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**Optimal acyclic edge-colouring  
of cubic graphs**

by

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# OPTIMAL ACYCLIC EDGE-COLOURING OF CUBIC GRAPHS

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## Abstract

An *acyclic edge-colouring* of a graph is a proper edge-colouring such that the subgraph induced by the edges of any two colours is acyclic. The *acyclic chromatic index* of a graph  $G$  is the smallest number of colours in an acyclic edge-colouring of  $G$ . We prove that the acyclic chromatic index of a connected cubic graph  $G$  is 4, unless  $G$  is  $K_4$  or  $K_{3,3}$ ; the acyclic chromatic index of  $K_4$  and  $K_{3,3}$  is 5.

## 1 Introduction

Various types of edge-colourings of graphs have occurred in graph theory for more than a century. Among these colourings the most important are *proper* colourings – those that do not allow two adjacent edges to have the same colour. Considerable effort has been devoted to proper edge-colourings where cycles that contain only two colours are forbidden.

A proper  $k$ -edge-colouring of  $G$  such that there are no two-coloured cycles in  $G$  is called an *acyclic  $k$ -edge-colouring* of  $G$ . The concept of acyclic colourings was introduced by Grünbaum in [8].

In this paper all the considered graphs are finite and simple, i. e. without multiple edges and loops. Let  $\Delta = \Delta(G)$  denote the maximum degree of a vertex in a graph  $G$ . Throughout the paper, any proper  $k$ -edge-colouring uses the colours denoted by  $1, 2, \dots, k$ . The *chromatic index* of a graph  $G$ , denoted by  $\chi'(G)$ , is the least number of colours needed to colour the edges of  $G$  by a proper edge-colouring. Similarly we define the *acyclic chromatic index* (also called acyclic edge chromatic

number) of a graph  $G$ , denoted by  $a'(G)$ , to be the least  $k$  such that  $G$  has an acyclic  $k$ -edge-colouring. Obviously,  $a'(G) \geq \chi'(G)$ . An alternative definition of the acyclic chromatic index gives a slightly different point of view: the acyclic chromatic index of a graph  $G$  is the minimum number of matchings which suffice to cover all edges of  $G$  in such a way that the union of any two matchings does not contain a cycle.

It is known that a simple graph  $G$  with maximal degree  $\Delta$  has chromatic index either  $\Delta$  or  $\Delta + 1$ . How many new colours do we have to use if we do not want to allow two-coloured cycles? It is known that  $a'(G) \leq 16\Delta$  (see [2] and [9]), but recent results (e. g. [1], [3], and [10]) suggest that this bound is far from tight. The conjecture of Fiamčík [7] and later by Alon, Sudakov, and Zaks [1] says that  $a'(G) \leq \Delta + 2$  for every graph  $G$ . Burnstein's result in [4] implies that for cubic graphs  $a'(G) \leq 5$ , therefore the conjecture is true for  $\Delta = 3$ . In 1980, Fiamčík [5] published a paper claiming that  $K_4$  is the only cubic graph requiring five colours in an acyclic edge-colouring, nevertheless both the result and the proof were incorrect. Four years later, in [6], he corrected one of the errors of the previous paper, and stated a correct result that there are two exceptional cubic graphs requiring five colours in an acyclic edge-colouring. However, the proof still contains a big gap: usage of Lemma 2 in [5] eliminates the cycle of colours 1 and 2, but it does not ensure that no two-coloured cycles of other pairs of colours are created. This may be the reason why the result of Fiamčík has fallen into obscurity.

In this paper we focus on graphs with  $\Delta = 3$ . In [3], Basavaraju and Chandran proved that  $a'(G) \leq 4$  for all subcubic graphs (graphs with maximal degree at most 3) containing a vertex of degree at most 2.

In every proper 3-edge-colouring of a cubic graph  $G$  the edges coloured by any of the colours form a perfect matching. Hence the subgraph of  $G$  induced by the edges of any two colours form a 2-factor, which is a nonempty set of two-coloured cycles. Therefore to have an acyclic edge-colouring of a cubic graph we need at least four colours. We prove that four colours are optimal for all connected cubic graphs with the exception of  $K_4$  and  $K_{3,3}$ , the two graphs mentioned by Fiamčík in [6], for which 5 colours are optimal.

Our main result is captured in the following theorem. The proof in Section 2 is algorithmic and does not rely on any probabilistic arguments. As a simple corollary we are able to determine the acyclic chromatic index of all cubic graphs.

**Theorem 1.1** *Let  $G$  be a connected graph with  $\Delta(G) \leq 3$  different from  $K_4$  and  $K_{3,3}$ . Then  $a'(G) \leq 4$ .*

**Corollary 1.2** *The acyclic chromatic index of a connected cubic graph  $G$  is 4 unless  $G$  is  $K_4$  or  $K_{3,3}$ , for which  $a'(K_4) = a'(K_{3,3}) = 5$ .*

**Proof.** For a cubic graph  $G$  we have proved that  $a'(G) \geq 4$ . An acyclic 5-edge-colouring of the graphs  $K_4$  and  $K_{3,3}$  is shown in Fig. 1.

It remains to prove that the acyclic chromatic index is at least 5 for both of these two graphs. Suppose that we have an acyclic 4-edge-colouring of  $K_4$ . The graph  $K_4$  has 6 edges, and the edges coloured by any of the colours form a matching, hence no colour is used on more than two edges. Therefore there are two colours such that both of them colour exactly two edges, and edges coloured by these colours form a 2-factor, which is a two-coloured cycle of length 4 in our case. We have derived a contradiction.

Suppose that we have an acyclic 4-edge-colouring of  $K_{3,3}$ . The graph  $K_{3,3}$  has 9 edges. If there are two colours such that both of them colour three edges, the edges coloured by these colours form a 2-factor, which is a two-coloured cycle of length 6 in our case. Otherwise exactly one colour, say 1, colours three edges and any other colour colours two edges. Choose a cycle  $C$  of length 4 in  $K_{3,3}$  such that two of its

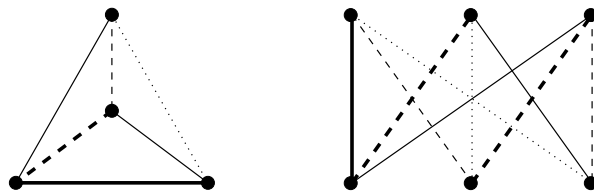


Figure 1: An acyclic 5-edge-colouring of  $K_4$  and  $K_{3,3}$

edges are coloured by the colour 1. The other edges in this cycle must be of different colours, say 2 and 3. Look at one of the edges not belonging to  $C$  and adjacent with the edge of  $C$  that is coloured by the colour 2. Its colour can be either 3 or 4. Since our colouring is proper and each colour different from 1 is used twice, the colours of all the other edges are determined. It is easy to check that in both cases there is a two-coloured cycle, hence the graph  $K_{3,3}$  cannot have an acyclic 4-edge-colouring.  $\square$

## 2 Acyclic edge-colouring of subcubic graphs

In this section we prove Theorem 1.1. For the sake of completeness we let the proof cover the case where the minimum degree  $\delta(G) \leq 2$ , although this case was proved in [3], as already noted.

