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Zambelli, Stefano

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Frisch's Propagation Problems
and Impulse Problems**

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*Department of Economics, Politics
and Public Administration.
Aalborg University
Fibigerstræde 3
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The Rocking Horse that Didn't Rock

Frisch's Propagation Problems and Impulse Problems

Exchanges between Frisch and Clark, Kalecki and Schumpeter on the modelling of economic fluctuations.

Stefano Zambelli

Department of Economics, Politics and Public Administration.

Aalborg University

Fibigerstræde 3

9220 Aalborg - Denmark.

e-mail: stefano@socsci.auc.dk

Tlf: office 0045-96358166

home 0045-98166750

Abstract. *Propagation Problems and Impulse Problems* in Dynamic Economics (Frisch, 1933) is one of the most influential papers in the development of econometrics, mathematical economics and business cycle analysis. In the present article it is claimed that *Propagation Problems and Impulse Problems* contains a fundamental blunder that has been unseen, regardless of the many reviews and of the importance of Frisch work, for more than 60 years. Most likely a prompt acknowledgment of this slip would have had a serious impact on the exchanges that Frisch had with J.M.Clark, Kalecki and Schumpeter and hence on the development of economic analysis. Why the gross error present in *Propagation Problems and Impulse Problems* went unnoticed for more than 60 years is a puzzle that should be of interest to scholars of mathematical economics, business cycle theory and history of economic thought.

Key words: Dynamic Economics, Business Cycles, Accelerator Principle, Econometric Society, Frisch, Clark, Schumpeter, Kalecki, Mitchell.

JEL classification: A11, B31, B40, E32

1. *Introduction.*
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“If you hit a wooden rocking-horse with a club, the movement of the horse will be very different from that of the club (Frisch, 1933, p.198)”ⁱ

Ragnar Frisch

Propagation Problems and Impulse Problems

1933, p.198

“The resulting change in outlook ([that came after the publication of *Propagation Problems and Impulse Problems*]) can be compared to that of the transition from classical to quantum mechanics”

Paul Samuelson

Foundations of Economic Analysis

1947, p.284

Introduction

Propagation Problems and Impulse Problems in Dynamic Economics by Ragnar Frisch (1933) is with no doubt one of the most celebrated and path breaking articles dealing with the mathematical analysis of economic fluctuationsⁱⁱ.

Propagation Problems and Impulse Problems has become a classic for the understanding of the development of economic thought for many good reasonsⁱⁱⁱ.

- First, *Propagation Problems and Impulse Problems* is the pioneer article where the distinction between micro-dynamics and macro-dynamics is explicitly formulated.
- Second, *Propagation Problems and Impulse Problems* is one of the first articles to attempt an explanation of economic fluctuations through the use of a mathematical model (applying the methodology of differential and difference equations).
- Third, in keeping the propagation mechanism substantially separated from the impulse mechanism, Frisch introduces an approach which is at the foundation of current Equilibrium Business Cycle Models and economic time series analysis, where the cycles are assumed to be damped, but constantly kept alive through erratic shocks.
- Fourth, the article suggests as an analytical method that of initiating the analysis with the most simple model, for which oscillations are not possible, and subsequently complicate the model so as to “*draw some conclusions about those properties of the system that may account for the cyclical character*”.
- Fifth, the model has served as an example of how mathematical reasoning could introduce rigour into the discipline of economics so as to promote it into a science^{iv}.

The above are just some of the well-recognized reasons that made *Propagation Problems and Impulse Problems* such an influential work.

Unfortunately and quite surprisingly, *Propagation Problems and Impulse Problems* contains a major error. Frisch major aim was to construct a propagation mechanism that would allow for free oscillations, i.e., a mechanism that when removed from equilibrium would return towards it in a fluctuating manner. The problem is that when searching for a complete (either analytical or numerical) solution, which Frisch did not, one discovers that the

system is not an oscillating one. His rocking horse is not rocking.

Why has this feature being unnoticed for so long is difficult to figure out^v. The reason might be that the paper is at first not easy to understand and one trusts Frisch to do things right, and this is, up to a point, also correct^{vi}. Moreover Frisch does fill the article with plots of what he calls the primary, secondary and tertiary cycles so that one has the impression that the model is in fact a cyclical one. But this is a pure illusion.

The discovery that the propagation mechanism of Frisch's celebrated article is not a cyclical one opens up for several questions. In particular it is remarkably puzzling to speculate on why economists have not noticed such a blunder and moreover it is equally interesting to wonder on the impact that the discovery of such a blunder would have had to the implementation of the mathematical method to economics and to the debates taking place in the thirties about the best way of modelling cyclical evolutions.

In section 2 a debate with J.M.Clark that preceded the publication of *Propagation Problems and Impulse Problems* on the proper understanding of the accelerator model and on the usefulness of mathematics applied to economics is discussed. In section 3 it is shown that the propagation mechanism of Frisch model is not cyclical and it is claimed that this represents an important methodological failure. In section 4 and 5 the importance that *Propagation Problems and Impulse Problems* had on the exchanges with Kalecki and Schumpeter on the proper way of modelling business cycles is assessed. In section 6 some conclusions are drawn. The Appendix explains the method used for the computation of the dynamical systems.

2. *The debate with J. M. Clark on the faults of literary reasoning as opposed to the stringency of mathematical rigour: the case of the accelerator principle.*

Frisch used *Propagation Problems and Impulse Problems* to show how things ought to be done in general, but also as a methodological example to be used with whom he had some public and/or private debates. In the pages of the *Journal of Political Economy* (Frisch, 1931, 1932a, 1932b, Clark, 1931, 1932), Frisch had an intellectual exchange with J.M.Clark where he showed, using the mathematical method, that Mitchel, A.Hansen and J.M.Clark had reached wrong conclusions with respect to the causality in the accelerator model, which was at the time a well accepted model explaining booms and depressions^{vii}. Obviously, Frisch, while extremely active in the attempt of founding the Econometric Society, wanted to show to well established ‘literary’ economists the power of mathematics when applied to pure as well as empirical economics^{viii}.

