Wave Propagation in Pipe-like Structures

Morsbøl, Jonas

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BY
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DISSERTATION SUBMITTED 2015

AALBORG UNIVERSITY
DENMARK

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Thesis submitted:	April, 2015
PhD supervisor:	Sergey V. Sorokin, Prof., Ph.D., M.Sc. Department of Mechanical and Manufacturing Engineering, Aalborg University, Denmark
PhD committee:	Julius Kaplunov, Prof., D.Sc., Ph.D., M.Sc. School of Computing and Mathematics, Keele University, UK
	Jakob Søndergaard Jensen, Prof., Dr.techn., Ph.D., M.Sc. Head of Centre for Acoustic-Mechanical Micro Systems, Department of Electrical Engineering, Technical University of Denmark, Denmark
	Jens Henrik Andreasen, Ass. Prof., Ph.D., M.Sc. Mechanical and Manufacturing Engineering, Aalborg University, Denmark
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Preface

This thesis has been submitted to the Faculty of Engineering and Science at Aalborg University in partial fulfillment of the requirements for the degree of Doctor of Philosophy (Ph.D.) in Mechanical Engineering. It covers the documentation of a sub-project under the overall research project entitled *Advanced Modelling of Wave Propagation in Spatially Curved Elastic Rods and Pipes*. The overall research project has been funded by *The Danish Council for Technology and Innovation*, under the grand no. 10-083896, as a result of a research application formulated in 2010 by Sergey V. Sorokin, Prof., D.Sci., Ph.D. and Niels Olhoff, Prof., Dr.Techn., Ph.D. both from Department of Mechanical and Manufacturing Engineering, Aalborg University. Within this research project the present Ph.D. project is formulated under the title *Advanced Modelling of Wave Propagation in Curved Elastic Fluid-Loaded Pipes*.

I will use this opportunity to express my gratitude to my always patient, inexhaustible, and knowledgeable supervisor, Sergey V. Sorokin. He has been a great master. I will also thank Prof. Nigel Peake whom I visited during a three mound stay at Department of Applied Mathematics and Theoretical Physics at Cambridge University, UK. His support, his ideas, and his mathematical capabilities have brought a significant amount of quality to the project. Thankfulness is also directed to my Ph.D. colleague, Rasmus B. Nielsen for all of our fruitful and challenging discussions. Finally I will thank my wife and our young children for constantly reminding me about other important things in life.

Aalborg, 2015

Jonas O. Morsbøl

Abstract

This Ph.D. thesis is concerned with wave propagation in pipe-like shell structures. Pipe-like shell structures are found in a wide range of practical applications such as: Wind turbine towers, jet engines, brass instruments, and piping systems. The thesis will in particular focus on how the curvedness of a thin-walled pipe, as well as a changing radius along a straight thin-walled pipe, affects the waveguide properties. It shows that the waveguide properties of curved pipes roughly can be divided into three regimes: The curved beam regime, the cylinder regime, and the torus regime. In the curved beam regime the waveguide properties of the pipe can be approximated by classical curved beam theory while in the cylinder regime the waveguide properties can be approximated by cylindrical shell theory. In the torus regime, on the other hand, none of the two other regimes apply, and a full-blown shell model, taking both the curvature and the flexibility of the pipe-wall into account, is needed. For the straight pipe with changing radius, which is also known as the shell of revolution, it is found that classical rod and beam theory, to some extent, can be used to approximate the fundamental modes of the torsional, axial, and breathing wave. However, by means of the shell model some remarkable effects are predicted when even these very fundamental waves are travelling along the shell of revolution. The effects covers modal changes and excitation of localised resonances and the effects are explained and predicted by relatively simple expressions which have not been available to the public before. For modes of higher order the model is solved numerically. By this approach similar excitations of localised resonances are also predicted.

Dansk Resumé

Denne Ph.D. afhandling omhandler bølgeudbredelse i rørlignende skalstrukturer. Rørlignende skalstrukturer findes i en bred vifte af praktiske anvendelser såsom: Vindmølletårne, jetmotorer, messinginstrumenter og rørsystemer. I afhandlingen fokuseres på hvordan krumningen af et tyndvægget rør, såvel som ændringen af radius langs et lige tyndvægget rør, påvirker bølgeudbredelsesegenskaberne. Det viser sig at bølgeudbredelsesegenskaberne for krumme rør i grove træk kan inddeles i tre regimer: Krumbjælkeregimet, cylinderregimet og torusregimet. I krumbjælkeregimet kan bølgeudbredelsesegenskaberne for røret approksimeres af klassisk krumbjælketeori mens i cylinderregimet kan bølgeudbredelsesegenskaberne approksimeres af cylinderskalteori. I torusregimet, derimod, finder ingen af de to andre regimer anvendelse, og en fuldstæ]ndig skalmodel, som tager både krumning og fleksibilitet af rørvæggen i betragtning, er nødvendig. For det lige rør med varierende krumning, som også er kendt som den rotationssymmetriske skal, er det vist at klassisk stang- og bjælketeori, i nogen grad, kan benyttes til at approksimere de fundamentale torsions-, aksielle- og vejrtrækningsbølger. Ved hjælp af skalmodellen er nogle bemærkelsesværdige effekter dog forudsagt, som viser sig når selv disse meget fundamentale bølger vandre langs den rotationssymmetriske skal. Effekterne påvirker modalformer og eksiterer lokaliserede resonanser, og effekterne er forklaret og forudsagt af relativt simple udtryk som ikke har været tilgængelige for offentligheden før. For højerordnede bølger er modellen løst numerisk. Med denne metode er lignende eksitasioner af lokaliserede resonanser også forudsagt.

Thesis Details

This thesis is made as a collection of papers and consists of an introduction to the area of research and two papers for publication in refereed scientific journals of which one is accepted and one is under review.

Thesis Title:	Wave Propagation in Pipe-like Structures
Ph.D. Student:	Jonas O. Morsbøl
PhD Supervisors:	Sergey V. Sorokin, Prof., Ph.D., M.Sc. Department of Mechanical and Manufacturing Engineering, Aalborg University

Publications in refereed journals

A) J.O. Morsbøl, S.V. Sorokin, 2015. "Elastic Wave Propagation in Curved Flexible Pipes", *International Journal for Solids and Structures.*

B) J.O. Morsbøl, S.V. Sorokin, N. Peake, 2015. "A WKB Approximation of Elastic Waves Travelling on a Shell of Revolution", *Journal of Sound and Vibration.*

This thesis has been submitted for assessment in partial fulfillment of the PhD degree. The thesis is based on the submitted or published scientific papers which are listed above. Parts of the papers are used directly or indirectly in the extended summary of the thesis. As part of the assessment, co-author statements have been made available to the assessment committee and are also available at the Faculty. The thesis is not in its present form acceptable for open publication but only in limited and closed circulation as copyright may not be ensured.

Contents

Preface iii

Abstract v

Dansk resumé vii

Contents xi

1 What is it All About? 1
- 1.1 Contextual, Historical, and Motivational Background 2
- 1.2 Tools to Aid the Reading of the Papers 7
  - 1.2.1 Thin Shell Theory 7
  - 1.2.2 Kinetic and Potential Energy Stored in the Shell . . . 11
  - 1.2.3 The WKB Method 14

2 Description and conclusions of the papers 19
- 2.1 Paper A 19
  - 2.1.1 Summary 19
  - 2.1.2 Main-conclusions 20
  - 2.1.3 Scientific Contribution 20
- 2.2 Paper B 21
  - 2.2.1 Summary 21
  - 2.2.2 Main-conclusions 22
  - 2.2.3 Scientific Contribution 22
- 2.3 Future Work 22
- 2.4 Review History 23

References 25

PAPERS

A Elastic Wave Propagation in Curved Flexible Pipes 29

B A WKB Approximation of Elastic Waves Travelling on a Shell of Revolution **67**

Chapter 1

What is it All About?