We proceed by induction on the number of vertices of  $G$ . If  $G$  is a 1-vertex graph, then the assertion is trivial. We therefore assume that  $G$  is a connected graph on at least two vertices with  $\Delta(G) \leq 3$ ,  $G \neq K_4$ ,  $G \neq K_{3,3}$ , and for every connected graph  $H$  of smaller order, with  $\Delta(H) \leq 3$  and different from  $K_4$  and  $K_{3,3}$ ,  $a'(H) \leq 4$ . The proof splits into several cases depending on whether  $G$  contains a bridge or not, and on  $\delta(G)$ .

**Case 1:  $G$  contains a bridge.** Let  $e$  be a bridge. Let  $V_1$  and  $V_2$  be the vertex-sets of the two components created by removing  $e$  and let  $G_i$  be the graph induced by  $V_i$ ,  $i = 1, 2$ . Both  $G_1$  and  $G_2$  have fewer vertices than  $G$  and each of them has at least one vertex of degree at most 2, therefore both these graphs are different from  $K_4$  and  $K_{3,3}$ , and, in turn, have an acyclic 4-edge-colouring. For  $i = 1, 2$ , let  $\varphi_i$  be an acyclic 4-edge-colouring of  $G_i$ . We can obtain an acyclic 4-edge-colouring  $\varphi$  of  $G$  by letting  $\varphi(e)$  be any colour and permuting the colours of  $\varphi_1$  and  $\varphi_2$  on  $G_1$  and  $G_2$  so that the edges of  $G$  adjacent to  $e$  have colours different from  $\varphi(e)$ .

**Case 2:  $G$  is bridgeless and  $\delta(G) = 2$ .** (When  $G$  is bridgeless  $\delta(G)$  is at least 2.) Let  $v$  be a vertex of degree 2 and let  $v_1$  and  $v_2$  be the two neighbours of  $v$ .

First suppose that  $v_1$  and  $v_2$  are not adjacent. We construct a new graph  $G'$  by removing the vertex  $v$  from  $G$  together with the incident edges  $v_1v$  and  $v_2v$  and adding a new edge  $v_1v_2$ ; clearly,  $G'$  is simple and connected. If  $G' = K_4$  or  $G' = K_{3,3}$ , then  $G$  together with an acyclic 4-edge-colouring is depicted in Fig. 2. If  $G' \neq K_4$  and  $G' \neq K_{3,3}$ , we have  $a'(G') \leq 4$  by the induction hypothesis. Let  $\varphi'$  be an acyclic 4-edge-colouring of  $G'$ . We construct an acyclic 4-edge-colouring  $\varphi$  of  $G$ . We set  $\varphi(e) = \varphi'(e)$  for every  $e \neq vv_1$  and  $e \neq vv_2$ .

If the degree of one of the neighbours of  $v$ , say  $v_2$ , is 2, denote by  $v_3$  the neighbour of  $v_2$  such that  $v \neq v_3$ . We choose  $\varphi(vv_1)$  to be  $\varphi'(v_1v_2)$ . To finish the colouring we chose  $\varphi(vv_2)$  to be any colour different from  $\varphi(vv_1)$  and  $\varphi(v_2v_3)$ . The resulting colouring is proper and clearly, from the induction hypothesis, if there is a 2-coloured cycle in  $\varphi$ , it has to contain the vertex  $v$ . But this is not possible because  $\varphi(vv_1) \neq \varphi(v_2v_3)$ .

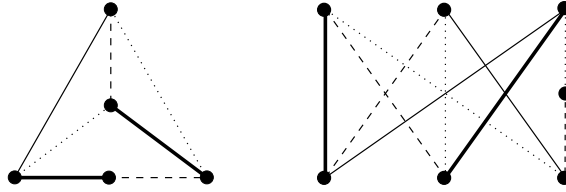


Figure 2: Graphs  $K_4$  and  $K_{3,3}$  with a subdivided edge and their acyclic 4-edge-colourings

If, on the other hand,  $\deg(v_1) = 3$  and  $\deg(v_2) = 3$ , let  $e_1$  and  $e_2$  be the edges incident with  $v_1$  but different from  $vv_1$  and let  $e_3$  and  $e_4$  be the edges incident with  $v_2$  but different from  $vv_2$ . Since  $\varphi'$  is a proper 4-edge-colouring of  $G'$  which includes the edge  $v_1v_2$ , we have  $\{\varphi(e_1), \varphi(e_2)\} \cap \{\varphi(e_3), \varphi(e_4)\} \neq \emptyset$ . If  $\{\varphi(e_1), \varphi(e_2)\} = \{\varphi(e_3), \varphi(e_4)\}$ , we choose for each of  $\varphi(vv_1)$  and  $\varphi(vv_2)$  one of the two remaining colours different from  $\varphi(e_1)$  and  $\varphi(e_2)$ , different from each other. If  $|\{\varphi(e_1), \varphi(e_2)\} \cap \{\varphi(e_3), \varphi(e_4)\}| = 1$ , and, say  $\varphi(e_1) = \varphi(e_3)$ , we set  $\varphi(vv_1) = \varphi(e_4)$  and  $\varphi(vv_2) = \varphi'(v_1v_2)$ . It can be easily seen that in both cases  $\varphi$  is an acyclic 4-edge-colouring of  $G$ .

Now suppose that  $v_1$  and  $v_2$  are adjacent. Since  $G$  is bridgeless, either  $G$  is a triangle,  $G$  is  $K_4$  minus an edge or the vertices  $v_1$  and  $v_2$  are of degree 3 and there is no vertex different from  $v$  adjacent to both  $v_1$  and  $v_2$ . If  $G$  is a triangle or  $K_4$  minus an edge, then  $a'(G) \leq 4$ ; in the latter case let  $u_1$  be the neighbour of  $v_1$  different from  $v$  and  $v_2$  and let  $u_2$  be the neighbour of  $v_2$  different from  $v$  and  $v_1$ . As noted,  $u_1 \neq u_2$ . We remove from  $G$  the vertices  $v$ ,  $v_1$ , and  $v_2$  together with the incident edges and add a new vertex  $w$  of degree 2, joined by an edge to  $u_1$  and  $u_2$ . We denote the resulting graph  $G'$ . The graph  $G'$  is simple and connected and has fewer vertices than  $G$ , and it is not  $K_4$  or  $K_{3,3}$  as it has a vertex of degree 2, therefore by the induction hypothesis  $a'(G') \leq 4$ . Let  $\varphi'$  be an acyclic 4-edge-colouring of  $G'$ . Define the colouring  $\varphi$  of  $G$  as follows:  $\varphi(e) = \varphi'(e)$  for every edge  $e$  that is not incident with any of  $v$ ,  $v_1$ , and  $v_2$ . Further set  $\varphi(u_1v_1) = \varphi(vv_2) = \varphi'(u_1w)$ ,  $\varphi(u_2v_2) = \varphi(vv_1) = \varphi'(u_2w)$  and colour the edge  $v_1v_2$  by any of the two remaining colours different from  $\varphi(vv_1)$  and  $\varphi(vv_2)$  (see Fig. 3). Clearly,  $\varphi$  is an acyclic 4-edge-colouring of  $G$ .