The logical problem in the Hansen-Mitchel-Clark theory was that it was claimed that “*In order to bring about an absolute shrinkage in the demand of intermediate product, all that may be needed is that the final demand should slacken its rate of growth* (Clark, 1917, p.222-3)”. This had to be so because according to them “*Every producer of things to be sold to producers has two demands to meet ... Both these demands come ultimately from the consumer, but they follow different laws. The demand for maintenance and replacement of existing capital varies with the amount of the demand for finished products, while the demand for new construction or enlargement of stocks depends upon whether or not the sales of the finished products are growing* (Clark, 1917, p.220)”. Frisch (1931) did elegantly summarize their argument, in particular Clark’s point of view, in the article *The Interrelation Between Capital Production and Consumer Taking* and showed that there was a logical inconsistency. If we define $x(t)$ as the yearly production of consumption goods, and $y(t)$ as the yearly

production of capital goods, the accelerator principle may be formalized by the following equation:

$$(1) \quad y(t) = mx(t) + \mu\dot{x}(t)$$

where m and μ are parameters^{ix}.

Frisch made very clear that the above is an equation with two unknowns and is, consequently, indeterminate. One would need either to specify the functional form of final demand for consumption goods, $x(t)$, or to introduce an additional linear equation. This was not to rule out the possibility that a slackening of the demand for final consumption goods *could* determine a decrease in the production of capital goods, but to rule out that it *would* necessarily do so.

Clark (1931) replied that Frisch was right only in the cases in which the changes in the rate of growth of consumer demand are very small. Frisch (1932a, p.254) rejoined by saying that “*any change in consumer-taking which is such that the effect of this change becomes dominating over the effect of the change in replacement production, will call forth an absolute decline in the total capital production*”. Frisch underlined also that the theory formulated with this qualification will not explain a turning point of a regular business cycle because “*there will be a little interval of time after the point of fastest increase in consumer-taking where total capital production continues to increase, although the rate of increase of consumer taking has slowed down. This little interval of time around the turning point in capital production is the critical interval in the business cycle. It is here that the enigma of the business cycle lies.*”

The issue of the turning point of the business cycles stimulated a very acute reply by Clark who, while admitting the complexity of the problem, urged mathematical economists to

find a solution. In relation to fluctuating magnitudes Clark (1932, p.693) wrote “... *the actual contractions (and the more rapid expansions), if they do not arise as original movements produced by ‘outside causes’, can be explained as results of an intensifying mechanism whereby a fluctuation in the rate of growth may be converted into alternations of rapid expansion and absolute contraction ... the challenging problem is not why there are cyclical fluctuations but why there is any limit to the fluctuations, short of zero on one side, or the full capacity of existing productive equipment on the other. ... The problem of defining limits of fluctuation on this assumption seems to be one to which the techniques of mathematical analysis are peculiarly adapted; and I sincerely hope that this discussion may stimulate some mathematical economists to produce a solution*”.

It is only much later (Hicks, 1950, Goodwin, 1951) that non-linear models could be shown to explain the ‘*limit*’ and persistence of fluctuations without relying on ‘*outside causes*’. Frisch preferred to take another direction, which was sketched in the final article of the Frisch-Clark debate and was to become the famous Propagation and Impulse article.

3. *Propagation problems and impulse problems: the methodological paper ... but the rocking horse never rocked.*

The article *Propagation Problems and Impulse Problems* is in many ways also a continuation of the debate with Clark-Mitchel and Hansen and in part is an attempt to solve Clark’s challenge. In section III of his article - “*Simplified Systems Without Oscillations*”- Frisch referred explicitly to the debate with Clark and the above equation (1) is a starting point of the construction of his model. Having recognized that the model is undetermined (two unknowns with only one equation) and using the Walrasian idea of the *encaisse désirée*, and after some manipulations^x, Frisch derived another relation that would link consumption and capital

production:

$$(2) \quad \dot{x}(t) = c - \lambda(rx(t) + sy(t))$$

This relation^{xi} together with relation (1) gives a first order linear differential equation:

$$\dot{x}(t) = \frac{1}{(1 + \lambda s \mu)} (c - \lambda (r + sm) x(t))$$

Frisch (*Propagation Problems and Impulse Problems*, p.162) pointed out, correctly, that “... *the system may easily be solved in explicit form. By doing so we see that the system is too simple to give oscillations. ... This means that the variables will develop monotonically as exponential functions*”. He also underlined in a footnote that “*the fact pointed out by Clark does not necessarily lead to a development giving a turning point*”.

Having assessed that the system was too simple to originate oscillations, Frisch went on introducing “... *little by little more complications into the picture, remembering, however, all the time to keep the system determinate. This procedure [...] enables us to draw some conclusion about those properties of the system that may account for the cyclical character of the variations* (*Propagation Problems and Impulse Problems*, p. 158).

In line with this methodological prescription Frisch went on complicating the system so as to obtain a mixed difference-differential equation (first order difference equation and second order differential equation).

He did so by introducing the idea that the amount of consumption would be related not to the instantaneous decision of production of capital goods, $y(t)$, but to the amount of

investment projects not yet completed, the “carry-on-activity”, $z(t)$.

$$(3) \quad z(t) = \int_0^{\varepsilon} D(\tau) y(t - \tau) d\tau = \int_0^{\varepsilon} \frac{y(t - \tau)}{\varepsilon} d\tau$$

where ε is a technically given constant, the time to completion of an investment project, and $D(\tau)$ is equal to $1 / \varepsilon$ (Frisch’s own assumption)..

Differentiating (3) we obtain:

$$(4) \quad \dot{z}(t) = (y(t) - y(t - \varepsilon)) / \varepsilon$$

The consumption function expressed in the simplified model by equation 2 is adjusted to take account of the introduction of this new variable

$$(2') \quad \dot{x}(t) = c - \lambda(rx(t) + sz(t))$$

According to Frisch equations (1), (2') and (3) represents ‘A Macro-Dynamic system giving rise to oscillations’ as opposed to the ‘Simplified system without oscillations’ represented by equations (1) and (2). Obviously the ‘carry-on-activity’ $z(t)$ is a magnitude that could be computed also in the case of the ‘simplified system’, but it would be a dependent variable not exerting any effect on the variables $y(t)$ and $x(t)$. The situation is different in the case of the so-called ‘Macro-Dynamic system giving rise to oscillations’. Here the variable $z(t)$ exerts an

effect on the demand function for consumption goods $x(t)$ and consequently also to the production of capital goods $y(t)$. Is this change sufficient in order to allow for free oscillations?

