb{T}$ . That is*

$$E_2^{*,*} = H^*(B\mathbb{T}; \mathbb{F}_p) \otimes H^*(L\mathbb{C}P^r; \mathbb{F}_p) \Rightarrow H^*((L\mathbb{C}P^r)_{h\mathbb{T}}; \mathbb{F}_p).$$

*For any positive integer  $r$  and any prime  $p$  one has that  $E_3 = E_\infty$ . Furthermore, the Poincaré series  $P_{r,p}(t)$  for  $H^*((L\mathbb{C}P^r)_{h\mathbb{T}}; \mathbb{F}_p)$  is given by*

$$P_{r,p}(t) = \frac{1}{1-t} \left( 1 + \sum_{k \in IF(r,p,2)} t^k \right).$$

*If  $p$  divides  $r+1$  we can rewrite this as*

$$P_{r,p}(t) = \frac{1-t^{2(r+1)}}{(1-t)(1-t^{2r})(1-t^{2p})}.$$

*Remark 13.10.* We have described the  $\mathbb{F}_p$ -algebra structure of the  $E_\infty$  page in proposition 12.2 with  $\alpha = 2$ .

*Proof.* We first use theorem 13.7 to prove that the Poincaré series is as stated. By this theorem we have that

$$P_{r,p}(t) = \frac{1}{1-t^2} \left( 1 + P_{IF(r,p,2)}(t) + \frac{1}{t} P_{IF'(r,p,2)}(t) \right) + \frac{1}{t} P_{IT(r,p,2)}(t). \quad (10)$$

By using equation (9) and lemma 11.7 we can rewrite this as

$$P_{r,p}(t) = \frac{1}{1-t^2} \left( 1 + P_{\mathcal{IF}(r,p,2)}(t) + \frac{1}{t} P_{\mathcal{IF}(r,p,2)}(t) \right) - \frac{1}{t} P_{\mathcal{IF}(r,p,2)}(t) + \frac{t}{1-t^2}.$$

The desired result follows by a small reduction.

When  $p$  divides  $r+1$ , we can write the index set as

$$\mathcal{IF}(r,p,2) = \{2(ri + pm) \mid 0 \leq i, 0 \leq m \leq \frac{r+1}{p} - 1\} \setminus \{0\}.$$

The last formula for  $P_{r,p}(t)$  follows.

Since corollary 13.8 holds, we have a formula for the Poincaré series of the  $E_3$  term in corollary 12.3 with  $\alpha = 2$ . This series is the same as  $P_{r,t}(t)$  when  $p$  does not divide  $r+1$  by (10). When  $p$  divides  $r+1$  the two series also agree by our last formula for  $P_{r,p}(t)$ . Thus, the Serre spectral sequence collapses from the  $E_3$  page.  $\square$

The proof of the main theorem involves the existence of non-trivial differentials in the Morse spectral sequence. This fact has a certain geometrical content which we describe below for  $r \geq 2$ . By similar methods one can get the same result for  $r = 1$ , but we will not go into this here.

**Corollary 13.11.** *For any  $r \geq 2$  and any  $n \geq 2$  there is a trajectory of loops on  $\mathbb{C}P^r$  which converges in positive time towards a geodesic with period  $n$  and in negative time towards a non-constant geodesic with period  $n+1$ .*

*Proof.* Assume that there are no such trajectories. Then the geometric map

$$(\mathcal{F}_{n+1}/\mathcal{F}_n)_{h\mathbb{T}} \rightarrow \Sigma(\mathcal{F}_n)_{h\mathbb{T}} \rightarrow \Sigma(\mathcal{F}_n/\mathcal{F}_{n-1})_{h\mathbb{T}},$$

which induces the  $d_1$  differential in the Morse spectral sequence, is nullhomotopic. So in the mod  $p$  Morse spectral sequence  $E_* = E_*(\mathcal{M})(LCP_{h\mathbb{T}}^r)$ , we have  $d_1 = 0 : E_1^{n,*} \rightarrow E_1^{n+1,*}$  for any prime  $p$ . We intend to show that you can always chose a prime  $p$ , such that this  $d_1$  must be non trivial, leading us to the desired contradiction.