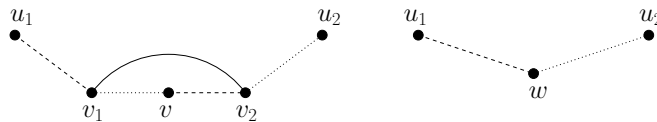


Figure 3: Creating  $G'$  from  $G$  in Case 2,  $v_1$  and  $v_2$  adjacent

**Case 3:  $G$  is a bridgeless cubic graph.** Here the proof splits into several subcases depending on the girth of  $G$  (denoted by  $g(G)$  in what follows).

**Case 3a: The girth of  $G$  is even and  $g(G) \geq 6$ .**

Let  $C$  be any of the shortest cycles in  $G$ . Let the vertices of  $C$  be  $u_0, v_0, u_1, v_1, \dots, u_{k-1}, v_{k-1}$  where  $k = g(G)/2 \geq 3$  (the indices will be taken mod  $k$  in what follows). Let  $u'_i$  and  $v'_i$  be the neighbours of  $u_i$  and  $v_i$ , respectively, which do not belong to  $C$  (the vertices  $u'_0, v'_0, u'_1, v'_1, \dots$  are distinct, because if a vertex outside  $C$  had two edges to  $C$  there would be a cycle of shorter length than  $C$ ). Let  $G'$  be the graph constructed in the following way: we remove from  $G$  the cycle  $C$  together with all the edges adjacent to this cycle and join  $u'_i$  to  $v'_i$  for  $i = 0, 1, 2, \dots, k-1$  (see Fig. 4). The resulting graph  $G'$  is cubic and simple because  $g(G) > 4$ : a multiple edge in  $G'$  yields a cycle of length 4 in  $G$ . Note that  $G'$  might not be connected but it contains



colours. Consider a cycle of intersection 1 entering  $C$  in  $u_i$ . This cycle contains at least three colours by the induction hypothesis (even if it enters  $C$  several times).

Now we modify  $\varphi$  to be a proper 4-edge-colouring of  $G$  such that any cycle other than  $C$  contains at least three colours. The only problematic cycles are those of intersection at least 2. Such a cycle containing only two colours cannot enter  $C$  in  $u_i$ : the colours of  $u_i u'_i$  and  $v_i v'_i$  are the same, hence the colours  $\varphi(u_i u'_i)$ ,  $\varphi(u_i v_i)$ , and  $\varphi(v_i u_{i+1})$  are three different colours contained in this cycle.

Let  $D$  be a two-coloured cycle which enters  $C$  in  $v_i$ . Then  $D$  cannot leave  $C$  in  $v_{i+1}$ , because the colours of  $v_i u_{i+1}$  and  $v_{i+1} v'_{i+1}$  cannot be the same. Hence the colours of  $v_i u_{i+1}$  and  $v_{i+1} u_{i+2}$  are the same. Now set  $\varphi(u_{i+1} v_{i+1})$  to be the colour different from  $\varphi(v_i v'_i)$ ,  $\varphi(v_i u_{i+1})$  and  $\varphi(u_{i+1} u'_{i+1})$ . Any cycle of intersection 1 containing the edge  $u_{i+1} v_{i+1}$  contains at least three colours by the induction hypothesis, thus we did not introduce a new two-coloured cycle.

After these modifications the only possibly two-coloured cycle is  $C$ . Assume that it contains only two colours, say 1 and 2. The edges incident with vertices of  $C$  but not belonging to  $C$  are coloured by colours 3 and 4. If  $\varphi(u_1 u'_1) = \varphi(u_2 u'_2) = 3$ , we can set  $\varphi(u_2 v_2) = 4$  to obtain an acyclic 4-edge-colouring of  $G$ . Otherwise we may assume that the edges  $u_i u'_i$  for  $i = 1, 2, \dots, k$  are coloured alternately by colours 3 and 4 (this assumption allows us to replace colour 1 on the edges of  $C$  by colour 2 and vice versa without introducing two-coloured cycles of intersection 1). Moreover we may assume that  $\varphi(u_1 u'_1) = \varphi(v_1 v'_1) = 3$ ,  $\varphi(u_2 u'_2) = \varphi(v_2 v'_2) = 4$ ,  $\varphi(u_1 v_1) = 2$  and  $\varphi(v_1 u_2) = 1$ . There are three cases for the colours  $f_1$  and  $f_2$  of the two edges incident with  $u'_2$  and not incident with  $u_2$ :

- if  $\{f_1, f_2\} = \{2, 3\}$  we set  $\varphi(v_1 u_2) = 4$  and  $\varphi(u_2 u'_2) = 1$ ,
- if  $\{f_1, f_2\} = \{1, 2\}$  we set  $\varphi(v_1 u_2) = 4$  and  $\varphi(u_2 u'_2) = 3$ ,
- if  $\{f_1, f_2\} = \{1, 3\}$  we set  $\varphi(v_1 u_2) = 4$  and  $\varphi(u_2 u'_2) = 2$  and colour the edges of  $C$  different from  $v_1 u_2$  by colours 1 and 2 alternatively to obtain a proper 4-edge-colouring.

The resulting colouring  $\varphi$  is an acyclic 4-edge-colouring of  $G$ .

**Case 3b: The girth of  $G$  is at least 5 and is odd.**

Let  $C$  be one of the shortest cycles of  $G$ . Let  $u_0, v_0, u_1, v_1, \dots, u_{k-1}, v_{k-1}, w$  be the vertices of this cycle in the order in which they lie on the cycle. Let  $u'_i, v'_i$ , and  $w'$  be the neighbours of  $u_i, v_i$ , and  $w$ , respectively, which do not lie on the cycle  $C$ .

