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STRUCTURAL RELIABILITY THEORY
PAPER NO. 93

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Aalborg, June 1992 Leo Thesbjerg

Contents

damaged during previous excitations, so that the strength has deteriorated, and no longer fulfils the code specifications. Active vibration control can then protect against the specified design loading for the structure which, without it, would cause a collapse.

Substantial advantages may be obtained by utilizing active control for vibration suppression, but some factors tend to favour passive structural design over active control techniques. The safety of a civil engineering structure is usually governed by its response to an infrequent event, such as a severe earthquake. As a result, an active earthquake control system would have to remain in stand-by mode for many years and perhaps several decades without being activated. The reliability of an infrequently used equipment then becomes a serious problem. An additional concern for active control systems is that the time at which the control power is most needed often coincides with the time at which failure of most utility systems can be expected. Hence, such active vibration control devices must be equipped with a local power source. In contradiction to control systems designed for safety requirements, those utilized for comfort operate relatively frequently. For continuously active controllers economics remains a major concern since energy consumption may become excessive. For these and other reasons, attention still has to be given to passive control measures, while research is performed in producing more reliable and effective active control devices.

Design of active control systems for vibration suppression of civil engineering structures may be considered as the third component in a general structural analysis, which also comprise system modelling and identification. These 3 topics are all treated in this study, but with the emphasis placed on the design of control algorithms. A concise description of model development in structural dynamics is presented, (Chapter 2). In addition, different methods for designing the control force algorithm are reviewed in this chapter. One of these employs an optimal control strategy, where the control forces are chosen such that a given performance index is minimized. Optimal control algorithms for linear structures based on a performance index being quadratic in the state vector and control forces are presented, (Chapter 3). Further to the classical solution techniques known from the literature on control theory, a new solution method based on the invariant embedding technique is presented. Implementation of active control possesses some practical problems which are discussed, (Chapter 4).

Next, the feasibility in controlling nonlinear structures has been explored, (Chapter 5). Here, the control algorithm developed from the invariant embedding technique in connection with linear structural systems is extended to include the nonlinear case. The algorithm is developed from a criterion of minimizing a quadratic performance index, but because of approximations introduced, it will only provide a suboptimal solution. Another suboptimal control law based on Pearson's equivalent linearization is presented. The two algorithms are compared by application to simulated data generated from a hysteretic single-degree-of-freedom system subjected to a Gaussian white noise, and the proposed algorithm based on the invariant embedding technique showed the best performance properties. In the remainder of Chapter 5 a control algorithm for simultaneous determination of structural parameters - a so-called adaptive controller, is presented. This control algorithm is finally validated by application to the numerical model used previously. Finally, based on the theory for optimal control of linear structures introduced experimental tests have been carried out in order to study the effectiveness of an actively tuned mass damper, (Chapter 6). In this context, a system identification technique is derived to find the parameters in the equation of motion for the experimental model.

Chapter 1

Introduction

In designing large civil engineering structures, an important consideration is prospective dynamic loadings which may include earthquake ground motion, wind gusts, severe sea states and moving vehicles, rotating and reciprocating machinery and others. Successful design of such structures requires providing for the safety and integrity of the structure, and in some cases also providing for a measure of comfort for the occupants during such events. The design task is complicated by considerable uncertainty in defining the dynamic loading which the structure and its occupants must endure. Due to these uncertainties, the civil engineering community has traditionally adopted a very conservative design approach. Thus, buildings and other structures have relied on the strength of the structure and its ability to dissipate energy in order to survive during severe dynamic loadings.

During the last 50 years, however, much attention has been given to socalled passive vibration suppression of civil engineering structures for providing structural safety. In fact, passive control devices such as base isolation systems, viscoelastic dampers and tuned mass dampers have been installed in existing structures, resulting in improved structural performance. Passive control devices utilize the property that energy dissipation mechanisms can be activated by the structure itself. However, the passive control parameters are generally designed to suppress vibration response only at a single frequency. Due to limitations in the passive control and in consequence of an increasing applicability of sophisticated analysis techniques, active control has during the last 20 years been considered for attenuating vibrations of civil engineering structures. Unlike passive control devices, active control implies a modification of the structural vibration by means of force action provided through an external loading mechanism. Comparing active and passive control systems one of the main advantages by a properly designed active control system is that it can be effective over a wide frequency range. Besides this essential property, there are other motivating factors for making research in the field of active structural control.

Driven by economic disincentivities such as increasing relative cost for materials, by the development of high-strength materials, and by the advancement in structural analysis and design, the trend has been towards the design and constructing increasingly more slender and flexible civil engineering structures. This, in turn, could in case of large environmental loads lead to excessive vibrations, which may need to be suppressed to prevent the structure from oscillating beyond acceptable limits. The application of active control is one of the options in safeguarding such structures.

Active control systems can also be installed in existing structures which have been slightly

1.1 Review of Literature

Active vibration control of large civil engineering structures is of relatively recent novelty, but the concept of active control has been prominent in the aerospace, electrical and mechanical engineering fields for many decades. Much of the theory applied by civil engineering researchers is rooted in classical and modern control techniques, which basically have been developed within these fields. However, its application to civil engineering structures is unique in many ways and possesses new problems to be solved. It is the purpose of what follows to review briefly some of the scientific work accomplished within active structural control, and one of the basic problems within this field is dealt with.

The concept of active structural control was first presented to the structural engineering profession by Yao (1972), who made a more rigorous formulation of the structural control problem on the basis of the control theory developed at that time. Since he suggested the employment of appropriate applied forces to reduce structural vibrations and thus, improve the safety and reliability of a structure, comprehensive research has taken place according to his idea. Most of the work done has concentrated on developing different control algorithms and making numerical documentation, assuming ideal conditions under which active control is implemented. Experimental verification has, however, only been performed to a limited extent.

The most essential component in an active control device is perhaps the actuator used to transform the supplied external energy into control forces. Naturally, a number of different actuators have been proposed and one type is the *active tendon control.* This system consists of a set of prestressed tendons connected to the structure and to one or more electro-hydraulic servo-mechanisms. The tension in the tendons is then regulated by means of the electro-hydraulic servo-mechanism, and if the tendons are connected to suitable locations of the structure this system will be able to reduce vibrations. Active tendon control has been studied both analytically by Abdel-Rohman and Leipholz (1978a,1979,1981), Abdel-Rohman (1987a), Cheng and Pantelides (1987), Roorda (1975), Yang (1975), Yang and Giannopulos (1978), Yang and Mingchien (1982), Yang and Samali (1983a), Yang and Lin (1983b), Yang et al. (1987), and experimentally by Chung et al. (1988,1989).

Another category is the *active tuned mass damper.* This control mechanism consist of a conventional tuned mass damper, McNamara (1977), for which the secondary mass is connected to an electro-hydraulic servo-mechanism. By application of active tuned mass dampers it is provided that the combined control device operates like a tuned mass damper when there is no control force. Hence, both the inertia forces resulting from motions of the secondary mass and the generated control force are used to suppress the structural vibrations. A series of feasibility studies of active mass dampers has been made, Abdel-Rohman and Leipholz (1983), Abdel-Rohman (1987a), Chang and Soong (1980b), Cheng and Pantelides (1987), Samali et al. (1985), Yang and Mingchien (1982), Yang and Samali (1983a), Yang and Lin (1983b), Yang et al. (1987). A semi-active mass damper has been proposed by Hrovat et al. (1983) to control wind-induced vibrations in tall buildings. This control device is a tuned mass damper augmented by a special time-varying damper in which the damper parameter is varied with time and, as such, requires only a relatively small amount of active energy to modulate the damping. Another semi-active mass damper has been proposed by Desanghere and Vansevenant (1991), where the stiffness is adjusted.A few tall buildings have been built with active tuned mass dampers with the primary intention to reduce wind-induced vibrations. In the 274 m high Citicorp Center in New York an actively controlled secondary mass damper of 400 tons has been installed on the top of the building. In Tokyo, Japan an active mass driver system has been installed on the top floor of the eleven storey Kyobashi Seiwa building capable of controlling vibrations due to strong wind or moderate earthquakes. This actuator is a pendulum-type dual-mass system, where one mass, weighting approximately four tons, is used for lateral motion control, and a second mass, weighting approximately one ton, is used for torsional control.

The required excitation for vibration suppression may also be generated by *pulse actuators*, where a pulse force of relatively short duration is produced by the release of air jets from a gas pulse generator. By analytical investigations control algorithms have been developed to determine when the pulses are to be triggered and with what magnitude, Masri et al. (1982), Reinhorn et al. (1987), Udwadia and Tabaie (1981a,1981b) and experimental studies have been carried out by Miller et al. (1988).

Wind-induced vibrations can be reduced by mounting an *aerodynamic appendage* on a suitable place of the particular structure. Using a simple on-off control scheme, the appendage is fully extended when the velocity of the structure at the controller location is in the direction against the wind, and fully closed when the velocity is in the opposite direction. This control scheme has been tested experimentally with an appendage situated at the top of a scaled-down tall structure, Soong and Skinner (1981). However, as concluded by Abdel-Rohman (1987a) from a numerical analysis, its feasibility is questionable for tall buildings. At present, systems are under development for control of flutter vibrations of suspension bridges, where the wind flow around the oscillating bridge is controlled by aerodynamic appendages in a way, that the lift forces from the wind always oppose the velocity of the bridge section.

Finally, the application of *piezo-electric actuators* has been proposed for controlling flexible beams. The piezo-electric actuator is bonded to a beam and by varying an electric field applied across this film, it will expand or contract. Hence, it is possible to bend a beam in a manner very similar to a bimetallic thermostat. This control device has been verified analytically, Baz and Poh (1988), and by experimental tests with cantilevered beams with a maximum length of 1.2m, Baz and Poh (1990), Bailey and Hubbard (1985).

The feasibility of using active vibration control has been explored for different types of structures. In many cases the investigations have been accomplished with a certain combination of actuator and structure. But almost to the same extent, it is simply assumed in the numerical analysis that the required control force is available, not including reflections of how to generate the forces. In the formulation of a control strategy many researchers do not specify the structure, but simply consider a beam type structure, Abdel-Rohman et al. (1980a), Baba et al. (1989), Baz and Poh (1988), Ebrahimi (1989b), Meirovitch et al. (1984b), Sadek et al. (1987,1988), Sloss et al. (1988), Soong and Skinner (1981), Yang and Giannopulos (1978), King et al. (1991) Schäfer and Holzach (1985). Active control of a membrane has been considered by Sadek and Adali (1984) and of plate by Luzzato (1983). However, specific structures have also been modelled in the development of control strategies, e.g. tall buildings, Abdel-Rohman (1982,1987a), Baruh (1987), Cheng and Pantelides (1987), Roorda (1975), Yang and Mingchien (1982), Yang and Samali (1983a), and bridges, Abdel-Rohman and Leipholz (1978a,1978b,1980b,1981), Abdel-Rohman and Nayfeh (1987b). A simultaneous design of structures and active control devices has also been considered by some researchers, Bendsøe et al. (1987), Grandhi (1987).

The objective of an active control system for civil engineering structures is, basically, to reduce vibrations caused by environmental loads. Hence, the control force is needed during the period of intense external excitations. This is a unique problem compared with the classical control theory, where the intention usually is to control eigenvibrations caused by non-zero initial conditions of the displacement and/or velocity. However, the latter type of problem has also been considered by control of flexible structures, Meirovitch et al. (1983a), Sadek et al. (1988), Sinha (1988), and the corresponding problem with forced vibrations due to a deterministic external loading, Abdel-Rohman and Leipholz (1980b,1984a), Abdel-Rohman (1987c), Baba et al. (1989), Martin and Soong (1976), Roorda (1975). Against it, the environmental loads of practical concern are generally of stochastic nature. With such random external excitations control schemes have been developed to reduce vibrations from earthquake ground motion, Chung et al. (1988,1989), Meirovitch and Silverberg (1983b), Puh and Hsu (1988), Pu and Kelly (1991), Suhardjo and Spencer (1990), Yang (1975), Yang and Lin (1983b), Yang et al. (1987), from wind loadings, Abdel-Rohman (1987a), Cheng and Pantelides (1987), Meirovitch and Ghosh (1987), Sae-Ung and Yao (1978), Soong and Skinner (1981), Yang (1975), Yang and Giannopulos (1978), Yang and Samali (1983a), Thesbjerg (1990), Wong and Luco (1991), and a single concentrated moving load on a bridge deck with irregularities, Abdel-Rohman and Leipholz (1981).

In the brief review given above, the most obliging proposals in application of active structural control have been mentioned with respect to the choice of actuator, type of civil engineering structure to protect against excessive vibrations, and type environmental load causing the undesired vibrations. A prospect, which is, however, not mentioned, is active vibration control of off-shore structures subjected to severe sea states. More examples and references to literature may be found in review papers by Yang and Soong (1988), Soong (1988), Reinhorn and Manolis (1985,1989), Spencer and Suhardjo (1990), and in a book by T.T. Soong (1990). Concerning the proposed control strategies and the applied theory, further references are given in each chapter of this thesis.

Chapter 2

Modelling and Design of Actively Controlled Structures

Control systems for active vibration control can generally be classified into 3 groups. The distinction between these classes refers to which kind of information is used to make continuous correction to the applied control force. When only variables describing the structural behaviour are measured, the control configuration is referred to as *closed-loop control.* If measurements of the external loadings are utilized an *open-loop control* results. The terms feedback and feedforward control are also used in the literature to designate the two categories. According to the former specification the case where information on both the response quantities and excitation is utilized for control design, the term *open-closed loop control* is used. A general representation of such a control system is given in fig. 2.1.

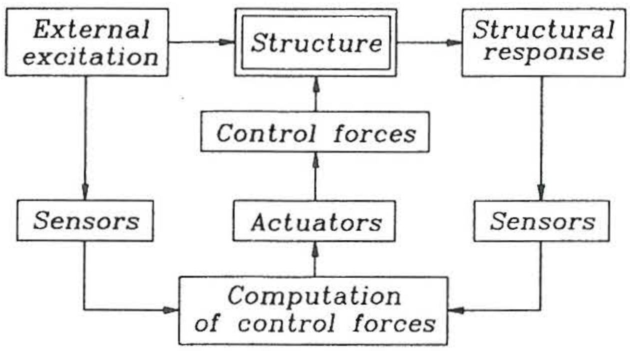

Figure 2.1: Schematic diagram of open-closed loop control system.

The advantage of a closed-loop system compared to an open-loop is that it is often more able to cope with unexpected external excitations and uncertainties about the structural behaviour. However, it need not be true that closed-loop control is always superior to open-loop control. When the measured response has sufficiently large errors, and when the unexpected external excitations are relatively unimportant, closed-loop control can have a performance which is inferior to open-loop control.

In active vibration control of building structures a lacking knowledge of the excitations subjected to the structure will often be dominant compared to measurement errors in the structural response. Therefore, closed-loop control is usually preferred, and if it is possible to measure the external excitations the open-closed loop control may be utilized to make use of as much information as possible.