Waves and vibrations – in a wide range of natural sciences and engineering technologies these terms are often used as soon as a system or quantity is oscillating periodically over time around some average configuration or mean value. In mechanics, which is known as the science concerned with the behaviour of physical structures and fluids subjected to forces and displacements, these two terms therefore come into play when forces and displacements start to oscillate over time. But frankly, what is the difference between waves and vibrations? At first glance, not much, but there are some good reasons to distinguish anyway. However, to illustrate one way of distinguishing, at least within mechanics, it makes sense to establish two fundamentally different situations: One where only waves are present and another where only vibrations are present. Firstly, imagine a homogeneous body, either structural or fluid-filled, which is unbounded in one, two, or three spatial dimensions. If an oscillating source, it could be a prescribed point force or point displacement, is acting somewhere in that body, *waves* will be created around the source and transport energy towards infinity. Secondly, imagine again a homogeneous one, two, or three dimensional body, but this time it is perfectly isolated from its surroundings by boundaries of either zero forcing or zero displacement. If this body is also subjected to a point source waves will just as well in this case be created. However, as time goes the waves will reach the boundaries and be perfectly reflected back into the body where they will create interference with those waves that are still outgoing. As time goes towards infinity, the interference between outgoing and ingoing waves will create and fill up the body with so-called standing waves, which could be defined as *vibrations* in their purest form. Now, to mix up this distinction again, it is fair to claim that, with these definitions of waves and vibrations, then the vibrations in the bounded body are simply a composition of the waves in the corresponding unbounded body. Thus, if it possible to determine the behaviour of all waves, that a able to exist in the unbounded body, these are the *waveguide properties*, it should in accordance to this logic be possible to represent any vibrational state by properly taking the boundary conditions into account of the corresponding bounded body. In other word, even though it indeed is possible to determine the vibrations in a bounded body without considering the waveguide properties, then these vibrations will still be composed of waves that are also supported by the unbounded body. Thus, the advantage of knowing the waveguide properties is that it gives a general overview of all the ingredients that possibly can cause the vibrations in any corresponding bounded body. Hence, with this logic in mind, it is exactly the *waveguide properties* of *unbounded* pipe-like structures, that are of interest in this Ph.D. thesis.

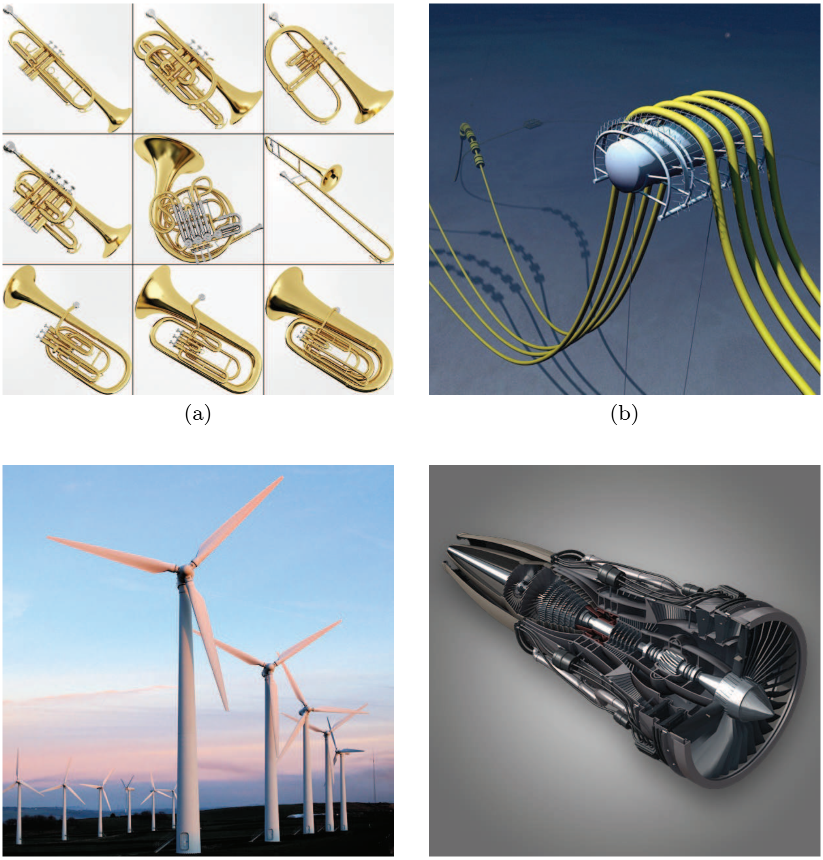
To determine the waveguide properties of pipe-like structures, the mechanisms governing the dynamic of such continuous bodies must be considered. To simplify these considerations, which still turn out to be fairly complicated anyway, the following assumptions are taken as preliminary throughout the entire work:

- The material, from which the structure is made of, is linear elastic.
- Likewise is the material homogeneous, isotropic, and non-damping.
- Finally, the displacements associated with the wave motion are small.

Within these assumptions, the dynamics of pipe-like structures are covered by the theory of linear elastodynamics. But before going deeper into that, a few contextual and historical aspects of the elastodynamics of pipe-like structures will be briefly discussed.

1.1 Contextual, Historical, and Motivational Background

Analysis of the elastodynamical aspects of pipe-like structures is a classical subject with many practical applications. In figure 1.1 a few examples where pipe-like structures are involved are given. In each of these examples sound and vibration, either structure borne or acoustical, play an important role for performance and/or safety. Also in each example the interaction between structure and fluid has a more or less heavy or light influence. The purpose of the brass instruments in (a) is obviously in some way to shape the sound produced by the lips of the musician, at the mouthpiece, and create the desired timber which is to be transferred acoustically to the surroundings. Within the scientific community related to that area it has been an ongoing discussion whether or not the vibration of the brass wall makes a difference for the far-field sound perception. However it seems like the current conviction is that altering the structural parameters, such as material selection and wall thickness, do have a significant effect on the far-field sound[1]. In itself, the fact that this discussion has kept on going over many centuries,



(a) (b) (c) (d)

Figure 1.1: (a) Collection of brass instruments, (b) submerged risers, (c) wind turbines, and (d) section view of a jet engine.

indicate that this is a case of very light fluid-structure interaction (FSI). Nevertheless, according to the current conviction, the brass instrument is not just a rigid duct that shapes the waves in the air, but the elastodynamic of the instrument itself also needs consideration. Risers, illustrated in (b), which are used in the oil and gas industry, are also highly affected by vibrations. One of the problems often referred to is flow induced pulsations[2]. While this phenomenon is primarily fluid dominated the elastodynamic of the rises themselves apparently become more and more important as the industry pushes the limit towards deeper and deeper waters[3]. Each of the wind turbines in (c) constitutes a highly dynamical system. The wind turbine is subjected to a broad spectrum of dynamic loads ranging over wind loads, seismic activities, noise generated by aerodynamic turbulence, and noise generated internally by gears and generator and emitted from the surfaces of the blades, nacelle, and tower[4]. Obviously the tower is a pipe-like structure, and because it is exposed to some of these dynamic loads its elastodynamic therefore needs attention. Hence it also becomes relevant for the context of this Ph.D. project. Final example is the jet engine in (d). From this section view several pipe-like components with rotational symmetry can be spotted. Without further documentation everyone knows that jet engines produce an enormous amount of noise. Moreover, the vibrations associated with the noise are of such magnitude that they must be addressed when the reliability of the engine and its components is assessed.

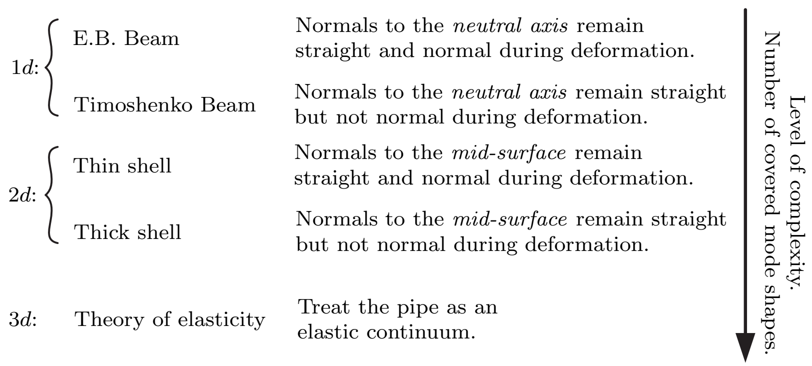


Figure 1.2: Different methods are available when modelling pipe-like structures.