Since  $n \geq 2$ , there is a prime  $p$  that divides  $n$ , say  $n = pm$ . We actually claim that any such  $p$  will do. Assume to the contrary, that  $d^1$  is trivial on  $E_1^{pm,*}$ . The generators of the  $\mathbb{F}_p[u]$ -module  $E_1^{pm,*}$  have degrees less or equal to  $2r(pm+1)$ . Since the lowest class in  $E_1^{pm+s,*}$  is in degree  $(2r)(pm+s-1)+1$ , the only possible non-trivial differential originating in  $E_*^{pm,*}$  would be a  $d_2$  hitting the non-trivial class in  $E_2^{pm+2, 2r(pm+1)-pm-1}$ . But comparison to the non-equivariant spectral sequence shows that this class is a permanent cycle, so every class in  $E_1^{pm,*}$  is a permanent cycle. (cf. the proof of theorem 9.11).

But this means that every generator of the free  $\mathbb{F}_p[u]$ -module  $E_1^{(pm,*)\text{even}}$  corresponds to a generator in the free  $\mathbb{F}_p[u]$ -module  $H^{\text{even}}(LCP_{h\mathbb{T}}^r)$ . So this modules has a generator in every degree  $2k$  where  $2rpm+2 \leq 2k \leq 2r(pm+1)$ . In case  $p \mid (r+1)$ , in addition to this, it has a generator in degree  $2k = 2rpm$ . We have listed the generators of this module in theorem 13.7, so this says that all these numbers  $2k$  are contained in  $\mathcal{IF}'(r,p,2)$ .

The rest of the proof is just very elementary number theory. As usual, there are two possible cases. We first deal with the case  $p \mid (r + 1)$ . According to the definition of  $\mathcal{IF}'(r, p, 2)$  just prior to theorem 13.7, this set does not contain any numbers divisible by  $2r$ . So we cannot have that  $2rpm \in \mathcal{IF}'(r, p, 2)$ , contradicting the assumption.

So now assume that  $p \nmid (r + 1)$ . We first show that there are not many pairs of consecutive even numbers in  $\mathcal{IF}(r, p, 2) = \mathcal{IF}'(r, p, 2)$ . Assume that both  $2k$  and  $2k + 2$  are contained in  $\mathcal{IF}(r, p, 2)$ . As in definition 11.3 we find four numbers  $i_1, i_2, j_1, j_2$  pairwise satisfying the conditions mentioned in that definition, and such that  $2k = 2ri_1 + 2j_1$  and  $2k + 2 = 2ri_2 + 2j_2$ . Then  $2 = 2r(i_2 - i_1) + 2(j_2 - j_1)$ . Since  $|j_1 - j_2| \leq 2(r - 1)$  it follows that either  $i_1 = i_2$  and  $j_2 = j_1 + 1$ , or  $i_2 = i_1 + 1$ ,  $j_2 = 0$  and  $j_1 = r - 1$ . The first possibility together with the congruence condition leads to  $p \mid ((r + 1)i_1 + j_1)$  and  $p \mid ((r + 1)i_1 + j_1 + 1)$ , which is a contradiction.

The second possibility leads to  $p \mid ((r + 1)i_1 + r - 1)$  and  $p \mid ((r + 1)(i_1 + 1))$ . Subtracting, we get that  $p \mid 2$ , that is  $p = 2$ . So the only possibility for that  $\{2k, 2k + 2\} \in \mathcal{IF}(r, p, 2)$  is that  $p = 2$ ,  $r$  is even,  $j_1 = r - 1$  is odd and  $k = ri_1 + j_1$  is odd. In particular, we can never have that  $\{2k, 2k + 2, 2k + 4\} \in \mathcal{IF}(r, p, 2)$ . Since the set of  $k$  such that  $2rpm + 2 \leq 2k \leq 2rp(m + 1)$  is a set of  $r$  consecutive numbers, we have that  $r = 2$ .

We are now reduced to showing that  $\mathcal{IF}(2, 2, 2)$  does not contain two numbers of the form  $8m + 2, 8m + 4$ . We already noted that if  $2k, 2k + 2 \in \mathcal{IF}(r, p, 2)$ , then  $k$  is odd. So if  $2k = 8m + 4$ , this particular  $k$  does not qualify. The proof is complete.  $\square$

## 14 Appendix: The circle transfer map

In this appendix we describe an elementary construction of a  $\mathbb{T}$  transfer map  $\tau$ . There are several discussions of transfer maps in the literature, and also of extensive refinements and generalizations. See for example [MS] or [MMM]. For our purposes, we need only a rather coarse version. We give a simple, self contained construction close to the one we gave in [BO1].