The objective of this chapter is to make a mathematical representation of a control system for active vibration control. The kernel of such a model is an equation of motion for the structure. In section 2.1 both a distributed and discrete parameter model are set up. In preparation for the following design and analysis of the control system the equations of motion are represented in state space form. Section 2.2 reviews the most common performance criteria for design of active vibration control systems.

2.1 Structural Models

In order to suppress vibrations by active control a system model is needed to describe the structural behaviour under external loadings, and the interconnection of the structure and the actuators. The method used here to develop such a model, consists of systematic application of basic physical laws to components of the structural system and the interconnection between these components.

Basically, civil engineering structures are distributed parameter continuous systems. Hence, the structural model will be obtained by considering a 3-dimensional continuously distributed structure, cf. fig. 2.2

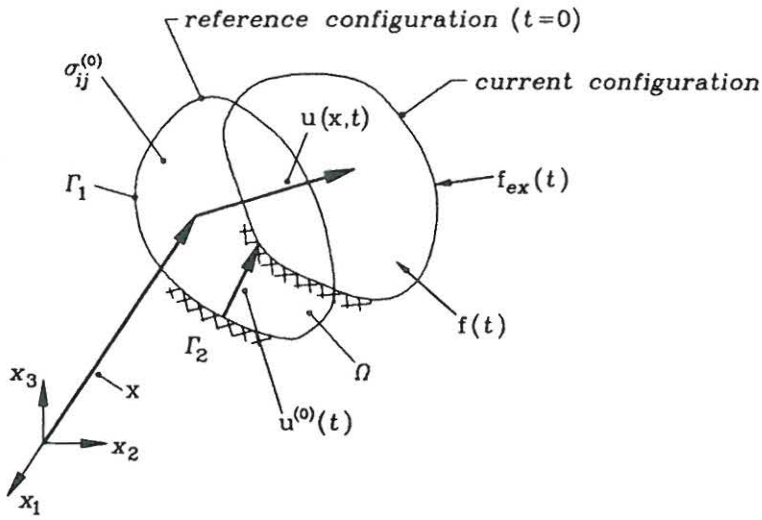

Figure 2.2: Reference and current configuration of an arbitrarily distributed parameter structure to be actively controlled.

To identify the motion of the structure, we introduce a fixed rectangular Cartesian coordinate system with origon O. Time is measured from a fixed reference time $t = 0$. Suppose that at $t = 0$ the structure occupies a fixed region of space Ω which has the boundary surface Γ_S. The position occupied by each particle of the structure at $t = 0$ is identified by the spatial coordinates $\mathbf{x}^T = [x_1\ x_2\ x_3]$ in the chosen coordinate system. The reference configuration is assumed to be a statical equilibrium state. Generally, boldface type denote a vector or a matrix.

On part of the surface Γ_1 mechanical boundary conditions are prescribed, while geometrical boundary conditions are given for the remaining part Γ_2 of the entire surface.

Suppose that, at $t = 0$, the structure is subjected to a time-varying displacement $\mathbf{u}^{(0)}(\mathbf{x}, t) = [u_1^{(0)}(\mathbf{x}, t)\; u_2^{(0)}(\mathbf{x}, t)\; u_3^{(0)}(\mathbf{x}, t)]$ of all particles at Γ_2. Simultaneously, the structure is subjected to the external dynamic forces per unit volume of the reference configuration $f_{ex,i}(\mathbf{x}, t)$. These excitations lead to a motion of the structure. It is assumed that the displacement of the material point occupying the position at point $\mathbf{x}_i \in \Omega \cup \Gamma_1$ at the reference time $t = 0$ is given by $u_i(\mathbf{x}, t)$. This displacement defines the current configuration at the time t. To counteract the motion a distributed system of control forces per unit volume of the referential configuration $f_i(\mathbf{x}, t)$ is activated.

In the statical reference configuration at the time $t = 0$, the stress configuration in the interior of the structure is given by the components $\sigma_{ij}^{(0)}$, which may be caused by residual stresses and external statical forces per unit volume, $f_i^{(0)}(\mathbf{x})$, of the reference state. As a consequence of dynamic excitations the stress is increased by $\Delta\sigma_{ij}$ leading to the total stress configuration

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \Delta\sigma_{ij} \tag{2.1}$$

In (2.1) the components σ_{ij} are interpreted as elements of the *second Piola-Kirchoff* stress tensor. The referential description of the Cauchy's equation of motion then becomes, see e.g. Spencer (1980)

$$\frac{\partial}{\partial x_j}\left[\left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right)\sigma_{kj}\right] + f_i^{(0)}(\mathbf{x}) + f_{ex,i}(\mathbf{x}, t) + f_i(\mathbf{x}, t) - f_{d,i}(\mathbf{x}, t)$$
$$= \rho(\mathbf{x})\frac{\partial^2}{\partial t^2}u_i(\mathbf{x}, t) \quad , \qquad \forall\,(\mathbf{x}, t) \in \Omega \times]0, \infty[\tag{2.2}$$

In (2.1) and below the index summation conventions will be applied to the indices $i, j, \ldots$. The symbol δ_{ij} is the Kronecker delta. To account for energy dissipation in the material besides possible hysteresis a structural damping force per unit volume of the reference state $-f_{d,i}(\mathbf{x}, t)$ has been included. $\rho(\mathbf{x})$ is the mass density in the reference configuration.

The mechanical boundary conditions are formulated in analogy with (2.2) by assuming that there is no surface traction. This leads to

$$\left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right)\sigma_{kj}n_j = 0 \quad , \qquad \forall\,(\mathbf{x}, t) \in \Gamma_1 \times]0, \infty[\tag{2.3}$$

where n_j are components of the outward normal unit vector to the undeformed surface.

In the reference configuration the structure is assumed to be in a static equilibrium state and hence it fulfils the equation of equilibrium

$$\frac{\partial}{\partial x_j}\sigma_{ij}^{(0)} + f_i^{(0)}(\mathbf{x}) = 0 \quad , \qquad \forall\,\mathbf{x} \in \Omega \tag{2.4}$$

and the mechanical boundary conditions

$$\sigma_{ij}^{(0)} n_j = 0 \quad , \qquad \forall\, \mathbf{x} \in \Gamma_1 \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.2) and (2.3), respectively, provides the incremental equations

$$\frac{\partial}{\partial x_j}\left[\Delta\sigma_{ij} + \frac{\partial u_i}{\partial x_k}\left(\sigma_{kj}^{(0)} + \Delta\sigma_{kj}\right)\right] + f_{ex,i}(\mathbf{x},t) + f_i(\mathbf{x},t) - f_{d,i}(\mathbf{x},t)$$
$$= \rho(\mathbf{x})\frac{\partial^2}{\partial t^2}u_i(\mathbf{x},t) \quad , \qquad \forall\,(\mathbf{x},t) \in \Omega \times\,]0,\infty[\tag{2.6}$$

$$\Delta\sigma_{ik}n_k + \frac{\partial u_i}{\partial x_j}\left(\sigma_{jk}^{(0)} + \Delta\sigma_{jk}\right)n_k = 0 \quad , \qquad \forall\,(\mathbf{x},t) \in \Gamma_1 \times\,]0,\infty[\tag{2.7}$$

The prescribed displacement on the subsurface Γ_2 is represented by

$$u_i(\mathbf{x},t) = u_i^{(0)}(t) \quad , \qquad \forall\,(\mathbf{x},t) \in \Gamma_2 \times\,]0,\infty[\tag{2.8}$$

As seen it has been assumed, that all points on the surface Γ_2 are subjected to the same displacement $u_i^{(0)}(t)$. Finally, the following initial conditions at $t = 0$ have to be satisfied

$$u_i(\mathbf{x},0) = 0 \quad , \quad \dot{u}_i(\mathbf{x},0) = 0 \quad , \qquad \forall\,\mathbf{x} \in \Omega \cup \Gamma_1 \tag{2.9}$$

For a specific problem the unknown stresses $\Delta\sigma_{ij}$ in (2.6) and (2.7) are eliminated by introducing appropriate constitutive laws and compatibility relations. Likewise, a constitutive model is introduced for $f_{d,i}(\mathbf{x},t)$ relating this quantity to the deformation state of the structure. Inserting these models into eqs. (2.6) and (2.7) a system model is obtained for an actively controlled continuum of material under external excitations, i.e. displacements and forces supplied by the environments.

2.1.1 Linear Structural Systems

A common approach in active control of civil engineering structures is to describe the motion by a linear mathematical model. Such a model is appropriate for small deformations of the structure, whose achievement is the aim of the control procedure.

To develop a linear model the constitutive equation for a linear elastic solid is introduced

$$\Delta\sigma_{ij}(\mathbf{x},t) = E_{ijkl}\big(\mathbf{x},\sigma_{mn}^{(0)}\big)\,\Delta\varepsilon_{kl}(\mathbf{x},t) \quad , \qquad \forall\,(\mathbf{x},t) \in \Omega \cup \Gamma_1 \times\,]0,\infty[\tag{2.10}$$

where $\Delta\varepsilon_{kl}$ and E_{ijkl} are, respectively, the components of the incremental strain tensor and the elasticity tensor. E_{ijkl} is the tangent modulus which depends on the linearization point $\sigma_{mn}^{(0)}$ in the referential configuration. Since $\Delta\sigma_{ij}$ and $\Delta\varepsilon_{kl}$ are symmetric tensors the number of independent elastic constants in E_{ijkl} is at most 21. According to an assumption of small deformations the incremental strain tensor is defined as

$$\Delta\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad , \qquad \forall\ (\mathbf{x},t)\ \in\ \Omega \cup \Gamma_1 \times\ ]0,\infty[\tag{2.11}$$

Finally, a linearization of (2.6) and (2.7) followed by an insertion of (2.10) and (2.11) yields

$$Lu_i + \rho(\mathbf{x})\frac{\partial^2 u_i}{\partial t^2} = f_{ex,i}(\mathbf{x},t) + f_i(\mathbf{x},t) - f_{d,i}(\mathbf{x},t) \quad , \quad \forall\ (\mathbf{x},t)\ \in\ \Omega \times\ ]0,\infty[\tag{2.12}$$

$$B_1 u_i = 0 \quad , \qquad \forall\ (\mathbf{x},t)\ \in\ \Gamma_1 \times\ ]0,\infty[\tag{2.13}$$

L is a self-adjoint differential operator of order 2 expressing the stiffness. L is defined as

$$Lu_i = -\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_k}\sigma_{kj}^{(0)} + E_{ijkl}\frac{\partial u_k}{\partial x_l}\right) \tag{2.14}$$

and B_1 is a differential operator of order 1, expressing the mechanical boundary conditions. B_1 is defined as

$$B_1 u_i = \left(\frac{\partial u_i}{\partial x_k}\sigma_{kj}^{(0)} + E_{ijkl}\frac{\partial u_k}{\partial x_l}\right) n_j(\mathbf{x}) \tag{2.15}$$

For the damping forces per unit volume $f_{d,i}(\mathbf{x},t)$ a linear viscous model is assumed in the form

$$f_{d,i}(\mathbf{x},t) = c(\mathbf{x})\left(\dot{u}_i(\mathbf{x},t) - \dot{U}_i^{(0)}(\mathbf{x},t)\right) \quad , \qquad (\mathbf{x},t)\ \in\ \Omega \times\ ]0,\infty[\tag{2.16}$$

where $c(\mathbf{x})$ is a non-negative function. $U_i^{(0)}(\mathbf{x},t)$ is a quasi-static displacement field caused by the boundary displacements $u_i^{(0)}(t)$. $U_i^{(0)}(\mathbf{x},t)$ is then determined by the following boundary value problem

$$LU_i^{(0)} = 0 \quad , \qquad \forall\ (\mathbf{x},t)\ \in\ \Omega \times\ ]0,\infty[\tag{2.17}$$

$$B_1 U_i^{(0)} = 0 \quad , \qquad \forall\ (\mathbf{x},t)\ \in\ \Gamma_1 \times\ ]0,\infty[\tag{2.18}$$

$$U_i^{(0)}(\mathbf{x},t) = u_i^{(0)}(t) \quad , \qquad \forall\ (\mathbf{x},t)\ \in\ \Gamma_2 \times\ ]0,\infty[\tag{2.19}$$

$U_i^{(0)}(\mathbf{x},t)$ may satisfy arbitrary initial conditions at $t = 0$. Substituting (2.16) into (2.12) yields

$$Lu_i + c(\mathbf{x})\frac{\partial u_i}{\partial t} + \rho(\mathbf{x})\frac{\partial^2 u_i}{\partial t^2} = c(\mathbf{x})\frac{\partial}{\partial t}U_i^{(0)}(\mathbf{x},t) + f_{ex,i}(\mathbf{x},t) + f_i(\mathbf{x},t) \quad ,$$
$$\forall\ (\mathbf{x},t)\ \in\ \Omega \times\ ]0,\infty[\tag{2.20}$$

Equation (2.20) expresses the required equation of motion for analysing the effect of applying control forces to a linear elastic distributed parameter structure subjected to external loadings and displacements of the surface. (2.20) must be solved using the initial conditions (2.9) and the boundary conditions (2.8), (2.13).

The preceding equation of motion (2.20) implies that control forces are applied at every point of the structure. This is an ideal situation, which cannot be realized in practice. Indeed, in practice actuators tend to be discrete elements. These actuator forces can formally be treated within the presented continuous formulation by writing

$$f_i(\mathbf{x},t) = \sum_{\alpha=1}^{n_m} \delta(\mathbf{x}-\mathbf{x}_\alpha) e_{\alpha i} F_\alpha(t) \tag{2.21}$$

In (2.21) n_m is the number of discrete control forces, and $F_\alpha(t)$, $\alpha = 1, 2, \ldots, n_m$, is the discrete control force applied at the material point $\mathbf{x}_\alpha^T = [x_{\alpha 1}\ x_{\alpha 2}\ x_{\alpha 3}]$ in a direction determined by the unit vector $\mathbf{e}_\alpha^T = [e_{\alpha 1}\ e_{\alpha 2}\ e_{\alpha 3}]$. $\delta(\mathbf{x}-\mathbf{x}_\alpha) = \delta(x_1 - x_{\alpha 1})\delta(x_2 - x_{\alpha 2})\delta(x_3 - x_{\alpha 3})$ is a spatial Dirac delta function.

2.1.2 Discretized Linear Structural Systems

To simplify the design of an active control system modelled by partial differential equations as (2.20), these will often be approximated by discrete-parameter models described by ordinary differential equations. For a discrete-parameter system all energy storage or energy dissipation is lumped into a finite number of discrete spatial locations.

The differential operators L in (2.14) and B_1 in (2.15) can be discretized in various ways, e.g. by a finite element scheme or by a finite difference scheme. In both cases the initial and boundary value problem formulated by (2.8), (2.9), (2.13) and (2.20) is transformed into a system of ordinary differential equations. Using (2.21), this yields

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}\dot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{C}\dot{\mathbf{U}}^{(0)}(t) + \mathbf{F}_{ex}(t) + \mathbf{I}_1\mathbf{F}(t) \quad , \qquad t > 0 \tag{2.22}$$
$$\mathbf{u}(0) = \mathbf{0} \quad , \quad \dot{\mathbf{u}}(0) = \mathbf{0}$$

$\mathbf{u}(t)$ is an n-dimensional vector of the absolute displacements and rotations in the n degrees of freedom of the structure. $\mathbf{U}^{(0)}(t)$ is an n-dimensional vector specifying the quasi-static displacements and rotations relative to the reference configuration, caused by the base motion.