From the graph  $G$  we construct a graph  $G'$  by removing the cycle  $C$  with all the edges adjacent to this cycle and joining  $u'_i$  to  $v'_i$  for  $i = 0, 1, 2, \dots, k-1$ . As the girth of  $G$  is at least 5 and  $C$  is a shortest cycle, the resulting graph  $G'$  contains no loops and no multiple edges. Moreover  $G'$  has  $\delta = 2$ ,  $\Delta = 3$  and no component of  $G$  is isomorphic to  $K_4$  or  $K_{3,3}$ . The last assertion can be proved in almost the same way as we have done it in the Case 3a, the difference occurs only for  $K = K_{3,3}$ , where we have three added edges forming a 1-factor: since the graph  $G$  in this case ( $K_{3,3}$  without a 1-factor) has girth at most 6 and the girth is odd, it is equal to 5, hence  $C$  has only 5 vertices. This is a contradiction to the fact that  $K$  has 6 distinct vertices that have 6 pairwise distinct neighbours lying on  $C$ . The induction hypothesis applied to each component of  $G'$  gives an acyclic 4-edge-colouring  $\varphi'$  of  $G'$ . We extend this colouring to an acyclic 4-edge-colouring  $\varphi$  of  $G$ .

First, set  $\varphi(e) = \varphi'(e)$  for each edge  $e$  of  $G$  that have no vertex in common with  $C$  and set  $\varphi(u_i u'_i) = \varphi(v_i v'_i) = \varphi'(u'_i v'_i)$ .

Next, we describe the colouring of the three edges adjacent with  $w$ . In what follows we may assume that the colours used in  $\varphi'$  are 1, 2, 3, 4 and the colours of the edges incident with  $w'$  but not incident with  $w$  are 1 and 2 and  $\varphi(v_{k-1} v'_{k-1}) \leq \varphi(u_0 u'_0)$ . There are several possibilities for the colours of  $u_0 u'_0$  and  $v_{k-1} v'_{k-1}$ , all of them with the desired colouring of the edges incident with  $w$  are in Fig. 5. Note that in any possibility there is no two-coloured cycle passing through any two of the three



edges  $ww'$ ,  $u_0u'_0$  and  $v_{k-1}v'_{k-1}$ . The direction  $u_0, v_0, u_1, \dots$  along  $C$  is referred to as *clockwise*. Recall the definitions of the terms *enter*, *leave* and *to be of intersection* for a cycle that has an edge in common with  $C$ , they apply also in this case.

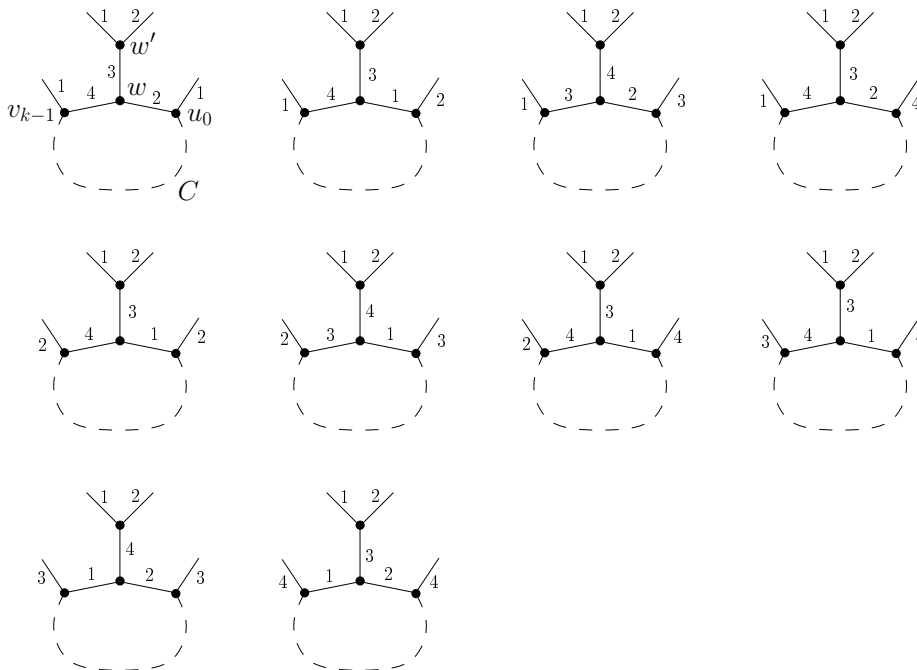


Figure 5: Colouring of the edges incident with  $w$  in Case 3b

As the next step we colour one edge of  $C$  so that there will be no two-coloured cycle that enters  $C$  at  $w$ . Any two-coloured cycle in a proper edge-colouring has its edges coloured alternately by two colours. We utilize this fact and choose the colour of one of the edges of  $C$  so that this edge together with the already coloured edge  $ww'$  will exclude the possibility of a two-coloured cycle entering  $C$  in  $w$ .

Consider one of the possibilities depicted in Fig. 5. Assume that there exists a smallest integer  $j \in \{0, 1, \dots, k-1\}$  such that  $v_jv'_j$  and  $u_{j+1}u'_{j+1}$  are not of the same colour. Any two-coloured cycle entering  $C$  at  $w$  would have to use the path  $wu_0v_0 \dots v_ju_{j+1}$ . There are two possible colours for the edge  $v_ju_{j+1}$ , we choose one of them so that this edge will break the possibly two-coloured cycle entering  $C$  in  $w$ . If there is no integer  $j$  with the desired property, there is no problem with a cycle entering  $C$  in  $w$ : all the edges  $u_iu'_i$  and  $v_iv'_i$  are of the same colour that is not contained in a two-coloured cycle entering  $C$  in  $w$ , hence this cycle cannot leave  $C$ .

We continue by colouring the edges  $v_iu_{i+1}$  which have not been coloured yet, we do this in the same manner as we have done in the Case 3a. Then we colour the edges  $u_iv_i$  to get a proper 4-edge-colouring  $\varphi$  of  $G$ . The induction hypothesis assures that any two-coloured cycle in  $G$  has to contain at least one edge from  $C$ . The cycle  $C$  itself has odd length, hence it cannot be two-coloured in a proper edge-colouring. From the construction of  $\varphi$  we know that no cycle of intersection 1 is two-coloured. We have dealt with cycles entering  $C$  in  $w$  in the previous paragraph. Since  $\varphi$  is proper, no two-coloured cycle can enter  $C$  in  $u_i$ . Put together this means that the only possibly problematic cycles is of intersection at least 2 and enter  $C$  in  $v_i$  for some  $i$ . For  $i = 0, 1, 2, \dots, k-2$  we successively apply the same recolouring as we have done for graphs of even girth to change  $\varphi$  so that any such cycle will contain at least three colours. Finally, almost the same recolouring applies also for a cycle entering  $C$  in  $v_{k-1}$  — which has to reach  $u_0$ , the only difference is that the path  $v_{k-1}u_0$  has length 2 instead of 1 and we possibly change the colour of  $u_0v_0$ . After this step  $\varphi$  is an acyclic 4-edge-colouring of  $G$ , hence  $a'(G) \leq 4$ .