Obviously Frisch believed that this change was sufficient to allow for fluctuations. But from an inspection of equation 3 or equation 4, for example, it is clear that as ε tends to zero the '*Macro-dynamic system giving rise to oscillations*' converges to '*the simplified system without oscillations*', because in that case $z(t)$ converges to $y(t)$. This simple consideration leads to infer that there should be positive values, however small, of ε for which the behaviour of the dynamic variables will be monotonic and not oscillating. The fact that there would be values of ε big enough so that the system would fluctuate has to be shown, which Frisch did not.

The equation resulting from his theoretical evaluation about the functioning of the hypothetical economy is a second order linear differential equation and first order linear difference equation. Being a second order differential equation the system can potentially account for free oscillations, but it is not necessarily so. The parameter space must be investigated and it so happens that Frisch model allows, for the set of economically relevant values of the parameters, only for a monotonic return towards the equilibrium (See the Appendix).

Numerical step-by-step solution of Frisch's model^{xiii} shows that the system is not at all oscillating. To repeat, when the system is removed away from equilibrium (impulse) the return towards equilibrium (propagation) is not an oscillating one.

In *Propagation Problems and Impulse Problems* Frisch confronts the reader with an example where a sample (trend, primary, secondary and tertiary) of the infinite number of harmonics that make the total solution of a mixed difference-differential equation is shown to

exhibit oscillating evolutions. He does not show the general solution, i.e. the solution where all the harmonics are summed up. In summing harmonics it is well known that the result of the summation might not be cyclical at all. A trivial example being represented by two sinusoidal functions having the same amplitude and being out of phase for 180 degrees: the sum of the two harmonics is a constant function, a straight line. What Frisch did was to show the fluctuating behaviour of the individual components of the general solution, but he did not sum them up. If he had done he would have discovered that his was not at all a cyclical model, quite the contrary. He would have discovered something very similar to a straight line^{xiii}.

[Figure 1: '*Simplified system without oscillations*'.]

[Figure 2: '*A Macro-Dynamic system giving rise to oscillations*']

Figure 1 reports the return towards equilibrium of consumption demand, $x(t)$, and of capital production, $y(t)$, for the case of the '*Simplified system without oscillations*'.

Figure 2 reports the same case applied to what Frisch defined to be '*A Macro-Dynamic system giving rise to oscillations*'. Note that in both figures the parameters are exactly the ones used by Frisch in his numerical examples. In both cases the system moves back towards equilibrium without oscillating around it. Moreover it is clear that the two evolutions are practically the same. The only difference is to be found in the little hump in the capital production function that is due to the presence of a delay mechanism and the same conclusion would be reached for all possible combinations of different economically significant set of parameters (see the Appendix).

Therefore, we have to conclude that Frisch's system and the formalization of Clark's

system, as they stood, were both not capable of accounting for free oscillations and were consequently not suitable to explain ‘*turning points*’. Obviously Frisch did not introduce the right ‘*complications*’ to the system and, seen under the light of this result, he could not, as he thought he could, “ ... *draw some conclusion about those properties of the system that may account for the cyclical character of the variations (Propagation Problems and Impulse Problems, p. 175)*. In other words Frisch did not construct an oscillating mechanism, quite the contrary. Hence Frisch did not show or prove what he said he was.

4. *Frisch’s critique of Kalecki’s model on the persistence of fluctuations.*

At the second meeting of the Econometric Society, Leyden (30 Sept. - 2 Oct. 1933) both Frisch and Kalecki (and Tinbergen) presented a model of fluctuations using Aftalion’s idea that a major source explaining booms and depressions is due to ‘time to build’. They presented their models that by that time had been already published^{xiv} and used difference-differential equations, but had very different views about the mechanism maintaining cycles. Kalecki had the view that cyclical evolution was an intrinsic feature of the functioning of a capitalistic system, basically a Marxist approach, while Frisch had the view that economic cycles are highly damped so that the system would return towards equilibrium if it was not for the existence of erratic shocks always disturbing the system out of equilibrium. In other words Kalecki was attributing the painful depressions to the capitalist mode of production, while Frisch only partly to the propagation mechanism, but mostly to chance (the impulses).

At the meeting in Leyden Frisch criticized Kalecki’s model because ‘*one must not assume a priori such values of the parameters of the business cycle equation as to render cyclical fluctuations of a constant amplitude, to the exclusion of damped or explosive fluctuations (Osityński, 1990, p. 445)*’. The persistence of the fluctuation in Kalecki’s model

did indeed rely on the restriction of the parameters to very specific values. Any divergence from such values, however small, would have generated either damped or explosive fluctuations. Frisch made his point at the meeting and did substantiate it with an article that was published a couple of years later in *Econometrica* (Frisch and Holme, 1935). In the forward to that article Frisch wrote “*The imposition of the condition that the solution shall be undamped is in my opinion not well founded. It is more correct, I think, to be prepared to accept any damping which the empirically determined constants will entail, and then explain the maintenance of the swings by erratic shocks. This would be an explanation along the lines indicated in my paper in the Cassel volume* (Frisch and Holme, 1935, p.225)^{xv}.”

Kalecki (1936, p. 359-60) admitted that it was inappropriate to assume constant amplitude, but explained also that the parameters would change values during the cycles so that as the system would converge towards equilibrium (damped behaviour) the parameters would change back to the previous values so as to restore persistence (undamped evolution)^{xvi}. This justification falls outside the actual formal model and cannot be considered a proof of persistence, but it is certainly the case that the macro variables in Kalecki’s model would fluctuate around equilibrium for a wide range of parameter values.

It is well recognized that Kalecki played an important role in the development of economic theory and in particular he belonged, during the thirties, to that small group of economists that were heavily relying on mathematical methods for the understanding of the business cycles.

While he was establishing intellectual relations with quantitative economists participating to the establishment of the Econometric Society (such as Tjallingii and Frisch) he was also establishing connections with the Cambridge economists (such as Robinson, Kaldor and Keynes). His methods and his conclusions were at first over shaded for several

reasons and from several points of view. Those relating to him from the mainstream were contrary to his distribution and demand approach for which he has been associated with Keynes' *General Theory*, while others tended to consider his methods of investigations apologetic.