From these examples, but also in general, pipes can usually be characterised as long, slender objects with a circular hollow cross section. Because they are often long and slender, the apparent point of departure in the mathematical formulation of their elastodynamics is to treat them as one-dimensional objects. On the other hand, the hollow cross section brings flexibility to the system meaning that the deformation of the cross section perhaps also in some way needs consideration. Thus, depending on the situation, pipe structures have typically been modelled either as beams, as shells, or by means of three-dimensional theory of elasticity. The hierarchy of complexity of these approaches is illustrated in the chart in figure 1.2. The benefit of a more complicated model, is that more information, here expressed as number of covered mode shapes[1], can be extracted. The beam models are valid when only the very fundamental mode shapes are active, that covers axial waves, torsional waves, shearing waves, and bending waves. Among beam models the two classical models based on the kine-

[1] As the wording says, the mode shape expresses the shape of the deformation associated with the wave. The number of degrees of freedom that are built into a model equals the number of mode shapes that it will predict.

matic assumptions formulated by, respectively, Kirchhoff, but named after Euler-Bernoulli, and Timoshenko are dominating. The additional degree of freedom in the Timoshenko model, on one hand, makes it able to predict the shearing wave, and, on the other hand, makes it able to perform exceptionally well in predicting the behaviour of the bending wave at higher frequencies. When more complicated mode shapes need to be accounted for, that happens typically at higher frequencies or if the flexibility of the pipe's cross section is more pronounced, a model which is able to attain these more complicated shapes is needed. At first instance this calls e.g. for a cylindrical shell model. Again, the choice stands between a shell theory suitable for thin shells, which is based on Kirchhoff kinematic, or a shell theory which is also suitable for thicker shell, which is based in Timoshenko kinematic. Due to the increased complexity of the shell models, compared to beam models, several more or less successful attempts have been made on simplifying the governing equations of these theories. Hence simplified models have been obtained either by ignoring terms of higher order of smallness, e.g. by systematically cancelling bending terms, or by taking into account that the shell e.g. is cylindrical. Unfortunately many of these attempts turn out not to perform well when comparing with the general un-simplified shell models. Taking the abilities of modern symbolic manipulation software into account it may therefore be worth to take departure in one of the general and un-simplified shell theories, and adapt that to the specific problem instead. Then, when things get even more complicated, and the dynamic through the thickness of the pipe wall also becomes significant, a three-dimensional elasticity model is needed. Although analytical solutions for very idealised cases do exist, such three-dimensional problems usually requires some sort of discretisation and numerical solution algorithm such as the finite element (FE) method.

Besides the considerations about their flexibility, pipes and pipe-like structures, like those presented above, are also typically irregularly curved and/or may change cross sectional dimensions along their axial extension. In the meantime the elastodynamics, and in particular the waveguide properties, of curved flexible pipes and pipes with varying cross section are not nearly as thoroughly studied as for straight prismatic pipes. Nonetheless, the curvedness and/or varying cross section has some effects that fundamentally disrupt the waveguide properties of straight prismatic pipes. As an example, see figure 1.3. In a straight pipe, each unique wave mode, e.g. the axial mode, the bending mode, the torsional mode, etc. are all uncoupled. It means that if a pipe is harmonically excited by one of these unique mode shapes, then exactly this mode shape will be preserved as the waves travel away from the excitation. It also means that each linearly independent mode shape can be analysed without considering the others and they can be linearly combined to create any arbitrary mode shape. If then the pipe is curved the mode shapes, which are linearly independent in the straight pipe, start to couple. As illustrated, while the axial wave is

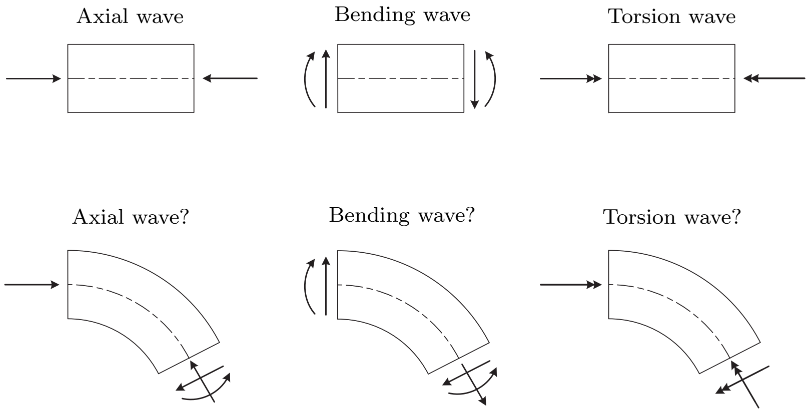


Figure 1.3: When the pipe is curved even the most fundamental wave modes couple.

propagating along the curved pipe it turns into a mixture of axial, in-plane[2] shearing, and in-plane bending waves. And obviously the in-plane bending wave, or in-plane shearing wave, also turns into a mixture of the same components. The torsion wave, on the other hand, turn into a combination of torsion and out-of-plane bending. Likewise, but not illustrated, will the out-of-plane shearing wave turn into a combination of torsion, out-of-plane bending, and out-of-plane shearing. Now, imagining a corresponding static counterpart where e.g. an axial force is balanced by an axial and shearing reaction force along with an in-plane reaction moment. This mixture of reactions will easily be determined by static equilibrium. But in case of dynamics the mixture will be frequency dependent and thus not as easily determined. Nevertheless, this will be the topic of paper A found later in this Ph.D. thesis. Turning to the varying cross sectional dimensions, their

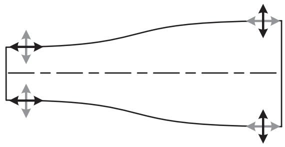


Figure 1.4: The coupled axial/breathing mode shifts from being dominated by axial motion where the pipe is narrow towards being dominated by breathing motion where the pipe is wide.

effects on the waveguide properties are perhaps not as intuitive. However it is still related to how waves are coupled. But now the concern is instead

[2] In relation to the plane of the paper on which the figure is printed.

on the wave modes which are already linearly dependent, i.e. coupled, in the straight pipe. This is e.g. the case for the axial wave and the breathing wave. Or rather, because they are coupled in the straight pipe, and therefore both always will be present to some extent, they together form a mode which can be designated as the axial/breathing mode. But as illustrated in figure 1.4 it turns out that the amount of, respectively, axial and breathing motion is affected by the radius of the pipe. Thus it can happen that a wave can be dominated by axial motion in one part of the pipe, but if the radius gradually increases, the wave will continuously shifts towards being dominated by breathing motion. Moreover, the wave amplitude is also affected. In fact, for a wave like the axial/breathing wave, but also for other coupled wave modes, it happens that in the vicinity of where some certain relation between the wave frequency, pipe radius, and thickness of the pipe wall is fulfilled a rather sharp peak in amplitude arises. The interpretation of this is that a local resonance in the pipe gets excited by the mode shape of the wave and thereby creates large displacements in that area. A thorough modelling and analysis of these effects are presented in paper B found later in this Ph.D. thesis.

With this motivational background the following section of theoretical models and mathematical tools are provided to ease the entry into the two papers of the Ph.D. thesis.

1.2 Tools to Aid the Reading of the Papers

In the following subsections, a careful selection of theoretical tools to aid the reading of the papers is provided. The tools perhaps seem a bit detached from one another and the motivation for providing exactly these tool, is only briefly explained in order not to paraphrase too much of what will be more thoroughly explained in the papers. One way of using this tool box could therefore be to skip it for now, but then return to it as a backup reference while reading the papers. Alternatively, stay tuned to get prepared for what foundation the papers are written on.

1.2.1 Thin Shell Theory

From the beginning of the Ph.D. project it has been chosen to treat the pipe structures as thin shells. As explained in connection with figure 1.2 the shell, on one hand, can better reflect the flexibility of a hollow pipe than a beam model, but on the other hand, the shell is simpler than treating the pipe as a three-dimensional continuum. Therefore it seems more likely to be able to formulate and solve, at least with a good approximation, the governing equations analytically.