**Theorem 14.1.** *Let  $X$  be a based  $\mathbb{T}$ -CW complex such that the action of  $\mathbb{T}$  is free away from the base point. Write  $q : X \rightarrow X/\mathbb{T}$  for the canonical projection. Let  $R$  be a principal ideal domain and  $M$  an  $R$ -module. There is a linear map which is natural in  $X$  and  $M$  as follows:*

$$\tau : \tilde{H}^n(X; M) \rightarrow \tilde{H}^{n-1}(X/\mathbb{T}; M).$$

*It is the connecting homomorphism in a long exact Gysin sequence*

$$\cdots \rightarrow \tilde{H}^{n-2}(X/\mathbb{T}; M) \xrightarrow{\gamma} \tilde{H}^n(X/\mathbb{T}; M) \xrightarrow{q^*} \tilde{H}^n(X; M) \xrightarrow{\tau} \tilde{H}^{n-1}(X/\mathbb{T}; M) \rightarrow \cdots$$

*in particular,  $\tau \circ q^* = 0$ . Assume that  $M = R$ . Then Frobenius reciprocity holds*

$$\tau(aq^*(b)) = \tau(a)b,$$

*and  $q^* \circ \tau = d$  where  $d$  denotes the action differential.*

*Remark 14.2.* We have  $H^*(B\mathbb{T}; R) = R[u]$  where  $\deg u = 2$  which gives us a class  $pr_1^*(u) \in H^2(X_{h\mathbb{T}}; R)$ . The map  $\gamma$  in the Gysin sequence is given by multiplication by this class:

$$\begin{array}{ccc} H^{n-2}(X_{h\mathbb{T}}, B\mathbb{T}; M) & \xrightarrow{pr_1^*(u)} & H^n(X_{h\mathbb{T}}, B\mathbb{T}; M) \\ \cong \uparrow & & \cong \uparrow \\ \tilde{H}^{n-2}(X/\mathbb{T}; M) & \xrightarrow{\gamma} & \tilde{H}^n(X/\mathbb{T}; M). \end{array}$$

*Proof.* The key point is to compare  $X$  to  $E\mathbb{T} \times X/E\mathbb{T}$ . So we start by considering the spherical fibration

$$\mathbb{T} \longrightarrow E\mathbb{T} \times X \xrightarrow{Q} E\mathbb{T} \times_{\mathbb{T}} X,$$

together with the two subspaces  $E\mathbb{T} = E\mathbb{T} \times *$  of  $E\mathbb{T} \times X$  and  $B\mathbb{T} = E\mathbb{T} \times_{\mathbb{T}} *$  of  $E\mathbb{T} \times_{\mathbb{T}} X$  with  $Q^{-1}(B\mathbb{T}) = E\mathbb{T}$ .

The fibration  $Q$  is a pullback of the fibration  $\mathbb{T} \rightarrow E\mathbb{T} \rightarrow B\mathbb{T}$  along the projection map  $pr_2$ . Since  $B\mathbb{T}$  is 1-connected it follows that  $Q$  is orientable. Thus there is a relative Gysin sequence for  $Q$  [W, VII.5.12]. We let  $\tilde{\tau}$  be the connecting homomorphism in this sequence.

Let  $X_0$  denote  $X$  with trivial  $\mathbb{T}$  action. Pick a point  $e \in E\mathbb{T}$  and define the  $\mathbb{T}$  map  $\theta : \mathbb{T} \times X_0 \rightarrow E\mathbb{T} \times X$  by  $(z, x) \mapsto (ez^{-1}, zx)$ . We have a commutative diagram

$$\begin{array}{ccc} (\mathbb{T} \times X_0, \emptyset) & \xrightarrow{pr_2} & (X, \emptyset) \\ \downarrow \theta & & \downarrow \\ (E\mathbb{T} \times X, E\mathbb{T}) & \xrightarrow{Q} & (E\mathbb{T} \times_{\mathbb{T}} X, B\mathbb{T}). \end{array}$$

By naturality of the Gysin sequence we have the following diagram, where the cohomology groups have coefficients in  $M$ :

$$\begin{array}{ccccccccc} \longrightarrow & H^{n-2}(X_{h\mathbb{T}}, B\mathbb{T}) & \xrightarrow{u} & H^n(X_{h\mathbb{T}}, B\mathbb{T}) & \xrightarrow{Q^*} & \tilde{H}^n(X) & \xrightarrow{\tilde{\tau}} & H^{n-1}(X_{h\mathbb{T}}, B\mathbb{T}) & \xrightarrow{u} \\ & \downarrow Q^* & & \downarrow Q^* & & \downarrow \theta^* & & \downarrow Q^* & \\ \longrightarrow & H^{n-2}(X) & \xrightarrow{0} & H^n(X) & \xrightarrow{pr_2^*} & H^n(\mathbb{T} \times X) & \xrightarrow{\bar{\tau}} & H^{n-1}(X) & \xrightarrow{0} \longrightarrow \end{array}$$