$\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ signify mass, damping, and stiffness matrices, respectively, all of dimensions $n \times n$. $\mathbf{F}_{ex}(t)$ is an n-dimensional vector of equivalent nodal forces obtained by discretizing the distributed external loading, and $\mathbf{F}(t)$ is an n_m-dimensional vector of the control forces $F_\alpha(t)$. $\mathbf{I}_1$ is a constant influence matrix of dimension $n \times n_m$ in which column α states the nodal loading in the degrees of freedom $\mathbf{u}$, when a unit force is applied at point $\mathbf{x}_\alpha$ in the direction $\mathbf{e}_\alpha$.

Let $\mathbf{u}^{(0)}(t)$ be an l-dimensional vector of nodal motions of the supports. On account of the linearity, we have

$$\mathbf{U}^{(0)}(t) = \mathbf{I}_2\mathbf{u}^{(0)}(t) \tag{2.23}$$

where $\mathbf{I}_2$ is a constant influence matrix of dimension $n \times l$. Column i states the displacement in the degrees of freedom $\mathbf{u}$, when $u_i^{(0)}$ is equal to 1, and the remaining components is equal to 0. $\mathbf{I}_2$ is determined by conventional statical methods.

The relative displacement $\mathbf{v}(t)$ with respect to the quasi-static displacement $\mathbf{U}^{(0)}(t)$ is given by

$$\mathbf{v}(t) \;=\; \mathbf{u}(t) \;-\; \mathbf{U}^{(0)}(t) \tag{2.24}$$

The quasi-static displacements and rotations $\mathbf{U}^0(t)$ caused by the boundary motions $\mathbf{u}^{(0)}(t)$ must in analogy with (2.17) satisfy the condition $\mathbf{K}\mathbf{U}^0(t) = \mathbf{0}$. Using this fact and substituting (2.23) and (2.24) into (2.9), the equation of motion becomes

$$\begin{aligned} &\mathbf{M}\ddot{\mathbf{v}} + \mathbf{C}\dot{\mathbf{v}} + \mathbf{K}\mathbf{v} \;=\; -\mathbf{M}\mathbf{I}_2\ddot{\mathbf{u}}^{(0)}(t) + \mathbf{F}_{ex}(t) + \mathbf{I}_1\mathbf{F}(t) \quad , \qquad t > 0 \\ &\mathbf{v}(0) = -\mathbf{I}_2\mathbf{u}^{(0)}(0) \quad , \quad \dot{\mathbf{v}}(0) = -\mathbf{I}_2\dot{\mathbf{u}}^{(0)}(0) \end{aligned} \tag{2.25}$$

These equations may, for example, model the behaviour of a frame-like structure subjected to an earthquake and discrete external forces.

2.1.3 State Space Representation

In the design and analysis of control systems the system equations will here be arranged as a set of equivalent first order differential equations. Use of this concept allows one to treat a distributed parameter system with the simple uniform notation

$$\begin{aligned} &\frac{\partial \mathbf{y}(\mathbf{x},t)}{\partial t} = \mathcal{L}\left(\mathbf{y}(\mathbf{x},t)\right) + \mathbf{b}_0(\mathbf{x},t)\mathbf{w}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t)\mathbf{f}(\mathbf{x},t) \quad , \qquad \forall\ (\mathbf{x},t)\ \in\ \Omega \times\ ]0,\infty[\\ &\mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}) \quad , \qquad \forall\, \mathbf{x}\ \in\ \Omega \end{aligned} \tag{2.26}$$

where $\mathbf{y}(\mathbf{x},t)$ is the state vector and $\mathcal{L}(\cdot)$ is a matrix differential operator of the order n with respect to $\mathbf{x}$, which in general may be non-linear. $\mathbf{w}(\mathbf{x},t)$ is the external excitation vector and $\mathbf{f}(\mathbf{x},t)$ is the applied control force vector. $\mathbf{w}(\mathbf{x},t)$ represents both the applied external load distributed over the domain Ω and the prescribed displacement of the surface Γ_2. $\mathbf{b}_0(\mathbf{x},t)$ and $\mathbf{b}(\mathbf{x},t)$ are coefficient matrices conjugating, respectively, the external excitations and the control forces to the state vector.

The kinematic and mechanical boundary conditions may be represented by the vector equations

$$\mathcal{G}_i\left(\mathbf{y}(\mathbf{x},t)\right) = \mathbf{0} \quad , \qquad i = 0,\ldots,n-1 \quad , \quad \forall\ (\mathbf{x},t)\ \in\ \Gamma \times\ ]0,\infty[\tag{2.27}$$

where $\mathcal{G}_i(\cdot)$ is in general a nonlinear vectorial spatial differential operator whose parameters may depend on $\mathbf{x}$ and t. These are ordered in ascending order of differentiability, so that $\mathcal{G}_i(\cdot)$ is a differential operator of order i. The indicated initial and boundary value problem is applicable to one, two and three-dimensional continua as well.

For many structures the equation of motion can be approximated by a linear mathematical model. In such cases the governing partial differential equations can be expressed in the following general form

$$\frac{\partial \mathbf{y}(\mathbf{x},t)}{\partial t} = \mathcal{A}(\mathbf{x},t)\mathbf{y}(\mathbf{x},t) + \mathbf{b}_0(\mathbf{x},t)\mathbf{w}(\mathbf{x},t) + \mathbf{b}(\mathbf{x},t)\mathbf{f}(\mathbf{x},t) \ , \quad \forall\ (\mathbf{x},t) \in \Omega \times]0,\infty[$$
$$\mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}) \quad , \qquad \forall\, \mathbf{x} \in \Omega \tag{2.28}$$

where $\mathcal{A}(\cdot)$ is a matrix, linear spatial differential operator.

For the linearly distributed parameter structure considered in section 2.1.1 it is quite easy to rearrange the governing partial differential equation (2.20) into state space form like (2.28),

$$\mathbf{y}(\mathbf{x},t) = \begin{bmatrix} \mathbf{u}(\mathbf{x},t) \\ \dot{\mathbf{u}}(\mathbf{x},t) \end{bmatrix} \quad , \quad \mathcal{A}(\mathbf{x}) = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\frac{1}{\rho(\mathbf{x})}\mathbf{L} & -\frac{c(\mathbf{x})}{\rho(\mathbf{x})}\mathbf{I} \end{bmatrix}$$
$$\mathbf{b}_0(\mathbf{x}) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \frac{c(\mathbf{x})}{\rho(\mathbf{x})}\mathbf{I} & \frac{1}{\rho(\mathbf{x})}\mathbf{I} \end{bmatrix} \quad , \quad \mathbf{w}(\mathbf{x},t) = \begin{bmatrix} \dot{\mathbf{U}}^{(0)}(\mathbf{x},t) \\ \mathbf{f}_{ex}(\mathbf{x},t) \end{bmatrix} \quad , \quad \mathbf{b}(\mathbf{x}) = \begin{bmatrix} \mathbf{0} \\ \frac{1}{\rho(\mathbf{x})}\mathbf{I} \end{bmatrix} \tag{2.29}$$

$\mathbf{u}(\mathbf{x},t)$, $\mathbf{f}_{ex}(\mathbf{x},t)$ and $\dot{\mathbf{U}}^{(0)}(\mathbf{x},t)$ are vectors of the corresponding components $u_i(\mathbf{x},t)$, $f_{ex,i}(\mathbf{x},t)$ and $\dot{U}_i^{(0)}(\mathbf{x},t)$. $\mathbf{Lu}(\mathbf{x},t)$ is a vector of the components $Lu_i(\mathbf{x},t)$, and $\mathbf{I}$ and $\mathbf{0}$ are, respectively, a unit matrix and a zero matrix of order 3×3.

The differential operators representing the boundary conditions (2.13) and (2.8) are given as

$$\mathcal{G}_0\mathbf{y}(\mathbf{x},t) = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{y}(\mathbf{x},t) - \mathbf{u}^{(0)}(t) \quad , \qquad \forall\ (\mathbf{x},t) \in \Gamma_2 \times]0,\infty[\tag{2.30}$$

$$\mathcal{G}_1\mathbf{y}(\mathbf{x},t) = \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} \end{bmatrix} \mathbf{y}(\mathbf{x},t) \quad , \qquad \forall\ (\mathbf{x},t) \in \Gamma_1 \times]0,\infty[\tag{2.31}$$

where $\mathbf{B}_1\mathbf{u}(\mathbf{x},t)$ is a vector with components $B_1u_i(\mathbf{X},t)$, and $\mathbf{u}^{(0)}(t)$ is a vector with components $u_i^{(0)}(t)$.

Considering a general non-linear discrete-parameter structure the equation of motion can be written in state space notation as

$$\dot{\mathbf{Y}}(t) = \mathbf{G}(\mathbf{Y}(t),t) + \mathbf{B}_0(t)\mathbf{W}(t) + \mathbf{B}(t)\mathbf{F}(t) \quad , \qquad \mathbf{Y}(0) = \mathbf{Y}_0 \tag{2.32}$$

where $\mathbf{Y}(t)$ is the state vector and $\mathbf{G}$ is a vector function accounting for structural dynamics. $\mathbf{F}(t)$ is a vector of control forces and $\mathbf{W}(t)$ is a vector of external excitations. $\mathbf{B}(t)$ and $\mathbf{B}_0(t)$ are general time-dependent coefficient matrices conjugating the control forces and external excitations to the state vector.

The nonlinearity of a structural model may for instance be due to a hysteretic restoring force. On condition that the constitutive law for the hysteretic component of the restoring force can be written in differential form then the model (2.32) may represent such a system. Furthermore, the governing differential equation (2.32) may include a safety or damage measure for the structure given in differential form. Sometimes, it is convenient to view the external loading as being generated by a dynamic system represented in differential form, which may also be included in (2.32). Hence, the state vector of an augmented structural model may include the state variables of a shaping filter generating the environmental loading, of a hysteretic system, and of a safety or damage measures.

A discretized linear structural system can be written in state space form as

$$\dot{\mathbf{Y}}(t) = \mathbf{A}(t)\mathbf{Y}(t) + \mathbf{B}_0(t)\mathbf{W}(t) + \mathbf{B}(t)\mathbf{F}(t) \quad , \qquad \mathbf{Y}(0) = \mathbf{Y}_0 \tag{2.33}$$

where $\mathbf{A}(t)$ is a general time-varying system matrix. For the discretized structural system considered in section 2.1.2, we have

$$\mathbf{Y}(t) = \begin{bmatrix} \mathbf{v}(t) \\ \dot{\mathbf{v}}(t) \end{bmatrix} \; , \; \mathbf{Y}_0 = \begin{bmatrix} -\mathbf{I}_2\mathbf{u}^{(0)}(0) \\ -\mathbf{I}_2\dot{\mathbf{u}}^{(0)}(0) \end{bmatrix} \; , \; \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}$$

$$\mathbf{B}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\mathbf{I}_2 & \mathbf{M}^{-1} \end{bmatrix} \; , \; \mathbf{W}(t) = \begin{bmatrix} \ddot{\mathbf{u}}^{(0)}(t) \\ \mathbf{F}_{ex}(t) \end{bmatrix} \; , \; \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{I}_1 \end{bmatrix} \tag{2.34}$$

where $\mathbf{0}$ and $\mathbf{I}$ denote, respectively, the null matrix and the identity matrix of appropriate dimensions.

2.2 Performance Specifications

Based on the developed equations of motion it is possible to see the effect of applying control forces to a given structure on ideal conditions. Consider a building structure modelled by the equation of motion (2.25) and represented in state space form by (2.33) and (2.34). Suppose that the open-closed loop configuration is used in which the control force is designed to be a linear function of the measured state vector $\mathbf{Y}(t)$ and the measured excitation vector $\mathbf{W}(t)$. The control force vector then takes the form

$$\mathbf{F}(t) = -\mathbf{G}_c(t)\mathbf{Y}(t) + \mathbf{I}_c(t)\mathbf{W}(t) \tag{2.35}$$

where $\mathbf{G}_c(t) = [\mathbf{K}_c(t)\ \mathbf{C}_c(t)]$ and $\mathbf{I}_c(t) = [\mathbf{I}_{c1}(t)\ \mathbf{I}_{c2}(t)]$ are respective gains which in general are time-dependent. Substituting (2.35) into (2.33) and rearranging yields

$$\mathbf{M}\ddot{\mathbf{v}} + (\mathbf{C} + \mathbf{I}_1\mathbf{C}_c)\,\dot{\mathbf{v}} + (\mathbf{K} + \mathbf{I}_1\mathbf{K}_c)\,\mathbf{v} \;=\; (-\mathbf{M}\mathbf{I}_2 + \mathbf{I}_1\mathbf{I}_{c1})\,\ddot{\mathbf{u}}^{(0)}(t) + (\mathbf{I} + \mathbf{I}_1\mathbf{I}_{c2})\,\mathbf{F}_{ex}(t) \quad , \quad t > 0 \tag{2.36}$$

Comparing equation (2.36) with equation (2.25) in the absence of control, it is seen that the effect of closed-loop control is to modify the structural damping and stiffness. The effect of the open-loop component is a modification of the excitation. The choice of the control gain matrices $\mathbf{G}_c$ and $\mathbf{I}_c$ depends on the selected control criteria.

2.2.1 Pole Assignment

Consider the state space model given by (2.33) and (2.34). In the absence of feedback control forces, the system matrix $\mathbf{A}$ defines the open-loop system dynamics, and its eigenvalues λ_i

provide the modal damping ratios ζ_i and the angular eigenfrequencies ω_i

$$\lambda_j = -\zeta_j\omega_j \pm i\omega_j\sqrt{1-\zeta_j^2} \tag{2.37}$$

where i is the complex imaginary unit. The open-loop eigenvalues are given as complex conjugated pairs. Let the control forces be determined by closed-loop feedback, i.e.

$$\mathbf{F}(t) = -\mathbf{G}_c(t)\mathbf{Y}(t) \tag{2.38}$$

where $\mathbf{G}_c(t)$ is a general time-varying control gain matrix. Substituting (2.38) into (2.33), and assuming that the system matrices $\mathbf{A}$, $\mathbf{B}$ and the gain matrix $\mathbf{G}_c$ are constant, leads to the closed-loop equation

$$\dot{\mathbf{Y}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{G}_c)\,\mathbf{Y}(t) + \mathbf{B}_0\mathbf{W}(t) \tag{2.39}$$

in which the closed-loop system matrix becomes $\mathbf{A} - \mathbf{B}\mathbf{G}_c$. The closed-loop eigenvalues η_i obtained from the modified system matrix provide the effective modal damping ratios and natural frequencies for the controlled structure. Since η_i, in part, defines the controlled system behaviour, a feasible control strategy is to choose the control gain $\mathbf{G}_c$ in such a way that the eigenvalues η_i take a set of values prescribed by the designer. Development of control algorithms according to this procedure is generally referred to as *pole assignment.*

The application of pole assignment algorithms to the control of civil engineering structures has been studied by Baruh (1987), Abdel-Rohman (1982,1987), Abdel-Rohman and Leipholz (1978), Martin and Soong (1976), and Sinha (1988). However, in general the application of this viable form of design is limited in structural control. One reason is the lack of criteria to prescribe the desired set of closed-loop eigenvalues. Furthermore, determination of the control gain $\mathbf{G}_c$ for prescribed eigenvalues generally requires the solution of highly nonlinear algebraic equations, Meirovitch et al. (1983a). Besides, this solution is generally not unique and hence, some other criteria must be incorporated, for instance the demand for certain eigenvectors.