In general, when a physical three-dimensional structure is treated as a shell, the fundamental assumption is that one of the structures dimensions is so small, compared to its other dimensions, that the structure can be modelled as a two-dimensional surface. In this way, the two-dimensional surface will typically approximate the mid-surface of the three-dimensional structure, while the remaining small dimension will just appear in the model as a fixed parameter referred to as the shell thickness. For a pipe structure the shell thickness is obviously the thickness of the pipe wall. The chosen thin shell theory is due to Gol'denweizer-Novozhilov[5]. This theory is almost identical to the perhaps more well-known shell theory by Flügge[6]. But as pointed out in the textbook by Novozhilov, Flügge keeps a few smaller terms which are beyond the accuracy of the fundamental assumptions of the thin shell theory itself, and hence it is more consistent to skip these. The thin shell theory of Gol'denweizer-Novozhilov is developed under the reservation that the ratio between the shell thickness and the smallest radius of curvature, at any location of the mid-surface, is no more than 1/20. Besides that, the shell can have any arbitrary shape. Though, the shape of the shell affects how stresses and deformations distribute throughout the shell, and therefore it is important to properly take that into account. The information about the shape is condensed in two, so-called, Lamé parameters, and two radii of curvature. These, in total four, parameters are obtainable by means of differential geometry.

1.2.1.1 Differential Geometry

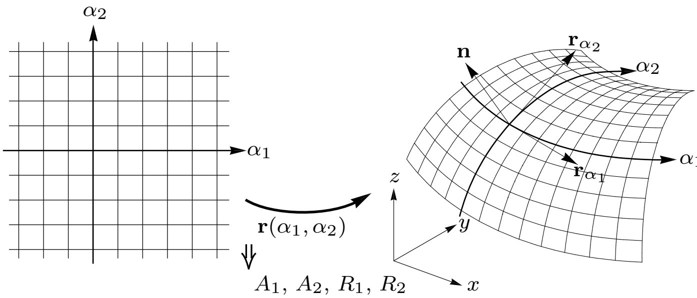


Figure 1.5: The parametrisation $\mathbf{r}(\alpha_1, \alpha_2)$ provides a mapping between a plane cartesian coordinate system and a curvilinear coordinated system which is wrapped on the three-dimensional mid-surface.

To calculate the Lamé parameters and the radii of curvature a parametrisation of the shells mid-surface is needed. Or more precisely, the starting point is the parametrisation which provides a mapping between a two-dimensional cartesian coordinate system, illustrated in the left-hand side of figure 1.5, to the orthogonal curvilinear coordinates on the three-dimensionally curved mid-surface, illustrated in the right-hand side of figure 1.5. Such parametrisation can take the form:

$$\mathbf{r}(\alpha_1,\alpha_2) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} f_1(\alpha_1,\alpha_2) \\ f_2(\alpha_1,\alpha_2) \\ f_3(\alpha_1,\alpha_2) \end{bmatrix} \tag{1.1}$$

The Lamé parameters are derived from the partial derivatives of the parametrisation:

$$\mathbf{r}_{\alpha_i} = \frac{\partial \mathbf{r}}{\partial \alpha_i} \quad \text{where} \quad i = \{1,2\} \tag{1.2}$$

As illustrated in the figure, $\mathbf{r}_{\alpha_i}$ are tangents to the curves generated by the curvilinear coordinates. Hence, by taking a small step in the cartesian coordinates system, $d\alpha_i$, the corresponding small step in the curvilinear system, ds_i, is:

$$ds_i = \left| \frac{\partial \mathbf{r}}{\partial \alpha_i} \right| d\alpha_i \tag{1.3}$$

From this, the Lamé parameters are simply defined as:

$$A_i \equiv \left| \frac{\partial \mathbf{r}}{\partial \alpha_i} \right| = \sqrt{\left(\frac{\partial x}{\partial \alpha_i}\right)^2 + \left(\frac{\partial y}{\partial \alpha_i}\right)^2 + \left(\frac{\partial z}{\partial \alpha_i}\right)^2} \tag{1.4}$$

In this way the Lamé parameters are a measure of how lengths are scaled between the cartesian and the curvilinear coordinate systems.

Now turning towards the radii of curvature. Compared to finding the radius of curvature of a curve, it is particularly more difficult to find it, or rather *them*, for a surface. In the shell theory it is exactly the radii of curvature in the principal directions which are of interest. To determine these, a series of calculations are needed. The background for these calculations can e.g. be found in [7], however the relevant results will be presented in the following. Thus, by taking departure in the parametrisation of the mid-surface, first step is to calculate the parameters of *1st* and *2nd fundamental form*:

1. fundamental form	2. fundamental form
$E(\alpha_1,\alpha_2) = \mathbf{r}_{\alpha_1} \cdot \mathbf{r}_{\alpha_1}$	$e(\alpha_1,\alpha_2) = -\frac{\mathbf{r}_{\alpha_1} \times \mathbf{r}_{\alpha_2}}{\lvert \mathbf{r}_{\alpha_1} \times \mathbf{r}_{\alpha_2} \rvert} \cdot \mathbf{r}_{\alpha_1,\alpha_1}$
$F(\alpha_1,\alpha_2) = \mathbf{r}_{\alpha_1} \cdot \mathbf{r}_{\alpha_2}$	$f(\alpha_1,\alpha_2) = -\frac{\mathbf{r}_{\alpha_1} \times \mathbf{r}_{\alpha_2}}{\lvert \mathbf{r}_{\alpha_1} \times \mathbf{r}_{\alpha_2} \rvert} \cdot \mathbf{r}_{\alpha_1,\alpha_2}$
$G(\alpha_1,\alpha_2) = \mathbf{r}_{\alpha_2} \cdot \mathbf{r}_{\alpha_2}$	$g(\alpha_1,\alpha_2) = -\frac{\mathbf{r}_{\alpha_1} \times \mathbf{r}_{\alpha_2}}{\lvert \mathbf{r}_{\alpha_1} \times \mathbf{r}_{\alpha_2} \rvert} \cdot \mathbf{r}_{\alpha_2,\alpha_2}$

(1.5)

Second step is to calculate the *Gaussian curvature* and *mean curvature* from the parameters above:

$$
\begin{aligned}
K(\alpha_1, \alpha_2) &= \frac{eg - f^2}{EG - F^2} \\
H(\alpha_1, \alpha_2) &= \frac{eG + gE - 2fF}{2\,(EG - F^2)}
\end{aligned}
\tag{1.6}
$$

Then in third step the principal curvatures are determined:

$$
\begin{aligned}
k_1(\alpha_1, \alpha_2) &= H + \sqrt{H^2 - K} \\
k_2(\alpha_1, \alpha_2) &= H - \sqrt{H^2 - K}
\end{aligned}
\tag{1.7}
$$

Last step is simply to invert the curvatures in order to obtain the radii:

$$
R_i(\alpha_1, \alpha_2) = \frac{1}{k_i} \tag{1.8}
$$

During the calculation of especially the radii of curvature, but also the Lamé parameters, it is indeed possible that mistakes occur. In the meantime it is possible to assess the consistency between the four parameters by means of the conditions of Codazzi and Gauss. These conditions are presented in the following subsection.