By the upper sequence,  $\tilde{\tau} \circ Q^* = 0$ . Write  $\eta : \mathbb{T} \times X \rightarrow X$  for the action map.  $\eta^*$  and  $\theta^*$  are the same in positive degrees. Assume that  $M = R$  Then  $\eta^*(y) = 1 \otimes y + v \otimes dy$  where  $v$  has degree 1. By the diagram  $\bar{\tau}(1 \otimes a) = 0$  and  $\bar{\tau}(v \otimes b) = b$  such that  $Q^* \circ \tilde{\tau} = \bar{\tau} \circ \eta^* = d$ . Finally, we have Frobenius reciprocity for  $\tilde{\tau}$  by [W, VII.5.16].

The map  $pr_2 : E\mathbb{T} \times X/E\mathbb{T} \rightarrow X$  is a homotopy equivalence if we restrict it to the fixed points for any closed subgroup of  $\mathbb{T}$ . So it is a  $\mathbb{T}$ -equivariant homotopy equivalence, see for instance [tD, II.2.7]. Thus, we have a natural isomorphism

$$pr_2^* : H^*(X/\mathbb{T}, *; M) \rightarrow H^*(X_{h\mathbb{T}}, B\mathbb{T}; M).$$

The result follows.  $\square$

## 15 Appendix: Rational coefficients

In this appendix we use theorem 11.1 to obtain results for rational coefficients.

**Proposition 15.1.** *Let  $X$  be a 1-connected space with  $H_*(X; \mathbb{Z})$  of finite type and assume that*

$$H^*(X; \mathbb{Z}) = \mathbb{Z}[x]/(x^{r+1}),$$

where  $\alpha = \deg(x)$  is even and  $r \geq 1$ . Put  $\rho = (r + 1)\alpha - 2$ . Then,

$$H^k(LX; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & k \in \{0\} \cup \{\rho i + \alpha j, \rho i + \alpha j - 1 \mid 0 \leq i, 1 \leq j \leq r\}, \\ 0, & \text{otherwise.} \end{cases}$$

The action differential  $d : H^k(LX; \mathbb{Q}) \rightarrow H^{k-1}(LX; \mathbb{Q})$  is an isomorphism when  $k \in \{\rho i + \alpha j \mid 0 \leq i, 1 \leq j \leq r\}$  and zero otherwise.

*Proof.* By the Serre spectral sequence for the fibration  $\Omega X \rightarrow PX \rightarrow X$  we see that  $H_*(\Omega X; \mathbb{Z})$  is of finite type. The Serre spectral sequence for  $\Omega X \rightarrow LX \rightarrow X$  then gives us that  $H_*(LX; \mathbb{Z})$  is of finite type.

Consider the universal coefficient sequence where  $A$  is an abelian group:

$$0 \longrightarrow \text{Ext}(H_{k-1}(LX; \mathbb{Z}), A) \longrightarrow H^k(LX; A) \longrightarrow \text{Hom}(H_k(LX; \mathbb{Z}), A) \longrightarrow 0.$$

By choosing  $A = \mathbb{Z}/p$  for  $p$  sufficiently large, we obtain that the Ext group is zero and that we can apply theorem 11.1 part 2). Thus,

$$H_k(LX; \mathbb{Z})/T_k(LX) \cong \begin{cases} \mathbb{Z}, & k \in \{0\} \cup \{\rho i + \alpha j, \rho i + \alpha j - 1 \mid 0 \leq i, 1 \leq j \leq r\}, \\ 0, & \text{otherwise,} \end{cases}$$

where  $T_k(LX)$  denotes the torsion subgroup of  $H_k(LX; \mathbb{Z})$ . We then choose  $A = \mathbb{Q}$  and obtain the stated result for  $H^k(LX; \mathbb{Q})$ .

Let  $\eta : \mathbb{T} \times LX \rightarrow LX$  denote the action map. By definition of the action differential  $d$  we have that  $\eta^*(y) = 1 \otimes y + v \otimes dy$  where  $\deg(v) = 1$  for  $\mathbb{Z}/p$  or  $\mathbb{Q}$  coefficients. There is also the projection  $pr_2 : \mathbb{T} \times LX \rightarrow LX$  with  $pr_2^*(y) = 1 \otimes y$ .