Schumpeter (1954) has also neglected Kalecki not at all citing him in his *History of Economic Analysis* although his contribution on business cycles was well known to him. On this point Andvig (1987, p.103) conjectures that "... *One of the major scientific reasons why Schumpeter, leaning himself on the group of mathematical economists who developed the mathematical models in the thirties, neglected Kalecki, was, I believe, precisely the way in which Kalecki manipulated the parameters in order to achieve exactly undamped cycles, thereby also trying to achieve more direct empirical relevance that was appropriate ... That the political tastes reflected in Kalecki's work were at odds with the majority opinion among economists at the time should, however, be added to any explanation of neglect*".

Whether Kalecki's macrodynamic theory of the business cycle was unduly neglected can certainly be a matter of debate. Nevertheless it is clear that Frisch played an important role in undermining his analytical method and did so also by proposing the superiority of his own method and model. If Kalecki as well as others had seen that Frisch model was in fact just an illusion and that it was not at all generating cycles - at least for the whole set of relevant parameters - he could have transformed his defensive position into an attaching one: Kalecki's model, after all, was allowing for propagation of impulses in a cyclical manner (free-oscillations), while Frisch's was not.

5. *The exchanges with Schumpeter on perpetual turbulence and the destruction of equilibria.*

It is well known that Frisch established, at the end of the twenties and beginning of the thirties several important connections, which led to the foundation of important institutions. In this process he met many economists, among which I. Fischer and Schumpeter. In particular he had several exchanges, by letter and with personal contacts, with Schumpeter (see Louçã, 2001) where he discussed, among other things, also the proper way to model business cycles.

Propagation Problems and Impulse Problems is well known to modern economists because of the introduction of Slutsky's idea that independent impulses could be transformed to originate correlations in the variables. He was actually attempting an explanation of why cycles are maintained through time. Section 5 of *Propagation Problems and Impulse Problems*, '*Erratic shocks as a source of energy maintaining oscillations*', contains an analysis of the *importance* of erratic impulses in providing energy to an otherwise damped cycle and his famous metaphor of the rocking horse. It must be underlined that section 5 was only an addendum to the more important previous sections where he constructed the (cyclical) propagation mechanism. If in this section he suggested a possible solution to the maintenance of cycles in the subsequent section he provided a suggestion for an alternative method, which should have allowed for maintenance of oscillations.

This alternative method was the result of personal and written exchanges with Schumpeter. In section 6 of *Propagation Problems and Impulse Problems* ('*The Innovations as a Factor Maintaining Oscillations*') Frisch himself proposed a different metaphor with respect to that of the rocking horse. Frisch wrote: '*Suppose that we have a pendulum freely suspended to a pivot. Above the pendulum is fixed a receptacle where there is water. A small pipe descends all along the pendulum, and at the pendulum the pipe opens with a valve which*

has a peculiar way of functioning. The opening of the valve points towards the left and is larger when the pendulum moves towards the right than when it moves towards the left. Concretely one may, for example, assume that the air resistance influences the valve or by some other factor that determines the opening of the valve as a function of the velocity of the pendulum. Finally we assume that the water in the receptacle is fed from a constantly running stream which is given as a function of time. The stream may, for instance, be a constant (Propagation Problems and Impulse Problems, p.203-4)'. This metaphor, which Frisch elaborated further with the addition of other complications, such as a-symmetric valves, is not as simple or appealing as that of the rocking horse (or pendulum) which is kept oscillating by erratic shocks, but it has the same feature of explaining the maintenance of cycles thanks to an exogenous force that provides energy to the system.

Frisch was suggesting that both erratic shocks and a non-linear conservative structure could together form an explanation for the maintenance of cycles. The opening sentences of section 6 expresses this view: *'The idea of erratic shocks represents one very essential aspect of the impulse problem in economic analysis, but probably it does not contain the whole explanation. There is also present another source of energy operating in a more continuous fashion and being more intimately connected with the permanent evolution in human societies, [Schumpeter theory of innovations]. ... It is like a force that is released during these phases, and this force is the source of energy that maintains oscillations. ... In mathematical language one could perhaps say that one introduces here the idea of an auto-maintained oscillation (Propagation Problems and Impulse Problems, p. 203, emphasis added)'* .

Although in his History of Economic Analysis Schumpeter refers neither to the rocking horse nor to the pendulum metaphor - but to that of a resonator as in the case of an 'electric clock placed upon a somewhat rickety table' or of 'a violin that 'reacts' in a

determined manner when 'irritated' as the player applies the bow' (Schumpeter, 1954, p.1167) - he was certainly influenced by Frisch's mathematical contributions. In discussing '*Dynamics and Business Cycle Research*' Schumpeter took a clear position in claiming that Mitchell - as well as '*Literary' students of the business cycles*' - would have sharpened and corrected their arguments if they had taken mathematical models of oscillating mechanisms more seriously: "*Problems of the mechanisms by which impulses are propagated through the economic systems may be cleared up by macrodynamic methods, which therefore may contribute substantially, among other things, to our understanding of turning points*".^{xvii}

If Schumpeter had known or seen himself that *Propagation Problems and Impulse Problems* was not propagating oscillations, and consequently was not able to explain turning points, he could not have referred directly to Frisch's work in the way in which he did. And it would have made him more careful in criticizing '*literary*' students of the business cycles for their '*mathematical disabilities*'. At the same time it would have reinforced his view that '*... Constructors of macrodynamic models, almost always, aim at explaining all the phases of cycles (ant the turning points) by a single 'final' equation. This is indeed not impossible. But it spells error to assume that it must be possible and to bend analysis to this requirement*' (Schumpeter, 1954, p.1168, emphasis added)^{xviii} .

6. Conclusions.

In this paper some aspects of *Propagation Problems and Impulse Problems* have been considered and the role that it played in the development of business cycles analysis has been stressed. It has been shown that **the propagation mechanism present in *Propagation Problems and Impulse Problems* is not a cyclical one** (section 3). **Therefore his model is**

not able to explain turning points and in this respect it has **the same logical faults for which Frisch criticized the Mitchel-Hansen-Clark** accelerator model (section 2). Moreover also the critique that Frisch addressed towards Kalecki, and for which he used also *Propagation Problems and Impulse Problems* as a good example on how things ought to be done, is to be viewed differently when we consider that **while Kalecki's model allowed for free oscillations for a wide range of plausible parameters Frisch's model allowed only for a monotonic return towards equilibrium.**

If both Clark and Kalecki had seen that *Propagation Problems and Impulse Problems* was an illusion, they would have had a better chance to defend their positions and models. Certainly Frisch's contribution would have been seen under a different light.