1.2.1.2 Conditions of Codazzi and Gauss

The conditions of Codazzi and Gauss are derived from the fundamental condition that the symmetry of mixed second derivatives must also apply for the vectors $\mathbf{r}_{\alpha_i}$ and $\mathbf{n}$. Hence:

$$
\frac{\partial^2 \mathbf{n}}{\partial\alpha_i \partial\alpha_j} = \frac{\partial^2 \mathbf{n}}{\partial\alpha_j \partial\alpha_i} \quad \text{and} \quad \frac{\partial^2 \mathbf{r}_{\alpha_i}}{\partial\alpha_i \partial\alpha_j} = \frac{\partial^2 \mathbf{r}_{\alpha_i}}{\partial\alpha_j \partial\alpha_i} \quad \text{where} \quad i, j = \{1, 2\} \tag{1.9}
$$

By expressing these vectors in the functions A_1, A_2, R_1, R_2 of the variables α_1 and α_2 the conditions of Codazzi can be derived from the mixed second derivatives of $\mathbf{n}$ to:

$$
\begin{aligned}
\frac{\partial}{\partial\alpha_1}\left(\frac{A_2}{R_2}\right) &= \frac{1}{R_1}\frac{\partial A_2}{\partial\alpha_1} \\
\frac{\partial}{\partial\alpha_2}\left(\frac{A_1}{R_1}\right) &= \frac{1}{R_2}\frac{\partial A_1}{\partial\alpha_2}
\end{aligned}
\tag{1.10}
$$

Likewise can the condition of Gauss be derived from the mixed second derivatives of $\mathbf{r}_{\alpha_i}$:

$$\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial A_2}{\partial\alpha_1}\right)+\frac{\partial}{\partial\alpha_2}\left(\frac{1}{A_2}\frac{\partial A_1}{\partial\alpha_2}\right)=-\frac{A_1A_2}{R_1R_2} \tag{1.11}$$

It should though be remarked that the conditions, in this form, rely on the assumption that the curves generated by the curvilinear coordinates, α_1 and α_2, coincide with the principal directions of the surface. If so, these conditions need to be satisfied for the functions A_1, A_2, R_1, R_2 to consistently determine a unique surface, except for its location in space [5].

1.2.2 Kinetic and Potential Energy Stored in the Shell

From the viewpoint of energies, a conservative elastodynamical system is occupied by kinetic and potential energy. The kinetic and potential energy will oscillate over time, whereas the total mechanical energy, which is the sum of kinetic and potential energy, will remain constant. In fact, the constant mechanical energy is in this way represented by a, literally, harmonic exchange between purely kinetic energy and purely potential energy. Not surprisingly, it can be mathematically shown that for such system the time-averaged kinetic and potential energy are identical[8]. That is:

$$\widehat{T}=\widehat{U} \tag{1.12}$$

where:

$\widehat{T}$: Time-averaged kinetic energy,

$\widehat{U}$: Time-averaged potential energy.

The time-average is obtained by integrating with respect to time over one full cycle, i.e. $\widehat{x}=\frac{1}{t_p}\int_0^{t_p}x dt$ where t_p is the time required to complete one cycle.

Now, the kinetic and potential energy in a piece of a shell are determined by point-wise calculating the velocity and strain components and then properly integrating them over the mid-surface of this piece of shell. Hence the kinetic energy integrated over the surface, S, is:

$$T=\int_S \xi dS=\frac{1}{2}\int_S \rho h\left|\frac{\partial\mathbf{u}}{\partial t}\right|^2 dS \tag{1.13}$$

where:

ρ : Density,
h : Shell thickness,
$\mathbf{n}$: Vector of the shell displacements u, v, and w.

while the potential energy integrated over the same surface is [5]:

$$
\begin{aligned}
U &= \int_S \upsilon dS \\
&= \frac{Eh}{2(1-\nu^2)} \int_S \left((\varepsilon_1 + \varepsilon_2)^2 - 2(1-\nu)\left(\varepsilon_1\varepsilon_2 - \frac{\gamma^2}{4}\right)\right) dS + \\
&+ \frac{Eh^3}{24(1-\nu^2)} \int_S \left((\kappa_1 + \kappa_2)^2 - 2(1-\nu)\left(\kappa_1\kappa_2 - \tau^2\right)\right) dS \qquad (1.14)
\end{aligned}
$$

where:

E :	Young's modulus,
ν :	Poisson's ratio,
ε_i :	Membrane normal strain in direction i,
γ :	Membrane shear strain,
κ_i :	Change of curvature in direction i,
τ :	Twist.

However, as a result of equation (1.12):

$$
\widehat{T} = \int_S \widehat{\xi} dS = \widehat{U} = \int_S \widehat{\upsilon} dS
$$

$$
\Updownarrow
$$

$$
\widehat{\xi} = \widehat{\upsilon} \qquad (1.15)
$$

Thus, the comparison between $\widehat{\xi}$ and $\widehat{\upsilon}$ can serve as a sanity check when a solution to a conservative elastodynamical problem of a shell is proposed. Hence it is useful to write out these two quantities. Firstly $\widehat{\xi}$:

$$
\widehat{\xi} = \frac{1}{t_p} \int_0^{t_p} \xi dt = \frac{1}{2t_p} \int_0^{t_p} \rho h \left| \frac{\partial \mathbf{u}}{\partial t} \right|^2 dt \qquad (1.16)
$$

It is assumed that the displacements can be expressed by a separable function of the form $\tilde{\mathbf{u}} e^{i\omega t}$ where $\tilde{\mathbf{u}}$ is independent of time, ω is the frequency, and i is defined as $\sqrt{-1}$. Moreover, the shell displacements, as well as the kinetic and potential energies, are after all real valued field variables. Hence $\mathbf{u} = Re\left\{\tilde{\mathbf{u}} e^{i\omega t}\right\}$. With this, the equation above can be written further out:

$$
\begin{aligned}
\widehat{\xi} &= \frac{\rho h}{2t_p} \int_0^{t_p} \left| \frac{\partial \mathbf{u}}{\partial t} \right|^2 dt \\
&= \frac{\rho h}{2t_p} \int_0^{t_p} \left| Re\left\{ i\omega \tilde{\mathbf{u}} e^{i\omega t} \right\} \right|^2 dt
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\rho h}{2t_p} \int_0^{t_p} \left| Re \left\{ i\omega \left(\begin{array}{l} Re\left\{\tilde{\mathbf{u}}\right\} cos(\omega t) - Im\left\{\tilde{\mathbf{u}}\right\} sin(\omega t) + \\ +i\left(Re\left\{\tilde{\mathbf{u}}\right\} sin(\omega t) + Im\left\{\tilde{\mathbf{u}}\right\} cos(\omega t)\right) \end{array} \right) \right\} \right|^2 dt \\
&= \frac{\rho h}{2t_p} \int_0^{t_p} \left| -\omega \left(Re\left\{\tilde{\mathbf{u}}\right\} sin(\omega t) + Im\left\{\tilde{\mathbf{u}}\right\} cos(\omega t)\right)\right|^2 dt \\
&= \frac{\rho h \omega^2}{2t_p} \int_0^{t_p} \left(\begin{array}{l} \left|Re\left\{\tilde{\mathbf{u}}\right\}\right|^2 sin(\omega t)^2 + \left|Im\left\{\tilde{\mathbf{u}}\right\}\right|^2 cos(\omega t)^2 + \\ +2Re\left\{\tilde{\mathbf{u}}\right\} Im\left\{\tilde{\mathbf{u}}\right\} sin(\omega t) cos(\omega t) \end{array} \right) dt \\
&= \frac{\rho h \omega^2}{2t_p} \frac{t_p}{2} \left(\left|Re\left\{\tilde{\mathbf{u}}\right\}\right|^2 + \left|Im\left\{\tilde{\mathbf{u}}\right\}\right|^2 \right) \\
&= \frac{\rho h \omega^2}{4} \left(Re\left\{\tilde{\mathbf{u}}\right\} + iIm\left\{\tilde{\mathbf{u}}\right\}\right) \left(Re\left\{\tilde{\mathbf{u}}\right\} - iIm\left\{\tilde{\mathbf{u}}\right\}\right) \\
&= \frac{\rho h \omega^2}{4} \tilde{\mathbf{u}} \overline{\tilde{\mathbf{u}}} \qquad (1.17)
\end{aligned}
$$

where $\overline{x}$ is the complex conjugated of x. Similarly can $\widehat{v}$ be written out. Though, for simplicity, this will only be shown for $(\epsilon_1 + \epsilon_2)^2$, c.f. equation (1.14):