By theorem 11.1 we have  $b_0^{j-1}b_i$  in  $H^*(LX; \mathbb{Z}/p)$  with  $\deg(b_0^{j-1}b_i) = \rho i + \alpha j$  and

$$d(b_0^{j-1}b_i) = (j + (r + 1)i)b_0^{j-1}a_i$$

for  $0 \leq i, 1 \leq j \leq r$ . Fix such  $i$  and  $j$  and put  $k = \rho i + \alpha j$ .

We use the universal coefficients sequence with abelian group  $A$  again. By naturality we get commutative diagrams for  $\eta$  and for  $pr_2$ . Choose  $A = \mathbb{Z}/p$  where  $p$  is so large that the Ext groups vanish,  $p > r + 1$  and  $p > j + (r + 1)i$ . Then,

$$0 \neq \eta_* - (pr_2)_* : H_k(\mathbb{T} \times LX; \mathbb{Z})/T_k(\mathbb{T} \times LX) \rightarrow H_k(LX; \mathbb{Z})/T_k(LX).$$

We then take  $A = \mathbb{Q}$  and find that  $0 \neq \eta^* - pr_2^*$  in degree  $k$  for  $\mathbb{Q}$  coefficients. Thus, the action differential  $d : H^k(LX; \mathbb{Q}) \rightarrow H^{k-1}(LX; \mathbb{Q})$  is an isomorphism as stated. Since  $d \circ d = 0$  the vanishing statement for  $d$  follows.  $\square$

The ring structure of  $H^*(LX; \mathbb{Q})$  was first computed by Švarc in [Š]. Combining his computation with the proposition above we obtain the following result:

**Proposition 15.2.** *Let  $X$  be as in proposition 15.1. Then,*

$$H^*(LX; \mathbb{Q}) = \mathbb{Q}[a_i, b_i | i \geq 0] / I,$$

where  $I$  is the ideal generated by the following elements for  $i, j \geq 0$ :

$$a_i a_j, \quad b_i b_j - b_0 b_{i+j}, \quad b_i a_j - b_0 a_{i+j}, \quad b_0^r b_i, \quad b_0^r a_i.$$

The degrees of the generators are  $|a_i| = \rho i + \alpha - 1$  and  $|b_i| = \rho i + \alpha$ . Furthermore,

$$H^*(LX_{h\mathbb{T}}; \mathbb{Q}) = \mathbb{Q}[u] \otimes \mathbb{Q}[w_i^{(j)} | 0 \leq i, 1 \leq j \leq r] / J,$$

where  $J$  is the ideal generated by all the products  $w_i^{(j)} w_k^{(\ell)}$ . The degrees of the generators are  $|u| = 2$  and  $|w_i^{(j)}| = \rho i + \alpha j - 1$ .

*Proof.* The result of Švarc's computation can be found in [Kl, §6]. Performing the substitutions  $a_i = g_{i+1}$  for  $i \geq 0$  and  $b_0 = x$ ,  $b_i = h_i$  for  $i \geq 1$  in Theorem 6.2 and 6.3 of [Kl], we obtain the stated description of the cohomology of  $LX$ .

Consider the Serre spectral sequence for the homotopy orbit space. Note that the elements  $u^k$  for  $k \geq 0$  are not hit by any differentials since we may factor  $id_{B\mathbb{T}}$  as  $B\mathbb{T} \rightarrow LX_{h\mathbb{T}} \rightarrow B\mathbb{T}$ , where we use a constant loop to define the first map and the second map is the projection  $pr_1$ .

The  $d_2$  differential is given by the action differential  $d$ . By proposition 15.1 we see that only the elements of the form  $b_0^{j-1} a_i$  in  $E_2^{0,*}$  and  $u^k$  in  $E_2^{*,0}$  survive to the  $E_3$  page and that  $E_3 = E_\infty$ . The unique class  $w_i^{(j)}$  in  $H^*(LX_{h\mathbb{T}}; \mathbb{Q})$  representing  $b_0^{j-1} a_i$  satisfies  $(w_i^{(j)})^2 = 0$  since it has odd degree. Thus any product  $w_i^{(j)} w_k^{(\ell)}$  has finite multiplicative order and hence cannot equal a nonzero constant times a power of  $u$ . Hence the multiplicative structure is as stated.  $\square$

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