2.2.2 Bounded State

The objective of active control is to suppress vibrations to obtain structural safety and human comfort. However, this is not tantamount to a design criterion of total vibration elimination and hence, within some small bounds oscillations may be tolerable. If a deadband region is prescribed a control algorithm can be designed to limit the state variables within these bounds. This approach is referred to as *bounded state control.*

One way to prevent a dangerous rhythmic build-up of the structural response is to apply force pulses of suitable magnitude and proper direction at several locations distributed throughout the structure. Such a pulse control strategy falls into the category of bounded state control, Soong (1988).

Assume that, on the basis of design considerations, threshold levels for the relative displacement $v_{r,i}$, $i = 1, 2, \ldots, n_m$, have been established for the location of the control force actuator i. The relative displacement v_i at the position of actuator i is continuously monitored. When the absolute value $|v_i|$ exceeds the threshold level $v_{r,i}$ a pulse control which

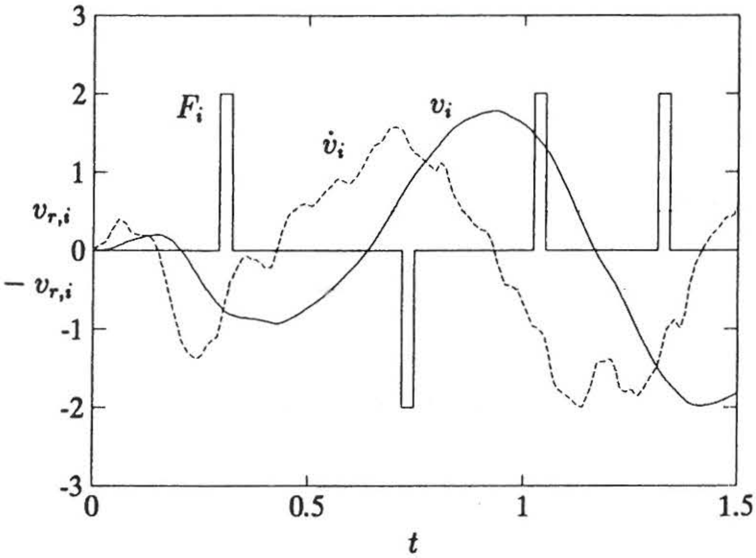

Figure 2.3: Pulse control force triggered by excess of prescribed threshold level.

opposes the velocity is applied as. The magnitude of the control pulse $F_i(t)$ is given by

$$F_i(t) = -G_i P_i(t)\text{sign}(\dot{v}_i) \quad , \qquad t_i \leq t \leq t_i + T_{d,i} \tag{2.40}$$

where

$$\text{sign}(x) = \begin{cases} 1 & , \quad x \geq 0 \\ -1 & , \quad x < 0 \end{cases} \tag{2.41}$$

G_i is the amplitude of the pulse at location i, $P_i(t)$ is the nominal pulse shape at location i, t_i is the threshold-crossing time at location i, and $T_{d,i}$ is the pulse duration of the control force at location i. The pulse amplitudes G_i are to be selected according to some criteria formulated by the designer. The pulse duration $T_{d,i}$ must be small compared to the fundamental period of the structure, e.g. $< 10\%$. Otherwise, the sign of the velocity will change during the effect of a particular pulse and hence, the vibration will be amplified.

Miller et al. (1988) and Udwadia (1981a,1981b) have determined the pulse magnitudes analytically so as to minimize a cost function which is quadratic in the state. These algorithms require that the equation of motion is given by a set of ordinary, linear differential equations with constant coefficients. However, the control algorithms based on (2.40), do not generally depend on any prior knowledge of the structural system and are not limited by nonlinearities. Masri et al. (1982) proposed a control period, which consists of application of pulses every time a zero-crossing of the displacement is detected to suppress vibrations of nonlinear structures. The pulse shape function is selected as a function of the instantaneous relative velocity at actuator i, $P_i(t) = |\dot{v}_i|^{n_i}$, where n_i is some appropriate power of the velocity.

Besides being suited for treatment of inelastic structures the pulse control procedures have the advantage that the control force need not be very large to counteract completely the energy being applied to the structure, Miller et al. (1988). Furthermore, the amount of control energy used is limited, since the control is only applied when the structural response exceeds a certain threshold related to the resistance of the structure.

2.2.3 Instantaneous Optimal Control

More complex design methods are concerned with developing control systems which are the best possible with respect to a standard, a so-called performance index. The category of optimal control algorithms described in this section, has been developed by Yang et al. (1987). A linear discretized structure described by the state space model (2.33) and (2.34) is considered. A category of optimal algorithms are established by using a time-dependent performance index $J(t)$ as follows

$$J(t) = \frac{1}{2}\mathbf{Y}^T(t)\mathbf{Q}(t)\mathbf{Y}(t) + \frac{1}{2}\mathbf{F}^T(t)\mathbf{R}(t)\mathbf{F}(t) \tag{2.42}$$

in which $\mathbf{Q}(t)$ is a symmetric, positive semi-definite matrix and $\mathbf{R}(t)$ is a symmetric, positive definite matrix. The control configuration is obtained by minimizing $J(t)$ at every instant of time t for all $t \in [0, T]$ subjected to the constraint given by the equations of motion (2.33). T is a fixed time to be longer than the external excitation causing the undesired vibrations. The optimal control thus developed is referred to as the *instantaneous optimal control* algorithm.

To obtain an optimal control law the evolution of the state vector $\mathbf{Y}(t)$ over a small interval Δt is first found by solving (2.33) and (2.34). The solution can be written

$$\mathbf{Y}(t) = e^{\mathbf{A}\Delta t}\mathbf{Y}(t-\Delta t) + \int_{t-\Delta t}^{t} e^{\mathbf{A}(t-\tau)}(\mathbf{B}_0\mathbf{W}(\tau) + \mathbf{B}\mathbf{F}(\tau))\mathrm{d}\tau \tag{2.43}$$

where

$$e^{\mathbf{A}t} = \boldsymbol{\Phi}e^{\boldsymbol{\Lambda}t}\boldsymbol{\Phi}^{-1} \tag{2.44}$$

$\boldsymbol{\Lambda}$ is a diagonal matrix consisting of complex eigenvalues λ_i, $i = 1, 2, \ldots, 2n$ of the system matrix $\mathbf{A}$, and $\boldsymbol{\Phi}$ is a matrix consisting of the associated eigenvectors. $e^{\boldsymbol{\Lambda}t}$ is a diagonal matrix with diagonal terms $e^{\lambda_i t}$, $i = 1, \ldots, 2n$.

Using a trapezoidal rule for the quadrature, the following difference equation is derived

$$\mathbf{Y}(t) = \mathbf{D}(t-\Delta t) + \frac{\Delta t}{2}\left[\mathbf{B}_0\mathbf{W}(t) + \mathbf{B}\mathbf{F}(t)\right] \tag{2.45}$$

where

$$\mathbf{D}(t-\Delta t) = e^{\mathbf{A}\Delta t}\left\{\mathbf{Y}(t-\Delta t) + \frac{\Delta t}{2}\left[\mathbf{B}_0\mathbf{W}(t-\Delta t) + \mathbf{B}\mathbf{F}(t-\Delta t)\right]\right\} \tag{2.46}$$

The necessary conditions for instantaneous optimal control is obtained by minimizing the performance index $J(t)$ subject to the constraint in eq. (2.45). The solution to this problem may be formulated in three different forms as an instantaneous open-loop, an instantaneous closed-loop, or an instantaneous open-closed loop solution, respectively. The control forces

$\mathbf{F}^*(t)$ that minimize (2.42) fulfil the minimum condition

$$\frac{\partial J(t)}{\partial \mathbf{F}(t)} = \mathbf{Y}^T(t)\mathbf{Q}(t)\frac{\partial \mathbf{Y}(t)}{\partial \mathbf{F}(t)} + \mathbf{F}^{*T}(t)\mathbf{R}(t)$$

$$= \mathbf{Y}^T(t)\mathbf{Q}(t)\frac{\Delta t}{2}\mathbf{B} + \mathbf{F}^{*T}(t)\mathbf{R}(t) = \mathbf{0} \quad \Rightarrow$$

$$\mathbf{F}^*(t) = -\frac{\Delta t}{2}\mathbf{R}^{-1}(t)\mathbf{B}^T\mathbf{Q}(t)\mathbf{Y}(t) \tag{2.47}$$

(2.47) represents the instantaneous optimal closed-loop control law. The instantaneous optimal open-loop control law is obtained by eliminating $\mathbf{Y}(t)$ in (2.47) by means of (2.45). By rearranging, this yields

$$\mathbf{F}^*(t) = -\left[\mathbf{R}(t) + \left(\frac{\Delta t}{2}\right)^2 \mathbf{B}^T\mathbf{Q}(t)\mathbf{B}\right]^{-1} \mathbf{B}^T\mathbf{Q}(t)\left[\mathbf{D}(t-\Delta t) + \frac{\Delta t}{2}\mathbf{B}_0\mathbf{W}(t)\right] \tag{2.48}$$

Alternatively, a control law can be derived by expressing $\mathbf{F}^*(t)$ as a weighted average of the right hand sides of (2.47) and (2.48), written as

$$\mathbf{F}^*(t) = -(1-\alpha)\left[\mathbf{R}(t) + \left(\frac{\Delta t}{2}\right)^2 \mathbf{B}^T\mathbf{Q}(t)\mathbf{B}\right]^{-1} \mathbf{B}^T\mathbf{Q}(t)\left[\mathbf{D}(t-\Delta t) + \frac{\Delta t}{2}\mathbf{B}_0\mathbf{W}(t)\right]$$
$$- \alpha\frac{\Delta t}{2}\mathbf{R}^{-1}(t)\mathbf{B}^T\mathbf{Q}(t)\mathbf{Y}(t) \quad , \qquad \alpha \in [0,1] \tag{2.49}$$

The control law obtained for the case of $\alpha = \frac{1}{2}$ is designated instantaneous optimal open-closed loop control.

It follows from the derivations that the control efficiencies for the three instantaneous optimal control algorithms are all identical under ideal conditions. On the other hand, the effectiveness will vary in practice because of measurement noise and uncertainties in the structural model and the loading. The selection of the best algorithm then depends on the relative importance of uncertainties and noise as discussed at the beginning of this chapter.

Some evaluation of these instantaneous optimal control algorithms has been carried out, both analytically and experimentally, for structures subjected to earthquake-type excitations, Yang et al. (1987a).

2.2.4 Optimal Control

The optimal control system introduced in the preceding section was developed within the last 5 years. However, optimality conditions in design of control systems have been applied during the last 30 years. According to the classical optimal control theory the design problem for a generally distributed parameter system, such as given in (2.26), is to minimize a general performance index in the form, Tzafestas and Stavroulakis (1983)

$$J[\mathbf{y},\mathbf{f}] = \int_{\Omega} L_0(\mathbf{y}(\mathbf{x},T),\mathbf{x},T)\,\mathrm{d}\mathbf{x} \;+\; \int_0^T \int_{\Omega} L_1(\mathbf{y}(\mathbf{x},t),\mathbf{f}(\mathbf{x},t),\mathbf{x},t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \tag{2.50}$$

where L_0 and L_1 are specified functions of their arguments. L_0 is the cost associated with the deviation of the terminal state from the desired state at the time T. L_1 is the cost function associated with the deviation of the transient state from the desired state and control effort. Hence, the functions L_0 and L_1 must be selected to put more or less emphasis on terminal accuracy, transient behaviour, and the expended control effort in the total cost functional J.

In active control of civil engineering structures it is usually desired to determine the system of control forces $\mathbf{f}(\mathbf{x},t)$ such that the structural response will correspond as closely as possible to the equilibrium state, when it is subjected to external excitations. In such a case the performance index is selected directly as in (2.50). This is known as the so-called *optimal regulator* problem.

The adjoint problem is to steer the structure from an initial state $\mathbf{y}_0(\mathbf{x})$ at t_0 to the equilibrium state in a minimum of time by means of an admissible control force $\mathbf{f}(\mathbf{x},t)$. This so-called *time-optimal control* problem is obtained by setting $L_0 = 0$ and using the restriction to the optimization problem $\int_{\Omega} L_1\,\mathrm{d}\mathbf{x} = 1$.

The latter performance specification would typically be used in case of a momentary pulse loading. However, in what follows the optimal regulator problem is considered, where the performance index is chosen in the classical quadratic form as

$$\begin{aligned} J[\mathbf{y},\mathbf{f}] &= \frac{1}{2}\int_{\Omega} \mathbf{y}^T(\mathbf{x},T)\mathbf{s}(\mathbf{x},T)\mathbf{y}(\mathbf{x},T)\,\mathrm{d}\mathbf{x} \\ &\quad + \frac{1}{2}\int_0^T \int_{\Omega} \left(\mathbf{y}^T(\mathbf{x},t)\mathbf{q}(\mathbf{x},t)\mathbf{y}(\mathbf{x},t) + \mathbf{f}^T(\mathbf{x},t)\mathbf{r}(\mathbf{x},t)\mathbf{f}(\mathbf{x},t)\right)\mathrm{d}\mathbf{x}\,\mathrm{d}t \end{aligned} \tag{2.51}$$

where $\mathbf{s}(\mathbf{x},t)$, $\mathbf{q}(\mathbf{x},t)$ are symmetric positive semi-definite matrix functions, and $\mathbf{r}(\mathbf{x},t)$ is a symmetric positive definite matrix. These weighting matrices are assigned according to the relative importance attached to the state variables and to the control forces in the minimization procedure. As the elements of $\mathbf{r}(\mathbf{x},t)$ increase, more weight is put on the reduction of the required control force. The opposite is true when these elements are reduced. Hence, by varying the relative magnitude of the weighting matrices, one can synthesize the controllers to achieve a proper trade-off between the control effectiveness and control energy consumption. There is no general strategy for selection of the weighting matrices.