$$
\begin{aligned}
\widehat{(\epsilon_1 + \epsilon_2)^2} &= \frac{1}{t_p} \int_0^{t_p} (\epsilon_1 + \epsilon_2)^2 dt \\
&= \frac{1}{t_p} \int_0^{t_p} \left(Re\left\{\tilde{\epsilon}_1 e^{i\omega t}\right\} + Re\left\{\tilde{\epsilon}_2 e^{i\omega t}\right\}\right)^2 dt \\
&= \frac{1}{t_p} \int_0^{t_p} \left(\begin{array}{l} Re\left\{\tilde{\epsilon}_1\right\} cos(\omega t) - Im\left\{\tilde{\epsilon}_1\right\} sin(\omega t) + \\ +Re\left\{\tilde{\epsilon}_2\right\} cos(\omega t) - Im\left\{\tilde{\epsilon}_2\right\} sin(\omega t) \end{array} \right)^2 dt \\
&= \frac{1}{t_p} \int_0^{t_p} \left(\begin{array}{l} \left(Re\left\{\tilde{\epsilon}_1\right\} + Re\left\{\tilde{\epsilon}_2\right\}\right) cos(\omega t) - \\ -\left(Im\left\{\tilde{\epsilon}_1\right\} + Im\left\{\tilde{\epsilon}_2\right\}\right) sin(\omega t) \end{array} \right)^2 dt \\
&= \frac{1}{t_p} \int_0^{t_p} \left(\begin{array}{l} \left(Re\left\{\tilde{\epsilon}_1\right\} + Re\left\{\tilde{\epsilon}_2\right\}\right)^2 cos(\omega t)^2 + \\ +\left(Im\left\{\tilde{\epsilon}_1\right\} + Im\left\{\tilde{\epsilon}_2\right\}\right)^2 sin(\omega t)^2 - \\ -2 \left(\begin{array}{l} \left(Re\left\{\tilde{\epsilon}_1\right\} + Re\left\{\tilde{\epsilon}_2\right\}\right) \cdot \\ \cdot \left(Im\left\{\tilde{\epsilon}_1\right\} + Im\left\{\tilde{\epsilon}_2\right\}\right) \end{array} \right) cos(\omega t) sin(\omega t) \end{array} \right) dt \\
&= \frac{1}{t_p} \frac{t_p}{2} \left(\left(Re\left\{\tilde{\epsilon}_1\right\} + Re\left\{\tilde{\epsilon}_2\right\}\right)^2 + \left(Im\left\{\tilde{\epsilon}_1\right\} + Im\left\{\tilde{\epsilon}_2\right\}\right)^2 \right)
\end{aligned}
$$

$$=\frac{1}{2}\left(Re\left\{\tilde{\epsilon}_1+\tilde{\epsilon}_2\right\}^2+Im\left\{\tilde{\epsilon}_1+\tilde{\epsilon}_2\right\}^2\right) \tag{1.18}$$

The entire expression for $\widehat{\upsilon}$ is can be found in paper B.

1.2.3 The WKB Method

In paper B the so-called WKB (Wentzel-Kramers-Brillouin) method is used to determine the waveguide properties of shells of revolution. In general terms, this method is particularly useful to asymptotically solve linear differential equations with slowly varying coefficients. The method appears to be rather well-known in quantum mechanics and theoretical physics where it is also known as the *phase-integral method*. In the meantime the method has only in a few cases found its way to classical mechanics, and therefor, to demonstrate how it is applied, a fundamental problem is solved below. The concepts behind the method are briefly explained in the paper, but more information can be found in [9]. Though still, in relation to the following example, the initiating assumption is that within just one or a few wave cycles the nonuniform waveguide does not change much. Therefore within these few wave cycles the nonuniform waveguide can be approximated by a uniform waveguide with the local properties. However, the effects of the nonuniform waveguide still accumulate in the waves as they propagate. The WKB method takes this accumulation into account. In the following example the waveguide properties of axial waves in a circular rod with slowly varying radius will be derived by means of the method. Similar example, but for a hollow thin-walled rod, is also commented in the paper, but here it will be presented in a lot more detail.

The partial differential equation governing the axial wave motion of a non-uniform rod can be found in [10]:

$$0=\frac{\partial}{\partial x^*}\left(E^*A^*(x^*)\frac{\partial u^*(x^*,t^*)}{\partial x^*}\right)-\rho^*A^*(x^*)\frac{\partial^2 u^*(x^*,t^*)}{\partial {t^*}^2} \tag{1.19}$$

where:

A^* : Cross-sectional area,
u^* : Axial displacement,
x^* : Axial coordinate.

Before actually solving the partial differential equation it is essential for the validity and robustness of the derived solution that the differential equation, firstly, becomes reformulated into non-dimensional quantities. In that connection dimensional quantities are, in this sub-section, marked with "$*$". The non-dimensionalisation is done by scaling. Thus lengths are scaled by the radius of the rod. But as the area is a function of the axial coordinate, so will the radius of the circular rod also be, i.e.

$A^*(x^*) = \pi r^*(x^*)^2$. It is in the meantime possible to express the radius as $r^*(x^*) = R^* r(x^*) = R^* R(x)$ where R^* is the measurable radius of the rod at the location $x^* = x_0^*$ where $r(x_0^*) = 1$ and $x^* = R^* x$. Moreover, like previously, the axial displacement is assumed to be a separable function taking the form $u^*(x^*, t^*) = \tilde{u}^*(x^*)e^{i\omega^* t^*} = R^* \tilde{u}(R^* x)e^{i\omega^* t^*} = R^* \tilde{U}(x)e^{i\omega^* t^*}$. The frequency is then scaled as $\omega^* = \frac{c_0^* \omega}{R^*} = \sqrt{\frac{E^*}{\rho^* R^{*2}}}\omega$ where c_0^* is the speed of sound of an axial wave in a uniform rod. Making these substitutions in the partial differential equation above the following is obtained:

$$
\begin{aligned}
0 =& \frac{1}{R^*}\frac{\partial}{\partial x}\left(E^* \pi \left(R^* R(x)\right)^2 \frac{R^*}{R^*}\frac{\partial \tilde{U}(x)}{\partial x} e^{i\omega^* t^*}\right) \\
& - \rho^* \pi \left(R^* R(x)\right)^2 i^2 \frac{E^*}{\rho^* {R^*}^2}\omega^2 R^* \tilde{U}(x) e^{i\omega^* t^*} \\
\Updownarrow & \\
0 =& \frac{\partial}{\partial x}\left(R(x)^2 \frac{\partial \tilde{U}(x)}{\partial x}\right) + R(x)^2\omega^2 \tilde{U}(x) \\
=& R(x)^2 \tilde{U}''(x) + 2R(x)R'(x)\tilde{U}'(x) + R(x)^2\omega^2\tilde{U}(x) \\
\Updownarrow & \\
0 =& R(x)\tilde{U}''(x) + 2R'(x)\tilde{U}'(x) + \omega^2 R(x)\tilde{U}(x)
\end{aligned}
\tag{1.20}
$$

Then, secondly, to get hold on the slowness of the varying radius a slow-scale parameter is introduced. By loosely defining a distance, D^*, over which the radius changes *perceivably* along the rod, then the slow-scale parameter defined as $\varepsilon = \frac{R^*}{D^*}$ will be small if the radius varies slowly. By means of this parameter, it is possible to distinguish between a so-called fast and slow axial coordinate:

$$
\varepsilon x = X \quad \Rightarrow \quad \frac{\partial}{\partial x} = \epsilon \frac{\partial}{\partial X} \tag{1.21}
$$

If R and $\tilde{U}$ are functions of the slow coordinate, X, instead of the fast coordinate, x, then the differential equation above becomes:

$$
0 = R(X)\varepsilon^2 \tilde{U}''(X) + 2\varepsilon^2 R'(X)\tilde{U}'(X) + \omega^2 R(X)\tilde{U}(X) \tag{1.22}
$$

After these manipulations the scene is set for introducing the actual WKB solution. With inspiration from [9] the WKB solution takes the form $\tilde{U}(X) = A(X)e^{\frac{i}{\varepsilon}\int_0^X k(\tau)d\tau} = \sum_{n=0}^{\infty} \varepsilon^n A_n(X) e^{\frac{i}{\varepsilon}\int_0^X k(\tau)d\tau}$ where $A_n(X)$ are the components of the asymptotic expansion of the amplitude function,