Samuelson (1974, p.10), while reviewing aspects of *Propagation Problems and Impulse Problems*, wrote:

"... let me point out that a great man's work can, in its impact on lesser men, have bad as well good effects. Thus, by 1940, Metzler and I as graduate students at Harvard fell into the dogma that all economic business-cycle models should have damped roots. We accepted Frisch's criticism of the Kalecki procedure of imposing constraints on his parameter-estimating equations so that roots would be neither damped nor undamped ... what was so bad about the dogma? Well, it slowed down our recognition of the importance of non-linear auto-relaxation models of the van der Pol-Rayleigh type, with their characteristic amplitude features lacked by linear systems. And, in my case, it led to suppressing development of the Harrod-Domar exponential growth aspects that kept thrusting themselves on anyone who worked with accelerator-multiplier systems (emphasis added)".

Whether the development of mathematical business cycle analysis and that of mathematical economics would have been substantially different is impossible to assess. But

one can certainly agree that subsequent works on the business cycles would have been written, at least in some of their parts, differently^{xix}. Would Frisch have played the same role in defining research directions in economic theory? Would he have gotten the same support from the Cowles and Rockefeller foundations? Would he have been able to support Tinbergen's United Nation's project in the same way as he did? Would he have had the same impact on its contemporaries as he did? Would research on non-linear business cycle theory have been different? These are all conjectural questions that imply conjectural answers. But an objective fact is, to stress the point once more, that *Propagation Problems and Impulse Problems* contains a major slip - and that the many economists that read the paper and/or reviewed it did not see this slip.

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Appendix

“Simplified Systems Without Oscillations”

$$(1) \quad y(t) = mx(t) + \mu\dot{x}(t)$$

$$(2) \quad \dot{x}(t) = c - \lambda(rx(t) + sy(t))$$

$$(3) \quad z(t) = \int_0^{\varepsilon} D(\tau)y(t-\tau)d\tau$$

‘A Macro-Dynamic System Giving Rise to Oscillations’

$$(1) \quad y(t) = mx(t) + \mu\dot{x}(t)$$

$$(2) \quad \dot{x}(t) = c - \lambda(rx(t) + sz(t))$$

$$(3) \quad z(t) = \int_0^{\varepsilon} D(\tau)y(t-\tau)d\tau$$

The above two systems can be approximated through numerical integration or *step-by-step* computation. Given the absence of singularities and the fact that the equations are linear the numerical integration is straight forward. Equations 2 and 2' can be computed with the Euler method (see for example Fröberg, 1985, p.322-3) while equation 4 can be computed with the first of the Newton-Cotes formulas, known as the trapezoidal algorithm (Fröberg, 1985, p.285)

The approximation of equation (2) is made by:

$$x(t+h) = (c - \lambda(rx(t) + sy(t)))h + x(t)$$

where h is the step-size.

The approximation of equation 2' is the same as the above where $y(t)$ is replaced by $z(t)$.

Equation 4 is computed by

$$z(t+h) = \frac{h}{2\epsilon} [y(t-\tau) + 2y(t-\tau+h) + 2y(t-\tau+2h) + \dots + 2y(t-h) + y(t)]$$

here h is the step-size.

The values of the parameters used by Frisch are

$$\epsilon = 6; \mu = 10; \lambda = 0.05; m = 0.5; r = 2; s = 1; c = 0.165; (h=1/6);$$

and the evolutions reported in the text are based on these parameter values.

The method is straight forward. First, the system is in equilibrium and a shock, impulse, removes the system from the equilibrium position (10% increase in consumer taking). Second, the return towards equilibrium is classified. The system is not cyclical when it does NOT oscillate around the equilibrium position. The figures reported in the text show that the return towards equilibrium is monotonic. Therefore the horse is NOT rocking.

The above procedure has been conducted also for other parameter values. The ranges for the computations have been the following:

ε	=	[1 , ..., 6 , ..., 30];	(10 equally spaced values);
μ	=	[1 , ..., 10 , ..., 20];	(10 equally spaced values);
λ	=	[0.01 , ..., 0.05 , ..., 1];	(10 equally spaced values);
m	=	[0. , ..., 0.5 , ..., 1];	(10 equally spaced values);
r	=	[0.01 , ..., 2 , ..., 3];	(10 equally spaced values);
s	=	[0.01 , ..., 1 , ..., 2];	(10 equally spaced values);
c	=	[0 , ..., 0.165 , ..., 1];	(10 equally spaced values);

(h=1/6);

All the combinations of the parameter values belonging to the above intervals have been tried out and **no instances of cyclical evolutions has been found.**