For discrete-parameter structures described in state-space form by (2.32) a general performance index is the following

$$J[\mathbf{Y},\mathbf{F}] = L_0(\mathbf{Y}(T),T) + \int_0^T L_1(\mathbf{Y}(t),\mathbf{F}(t),t)\mathrm{d}t \tag{2.52}$$

where L_0 and L_1 are real non-negative scalar-valued functions of their arguments. A classic form of (2.52) is the quadratic performance index specified as

$$J[\mathbf{Y},\mathbf{F}] = \frac{1}{2}\mathbf{Y}^T(T)\mathbf{S}(T)\mathbf{Y}(T) + \frac{1}{2}\int_0^T \left(\mathbf{Y}^T(t)\mathbf{Q}(t)\mathbf{Y}(t) + \mathbf{F}^T(t)\mathbf{R}(t)\mathbf{F}(t)\right)\, dt \qquad (2.53)$$

The symmetric weighting matrices $\mathbf{S}(T)$, $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ in (2.53) fulfil the same definite properties as the associated weighting matrix functions in (2.51).

Control design based on the minimization of the quadratic performance indices (2.51) and (2.53) are treated in Chapter 3.

2.3 Conclusions

The behaviour of civil engineering structures subjected to external excitations and control forces has been modelled for distributed and discrete parameter systems by linear and non-linear partial and ordinary differential equations, respectively. Next, for the purpose of control design the differential equations are represented in state space form for both classes.

Active vibration control systems can be designed from different performance criteria. In general, however, the effect of closed- and open-loop control is to modify the structural parameters (stiffness and damping) and the total excitation, respectively.

The decision of which one of the mentioned performance criteria to be used may depend on the particular problem. Concerning the pole assignment method a general drawback is the lack of criteria to prescribe the desired set of poles. According to the name, algorithms falling into the category of bounded state control, are designed to limit the state variables within prescribed bounds. Such an algorithm may then be used, when the purpose is to maintain a set of structural variables within an allowable region determined by the requirements of structural safety and human comfort.

The design criteria based on the optimality conditions are divided into 2 groups, instantaneous and classical optimal control. Common to these methods is that the control forces are designed by minimizing a performance index. The difference is that the performance index is time-dependent for the instantaneous optimal control problem, whereas the performance index is the integral of a function over the entire control period for the classical problem. Hence, the latter criterion is based on information over a specified period of time and may therefore be considered to be better for structural control. However, concerning the choice of performance criterion, it is generally impossible to settle which one is the best. In the remainder of this thesis the optimal control technique has been selected, and its applicability for structural control has been investigated.

Chapter 3

Optimal Control of Linear Elastic Structures

Design of an active control system according to the optimality criteria introduced in Chapter 2 requires the solution of a constrained optimization problem. For this purpose different techniques such as variational calculus and dynamic programming are available. These techniques will be applied in this chapter to derive necessary conditions for optimality of the solution. The problem equations obtained hereby are then solved by the matrix Ricatti equation and the invariant imbedding method.

Treating the optimal control problem it is first assumed that the external loading is known during the specified time of control. Next, the optimal control problem for structural systems subjected to external loadings described by stochastic processes is considered. At the end of this chapter the problem of incomplete measurements of the state variables for feedback control is treated. In the latter case control and estimation of the state variables must be performed concurrently.

3.1 Distributed Parameter Structures

The objective is to determine a continuously distributed optimal control law which minimizes the response of a distributed parameter structure according to a specified performance index. Let the equation of motion with associated boundary conditions be represented by (2.28) and (2.27). Notice that the boundary conditions (2.27) are linear in the present case. Further, the performance index is taken to be quadratic as given by (2.51).

The problem then is to determine the optimal field of control forces $\mathbf{f}^*(\mathbf{x}, t)$ that minimizes (2.51) with respect to the restrictions (2.27) and (2.28). Using the method of Lagrangian multipliers, the optimal control is determined from the unrestricted stationarity conditions of the extended functional

$$\bar{J}[\mathbf{y}, \mathbf{f}, \boldsymbol{\lambda}] = \frac{1}{2}\int_{\Omega} \mathbf{y}^T(\mathbf{x}, T)\mathbf{s}(\mathbf{x}, T)\mathbf{y}(\mathbf{x}, T)\,\mathrm{d}\mathbf{x}$$

$$+ \int_0^T \int_{\Omega} \Big[\frac{1}{2}\mathbf{y}^T(\mathbf{x}, t)\mathbf{q}(\mathbf{x}, t)\mathbf{y}(\mathbf{x}, t) + \frac{1}{2}\mathbf{f}^T(\mathbf{x}, t)\mathbf{r}(\mathbf{x}, t)\mathbf{f}(\mathbf{x}, t)$$

$$+\boldsymbol{\lambda}^T(\mathbf{x},t)\Big(\mathcal{A}(\mathbf{x},t)\mathbf{y}(\mathbf{x},t)+\mathbf{b}_0(\mathbf{x},t)\mathbf{w}(\mathbf{x},t)+\mathbf{b}(\mathbf{x},t)\mathbf{f}(\mathbf{x},t)-\dot{\mathbf{y}}(\mathbf{x},t)\Big)\Big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \tag{3.1}$$

In (3.1) $\boldsymbol{\lambda}(\mathbf{x},t)$ is a vector of Lagrange multipliers, (also called the co-state or adjoint state vector). In the last term (3.1) partial integration of $\boldsymbol{\lambda}^T(\mathbf{x},t)\dot{\mathbf{y}}(\mathbf{x},t)$ is performed with respect to time. Further, partial integration of the term $\boldsymbol{\lambda}^T(\mathbf{x},t)\mathcal{A}(\mathbf{x},t)\mathbf{y}(\mathbf{x},t)$ is performed n times, using the divergence theorem. Then (3.1) attains the form

$$\begin{aligned}
\bar{J}[\mathbf{y},\mathbf{f},\boldsymbol{\lambda}] =& \int_\Omega \left[\frac{1}{2}\mathbf{y}^T(\mathbf{x},T)\mathbf{s}(\mathbf{x},T)\mathbf{y}(\mathbf{x},T)-\boldsymbol{\lambda}^T(\mathbf{x},T)\mathbf{y}(\mathbf{x},T)+\boldsymbol{\lambda}^T(\mathbf{x},0)\mathbf{y}(\mathbf{x},0)\right]\mathrm{d}\mathbf{x} \\
&+\int_0^T\int_{\Gamma_S}\left(\sum_{i=0}^{n-1}(-1)^{i+1}\left(\mathcal{G}_i^*(\mathbf{x},t)\boldsymbol{\lambda}(\mathbf{x},t)\right)^T\mathcal{G}_{n-1-i}(\mathbf{x},t)\mathbf{y}(\mathbf{x},t)\right)\mathrm{d}A\,\mathrm{d}t \\
&+\int_0^T\int_\Omega\Big[\frac{1}{2}\mathbf{y}^T(\mathbf{x},t)\mathbf{q}(\mathbf{x},t)\mathbf{y}(\mathbf{x},t)+\frac{1}{2}\mathbf{f}^T(\mathbf{x},t)\mathbf{r}(\mathbf{x},t)\mathbf{f}(\mathbf{x},t) \\
&+\left(\left(\mathcal{A}^*(\mathbf{x},t)\boldsymbol{\lambda}(\mathbf{x},t)\right)^T+\dot{\boldsymbol{\lambda}}^T(\mathbf{x},t)\right)\mathbf{y}(\mathbf{x},t) \\
&+\boldsymbol{\lambda}^T(\mathbf{x},t)\Big(\mathbf{b}_0(\mathbf{x},t)\mathbf{w}(\mathbf{x},t)+\mathbf{b}(\mathbf{x},t)\mathbf{f}(\mathbf{x},t)\Big)\Big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t
\end{aligned} \tag{3.2}$$

in which $\mathrm{d}A$ is an area element on the surface of the structure, and $\mathcal{G}_i^*(\mathbf{x},t)$ and $\mathcal{A}^*(\mathbf{x},t)$ are the adjoint differential operators to $\mathcal{G}_i(\mathbf{x},t)$ and $\mathcal{A}(\mathbf{x},t)$, respectively. Stationarity of $\bar{J}(\mathbf{y},\mathbf{f},\boldsymbol{\lambda})$ implies

$$\begin{aligned}
\delta\bar{J}(\mathbf{y},\mathbf{f},\boldsymbol{\lambda}) =& \int_\Omega\left[(\mathbf{y}^T(\mathbf{x},T)\mathbf{s}(\mathbf{x},T)-\boldsymbol{\lambda}^T(\mathbf{x},T))\delta\mathbf{y}(\mathbf{x},T)+\boldsymbol{\lambda}^T(\mathbf{x},0)\delta\mathbf{y}(\mathbf{x},0)\right]\mathrm{d}\mathbf{x} \\
&+\int_0^T\int_{\Gamma_S}\left(\sum_{i=0}^{n-1}(-1)^{i+1}\left(\mathcal{G}_i^*(\mathbf{x},t)\boldsymbol{\lambda}(\mathbf{x},t)\right)^T\mathcal{G}_{n-1-i}(\mathbf{x},t)\delta\mathbf{y}(\mathbf{x},t)\right)\mathrm{d}A\,\mathrm{d}t \\
&+\int_0^T\int_\Omega\Big[\left(\mathbf{y}^T(\mathbf{x},t)\mathbf{q}(\mathbf{x},t)+\left(\mathcal{A}^*(\mathbf{x},t)\boldsymbol{\lambda}(\mathbf{x},t)\right)^T+\dot{\boldsymbol{\lambda}}^T(\mathbf{x},t)\right)\delta\mathbf{y}(\mathbf{x},t) \\
&+(\mathbf{f}^T(\mathbf{x},t)\mathbf{r}(\mathbf{x},t)+\boldsymbol{\lambda}^T(\mathbf{x},t)\mathbf{b}(\mathbf{x},t))\delta\mathbf{f}(\mathbf{x},t)\Big]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t=0
\end{aligned} \tag{3.3}$$

From this the Euler conditions for stationarity

$$\boldsymbol{\lambda}(\mathbf{x},T)=\mathbf{s}(\mathbf{x},T)\mathbf{y}(\mathbf{x},T)\quad,\qquad\forall\,\mathbf{x}\in\Omega \tag{3.4}$$

$$\mathcal{G}_i^*(\mathbf{x},t)\boldsymbol{\lambda}(\mathbf{x},t)=\mathbf{0}\quad,\qquad i=0,\ldots,n-1\quad,\quad\forall\,(\mathbf{x},t)\in\Gamma_S\times\,]0,\infty[\tag{3.5}$$

$$\dot{\boldsymbol{\lambda}}^T(\mathbf{x},t)=-\mathcal{A}^*(\mathbf{x},t)\boldsymbol{\lambda}(\mathbf{x},t)-\mathbf{q}(\mathbf{x},t)\mathbf{y}(\mathbf{x},t)\quad,\qquad\forall\,(\mathbf{x},t)\in\Omega\times\,]0,\infty[\tag{3.6}$$

$$\mathbf{f}^*(\mathbf{x},t) = -\mathbf{r}^{-1}(\mathbf{x},t)\mathbf{b}^T(\mathbf{x},t)\boldsymbol{\lambda}(\mathbf{x},t) \quad , \qquad \forall\,(\mathbf{x},t) \in \Omega \times]0,\infty[\tag{3.7}$$

The boundary conditions in (3.5) are prescribed on the part of Γ_S, where $\mathcal{G}_i(\mathbf{x},t)\mathbf{y}(\mathbf{x},t) \neq 0$. The control force field can be eliminated from the equation of motion (2.28) using (3.7), leading to the initial and boundary value problem

$$\begin{aligned}
&\dot{\mathbf{y}}(\mathbf{x},t) = \mathcal{A}(\mathbf{x},t)\mathbf{y}(\mathbf{x},t) + \mathbf{b}_0(\mathbf{x},t)\mathbf{w}(\mathbf{x},t) - \mathbf{b}(\mathbf{x},t)\mathbf{r}^{-1}(\mathbf{x},t)\mathbf{b}^T(\mathbf{x},t)\boldsymbol{\lambda}(\mathbf{x},t) \quad , \\
&\forall\,(\mathbf{x},t) \in \Omega \times]0,\infty[\\
&\mathcal{G}_i(\mathbf{x},t)\mathbf{y}(\mathbf{x},t) = \mathbf{0} \quad , \qquad i = 0,\ldots,n-1 \quad , \quad \forall\,(\mathbf{x},t) \in \Gamma_S \times]0,\infty[\\
&\mathbf{y}(\mathbf{x},0) = \mathbf{y}_0(\mathbf{x}) \quad , \qquad \forall \mathbf{x} \in \Omega
\end{aligned} \tag{3.8}$$

Notice, that most adjoint linear differential operators in structural mechanics fulfil

$$\mathcal{A}^*(\mathbf{x},t) = \mathcal{A}^T(\mathbf{x},t) \quad , \qquad \forall\,(\mathbf{x},t) \in \Omega \times]0,\infty[\tag{3.9}$$

This is mainly due to the self-adjointness of the differential operator L on the displacement field $\mathbf{u}(\mathbf{x},t)$.

Illustrative Example

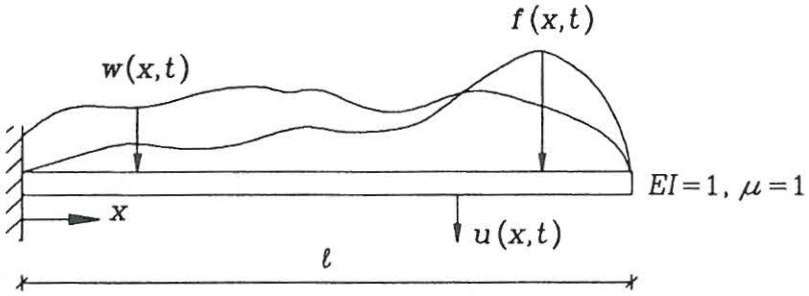

Figure 3.1: Linear elastic homogeneous cantilevered Bernoulli-Euler beam subjected to external loading, $w(x,t)$, and distributed control forces, $f(x,t)$.

As an example consider the Bernoulli-Euler beam of length l shown in fig 3.1. Due to the homogeneity the bending stiffness and the mass per unit length can, without loss of generality, be selected as $EI = \mu = 1$.