**Case 3c: The girth of  $G$  is 4.**

Let  $C_4 = v_1v_2v_3v_4$  be a 4-cycle in  $G$ . First assume that  $C_4$  contains two non-adjacent edges such that neither of them is contained in a 4-cycle different from  $C_4$ . For  $i = 1, 2, 3, 4$ , let  $v'_i$  denote the neighbour of  $v_i$  not belonging to  $C_4$ . There are no triangles in  $G$  and thus no two vertices of  $v'_1, v'_2, v'_3$ , and  $v'_4$  coincide. Moreover, since  $G$  is bridgeless, the graph  $G - C_4$  has either one or two components.

If  $G - C_4$  is disconnected, there are two vertices in any of the two components of  $G - C_4$  joined by an edge to  $C_4$  in  $G$ . Without loss of generality we can assume that  $v'_1$  and  $v'_2$  are in different components of  $G - C_4$ . We create the graph  $G'$  by removing the vertices of  $C_4$  together with the incident edges and adding edges joining  $v'_1$  with  $v'_2$  and  $v'_3$  with  $v'_4$ . The graph  $G'$  is a cubic, connected and of smaller order than  $G$  and it contains a 2-edge cut, therefore  $G' \neq K_4$  and  $G' \neq K_{3,3}$ . By the induction hypothesis,  $G'$  has an acyclic 4-edge-colouring  $\varphi'$ . We define an acyclic 4-edge-colouring  $\varphi$  of  $G$  by setting  $\varphi(e) = \varphi'(e)$  for every edge  $e$  not incident with a vertex of  $C_4$ ,  $\varphi(v_1v'_1) = \varphi(v_2v'_2) = \varphi'(v'_1v'_2)$ , and  $\varphi(v_3v'_3) = \varphi(v_4v'_4) = \varphi'(v'_3v'_4)$ . Further, if  $\varphi'(v'_1v'_2) = \varphi'(v'_3v'_4)$ , we set  $\varphi(v_2v_3)$  to be the colour not assigned to an edge incident with  $v'_2$  in  $G$ ,  $\varphi(v_1v_4)$  to be the colour not assigned to an edge incident with  $v'_1$  in  $G$  and if  $\varphi(v_2v_3) \neq \varphi(v_1v_4)$  we set  $\varphi(v_1v_2)$  and  $\varphi(v_3v_4)$  to be the colour different from  $\varphi(v_1v'_1)$ ,  $\varphi(v_2v_3)$ , and  $\varphi(v_1v_4)$ ; otherwise, if  $\varphi(v_2v_3) = \varphi(v_1v_4)$ , we set  $\varphi(v_1v_2)$  and  $\varphi(v_3v_4)$  to be the two different colours, both different from  $\varphi(v_1v'_1)$  and  $\varphi(v_2v_3)$ . If, on the other hand, the colours  $\varphi'(v'_1v'_2)$  and  $\varphi'(v'_3v'_4)$  are different, we set  $\varphi(v_1v_2) = \varphi(v_3v'_3)$ ,  $\varphi(v_3v_4) = \varphi(v_1v'_1)$  and set  $\varphi(v_2v_3)$  and  $\varphi(v_1v_4)$  to be two different colours, both different from  $\varphi(v_1v'_1)$  and  $\varphi(v_3v'_3)$ . In both cases it can be easily checked that the resulting colouring is acyclic.

Now suppose that  $G - C_4$  is connected and the edges  $v_1v_2$  and  $v_3v_4$  are in no other 4-cycle of  $G$  than  $C_4$ . We form the graph  $G'$  in the same way as if  $G - C_4$  was disconnected. If  $G' \neq K_4$  and  $G' \neq K_{3,3}$  we find an acyclic 4-edge-colouring of  $G$  analogously. If  $G' = K_4$  or  $G' = K_{3,3}$  the graph  $G$  together with an acyclic 4-edge-colouring is depicted in Fig. 11(a), 11(b) and 11(c).

Now we proceed to the case that  $C_4$  does not contain a pair of non-adjacent edges such that neither of them is contained in a 4-cycle different from  $C_4$ . Then  $C_4$  has to contain two adjacent edges, each of them contained in a 4-cycle different from  $C_4$ . Let  $v_1v_2$  and  $v_2v_3$  be two such edges.

We first deal with the case that  $G$  contains a 4-cycle  $D$  different from  $C_4$  which contains both  $v_1v_2$  and  $v_2v_3$ . Since  $G$  is simple, neither  $v_3v_4$  nor  $v_4v_1$  is contained in  $D$ . Therefore, there is a vertex  $v_5$  different from  $v_2$  and  $v_4$ , adjacent to both  $v_1$  and  $v_3$ . Let  $v'_2, v'_4$ , and  $v'_5$  be the neighbour of  $v_2, v_4$ , and  $v_5$ , respectively, that is not contained in  $\{v_1, v_2, \dots, v_5\}$ . Since  $G$  is bridgeless, the vertices  $v'_2, v'_4$ , and  $v'_5$  exist but they need not be pairwise distinct. Moreover, it could not happen  $v'_2 = v'_4 = v'_5$ , for otherwise  $G = K_{3,3}$ .

Assume that the vertices  $v'_2, v'_4$ , and  $v'_5$  are pairwise distinct. We form a new graph  $G'$  from  $G$  in the following way. We remove from  $G$  the vertices  $v_1, v_2, \dots, v_5$  and add a new vertex  $u$  linked by an edge with the vertices  $v'_2, v'_4$ , and  $v'_5$  (see Fig. 6). Clearly,  $G'$  is simple, cubic and connected and is distinct from  $K_4$  since  $G$  does not contain triangles. If  $G'$  is distinct from  $K_{3,3}$ , then  $a'(G') \leq 4$  by the induction hypothesis. Let  $\varphi'$  be an acyclic 4-edge-colouring of  $G'$ . An acyclic 4-edge-colouring  $\varphi$  of  $G$  can be constructed by setting  $\varphi(e) = \varphi'(e)$  for every edge  $e$  in  $G$  which is not incident with any of  $v_1, v_2, \dots, v_5$  and  $\varphi(v_1v_2) = \varphi(v_3v_5) = \varphi(v_4v'_4) = \varphi'(uv'_4)$ ,  $\varphi(v_1v_5) = \varphi(v_2v'_2) = \varphi(v_3v_4) = \varphi'(uv'_2)$ , and  $\varphi(v_1v_4) = \varphi(v_2v_3) = \varphi(v_5v'_5) = \varphi'(uv'_5)$ . If  $G' = K_{3,3}$ , an acyclic 4-edge-colouring of  $G$  is depicted in Fig. 11(d).