**Simplified System:
First Order Linear Differential Equation**

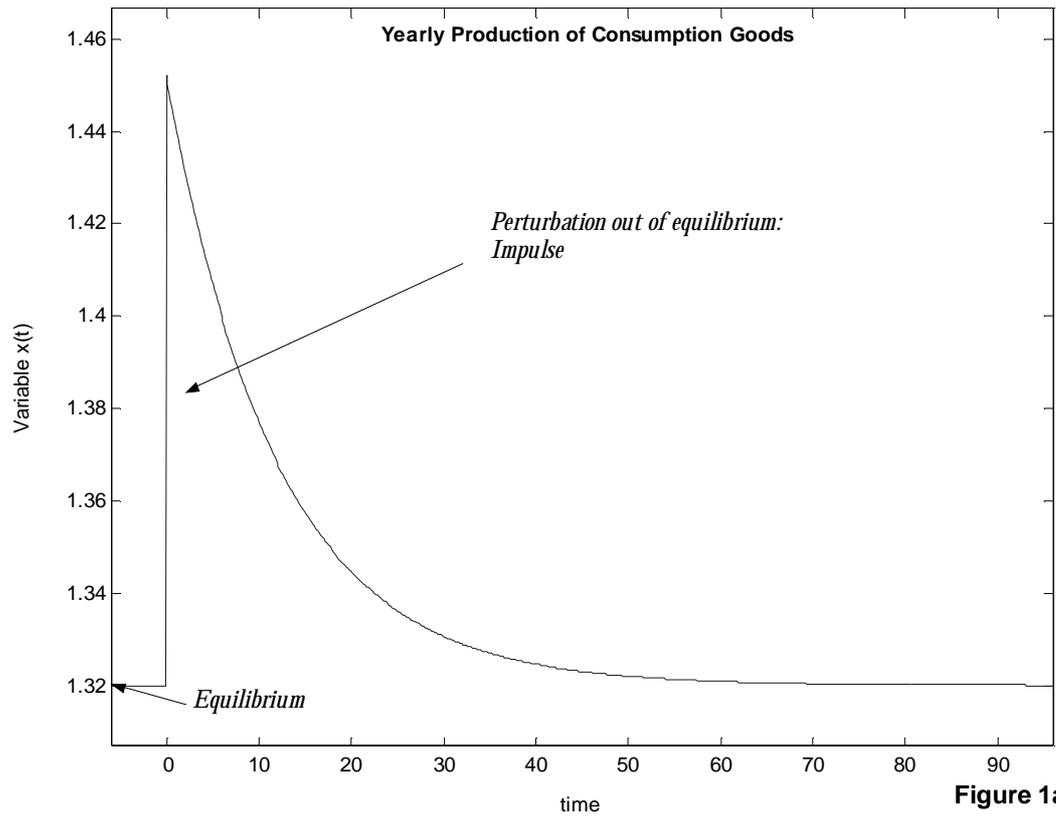
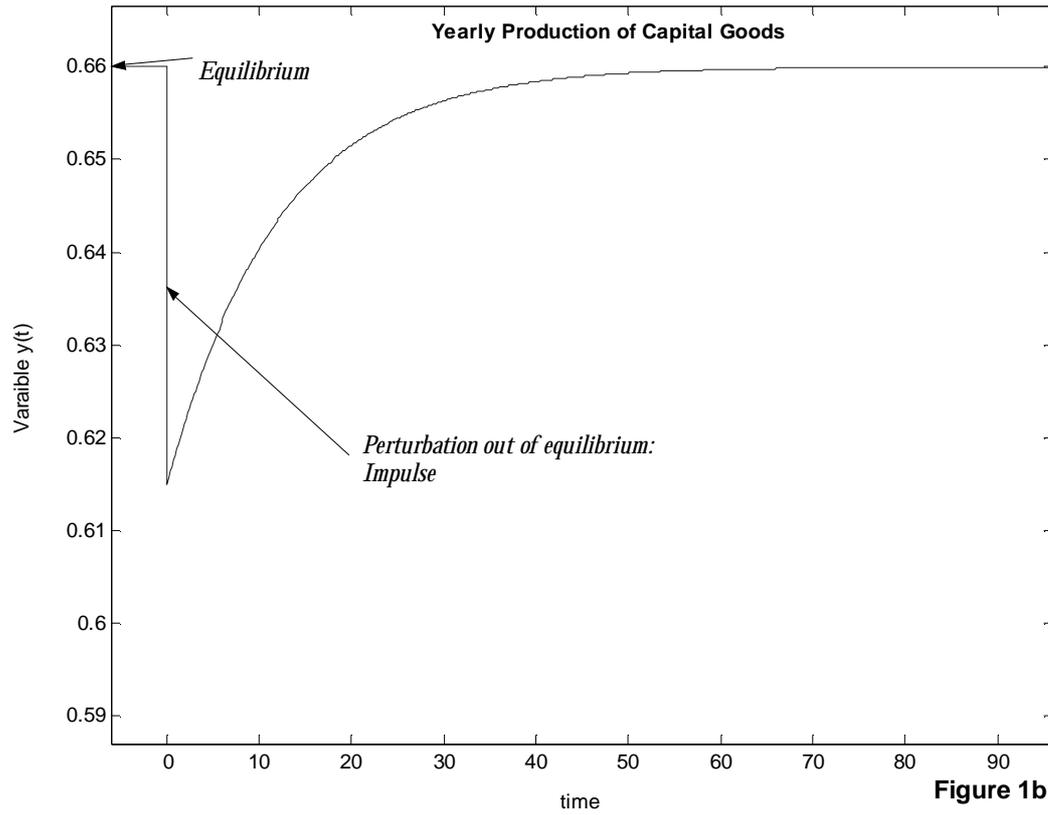