In this case the equations of motion with associated boundary and initial conditions become, corresponding to (2.27) and (2.28),

$$\begin{aligned}
&\dot{\mathbf{y}}(x,t) = \mathcal{A}\mathbf{y}(x,t) + \mathbf{b}_0 w(x,t) + \mathbf{b}f(x,t) \quad , \qquad \forall\,(x,t) \in]0,l[\times]0,\infty[\\
&\mathcal{G}_i\mathbf{y}(x,t) = 0 \quad , \qquad i = 0,1,2,3 \quad , \quad \forall\,(x,t) \in \{0,l\} \times]0,\infty[\\
&\mathbf{y}(x,0) = \mathbf{y}_0(x) \quad , \qquad \forall\, x \in]0,l[
\end{aligned}$$

where

$$\mathbf{y}(x,t)=\begin{bmatrix} u(x,t) \\ \dot{u}(x,t) \end{bmatrix} \quad , \quad \mathcal{A}=\begin{bmatrix} 0 & 1 \\ -\frac{\partial^4}{\partial x^4} & 0 \end{bmatrix} \quad , \quad \mathbf{b}_0=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad , \quad \mathbf{b}=\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.10}$$

$$\mathcal{G}_i=\begin{bmatrix} \frac{\partial^i}{\partial x^i} & 0 \end{bmatrix} \; , \; i=0,1,2,3 \quad , \quad \mathbf{y}_0(x)=\begin{bmatrix} u(x,0) \\ \dot{u}(x,0) \end{bmatrix} \tag{3.11}$$

$u(x,t)$ is the displacement field. In this case the extended functional (3.2) attains the form

$$\begin{aligned} \bar{J}[\mathbf{y},f,\boldsymbol{\lambda}] &= \frac{1}{2}\int_0^l \mathbf{y}^T(x,T)\mathbf{s}(x,T)\mathbf{y}(x,T)\,\mathrm{d}x \\ &+ \int_0^T\int_0^l \Big[\frac{1}{2}\mathbf{y}^T(x,t)\mathbf{q}(x,t)\mathbf{y}(x,t)+\frac{1}{2}f^2(x,t)r(x,t) \\ &+ \boldsymbol{\lambda}^T(x,t)\Big(\mathcal{A}\mathbf{y}(x,t)+\mathbf{b}_0 w(x,t)+\mathbf{b}f(x,t)-\dot{\mathbf{y}}(x,t)\Big)\Big]\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.12}$$

Let

$$\boldsymbol{\lambda}(x,t)=\begin{bmatrix} \lambda_1(x,t) \\ \lambda_2(x,t) \end{bmatrix} \tag{3.13}$$

where $\lambda_1(x,t)$ and $\lambda_2(x,t)$ represent the Lagrange multipliers adjoint to $u(x,t)$ and $\dot{u}(x,t)$, respectively, i.e $\boldsymbol{\lambda}^T(x,t)\mathcal{A}\mathbf{y}(x,t) = \lambda_1(x,t)\dot{u}(x,t) - \lambda_2(x,t)\frac{\partial^4}{\partial x^4}u(x,t)$. Partial integration with respect to time is then performed on the term $-\boldsymbol{\lambda}^T(x,t)\dot{\mathbf{y}}(x,t)$, and partial integration with respect to x is performed 4 times on the term $-\lambda_2(x,t)\frac{\partial^4}{\partial x^4}u(x,t)$. (3.12) then attains the form

$$\begin{aligned} \bar{J}(\mathbf{y},f,\boldsymbol{\lambda}) &= \int_0^l \Big[\Big(\frac{1}{2}\mathbf{y}^T(x,T)\mathbf{s}(x,T)\mathbf{Y}(x,T)-\boldsymbol{\lambda}^T(x,T)\mathbf{y}(x,T)+\boldsymbol{\lambda}^T(x,0)\mathbf{y}_0\Big)\Big]\,\mathrm{d}x \\ &+ \int_0^T \Bigg(-\lambda_2(x,t)\frac{\partial^3 u(x,t)}{\partial x^3}\bigg|_0^l + \frac{\partial\lambda_2(x,t)}{\partial x}\frac{\partial^2 u(x,t)}{\partial x^2}\bigg|_0^l \\ &- \frac{\partial^2\lambda_2(x,t)}{\partial x^2}\frac{\partial u(x,t)}{\partial x}\bigg|_0^l + \frac{\partial^3\lambda_2(x,t)}{\partial x^3}u(x,t)\bigg|_0^l\Bigg)\,\mathrm{d}t \\ &+ \int_0^T\int_0^l \Big[\frac{1}{2}\mathbf{y}^T(x,t)\mathbf{q}(x,t)\mathbf{y}(x,t)+\frac{1}{2}f^2(x,t)r(x,t)+\lambda_1(x,t)\dot{u}(x,t) \\ &- \frac{\partial^4\lambda_2(x,t)}{\partial x^4}u(x,t)+\dot{\boldsymbol{\lambda}}^T(x,t)\mathbf{y}(x,t)+\boldsymbol{\lambda}^T(x,t)\big(\mathbf{b}_0 w(x,t)+\mathbf{b}f(x,t)\big)\Big]\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.14}$$

Introduce the following differential operators

$$\mathcal{G}_i^* = \left[\begin{array}{cc} 0 & \frac{\partial^i}{\partial x^i} \end{array}\right] \quad , \qquad i = 0,1,2,3 \tag{3.15}$$

$$\mathcal{A}^* = \left[\begin{array}{cc} 0 & -\frac{\partial^4}{\partial x^4} \\ 1 & 0 \end{array}\right] \tag{3.16}$$

Notice, that the defined differential operator $\mathcal{A}^*$ satisfies (3.9). According to the definitions (3.15) and (3.16), the augmented performance index (3.31) can then be written in the same form as the general one, (3.2),

$$\begin{aligned}\bar{J}(\mathbf{y},f,\boldsymbol{\lambda}) = &\int_0^l \left[\frac{1}{2}\mathbf{y}^T(x,T)\mathbf{s}(x,T)\mathbf{Y}(x,T) - \boldsymbol{\lambda}^T(x,T)\mathbf{y}(x,T) + \boldsymbol{\lambda}^T(x,0)\mathbf{y}_0\right] dx \\ &+ \int_0^T \left(\sum_{i=0}^{3}(-1)^{i+1}\,\mathcal{G}_i^*\boldsymbol{\lambda}(x,t)\mathcal{G}_{3-i}\mathbf{y}(x,t)\Big|_0^l\right) dt \\ &+ \int_0^T\int_0^l \left[\frac{1}{2}\mathbf{y}^T(x,t)\mathbf{q}(x,t)\mathbf{y}(x,t) + \frac{1}{2}f^2(x,t)r(x,t)\right. \\ &+ \left.\left((\mathcal{A}^*\boldsymbol{\lambda}(x,t))^T + \dot{\boldsymbol{\lambda}}^T(x,t)\right)\mathbf{y}(x,t) + \boldsymbol{\lambda}^T(x,t)(\mathbf{b}_0 w(x,t) + \mathbf{b}f(x,t))\right] dx\, dt\end{aligned} \tag{3.17}$$

Hence, the Euler equations for the considered example can be derived as in the general formulation. This finishes the example.
Equations (3.8) and the adjoint initial and boundary value problem (3.4)-(3.6) constitute a linear two-point boundary-value problem (TPBVP). This TPBVP is very difficult to solve for analytic expressions for the costate $\boldsymbol{\lambda}(\mathbf{x},t)$, and following the control forces $\mathbf{f}(\mathbf{x},t)$ as given by (3.7). Even when one is able to solve these equations and thus, determine the optimal control law, one faces the difficulty of implementing it through continuously distributed control forces and measurements. Therefore, another simplified method is sought out, where the structure is first discretized in space by expanding the distributed dependent variables into a finite series of linear undamped eigenfunctions. Next, the discretized structure is controlled as if it were the distributed one. In order to develop this method, the general optimal control problem of discretized structures is first considered. The approach for optimal control of distributed structures follows in section 3.2.1.

3.2 Discretized Structures

An optimal control algorithm for active vibration suppression of linear discretized structures, developed by minimizing a quadratic performance index, is called an *LQ regulator*. The necessary conditions for optimal control is developed according to the technique of dynamic programming. It provides an efficient means for sequential decision making, based on the following *Bellman's principle of optimality*: An optimal control function has the property that whatever the initial state and the initial choice of control are, the remaining choices of control must constitute an optimal control function, with regard to the state resulting from the first choice of control, see e.g. Brogan (1985).

First, the dynamic programming equation for the minimum cost is developed, in the case where the equation of motion is represented by the general non-linear state space model (2.32) and the associated performance index is given by (2.52). Next, this equation is solved for the LQ problem leading to an optimal control law.

Consider the minimum cost of suppressing the structural vibrations from an arbitrary time t and state $\mathbf{Y}(t)$ to the terminal time T with the state $\mathbf{Y}(T)$, defined according to the general performance index (2.52) as

$$J^*(\mathbf{Y}(t),t) = \min_{\{\mathbf{F}(\tau),\tau\in[t,T]\}} \left\{ L_0(\mathbf{Y}(T),T) + \int_t^T L_1(\mathbf{Y}(\tau),\mathbf{F}(\tau),\tau)\,\mathrm{d}\tau \right\} \tag{3.18}$$

Equation (3.18) is rewritten by breaking the integral into two segments,

$$\begin{aligned} J^*(\mathbf{Y}(t),t) &= \min_{\{\mathbf{F}(\tau),\tau\in[t,T]\}} \left\{ L_0(\mathbf{Y}(T),T) + \int_{t+\Delta t}^T L_1(\mathbf{Y}(\tau),\mathbf{F}(\tau),\tau)\,\mathrm{d}\tau \right. \\ &\qquad \left. + \int_t^{t+\Delta t} L_1(\mathbf{Y}(\tau),\mathbf{F}(\tau),\tau)\,\mathrm{d}\tau \right\} \end{aligned} \tag{3.19}$$

According to the principle of optimality the cost $J(\mathbf{Y}(t+\Delta t), t+\Delta t)$, when the system initiates from the state $\mathbf{Y}(t+\Delta t)$ at the time $t+\Delta t$, must be minimum if the total cost is to be at a minimum. The first two terms on the right-hand side of (3.19) represent $J^*(\mathbf{Y}(t+\Delta t), t+\Delta t)$. Suppose that $J^*(\mathbf{Y}(t+\Delta t), t+\Delta t)$ and the corresponding optimal control force have been determined on the interval $[t+\Delta t, T]$ for each $\mathbf{Y}(t+\Delta t)$. Then it only remains to select the current control on the interval $[t, t+\Delta t]$. Hence

$$J^*(\mathbf{Y}(t),t) = \min_{\mathbf{F}(t)} \left\{ J^*(\mathbf{Y}(t+\Delta t), t+\Delta t) + \int_t^{t+\Delta t} L_1(\mathbf{Y}(\tau),\mathbf{F}(\tau),\tau)\,\mathrm{d}\tau \right\} \tag{3.20}$$

The following Taylor expansions to first order in Δt are introduced: $J^*(\mathbf{Y}(t+\Delta t), t+\Delta t)) = J^*(\mathbf{Y}(t),t) + \frac{\partial J^*(\mathbf{Y}(t),t)}{\partial t}\Delta t + \left(\frac{\partial J^*(\mathbf{Y}(t),t)}{\partial \mathbf{y}}\right)^T \dot{\mathbf{Y}}(t)\Delta t + \mathbf{o}(\Delta t)$ and $\int_t^{t+\Delta t} L_1(\mathbf{Y}(\tau),\mathbf{F}(\tau),\tau)\,\mathrm{d}\tau = L_1(\mathbf{Y}(t),\mathbf{F}(t),t)\Delta t + o(\Delta t)$, where $\lim_{\Delta t\to 0}\mathbf{o}(\Delta t) = \mathbf{0}$ and $\lim_{\Delta t\to 0} o(\Delta t) = 0$. Finally, $\dot{\mathbf{Y}}(t)$ is eliminated by means of (2.32). The following partial differential equation is then derived for $J^*(\mathbf{y},t)$

$$-\frac{\partial J^*(\mathbf{y},t)}{\partial t} = \min_{\mathbf{F}(t)} \left\{ L_1(\mathbf{y},\mathbf{F}(t),t) + \left(\frac{\partial J^*(\mathbf{y},t)}{\partial \mathbf{y}}\right)^T \Big(\mathbf{G}(\mathbf{y},t) + \mathbf{B}_0(t)\mathbf{W}(t) + \mathbf{B}(t)\mathbf{F}(t)\Big) \right\} \tag{3.21}$$

(3.21) is called the *Hamilton-Jacobi-Bellman* (H-J-B) equation. In (3.21) $\partial J^*/\partial \mathbf{y}$ designates the gradient of J^* with respect to $\mathbf{y}$. According to (2.52) and the definition (3.18) the boundary condition is $J^*(\mathbf{y}, T) = L_0(\mathbf{Y}(T), T)$. Equation (3.21) is solved backwards in time with this boundary condition. Notice, that the solution depends on the terminal value $\mathbf{Y}(T)$ of the vector, which in general is unknown. $\mathbf{Y}(T)$ must then be obtained by forward integration of (2.32), when the control forces $\mathbf{F}(t)$ have been expressed in the state vector by the minimal condition (3.21). This problem is in general quite difficult to solve analytically. Whether it can be solved or not depends on the class of structures, cost functions, admissible controls, and state constraints. The optimal control is the one which minimizes the right side of equation (3.21). If there are no restrictions on $\mathbf{F}(t)$ then the minimum can be found by differentiating with respect to $\mathbf{F}(t)$ and setting the resultant gradient equal to zero. This gives the necessary condition for optimality.

Linear Quadratic Control

Return to the class of linear structures represented in state space form by (2.33). If the performance index is quadratic, i.e. J is given by (2.53), then $\partial L_1/\partial \mathbf{F} = \mathbf{F}^T(t)\mathbf{R}(t)$ so that the optimal control $\mathbf{F}(t)$ for the LQ regulator problem is given by

$$\mathbf{F}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\frac{\partial J^*(\mathbf{Y}(t), t)}{\partial \mathbf{y}} \tag{3.22}$$

The H-J-B equation is solved by assuming a solution given by

$$J^*(\mathbf{y}, t) = \frac{1}{2}\mathbf{y}^T\mathbf{S}(t)\mathbf{y} + \mathbf{T}^T(t)\mathbf{y} + V(t) \tag{3.23}$$

where $\mathbf{S}(t)$ is an unknown positive semi-definite symmetric matrix, $\mathbf{T}(t)$ is an unknown vector, and $V(t)$ is an unknown scalar. Substituting (3.22) and (3.23) into (3.21), and collecting the quadratic terms in $\mathbf{y}$, the linear terms in $\mathbf{y}$ and the terms not involving $\mathbf{y}$, equations for $\mathbf{S}(t)$, $\mathbf{T}(t)$ and $V(t)$ are found from the requirement that each term must balance individually. After some algebra, the following equations are obtained

$$\dot{\mathbf{S}}(t) + \mathbf{S}(t)\mathbf{A}(t) + \mathbf{A}^T(t)\mathbf{S}(t) - \mathbf{S}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{S}(t) + \mathbf{Q}(t) = \mathbf{0} \tag{3.24}$$

$$\dot{\mathbf{T}}(t) + \mathbf{A}^T(t)\mathbf{T}(t) - \mathbf{S}(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{T}(t) + \mathbf{S}(t)\mathbf{B}_0(t)\mathbf{W}(t) = \mathbf{0} \tag{3.25}$$

$$\dot{V}(t) + \mathbf{T}^T(t)\mathbf{B}_0(t)\mathbf{W}(t) - \frac{1}{2}\mathbf{T}^T(t)\mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}_0^T(t)\mathbf{T}(t) = 0 \tag{3.26}$$

(3.24) is a matrix *Ricatti differential equation*. The boundary conditions for the three sets of differential equations are $\mathbf{S}(T) = \mathbf{S}_T$, $\mathbf{T}(T) = \mathbf{0}$ and $V(T) = 0$. For the assumed solution (3.23) the partial derivative in (3.22) is $\frac{\partial}{\partial \mathbf{y}}J^*(\mathbf{Y}(t), t) = \mathbf{S}(t)\mathbf{Y}(t) + \mathbf{T}(t)$ and hence, the optimal control law (3.22) becomes

$$\mathbf{F}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{S}(t)\mathbf{Y}(t) - \mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{T}(t) \tag{3.27}$$

The first term on the right-hand side of (3.27) represents the *closed-loop* control, i.e. the term depending on the current state $\mathbf{Y}(t)$. The second term represents the *open-loop* control, which is independent of the state vector of the structure, but depends on the external loading through $\mathbf{T}(t)$.