Assume that  $v'_2 \neq v'_4 = v'_5$ . Let  $v_6$  be the neighbour of  $v'_4$  distinct from  $v_4$  and  $v_5$ ; as  $G$  is bridgeless,  $v_6 \neq v'_2$ . We create a new graph  $G'$  by deleting the vertices  $v_1, v_2, \dots, v_5, v'_4$  and adding a new vertex  $v$  joined to  $v'_2$  and  $v_6$ . Then  $G'$  has fewer

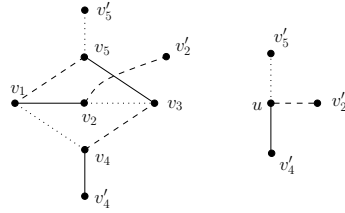


Figure 6: Creating  $G'$  from  $G$  in Case 3c if there is a 4-cycle  $D$  containing both  $v_1v_2$  and  $v_2v_3$  and  $v'_2, v'_4,$  and  $v'_5$  are pairwise distinct

vertices than  $G$ , it is connected, contains a vertex of degree 2 and has maximum degree 3. Hence  $a'(G') \leq 4$  by the induction hypothesis. Let  $\varphi'$  be an acyclic 4-edge-colouring of  $G'$ . We extend  $\varphi'$  to an acyclic 4-edge-colouring of  $G$  in the following way. Without loss of generality we can assume that  $\varphi'(vv'_2) = 1$  and  $\varphi'(vv_6) = 4$ .

We set  $\varphi(e) = \varphi'(e)$  for every edge  $e$  that is not incident with any of  $v_1, v_2, \dots, v_5, v'_4$  and let  $\varphi(v_1v_5) = \varphi(v_3v_4) = \varphi(v_2v'_2) = 1$ ,  $\varphi(v_3v_5) = \varphi(v_4v'_4) = 2$ ,  $\varphi(v_1v_4) = \varphi(v_2v_3) = \varphi(v_5v'_4) = 3$ , and  $\varphi(v_1v_2) = \varphi(v'_4v_6) = 4$ , see Fig. 7. If  $v'_4 \neq v'_2 = v'_5$  or  $v'_5 \neq v'_2 = v'_4$  the proof can be done in a similar way.

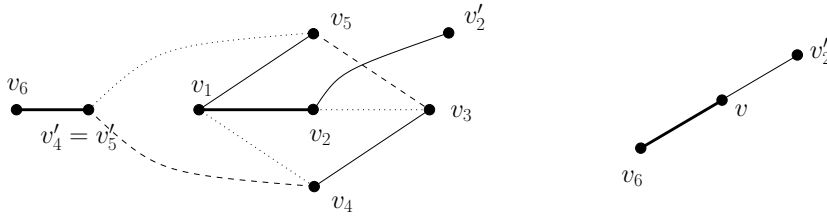


Figure 7: Creating  $G'$  from  $G$  in Case 3c if there is a 4-cycle  $D$  containing both  $v_1v_2$  and  $v_2v_3$  and  $v'_2 \neq v'_4 = v'_5$

Now assume that there is no 4-cycle different from  $C_4$  containing two adjacent edges of  $C_4$ , and  $C_4$  contains two adjacent edges, say  $v_1v_2$  and  $v_2v_3$ , each contained in a 4-cycle different from  $C_4$ . Let  $D_1 = v_1v_2v_5v_6$  be a 4-cycle such that  $\{v_3, v_4\} \cap \{v_5, v_6\} = \emptyset$  and let  $D_2$  be a 4-cycle different from  $C_4$  containing  $v_2v_3$ . Since  $G$  is cubic and simple,  $D_1$  and  $D_2$  share either one or two edges.

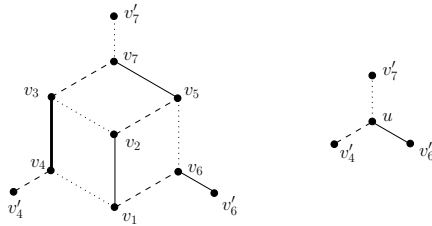


Figure 8: Creating  $G'$  from  $G$  in Case 3c if there is a 4-cycle  $D_1$  containing  $v_1v_2$  and a 4-cycle  $D_2$  containing  $v_2v_3$ ,  $D_1$  and  $D_2$  share exactly one edge, and  $v'_4, v'_6,$  and  $v'_7$  are pairwise distinct

If  $D_1$  and  $D_2$  share exactly one edge then there exists a vertex  $v_7$  different from  $v_1, v_2, \dots, v_6$  and adjacent to both  $v_3$  and  $v_5$ . Note that no two vertices of  $\{v_4, v_6, v_7\}$  are joined by an edge, for otherwise  $G$  would contain a triangle. Let  $v'_4, v'_6,$  and  $v'_7$  be the neighbour of  $v_4, v_6,$  and  $v_7$ , respectively, different from all the vertices  $v_1, v_2, \dots, v_7$ ; the vertices  $v'_4, v'_6,$  and  $v'_7$  are not necessarily distinct. If  $v'_4, v'_6,$  and  $v'_7$  are pairwise distinct, we obtain  $G'$  from  $G$  by removing the vertices  $v_1, v_2, \dots, v_7$  together with the incident edges and adding a new vertex  $u$  joined by

an edge to  $v_4$ ,  $v_6$ , and  $v_7$ . Since  $G$  is triangle-free,  $G'$  is different from  $K_4$ . If  $G'$  is also different from  $K_{3,3}$ , then since it is connected and cubic and has fewer vertices than  $G$ ,  $G'$  has an acyclic 4-edge-colouring  $\varphi'$  by the induction hypothesis. We construct an acyclic 4-edge-colouring  $\varphi$  of  $G$  by setting  $\varphi(e) = \varphi'(e)$  for every edge  $e$  which is incident with no vertex from  $\{v_1, v_2, \dots, v_7\}$  and further set  $\varphi(v_1v_2) = \varphi(v_6v'_6) = \varphi(v_5v_7) = \varphi'(uv'_6)$ ,  $\varphi(v_1v_4) = \varphi(v_2v_3) = \varphi(v_5v_6) = \varphi(v_7v'_7) = \varphi'(uv'_7)$ ,  $\varphi(v_1v_6) = \varphi(v_2v_5) = \varphi(v_3v_7) = \varphi(v_4v'_4) = \varphi'(uv'_4)$ , and  $\varphi(v_3v_4)$  be the colour different from the colours of edges incident with  $u$  in  $G'$  (see Fig. 8). If  $G'$  is  $K_{3,3}$ , an acyclic 4-edge-colouring of  $G$  can be found in Fig. 11(f).