Figure 1a

**Simplified System:
First Order Linear Differential Equation**



**Macro-Dynamic System Giving Rise to Oscillations:
Mixed Difference-Differential Equation**

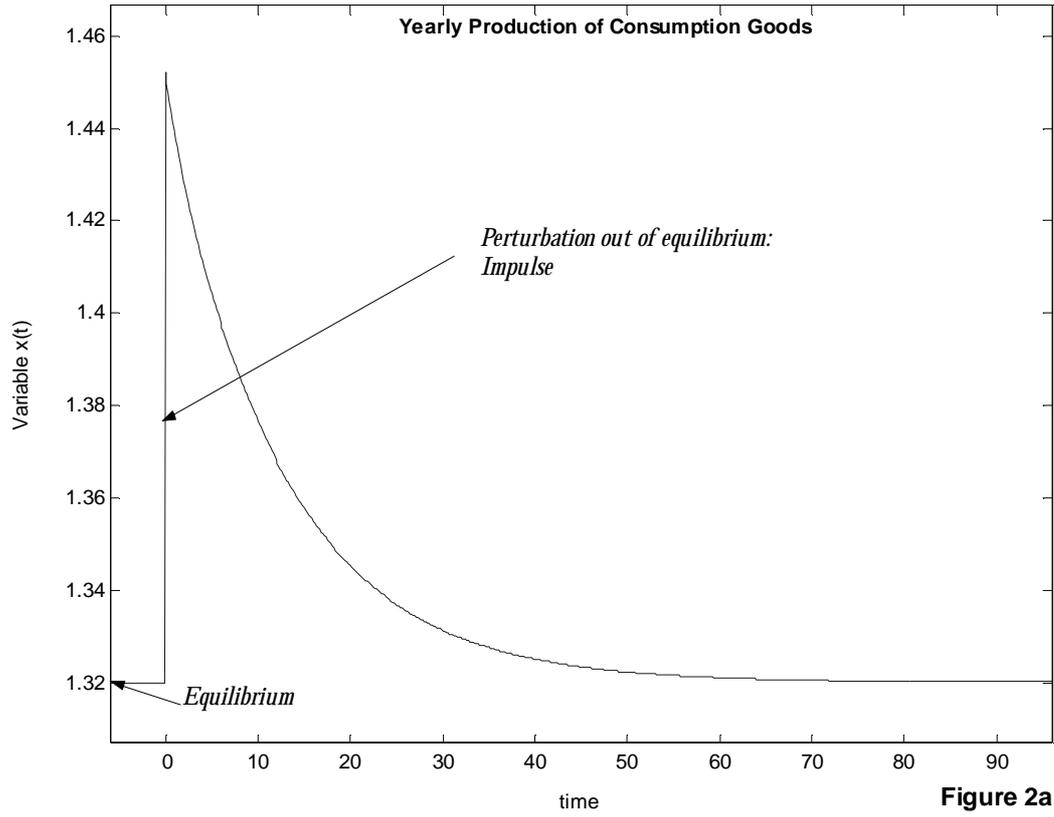


Figure 2a

**Macro-Dynamic System Giving Rise to Oscillations:
Mixed Difference-Differential Equation**

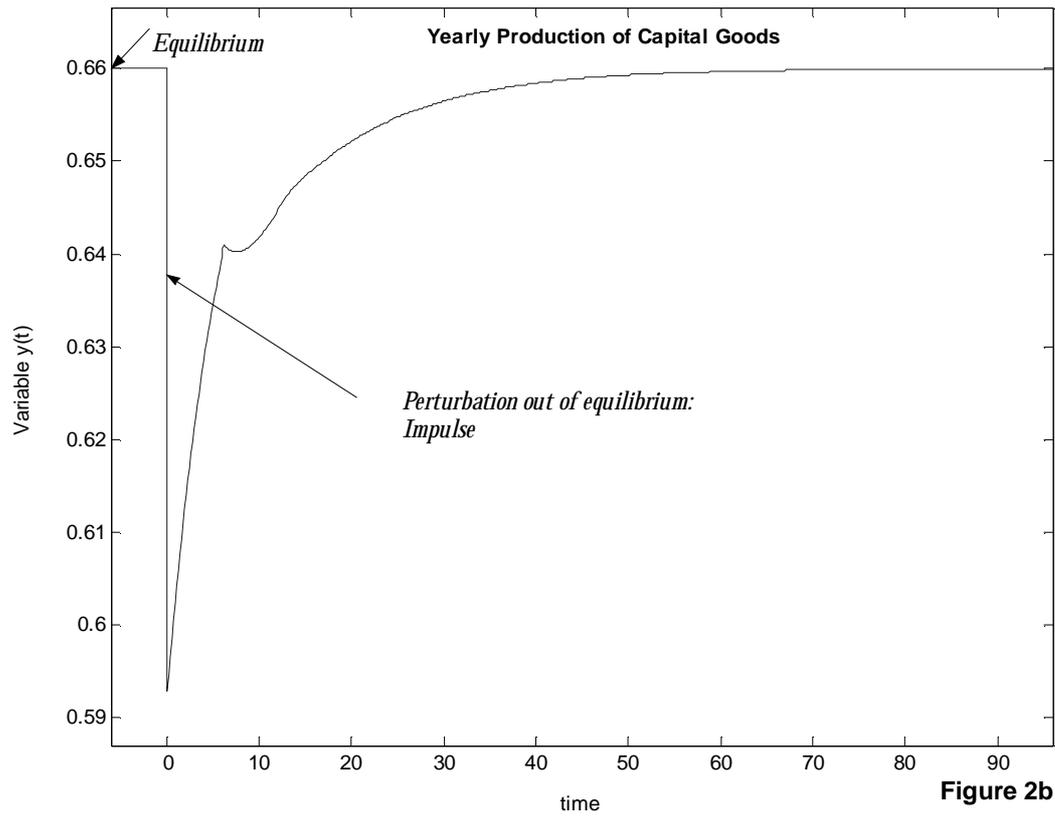


Figure 2b

Endnotes

ⁱThis famous phrase, which is recognized as a good metaphor for the understanding of economic fluctuations, became popular thanks to Ragnar Frisch who quoted it in his celebrated article on *propagation and impulses*. Although it is attributed to Wicksell, there is no explicit trace of it in Wicksell's own work. For the whole and detailed story see Velupillai, 1992.

ⁱⁱIt is generally credited by economists that Frisch was awarded the Nobel Prize precisely for this important contribution. Olav Bjerkholt (1995, p.xxxi) in the introduction to the two volumes dedicated to the centenary of Frisch's birth acknowledges that "*The Propagation and impulse article became Frisch's most celebrated article, and the one for which he was awarded the Nobel Prize (although the article itself is not explicitly mentioned in the official announcement)*". Louçã (2001, p.28) writes "... *Frisch's paper (Propagation Problems and Impulse Problems) was to win him the first Nobel Prize to be awarded in Economics*". MacGrattan (1990, p.130) in her History of Econometrics Ideas writes that "*We should also note that it was for developing and applying macrodynamic models that Tinbergen and Frisch were awarded the first Nobel Memorial Prize in Economics in 1969*". Strøm (1998, xi) also writes "*Frisch was awarded the Nobel Prize in economics primarily for his 'Propagation and impulse' article*". And in fact from the Official Announcement of the Royal Academy of Sciences (1969, p.300) one reads: "*Since the beginning of the nineteen-thirties he has, in a series of papers, developed mathematically specified dynamic models for the analysis of economic courses of events; he has among other things shown how a combination of equations, giving hypothetic connections to the accumulation of capital, consumption and supply of money, tends to lead to cyclical fluctuations, which can be maintained by randomly occurring disturbances*". And this is an indirect but obvious reference to *Propagation Problems and Impulse Problems*.

ⁱⁱⁱHere a thorough discussion of *Propagation Problems and Impulse Problems* will not be attempted. Among many contributions dealing with some aspects of *Propagation Problems and Impulse*

Problems see Tinbergen (1935), Arrow (1960), Samuelson (1974) Bjerkholt (1995), Morgan (1990), Velupillai (1992), Thalberg (1990, 1998), Zambelli (1992), Klein (1998).