Implementation of the open-closed loop control law (3.27) implies that the Ricatti equation (3.24) and (3.25) are solved backwards in time for $\mathbf{S}(t)$ and $\mathbf{T}(t)$, respectively, because of the corresponding terminal conditions. However, solving (3.25) requires a priori knowledge of the external excitation history $\mathbf{W}(t)$. Unfortunately, many environmental loads, including earthquake ground acceleration, wind gusts, etc., are not known a priori. Thus, open-loop control is generally infeasible in structural control application.

A simple approach in control of civil engineering structures under random environmental loads is to neglect the open-loop term in (3.27) corresponding to $\mathbf{T}(t) = \mathbf{0}$, see e.g. Chung et al. (1988,1989) and Pu and Hsu (1988). According to (3.25) and the terminal condition $\mathbf{T}(T) = \mathbf{0}$, this solution is only optimal if $\mathbf{W}(t) = \mathbf{0}$. In this case the control vector $\mathbf{F}(t)$ is given by

$$\mathbf{F}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{S}(t)\mathbf{Y}(t) \tag{3.28}$$

Application of this solution is referred to as *optimal closed-loop control.* Defining a feedback as

$$\mathbf{G}_c(t) = \mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{S}(t) \tag{3.29}$$

Equation (3.28) can be written in the form as the general linear feedback control law (2.38). Furthermore, assume time-invariant matrices $\mathbf{A}, \mathbf{B}, \mathbf{Q}$ and $\mathbf{R}$. Then for many structures the Ricatti matrix $\mathbf{S}(t)$ is known to remain constant over the entire control interval, and it drops rapidly to the boundary value $\mathbf{S}_T$ near T, see e.g. Lewis (1986a) and Yang et al. (1987a). In other words, $\mathbf{S}(t)$ converges to a limit matrix $\mathbf{S}$ in a very short time starting from T backwards. If so, the effectiveness of the control system is not affected significantly by utilizing the steady-state solution $\mathbf{S}$ obtained by solving (3.24) for $\dot{\mathbf{S}}(t) = \mathbf{0}$,

$$\mathbf{SA} + \mathbf{A}^T\mathbf{S} - \mathbf{SBR}^{-1}\mathbf{B}^T\mathbf{S} + \mathbf{Q} = \mathbf{0} \tag{3.30}$$

Hence, application of the steady-state solution to the Ricatti equation correspond to, that the performance index is defined over an infinite time interval, i.e $T = \infty$. Readers who are interested in knowing on what conditions the stationary solution exist, and in how to determine the correct solution to (3.30) among the existing solutions, may for instance see in Lewis (1986a).

Since all parameters of the algebraic Ricatti equation (3.30) are presumably known, the matrix $\mathbf{S}$ and hence $\mathbf{G}_c$ can be determined off-line, and the only on-line computation required during control execution involves matrix multiplication as indicated by (2.38).

3.2.1 Reduced-Order Distributed-Parameter Structures

As an approach to the optimal control of distributed-parameter structures with an infinite number of degrees-of-freedom a method based on the control of a finite number of modes is introduced. It is assumed that the structural system to be controlled can be modelled by a set of linear differential equations with constant coefficients. Furthermore, the derivation will be confined to comprise the distributed-parameter system modelled in Chapter 2 by the equations of motion (2.20). If it is desired, the method can easily be enlarged to include systems described by the more general linear state-space model (2.28).

Consider the linear eigenvalue problem associated with the undamped eigenvibrations of (2.20)

$$L\Phi_i^{(m)}(\mathbf{x}) - \omega_m^2 \rho(\mathbf{x})\Phi_i^{(m)}(\mathbf{x}) = 0 \quad , \qquad \forall\, \mathbf{x} \in \Omega \quad , \quad m = 1,2,\ldots \tag{3.31}$$

subject to the boundary conditions

$$B_1\Phi_i^{(m)}(\mathbf{x}) = 0 \quad , \qquad \forall\, \mathbf{x} \in \Gamma_1 \quad , \quad m = 1,2,\ldots \tag{3.32}$$

$$\Phi_i^{(m)}(\mathbf{x}) = 0 \quad , \qquad \forall\, \mathbf{x} \in \Gamma_2 \quad , \quad m = 1,2,\ldots \tag{3.33}$$

The solution of (3.31)-(3.33) consists of a denumerably infinite set of eigenvalues ω_m^2, $m = 1,2,\ldots$, and the associated eigenmodes $\Phi_i^{(m)}$, $m = 1,2,\ldots$. Since the differential operator L is a self-adjoint and positive definite, all eigenvalues are real and positive. Further, the eigenfunctions fulfil the following orthogonality properties

$$\int_\Omega \rho(\mathbf{x})\Phi_i^{(m)}(\mathbf{x})\Phi_i^{(n)}(\mathbf{x})\,d\mathbf{x} = \begin{cases} 0\,, & m \neq n \\ M_m\,, & m = n \end{cases} \tag{3.34}$$

$$\int_\Omega \Phi_i^{(m)}(\mathbf{x})L\Phi_i^{(n)}(\mathbf{x})\,d\mathbf{x} = \begin{cases} 0\,, & m \neq n \\ \omega_m^2 M_m\,, & m = n \end{cases} \tag{3.35}$$

In the above equations (3.34) and (3.35) M_m signifies the modal mass defined as

$$M_m = \int_\Omega \rho(\mathbf{x})\Phi_i^{(m)}(\mathbf{x})\Phi_i^{(m)}(\mathbf{x})\,d\mathbf{x} \quad , \qquad m = 1,2,\ldots \tag{3.36}$$

Define the non-dimensional damping parameters ζ_m and ζ_{mn} according to the following equations

$$\int_\Omega c(\mathbf{x})\Phi_i^{(m)}(\mathbf{x})\Phi_i^{(n)}(\mathbf{x})\,d\mathbf{x} = \begin{cases} 2\zeta_{mn}\sqrt{\omega_m\omega_n M_m M_n}\,, & m \neq n \\ 2\zeta_m\omega_m M_m\,, & m = n \end{cases} \tag{3.37}$$

Let the solution of (2.20) be represented by the following infinite series

$$u_i(\mathbf{x},t) = U_i^{(0)}(\mathbf{x},t) + \sum_{m=1}^{\infty} \Phi_i^{(m)}(\mathbf{x})q_m(t) \tag{3.38}$$

where $q_m(t)$ are modal coordinates and $U_i^{(0)}(\mathbf{x},t)$ is the quasi-static displacement field defined by (2.17)-(2.19). This solution for $\mathbf{u}(\mathbf{x},t)$ fulfils the associated boundary conditions (2.8) and (2.13), since $U_i^{(0)}(\mathbf{x},t)$ and $\Phi_i^{(m)}(\mathbf{x})$ satisfy, respectively, the boundary conditions (2.18), (2.19) and (3.32), (3.33). Introducing (3.38) into (2.20), multiplying both sides of the result by $\Phi_i^{(n)}(\mathbf{x})$, integrating over the domain Ω and employing (3.34)-(3.37), the following ordinary differential equations are obtained for the modal coordinates

$$\ddot{q}_m + 2\omega_m\left(\zeta_m\dot{q}_m + \sum_{n=1,n\neq m}^{\infty}\sqrt{\frac{\omega_n M_n}{\omega_m M_m}}\zeta_{mn}\dot{q}_n\right) + \omega_m^2 q_m = w_m(t) + f_m(t) \quad ,$$
$$m = 1,2,\ldots \quad , \quad t \in]0,\infty[\tag{3.39}$$

In (3.39) $w_m(t)$ are modal excitations caused by support displacements and external loadings given by

$$w_m(t) = -\frac{1}{M_m}\int_{\Omega}\Phi_i^{(m)}(\mathbf{x})\left(\rho(\mathbf{x})\frac{\partial^2}{\partial t^2}U_i^{(0)}(\mathbf{x},t) - f_{ex,i}(\mathbf{x},t)\right)d\mathbf{x} \quad , \quad m = 1,2,\ldots \tag{3.40}$$

and $f_m(t)$ are the modal control forces given by

$$f_m(t) = \frac{1}{M_m}\sum_{\alpha=1}^{n_m}\Phi_i^{(m)}(\mathbf{x}_\alpha)e_{\alpha i}F_\alpha(t) \quad , \qquad m = 1,2,\ldots \tag{3.41}$$

Even in the absence of feedback control forces, equations (3.39) are coupled in case some of the off-diagonal damping parameters ζ_{mn} are non-zero. This coupling is referred to as *internal*. However, if feedback control forces are present, and the modal feedback control forces $f_m(t)$ depend on all modal coordinates and their velocities, i.e. $f_m = f_m(q_1, q_2, \ldots; \dot{q}_1, \dot{q}_2, \ldots)$ then equations (3.39) are *externally* coupled. In the remainder of this section the problem of coupled control is considered.

It is not practical to control the entire infinity of modes, and hence a truncation of the series expansion (3.37) is performed. This amounts to replacing the distributed parameter model with a discrete one. With a suitable numbering of the modes, it can without any restriction be assumed, that the n_c lowest modes should be controlled. Then the corresponding n_c modal equations can be expressed in matrix form as

$$\ddot{\mathbf{q}}_c + \mathbf{\Xi}_c\dot{\mathbf{q}}_c + \mathbf{\Lambda}_c\mathbf{q}_c = \mathbf{w}_c(t) + \mathbf{f}_c(t) \tag{3.42}$$

where the following vectors are introduced

$$\begin{aligned}
\mathbf{q}_c &= \left[\begin{array}{cccc} q_1 & q_2 & \ldots & q_{n_c}\end{array}\right]^T \\
\mathbf{w}_c &= \left[\begin{array}{cccc} w_1 & w_2 & \ldots & w_{n_c}\end{array}\right]^T \\
\mathbf{f}_c &= \left[\begin{array}{cccc} f_1 & f_2 & \ldots & f_{n_c}\end{array}\right]^T
\end{aligned} \tag{3.43}$$

The matrices in (3.42) are defined as follows

$$\Xi_{c,mn} = \begin{cases} 2\omega_m \sqrt{\frac{\omega_n M_n}{\omega_m M_m}} \zeta_{mn} \ , & m \neq n \\ 2\omega_m \zeta_m \ , & m = n \end{cases} \quad , \qquad m,n = 1,2,\ldots,n_c \tag{3.44}$$

$$\Lambda_{c,mn} = \begin{cases} 0 \ , & m \neq n \\ \omega_m^2 \ , & m = n \end{cases} \quad , \qquad m,n = 1,2,\ldots,n_c \tag{3.45}$$

Define a vector $\mathbf{\Phi}^{(m)}(\mathbf{x}_\alpha) = \begin{bmatrix} \Phi_1^{(m)}(\mathbf{x}_\alpha) & \Phi_2^{(m)}(\mathbf{x}_\alpha) & \Phi_3^{(m)}(\mathbf{x}_\alpha) \end{bmatrix}^T$ and next a matrix $\mathbf{b}_c$ with the components

$$b_{c,m\alpha} = \frac{1}{M_m} \mathbf{e}_\alpha^T \mathbf{\Phi}^{(m)}(\mathbf{x}_\alpha) \quad , \qquad m = 1,2,\ldots,n_c \quad , \quad \alpha = 1,2,\ldots,n_m \tag{3.46}$$

According to (3.46) the modal control forces defined by (3.41) can be written in matrix form, too, as

$$\mathbf{f}_c(t) = \mathbf{b}_c \mathbf{F}(t) \tag{3.47}$$

The modal equations (3.42) and (3.47) can finally be written in state space form as

$$\dot{\mathbf{Y}}_c(t) = \mathbf{A}\mathbf{Y}_c(t) + \mathbf{B}\mathbf{F}(t) + \mathbf{B}_0 \mathbf{w}_c(t) \tag{3.48}$$

where

$$\mathbf{Y}_c = \begin{bmatrix} \mathbf{q}_c \\ \dot{\mathbf{q}}_c \end{bmatrix} \ , \quad \mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{\Lambda}_c & -\mathbf{\Xi}_c \end{bmatrix} \ , \quad \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{b}_c \end{bmatrix} \ , \quad \mathbf{B}_0 = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \end{bmatrix} \tag{3.49}$$

The optimal control forces for the discretized system (3.48) is then designed by minimizing the quadratic performance index (2.53) where $\mathbf{Y}(t)$ is replaced by $\mathbf{Y}_c(t)$. This optimal control problem belongs to the general class of optimal control of systems represented by discrete-parameter models, which were considered in section 3.2. For the case of coupled control, minimization of the quadratic performance index in conjunction with equation (3.48) then leads to, cf. (3.27)

$$\mathbf{F}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T\mathbf{S}(t)\mathbf{Y}_c(t) - \mathbf{R}^{-1}(t)\mathbf{B}^T\mathbf{T}(t) \tag{3.50}$$

In the above equation the symmetric matrix $\mathbf{S}(t)$ satisfies the matrix Ricatti equation (3.24), and the vector $\mathbf{T}(t)$ satisfies (3.25), where $\mathbf{W}(t)$ is replaced by $\mathbf{w}_c(t)$.

3.2.2 Independent Modal Space Control

A much simpler control device can be obtained if equations (3.39) become internally and externally decoupled. This is the basis of the *independent modal space control* (IMSC) method, which has been extensively developed by Meirovitch et al. (1982,1983a).

The requirement of internal decoupling means $\zeta_{mn} = 0$ for all $m, n = 1, 2, \ldots, n_c$ in the modal equations (3.39). In the special case of external decoupling the modal control force in the mth mode $f_m(t)$ only depends on q_m and $\dot{q}_m$, i.e. $f_m = f_m(q_m, \dot{q}_m)$. The equations of motion (3.39) for the mth mode can then be written in state space form as

$$\dot{\mathbf{Y}}_m(t) = \mathbf{A}_m \mathbf{Y}_m(t) + \mathbf{B}_m \left(w_m(t) + f_m(\mathbf{Y}_m(t)) \right) \tag{3.51}$$

where

$$\mathbf{Y}_m = \begin{bmatrix} q_m \\ \dot{q}_m \end{bmatrix} \quad , \quad \mathbf{A}_m = \begin{bmatrix} 0 & 1 \\ -\omega_m^2 & -2\zeta_m \omega_m \end{bmatrix} \quad , \quad \mathbf{B}_m = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{3.52}$$

For optimal control the modal control forces can be determined by minimizing a performance index in the form

$$J = \sum_{m=1}^{n_c} J_m \tag{3.53}$$

where J_m is the modal loss functional chosen as

$$J_m[\mathbf{Y}_m, f_m] = \frac{1}{2} M_m \mathbf{Y}_m^T(T) \mathbf{S}_m(T) \mathbf{Y}_m(T) + \frac{1}{2} M_m \int_{t_0}^{T} \left(\mathbf{Y}_m^T(t) \mathbf{Q}_m \mathbf{Y}_m(t) + r_m f_m^2(t) \right) dt \tag{3.54}$$

$\mathbf{S}_m(T)$ and $\mathbf{Q}_m$ are symmetric, positive semidefinite weighting matrices of dimension 2×2. r_m is a positive weighting coefficient. A logical choice for $\mathbf{Q}_m$ is

$$\mathbf{Q}_m = \begin{bmatrix} \omega_m^2 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.55}$$

so that the first term of the integral in (3.54) represents the total mechanical energy associated with the mth mode.