$A(X)$, and $k(\tau)$ is the wavenumber. It may here be remarked that the dimensional wavenumber then is: $k^*(X) = \frac{k(X)}{R^*}$. The integral form of this trial solution reflects the accumulation of how the non-uniform waveguide affects the wave. In order to determine the wavenumber and the amplitude function, at least asymptotically, this trial solution has to be substituted into the differential equation above. Hence, it needs to be differentiated:

$$\begin{aligned}\tilde{U}'(X) &= \left(A(X)e^{\frac{i}{\varepsilon}\int_0^X k(\tau)d\tau}\right)' = \left(A(X)e^{\frac{i}{\varepsilon}(K(X)-K(0))}\right)' \\ &= A'(X)e^{\frac{i}{\varepsilon}(K(X)-K(0))} + A(X)\frac{i}{\varepsilon}K'(X)e^{\frac{i}{\varepsilon}(K(X)-K(0))} \\ &= \left(A'(X) + A(X)\frac{i}{\varepsilon}k(X)\right)e^{\frac{i}{\varepsilon}\int_0^X k(\tau)d\tau} \qquad (1.23)\end{aligned}$$

$$\begin{aligned}\tilde{U}''(X) &= \left(A''(X) + A'(X)\frac{i}{\varepsilon}k(X) + A(X)\frac{i}{\varepsilon}k'(X)\right)e^{\frac{i}{\varepsilon}\int_0^X k(\tau)d\tau} + \\ &+ \left(A'(X) + A(X)\frac{i}{\varepsilon}k(X)\right)\frac{i}{\varepsilon}K'(X)e^{\frac{i}{\varepsilon}\int_0^X k(\tau)d\tau} \\ &= \left(\begin{array}{l} A''(X) + 2A'(X)\frac{i}{\varepsilon}k(X) + \\ +A(X)\frac{i}{\varepsilon}k'(X) - A(X)\frac{1}{\varepsilon^2}k(X)^2 \end{array}\right)e^{\frac{i}{\varepsilon}\int_0^X k(\tau)d\tau} \qquad (1.24)\end{aligned}$$

Substituting this into the differential equation, dividing through with the exponential function, and reminding that $A(X) = \sum_{n=0}^{\infty}\varepsilon^n A_n(X)$ the following is obtained:

$$\begin{aligned}0 &= \left(-R(X)A_0(X)k(X)^2 + \omega^2 R(X)A_0(X)\right) + \\ &+ \varepsilon\left(\begin{array}{l} i\left(R(X)\left(2A_0'(X)k(X) + A_0(X)k'(X)\right) + 2R'(X)A_0(X)k(X)\right) - \\ -R(X)A_1(X)k(X)^2 + \omega^2 R(X)A_1(X) \end{array}\right) + \\ &+ O(\varepsilon^2) \qquad (1.25)\end{aligned}$$

Because ε is small then $1 \gg \varepsilon \gg \varepsilon^2$. Hence the above equation can be regarded as series of individual equations separated by the order of ε. The $O(1)$-equation then gives:

$$\begin{aligned}& 0 = -R(X)A_0(X)k(X)^2 + \omega^2 R(X)A_0(X) \\ \Downarrow \\ & k(X) = \pm\omega \qquad (1.26)\end{aligned}$$

This result shows that even though the rod is non-uniform then the wavenumber is to the leading order simply given by the frequency anywhere along the rod. Turning towards the $O(\varepsilon)$-equation it states:

$$\begin{aligned} &- R(X)A_1(X)k(X)^2 + \omega^2 R(X)A_1(X) \\ &= - i\left(R(X)\left(2A_0'(X)k(X) + A_0(X)k'(X)\right) + 2R'(X)A_0(X)k(X)\right) \end{aligned} \quad (1.27)$$

The left-hand side of this equation is obviously of same form as the $O(1)$-equation. Thus, for any arbitrary choice of A_1 the left-hand side will vanish. The right-hand side then gives:

$$\begin{aligned} 0 &= -i\left(R(X)\left(2A_0'(X)k(X) + A_0(X)k'(X)\right) + 2R'(X)A_0(X)k(X)\right) \\ &\Downarrow \\ 0 &= 2R(X)k(X)A_0'(X) + \left(R(X)k'(X) + 2R'(X)k(X)\right)A_0(X) \end{aligned} \quad (1.28)$$

This equation, which in fact can be recognised as a differential equation in $A_0(X)$, has the general solution:

$$A_0(X) = C_0 e^{-\int \frac{Q(X)}{P(X)} dX} \quad (1.29)$$

where $Q(X)$ and $P(X)$ are the coefficients in front of, respectively, $A_0(X)$ and $A_0'(X)$, and C_0 is an arbitrary constant. Thus:

$$\begin{aligned} A_0(X) &= C_0 e^{-\int \frac{R(X)k'(X)+2R'(X)k(X)}{2R(X)k(X)} dX} \\ &= C_0 e^{-\int \left(\frac{k'(X)}{2k(X)} + \frac{R'(X)}{R(X)}\right) dX} \\ &= C_0 e^{-\left(\frac{1}{2} ln(k(X)) + ln(R(X))\right)} \\ &= C_0 e^{ln\left(\frac{1}{\sqrt{k(X)}R(X)}\right)} \\ &= C_0 \frac{1}{\sqrt{k(X)}R(X)} \end{aligned} \quad (1.30)$$

And because $k(X) = \pm\omega$ then $A_0(X) \propto \frac{1}{R(X)}$. In principle the $O(\varepsilon^2)$-equation may also provide additional information. But within the framework of the WKB method, these terms are not considered. Hence, if expressed in dimensional parameters, the wavenumber is $k^* = \frac{\omega^*}{c_0^*}$ and the amplitude function is $a^*(x^*) = \frac{R^*}{r\left(\frac{\varepsilon x^*}{R^*}\right)}$.

Chapter 2

Description and conclusions of the papers

To round of the introduction to the two papers of this thesis, the following chapter gives a short summary of each of them along with their conclusions and scientific contributions. At this stage the conclusions will naturally jump a bit out of nothing without much prior explanation. However, it gives an idea of the outcome of the two papers. Finally a few suggestions to future work are presented.

2.1 Paper A

2.1.1 Summary

Under the title *Elastic Wave Propagation in Curved Flexible Pipes* paper A deals with the elastodynamics of an unbounded toroidal shell. Two models, describing the elastodynamics, have been developed. One is derived analytically from classical thin shell theory while the other is a wave fine element (WFE) model. The analytically derived model is based on the general thin shell theory presented in [5]. The governing equations of this model constitute a system of three partial differential equations with periodic coefficients to which the solution provides the displacement field of the shell. The governing equations are formulated in non-dimensional parameters and thereby only depend on five quantities: A thickness parameter, a curvedness parameter, Poisson's ratio, and two wave-related variables which are the wavenumber and the frequency. In order to solve the governing equations, at least approximately, a trial solution, based on assumptions about the wave field supported by the toroidal shell, is suggested and adopted to the equations by means of the Galerkin method. This provides an eigenvalue problem from which the dispersion relation of the waveguide, as well as the modal vectors, can be calculated numerically. By altering the curvedness of the torus and the shell thickness it is possible to study how these geometrical properties affect the waveguide properties of such pipe-like shell. Likewise can the dispersion relation be extracted from the WFE model. By comparing the results of each model it is possible to uncover some of the strengths and limitations of each of the two modelling approaches. Besides comparing the two toroidal shell models they have also been compared to curved beam theory and cylindrical shell theory.

2.1.2 Main-conclusions

- The comparisons between the models reveal that the predictions of both toroidal shell models agree well within a rather large subspace of curvedness, thickness, and frequency combinations.