^{iv}Samuelson (1947, p.284) on the revolutionary impact of *Propagation Problems and Impulse Problems* on the relation between Static and Dynamic notions writes that “*The resulting change in outlook ([that came after the publication of Propagation Problems and Impulse Problems]) can be compared to that of the transition from classical to quantum mechanics. And just as in the field of Physics it was well that the relationship between the old and the new theories could be in part clarified, so in our field a similar investigation seems in order*”.

^vRecall that among the many economists that have explicitly reviewed the propagation mechanism of Frisch model one finds at least four Nobel Price laureates (Tinbergen, 1935, Arrow, 1960, Samuelson, 1974, Klein, 1998).

^{vi}For example all the all the numerical computation presented in *Propagation Problems and Impulse Problems* were extremely accurate. The computers of his time, that is Mr. Holme and Mr. Thorbjørnsen (Frisch, 1933, p.186), were almost as precise as today’s digital computers.

^{vii}Clearly the accelerator model was playing an important role in the explanation of business cycles. Two of the very first mathematical models of the business cycles, Frisch’s Propagation and Impulse model and Kalecki’s outline of a business cycles are based on the work of Aftalion (1927) which, together with Clark (1917), is considered one of the most representative contribution for the development of the accelerator model.

^{viii}The Frisch ambition to transform the discipline of economics into a ‘science’ is clear from his printed work, the minutes of several meetings and from his correspondence. In particular see Strøm

(1998), Bjerkholt (1995, 1998), Andvig and Thonstad (1998), Louçã (2001). Bjerkholt (1998, p.36) reports a citation from a letter send by Frisch to Divisia, June 26, 1930, where it is stated that “*what we want is more to penetrate the whole body of economic theory with the keenness of mathematical thought ... I therefore believe that by acting wisely now we could be able, so to speak, to swallow the whole body of economic theory*”. And so it went!

^{ix}This equation corresponds exactly to equation (3) of *Propagation Problems and Impulse Problems* and with a change in notation corresponds also to equation (5) of Frisch (1931). For an estimation of the elegance with which Frisch arrived at the above equation see Rasmussen (1987).

^xVelupillai (1992) points out quite convincingly that Frisch did simplify his own theoretical model so as to obtain a linear relation from an otherwise non-linear one. This approximating procedure has also, among other things, kept economists away from the analysis of non-linear models and away from the development of proper mathematical tools.

^{xi}Equation (2) is equation (6) in *Propagation Problems and Impulse Problems*.

^{xii}Frisch used the numerical step-by-step solution to show an alternative way to performe computation. Quite surprisingly, he applied the method only to compute what he called the primary cycle. If he had applied it to the general solution, i.e., directly on the system of equations 1, 2' and 3 he would have seen immediately that his system was not giving rise to oscillations.

^{xiii}In the Appendix a numerical approximation of the two systems is made and it is shown, using Frisch's own definition of the parameter values that his system does not at all differ from his formalization of the Clark-Mitchell-Hansen accelerator principle.

^{xiv} In July 1933, Kalecki published in Polish the “Essay on the Business Cycles (Próba teorii konjunktury)”, see Osityński (1990), while Frisch had published his article *Propagation Problems and Impulse Problems* in the Essays in honour of Cassel.

^{xv}Frisch’s ‘*editorial oddities*’ have been the sources of many discussions with and critiques from the members of the Econometric Society (Bjerkholt , 1995, p.xxxiv-v). In this case Frisch and Holme’s critique to Kalecki’s model was published in April 1933 that is before the actual publication of Kalecki’s model, which was published in the July issue.

^{xvi}Kalecki (1936, pp.359-60) defended his model by writing: “*I have obtained for m and n the numerical values: $m=0.95$, $n=0.12$ and 10 years as the duration of the cycle, assuming constant amplitude of fluctuations. This assumption was based on the fact that in real life this amplitude does not exhibit any steady progression or degression. Frisch and Holme object to the above assumption of constant amplitude. They are right, for it is by no means sufficient to say that an assumption is correct just because it is confirmed by the conditions of real life. It must be made clear why the real life is like that, otherwise the particular predilection it shows for a constant amplitude might appear metaphysical (emphasis added).” Kalecki continued his defence by explaining that some parameters would be themselves a function of the state of the system so as to justify persistence (that is, non-linearity). For a thorough discussion of Kalecki’s model with respect to non-linearity see Velupillai (1997).*

^{xvii}In the footnote Schumpeter (1954, p.1167, n.17) refers to the Clark-Frisch debate as an ‘*instructive example*’. Here we find another example of how influential Frisch and his article *Propagation Problems and Impulse Problems* was for the explanation of business cycles.

^{xviii}It is also well known that Schumpeter had the view that the representative macro-variable could be separated into cyclical components of different frequency and this feature is apparently well captured, because of the separation into harmonics of the solution equation, in *Propagation Problems and Impulse Problems*. But again, this is not the case. What Schumpeter had in mind was a precise causal connection between the type of innovation, which has its own specific character, and the length of the cycle (Schumpeter distinguished between the approximately forty months Kitchin cycles, ten year for the Juglar cycles and sixty year for the Kondratieff cycles.). But this connection is not possible when we solve for a differential(-difference) equation of a low order. For example, the analytical solution of a differential equation allows for cyclical development when the solution of the characteristic equation has complex roots and this is all one can say. The separation made by Frisch into trend, primary, secondary and tertiary cycles does not add any new causal understanding to the actual dynamical evolution of the macro-variable, Obviously the qualitative evolution of the dynamical system would depend on the fact that the component of the general solution are imaginary or not. But whether this is so depends on the parameter values, i.e., the structure of the supposedly rocking horse. The general solution for the evolution of the rocking horse is a real variable, but the components of the solution can be made of complex conjugates. The actual evolution after displacement out of equilibrium depends on the initial conditions.

^{xix}As the result of the great depression during the thirties a great deal of intellectual effort has been addressed towards the understanding of booms and depressions. The League of Nations supported research on business cycles (Tinbergen, 1939 and Haberler, 1937) and several important contributions such as that of Schumpeter (1939) and Samuelson (1939), just to name a few, were published.