Now we deal with the case that two of the vertices  $v'_4$ ,  $v'_6$ , and  $v'_7$  are equal and different from the third. Assume that  $v'_4 = v'_6 \neq v'_7$ , the proof for other pairs can be done analogously (in fact the situations are completely symmetric). Let  $u$  be the neighbour of  $v'_4$  different from  $v_4$  and  $v_6$ . Since  $G$  is bridgeless,  $u$  is different from  $v'_7$ . We form the graph  $G'$  by removing the vertices  $v_1, v_2, \dots, v_7$  and  $v'_4$  from  $G$  and adding a new vertex  $v$  joined to each of  $v'_7$  and  $u$  by an edge. The graph  $G'$  is different from  $K_4$  and  $K_{3,3}$  as it has a vertex of degree 2, and it is connected, has maximum degree 3 and has fewer vertices than  $G$ . We take an acyclic 4-edge-colouring  $\varphi'$  of  $G'$  and extend it to an acyclic colouring  $\varphi$  of  $G$ . Without loss of generality we may assume that  $\varphi'(vv'_7) = 1$  and  $\varphi'(vu) = 4$ . We set  $\varphi(e) = \varphi'(e)$  for every edge not incident with any of  $v_1, v_2, \dots, v_7$  and  $\varphi(v_1v_6) = \varphi(v_2v_5) = \varphi(v_3v_4) = \varphi(v_7v'_7) = 1$ ,  $\varphi(v_1v_4) = \varphi(v_2v_3) = \varphi(v_5v_7) = \varphi(v_6v'_4) = 2$ ,  $\varphi(v_1v_2) = \varphi(v_3v_7) = \varphi(v_4v'_4) = 3$  and  $\varphi(v_5v_6) = \varphi(v'_4u) = 4$  (see Fig. 9).

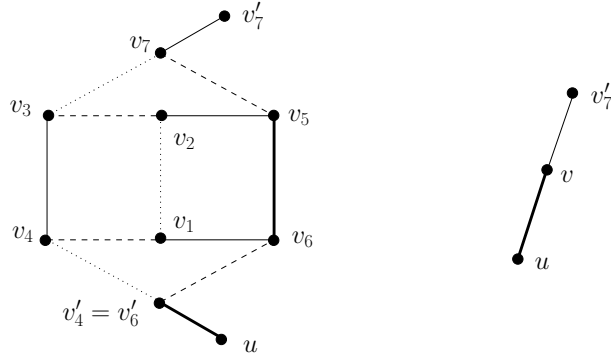


Figure 9: Creating  $G'$  from  $G$  in Case 3c if there is a 4-cycle  $D_1$  containing  $v_1v_2$  and a 4-cycle  $D_2$  containing  $v_2v_3$ ,  $D_1$  and  $D_2$  share exactly one edge, and  $v'_4 = v'_6 \neq v'_7$

If all three vertices  $v'_4$ ,  $v'_6$ , and  $v'_7$  coincide, graph  $G$  is the 3-dimensional cube  $Q_3$ . An acyclic 4-edge-colouring of  $Q_3$  is shown in Fig. 11(g).

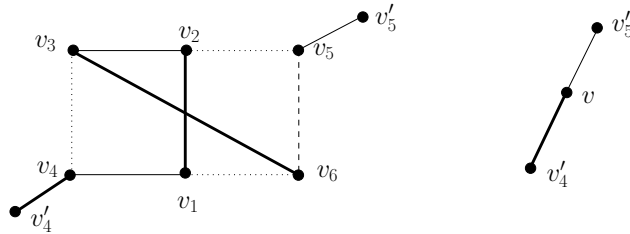


Figure 10: Creating  $G'$  from  $G$  in Case 3c if there is a 4-cycle  $D_1$  containing  $v_1v_2$  and a 4-cycle  $D_2$  containing  $v_2v_3$ ,  $D_1$  and  $D_2$  share two edges

Assume finally that  $D_1$  and  $D_2$  share two edges. Then,  $G$  has to contain the edge  $v_3v_6$ . Let  $v'_4$  be the neighbour of  $v_4$  different from  $v_1$  and  $v_3$  and let  $v'_5$  be the neighbour of  $v_5$  different from  $v_2$  and  $v_6$ . Since  $G \neq K_{3,3}$ , the vertices  $v'_4$  and  $v_5$

are distinct and since  $G$  is bridgeless we have  $v'_4 \neq v'_5$ . We construct a connected graph  $G'$  with fewer vertices than  $G$  by removing the vertices  $v_1, v_2, \dots, v_6$  together with the incident edges from  $G$  and adding a new vertex  $v$  joined to  $v'_4$  and  $v'_5$ . The graph  $G'$  is different from  $K_4$  and  $K_{3,3}$ , since it has a vertex of degree 2. We extend an acyclic 4-edge-colouring  $\varphi'$  of  $G'$  to an acyclic 4-edge-colouring  $\varphi$  of  $G$  according to Fig. 10.

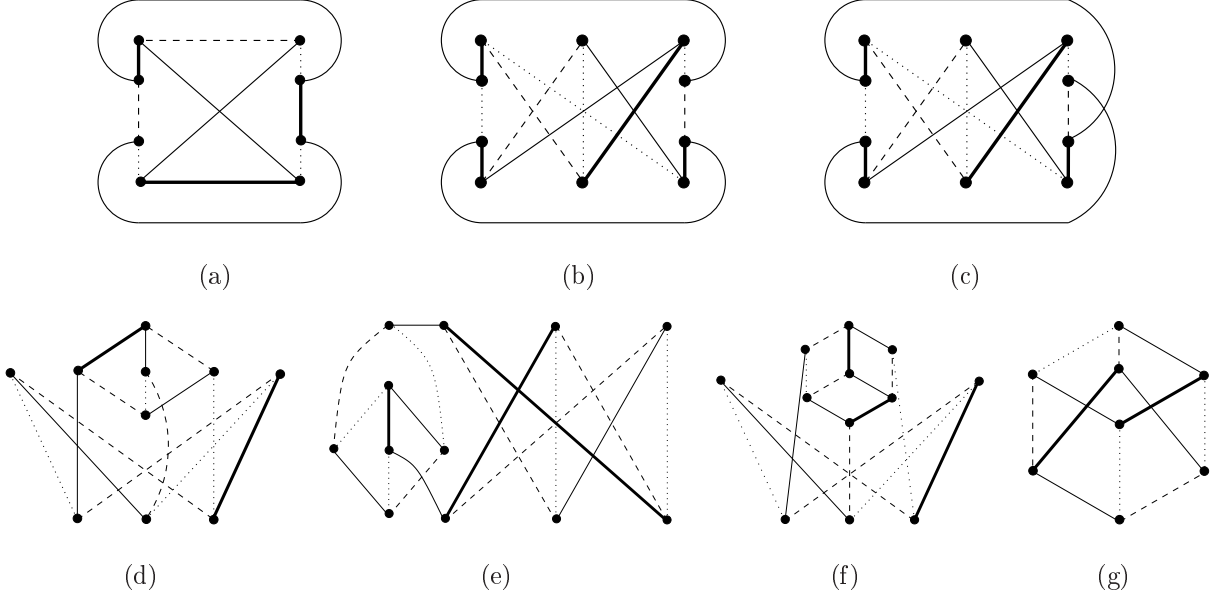


Figure 11: Acyclic 4-edge-colourings of small graphs from Case 3c

**Case 3d: The girth of  $G$  is 3.** First, suppose that  $G$  contains a triangle with no edge included in another triangle; denote one such triangle  $C_3$  and let  $C_3 = v_1v_2v_3$ . Let  $v'_i$  be the neighbour of  $v_i$  not contained in  $C_3$ . Create a new graph  $G'$  by removing the vertices of  $C_3$  together with the incident edges and introducing a new vertex  $u$  connected to  $v'_1, v'_2$ , and  $v'_3$ . Since no edge of  $C_3$  was included in another triangle,  $G'$  is simple.