- When comparing the toroidal shell models to curved beam theory and cylindrical shell theory it turn out convenient to divide the waveguide properties into three regimes:

  - **Curved beam regime:** At very low frequencies (i.e. in the low-frequency range), when the torus is only slightly curved, and the shell wall is about as thick as the thin shell theory allows, then the waveguide properties of the toroidal shell and of the curved beam become very close. Thus, in this regime the waveguide properties of the toroidal shell can just as well be predicted by the curved beam theory.
  - **Cylindrical shell regime:** At higher frequencies (and shorter wavelengths), where the first few high order modes have become propagating, the toroidal shell start to behave as a classical straight cylindrical shell. The merging between the toroidal shell theory and the cylindrical shell theory is only weakly affected by the curvedness of the torus and the thickness of the shell. Therefore, even when the torus is strongly curved, its waveguide properties can be approximated by the cylindrical shell theory as soon as the first few high order modes have cut on.
  - **Torus regime:** The last regime falls in-between the two others. In this regime the behaviour of the toroidal shell is too complicated to be captured by the beam-approximation and the frequency is still too low (and the waves too long) to propagate along the torus as if it was a straight cylinder.

2.1.3 Scientific Contribution

The apparent scientific contribution of paper A is that it demonstrates two ways of predicting the waveguide properties of unbounded toroidal shells.

Hence it provides insight about how the curvedness of such pipe-like structure affects its waveguide properties. In addition to this, paper A demonstrates that in many cases it is possible to approximate the waveguide properties of such a complicated structure by means of classical tools like curved beam theory and cylindrical shell theory. But it also clearly demonstrates that in other cases the behaviour of the torus is much too complicated to be captured by these simpler methods. Finally, qualitative guidelines about when the waveguide properties should be expected to fall into the curved beam regime or the cylinder regime are provided. However, deterministic and quantitative guidelines will depend on the specific application and error tolerances, and are therefore not considered here.

2.2 Paper B

2.2.1 Summary

This paper is entitled *A WKB Approximation of Elastic Waves Travelling on a Shell of Revolution*. Thus the concern is no longer on curved pipes, but on straight pipes with changing radius. An example of such pipe is illustrated in figure 1.4. Again the pipe is treated as a thin classical shell, but in this paper the geometry is more loosely defined than in previous paper. The radius of the pipe is namely just represented by an unknown function of the axial coordinate. In this manner it is up to the reader to pick a favourite radius function and substitute it into the results. The fundamental reservation is though that the radius function must be slowly varying. It is explained how the dispersion relation and modal vector depend on the radius function and how they thereby also become functions of the axial coordinate. Moreover, closed-form expressions predicting how the amplitude of torsional, axial, and breathing waves are modulated, as these fundamental waves are travelling along the shell, are derived. The validity of the derived expressions is challenged by comparing them to classical rod and beam theory and by calculating the associated power flow, which must be conserved. Each of the expressions passes these sanity checks. The amplitude modulation of waves of higher order may in principle also be predicted by the prescribed method. Though, in these cases a numerical approach is suggested and explained. To illustrate how waves can be affected by the changing radius two characteristic radius functions are selected, and the dispersion relation, modal vector, and amplitude function for these examples are studied. As explained in previous chapter, it is from classical cylindrical shell theory known in advance that the axial wave and the breathing wave will be coupled due to Poisson's coupling, and this will also be the case in the shell of revolution. In the meantime the fraction between axial and breathing motion, in this coupled wave, will depend on the local radius. Hence it is demonstrated, in the paper, how a so-called modal change can occur over a rather short distance along the shell. On top of this, it turns out that for this particular coupled wave the modal change is associated with a sharp localised amplitude increase – i.e. a localised resonance. The location of this amplitude peak appears exactly where the local radius and the frequency of the wave match the conditions for the ring-frequency. Thus, the structure will be very compliant towards breathing motion at this location and at this frequency, and therefore the coupled axial/breathing wave is able to excite the localised resonance. Similar localised amplitude peaks, in association with a modal change, are demonstrated for other wave modes as well, but cases are also found where a modal change appears without any noticeable amplitude peak.

2.2.2 Main-conclusions

- Despite the complexity of a structure like the shell of revolution its waveguide properties for the torsional wave, and to some extend also the axial and breathing wave, can be well approximated by classical rod and beam theory.

- As a wave is travelling along the shell, modal changes appear and sharp localised peaks in amplitude may arise. For the coupled axial/breathing wave the presence and location of such peak is physically explained by means of the conditions of the ring-frequency.

- Even though these amplitude peaks are found in association with modal changes of other waves as well, it cannot be regarded as a general outcome of a modal change.

2.2.3 Scientific Contribution

Like in paper A the apparent scientific contribution of paper B is that it presents a way of analytically estimating the waveguide properties of a shell of revolution. It presents closed-form expressions for the amplitude function for the torsional, axial, and breathing wave which have not been publicly available before, and it explains how the amplitude modulation can be calculated numerically for waves of any order. Out of the expressions jumps the prediction of localised amplitude peaks. At first glance these peaks may seem rather unexpected. Nevertheless a clear physical interpretation is given which turn these peaks into rather intuitive and probable events.

2.3 Future Work

Suggestions to future work is:

- As explained in previous chapter, pipe-like structures are often found in contact with fluids such as oil, air or water. Thus inclusion of fluid interaction, both internally, and externally, and with and without mean flow, is of high relevance.

- Incorporation of boundary conditions may also be relevant. In such case it will be possible to study the response to a forced excitation at one end of the pipe while the other end may be clamped, free, subjected to some kind of impedance, or so something else.

- In principle it is possible to combine the two analytical methods in paper A and B. Thus the waveguide properties of a curved shell of revolution may be determined by accounting for the curvedness by means of the Galerkin method and the changing cross section by means of the WKB method.

- Non-linear effects such as membrane stiffening could be of interest. The curvature on a flexible pipe may have appeared by bending a straight pipe within its elastic range and then more or less statically fixing it in that curved configuration. In such case the pipe wall will be pre-stressed by compression on the inside of the curvature and tension on the outside (, and the cross section will be forced into an oval shape). These pre-stresses will affect the waves travelling along the pipe, but a non-linear analysis is required to determine how much.

- Experimental validation, in particular experimental validation of the localised amplitude peaks in the shell of revolution will be appreciated.

2.4 Review History

Paper A – Elastic Wave Propagation in Curved Flexible Pipes

- **12. September 2014:** Paper submitted to International Journal of Solids and Structures

- **22. February 2015:** Paper accepted with minor revisions.

- **18. March 2015:** Paper resubmitted.

Paper B – A WKB Approximation of Elastic Waves Travelling on a Shell of Revolution

- **22. April 2015:** Paper submitted to Journal of Sound and Vibration.

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Paper A

Elastic Wave Propagation in Curved Flexible Pipes

Jonas Morsbøl, Sergey V. Sorokin
Department of Mechanical and Manufacturing Engineering, Aalborg University, Fibigerstraede 16, DK-9220 Aalborg East, Denmark

The paper is submitted and under review in *International Journal for Solids and Structures*.

Paper B

A WKB Approximation of Elastic Waves Travelling on a Shell of Revolution

J. O. Morsbøl[1], S. V. Sorokin[1], N. Peake[2]

[1] *Department of Mechanical and Manufacturing Engineering, Aalborg University, Fibigerstraede 16, DK-9220 Aalborg East, Denmark*

[2] *cambridge University, DAMTP, Centre for Mathematical Sciences, Wilberforce Road, Cambridge, United Kingdom*

The paper is submitted and under review in *Journal of Sound and Vibration.*

SUMMARY

Pipe-like shell structures are found in a wide range of practical applications such as: Wind turbine towers, jet engines, brass instruments, and piping systems. For many of these applications, structural vibrations play an important role for their performance. This thesis will in particular focus on how the curvature on a thin-walled pipe, as well as a changing radius along a straight thin-walled pipe, affects the waveguide properties. It shows that the waveguide properties of curved pipes roughly can be divided into three regimes: The curved beam regime, the cylinder regime, and the torus regime. In the curved beam regime the waveguide properties of the pipe can be approximated by classical curved beam theory while in the cylinder regime they can be approximated by cylindrical shell theory. In the torus regime none of the two other regimes apply, and a full-blown shell model is needed. For the straight pipe with changing radius, which is known as the shell of revolution, it is found that classical rod and beam theory, to some extent, can be used to approximate the fundamental modes of the torsional, axial, and breathing wave. However, by means of the shell model some remarkable effects are predicted when even these very fundamental waves are travelling along a shell of revolution. The effects cover modal changes and excitation of localised resonances. For modes of higher order similar excitations of localised resonances are also predicted.

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