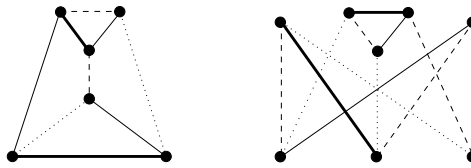


Figure 12: Acyclic 4-edge-colouring of  $K_4$  and  $K_{3,3}$  with a vertex expanded to a triangle

If  $G' = K_4$  or  $G' = K_{3,3}$  then  $G$  is isomorphic to one of the graphs in Fig. 12; an acyclic 4-edge-colouring of  $G$  is given in the figure. Otherwise,  $G'$  is connected and has fewer vertices than  $G$ , thus by the induction hypothesis,  $a'(G') \leq 4$ . Let  $\varphi'$  be an acyclic 4-edge-colouring of  $G'$ . We extend  $\varphi'$  to an acyclic 4-edge-colouring of  $G$ : we set  $\varphi(e) = \varphi'(e)$  for every edge  $e$  not incident with a vertex of  $C_3$ , set  $\varphi(v'_i v_i) = \varphi'(v'_i u)$ ,  $\varphi(v_1 v_2) = \varphi'(v_3 v'_3)$ ,  $\varphi(v_1 v_3) = \varphi'(v_2 v'_2)$ , and  $\varphi(v_2 v_3) = \varphi'(v_1 v'_1)$ .

Now suppose that each triangle in  $G$  has an edge in common with another triangle. Fix a triangle  $C_3 = v_1v_2v_3$  in  $G$ . Since  $G \neq K_4$ ,  $C_3$  has exactly one edge, say  $v_1v_2$ , included in a triangle different from  $C_3$ . Let  $v_4$  be the common neighbour of  $v_1$  and  $v_2$  different from  $v_3$ . Let  $v'_3$  be the neighbour of  $v_3$  different from  $v_1$  and  $v_2$  and let  $v'_4$  be the neighbour of  $v_4$  different from  $v_1$  and  $v_2$ . Since  $G$  does not contain a bridge, we have  $v_3 \neq v_4$ . Construct a new graph  $G'$  by removing the vertices  $v_1$ ,

$v_2$ ,  $v_3$ , and  $v_4$  from  $G$  and introducing a new vertex  $u$  of degree 2 adjacent to  $v'_3$  and  $v'_4$ . Let  $\varphi'$  be an acyclic 4-edge-colouring of  $G'$ ;  $G'$  admits such a colouring by the induction hypothesis. We construct a colouring  $\varphi$  of  $G$  as follows: we set  $\varphi(e) = \varphi'(e)$  for every edge  $e$  which is not incident with either of  $v_i$  for  $i = 1, 2, 3, 4$  and  $\varphi(v'_3v_3) = \varphi(v_2v_4) = \varphi'(v'_3u)$ ,  $\varphi(v'_4v_4) = \varphi(v_1v_3) = \varphi'(v'_4u)$ . Further, let  $c_1$  and  $c_2$  be the two colours different from  $\varphi'(uv'_3)$  and  $\varphi'(uv'_4)$ . To finish the definition of the colouring set  $\varphi(v_2v_3) = \varphi(v_1v_4) = c_1$ , and, finally  $\varphi(v_1v_2) = c_2$  (see Fig. 13). Clearly  $\varphi$  is acyclic. This concludes the proof.

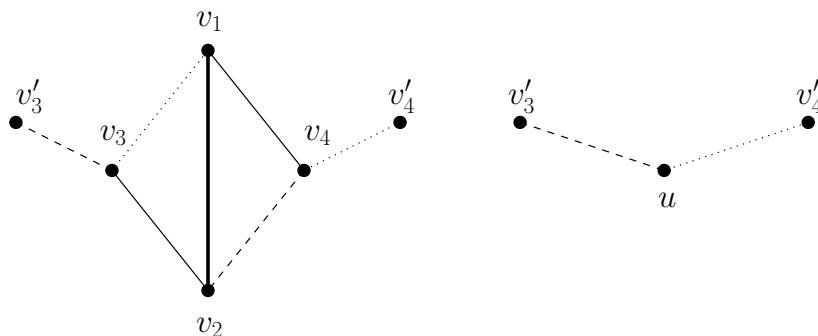


Figure 13: Creating  $G'$  from  $G$  in Case 3a if every triangle shares an edge with another triangle

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## References

- [1] N. Alon, B. Sudakov and A. Zaks, Acyclic edge-colorings of graphs, *J. Graph Theory* 37 (2001), 157–167.
- [2] N. Alon, C. J. H. McDiarmid, and B. A. Reed, Acyclic coloring of graphs, *Random Struct Algorithms* 2 (1991), 277–288.
- [3] M. Basavaraju and L. S. Chandran, Acyclic edge coloring of subcubic graphs, *Discrete Math.*, In Press.
- [4] M. I. Burstein, Every 4-valent graph has an acyclic 5-coloring (in Russian), *Soobšč. Akad. Nauk Gruzin. SSR* 93, 21–24, 1979 (Georgian and English summaries).
- [5] J. Fiamčík, Acyclic chromatic index of a graph with maximum valency three, *Arch. Math. (Brno)* 16 (1980), 81–88
- [6] J. Fiamčík, Acyclic chromatic index of subdivided graph, *Arch. Math. (Brno)* 20 (1984), 69–82
- [7] J. Fiamčík, The acyclic chromatic class of a graph (in russian), *Math. Slovaca*, 28 (1978), 139–145
- [8] B. Grünbaum, Acyclic colorings of planar graphs, *Israel Journal of Mathematics* (1973) 390–408.

- [9] M. Molloy, B. Reed, Further algorithmic aspects of the local lemma, Proceedings of the 30th Annual ACM Symposium on the Theory of Computing, May 1998, 524–529.
- [10] J. Nešetřil, N. C. Wormald, The acyclic edge chromatic number of a random  $d$ -regular graph is  $d + 1$ , J Graph Theory 49 (2005), 69–74.