Due to the modal decoupling the minimization of the performance index J can be achieved by minimizing each modal cost function J_m independently. The minimization of J_m leads to the modal feedback force, cf. (3.27)

$$f_m(t) = -\frac{1}{r_m} \mathbf{B}_m^T \mathbf{S}_m(t) \mathbf{Y}_m(t) - \frac{1}{r_m} \mathbf{B}_m^T \mathbf{T}_m(t) \tag{3.56}$$

where the 2×2 symmetric matrix $\mathbf{S}_m(t)$ satisfies the matrix Ricatti equation, cf. (3.24)

$$\dot{\mathbf{S}}(t)_m + \mathbf{S}_m(t) \mathbf{A}_m + \mathbf{A}_m^T \mathbf{S}_m(t) - \frac{1}{r_m} \mathbf{S}_m(t) \mathbf{B}_m \mathbf{B}_m^T \mathbf{S}_m(t) + \mathbf{Q}_m = \mathbf{0} \tag{3.57}$$

and the 2×1 vector $\mathbf{T}(t)$ satisfies, cf. (3.25)

$$\dot{\mathbf{T}}_m(t) + \mathbf{A}_m^T \mathbf{T}_m(t) - \frac{1}{r_m}\mathbf{B}_m \mathbf{B}_m^T \mathbf{T}_m(t) + \mathbf{S}_m(t)\mathbf{B}_0 w_m(t) = \mathbf{0} \tag{3.58}$$

The above equations are subject to the terminal conditions $\mathbf{S}_m(T) = \mathbf{S}_{T,m}$ and $\mathbf{T}_m(T) = \mathbf{0}$. The solutions of (3.57) and (3.58) can only be obtained numerically. However, the closed-loop modal control force is easy to develop analytically when the steady-state solution of the Ricatti equation is used. Letting $\dot{\mathbf{S}}_m(t) = \mathbf{0}$ in (3.57), three nonlinear algebraic equations are obtained. Using the extra condition, that $\mathbf{S}_m$ must be positive semi-definite for an arbitrary positive choice of r_m, these can be solved uniquely

$$\begin{aligned}
s_{11,m} &= 2\zeta_m \omega_m s_{12,m} + \omega_m^2 s_{22,m} + \frac{s_{12,m}s_{22,m}}{r_m} \\
s_{12,m} &= s_{21,m} = \omega_m^2 r_m \left(\sqrt{1 + \frac{1}{\omega_m^2 r_m}} - 1 \right) \\
s_{22,m} &= 2\zeta_m \omega_m r_m \left(\sqrt{1 + \frac{1 + 2s_{12,m}}{4\zeta_m^2 \omega_m^2 r_m}} - 1 \right)
\end{aligned} \tag{3.59}$$

where $s_{ij,m}$, $i,j = 1,2$ are the components of $\mathbf{S}_m$. Hence, by inserting (3.59) into (3.56) a closed-form solution is obtained for the modal control force $f_m(t)$,

$$f_m(t) = -\frac{1}{r_m}\left(s_{12,m} q_m(t) + s_{22,m} \dot{q}_m(t)\right) \tag{3.60}$$

The actual control forces $\mathbf{F}(t)$ are synthesized from the modal control forces contained in $\mathbf{f}_c(t)$ by solving (3.47),

$$\mathbf{F}(t) = \mathbf{b}_c^{-1}\mathbf{f}_c(t) \tag{3.61}$$

which requires that $\mathbf{b}_c$ is quadratic and non-singular. The former of these requirements is fulfilled if the number of actuators is equal to the number of controlled modes, $n_m = n_c$, cf. (3.46). The latter is satisfied if the locations and directions of the control forces are chosen appropriately.

3.2.3 Invariant Embedding Technique

In the beginning of section 3.2 the general optimal control problem was solved by means of dynamic programming, which lead to the Hamilton-Jacobi-Bellman partial differential equation for the minimum cost. In this section an alternative approach to the optimal control problem is obtained by means of calculus of variations. This leads to a two-point boundary-value problem in the state vector and the co-state vector, analogous to the two-point boundary-value problem for the state vector and the cost function in the dynamic programming approach. Analytical solutions for two-point boundary-value problems can only be obtained for simple systems. Thus, in general, numerical methods must be used. The numerical methods considered in this section are the matrix Ricatti equation and invariant embedding.

The objective is to determine the optimal control forces $\mathbf{F}^*(t)$, $t \in [0,T]$, that minimize the general performance index (2.52), subjected to the restriction that the state vector is required to fulfil the non-linear equations of motion (2.32). Using the method of Lagrangian multipliers, the optimal control is determined from the unrestricted stationarity conditions of the extended functional

$$\bar{J}[\mathbf{Y},\mathbf{F},\boldsymbol{\lambda}] = L_0(\mathbf{Y}(T),T) + \int_0^T \Big[L_1(\mathbf{Y}(t),\mathbf{F}(t),t) + \boldsymbol{\lambda}^T(t)\Big(\mathbf{G}(\mathbf{Y}(t),t) + \mathbf{B}_0(t)\mathbf{W}(t) + \mathbf{B}(t)\mathbf{F}(t) - \dot{\mathbf{Y}}(t)\Big)\Big]\mathrm{d}t \tag{3.62}$$

$\boldsymbol{\lambda}^T(t)$ is a vector of Lagrange multipliers. In the integral, partial integration of $-\boldsymbol{\lambda}^T(t)\dot{\mathbf{Y}}(t)$ is performed. Then (3.28) attains the form

$$\bar{J}[\mathbf{Y},\mathbf{F},\boldsymbol{\lambda}] = L_0(\mathbf{Y}(T),T) - \boldsymbol{\lambda}^T(T)\mathbf{Y}(T) + \boldsymbol{\lambda}^T(0)\mathbf{Y}_0 + \int_0^T \Big[L_1(\mathbf{Y}(t),\mathbf{F}(t),t) + \boldsymbol{\lambda}^T(t)\Big(\mathbf{G}(\mathbf{Y}(t),t) + \mathbf{B}_0(t)\mathbf{W}(t) + \mathbf{B}(t)\mathbf{F}(t)\Big) + \dot{\boldsymbol{\lambda}}^T(t)\mathbf{Y}(t)\Big]\mathrm{d}t \tag{3.63}$$

$\mathbf{Y}(t)$, $\mathbf{F}(t)$, $t \in]0,T]$ are varied, whereas the Lagrange multipliers are constants. Using the stationarity condition $\delta\bar{J}[\mathbf{Y},\mathbf{F},\boldsymbol{\lambda}] = 0$, the following Euler equations are easily derived

$$\boldsymbol{\lambda}(T) = \left(\frac{\partial L_0(\mathbf{Y}(T),T)}{\partial \mathbf{Y}}\right)^T \tag{3.64}$$

$$\dot{\boldsymbol{\lambda}}(t) = -\left(\frac{\partial \mathbf{G}(\mathbf{Y}(t),t)}{\partial \mathbf{Y}}\right)^T \boldsymbol{\lambda}(t) - \left(\frac{\partial L_1(\mathbf{Y}(t),\mathbf{F}(t),t)}{\partial \mathbf{Y}}\right)^T \tag{3.65}$$

$$\frac{\partial L_1(\mathbf{Y}(t),\mathbf{F}(t),t)}{\partial \mathbf{F}} + \boldsymbol{\lambda}^T(t)\mathbf{B}(t) = \mathbf{0} \tag{3.66}$$

(3.66) gives a set of algebraic equations which allow the determination of the optimal control forces $\mathbf{F}^*(t)$ in terms of the still unknown $\boldsymbol{\lambda}(t)$ and $\mathbf{Y}(t)$. Subsequently, $\mathbf{F}(t)$ can be eliminated from the equation of motion (2.32) and the adjoint state equation (3.65).

The *Hamiltonian* is defined as the scalar function

$$\mathcal{H}(\mathbf{Y}(t),\boldsymbol{\lambda}(t),\mathbf{F}(t),t) = L_1(\mathbf{Y}(t),\mathbf{F}(t),t) + \boldsymbol{\lambda}^T(t)\big(\mathbf{G}(\mathbf{Y}(t),t) + \mathbf{B}_0(t)\mathbf{W}(t) + \mathbf{B}(t)\mathbf{F}(t)\big) \tag{3.67}$$

Then equations (2.32), (3.64), (3.65) and (3.66) can be given the following canonical formulation

$$\dot{\mathbf{Y}}(t) = \frac{\partial \mathcal{H}(\mathbf{Y}(t), \boldsymbol{\lambda}(t), \mathbf{F}(t), t)}{\partial \boldsymbol{\lambda}} \quad , \qquad \mathbf{Y}(0) = \mathbf{Y}_0 \tag{3.68}$$

$$\dot{\boldsymbol{\lambda}}^T(t) = -\frac{\partial \mathcal{H}(\mathbf{Y}(t), \boldsymbol{\lambda}(t), \mathbf{F}(t), t)}{\partial \mathbf{Y}} \quad , \qquad \boldsymbol{\lambda}^T(T) = \frac{\partial L_0(\mathbf{Y}(T), T)}{\partial \mathbf{Y}} \tag{3.69}$$

$$\frac{\partial \mathcal{H}(\mathbf{Y}(t), \boldsymbol{\lambda}(t), \mathbf{F}(t), t)}{\partial \mathbf{F}} = \mathbf{0} \tag{3.70}$$

Equations (3.68) and (3.69) constitute a non-linear two-point boundary-value problem for the optimal controlled state $\mathbf{Y}^*(t)$.

Linear Quadratic Control

In order to eliminate $\mathbf{F}(t)$ from the TPBVP, it is necessary to introduce an expression for the function L_1 in the performance index (2.52). Assuming L_1 is quadratic as specified in (2.53) and the structure is represented by the linear state space model (2.33), the Hamiltonian becomes

$$\begin{aligned} \mathcal{H}(\mathbf{Y}(t), \boldsymbol{\lambda}(t), \mathbf{F}(t), t) \;=\; & \frac{1}{2}\mathbf{Y}^T(t)\mathbf{Q}(t)\mathbf{Y}(t) + \frac{1}{2}\mathbf{F}^T(t)\mathbf{R}(t)\mathbf{F}(t) \\ & + \boldsymbol{\lambda}^T(t)\Big(\mathbf{A}(t)\mathbf{Y}(t) + \mathbf{B}_0(t)\mathbf{W}(t) + \mathbf{B}(t)\mathbf{F}(t)\Big) \end{aligned} \tag{3.71}$$

Using (3.71), the control forces $\mathbf{F}(t)$ are obtained from (3.70) as follows

$$\begin{aligned} & \frac{\partial \mathcal{H}}{\partial \mathbf{F}} = \mathbf{F}^T(t)\mathbf{R}(t) + \boldsymbol{\lambda}^T(t)\mathbf{B}(t) = \mathbf{0} \qquad \Rightarrow \\ & \mathbf{F}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\boldsymbol{\lambda}(t) \end{aligned} \tag{3.72}$$

Substituting (3.71) into (3.69) the adjoint vector $\boldsymbol{\lambda}(t)$ becomes the solution of the differential equation

$$\dot{\boldsymbol{\lambda}}(t) = -\mathbf{Q}(t)\mathbf{Y}(t) - \mathbf{A}^T(t)\boldsymbol{\lambda}(t) \tag{3.73}$$

Inserting (3.72) into (2.33) gives

$$\dot{\mathbf{Y}}(t) = \mathbf{A}(t)\mathbf{Y}(t) + \mathbf{B}_0(t)\mathbf{W}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\boldsymbol{\lambda}(t) \tag{3.74}$$

According to (3.64), the boundary conditions for (3.73) are $\boldsymbol{\lambda}(T) = \mathbf{S}(T)\mathbf{Y}(T)$. However, the importance of the terminal state $\mathbf{Y}(T)$ will not be given any special weighting and hence, we set $\mathbf{S}(T) = \mathbf{0}$. Then equations (3.73) and (3.74) constitute a linear two-point boundary-value problem with boundary conditions

$$\mathbf{Y}(0) = \mathbf{Y}_0 \quad , \qquad \boldsymbol{\lambda}(T) = \mathbf{0} \tag{3.75}$$

Linear TPBVP's are easily solved, at least in principle. But for large structures with many degrees-of-freedom, and in case of time-varying system matrices and/or weighting matrices it may be quite complicated to solve analytically. Instead, we will present a solution based on *the sweep method*, see Bryson (1975). Thus, assume that $\mathbf{Y}(t)$ and $\boldsymbol{\lambda}(t)$ satisfy the linear relation

$$\boldsymbol{\lambda}(t) = \mathbf{S}(t)\mathbf{Y}(t) + \mathbf{T}(t) \tag{3.76}$$

where $\mathbf{S}(t)$ is an unknown matrix function and $\mathbf{T}(t)$ is an unknown vector function. Using equations (3.73)-(3.76) it is found that the assumption (3.76) is valid if $\mathbf{S}(t)$ and $\mathbf{T}(t)$ satisfy, respectively, (3.24) and (3.25). Hence, if (3.76) is inserted into (3.72) the same optimal control law is obtained as the one given by (3.27). As already mentioned this solution requires a priori knowledge of the external excitation history $\mathbf{W}(t)$, which is not the case for many environmental loads affecting civil engineering structures.

To circumvent this problem an algorithm is developed, where the TPBVP is solved by using the invariant embedding technique. The basic concept of this method is to change the original TPBVP into a class of more general initial value problems, see Bellman et al. (1960,1961,1963,1967), Kagiwada and Kalaba (1968), Kalaba and Spingarn (1977).

The TPBVP of equations (3.73) and (3.74) is generalized by letting the terminal boundary condition on $\boldsymbol{\lambda}(T)$ take a general value $\mathbf{c}$ rather than $\mathbf{0}$, i.e.

$$\boldsymbol{\lambda}(T) = \mathbf{c} \tag{3.77}$$

In addition, it is assumed that both the boundary value $\mathbf{c}$ and the terminal time T are independent variables. For the trajectory which satisfies (3.77), let the missing terminal value for $\mathbf{Y}(t)$ be given by

$$\mathbf{Y}(T) = \mathbf{r}(\mathbf{c}, T) \tag{3.78}$$

In other words, the function $\mathbf{r}(\mathbf{c}, T)$ represents the relation between the boundary condition $\boldsymbol{\lambda}(T) = \mathbf{c}$ and the terminal value of $\mathbf{Y}(T)$. According to the invariant embedding technique an equation for the unknown function $\mathbf{r}(\mathbf{c}, t)$ is derived. The basic idea in the derivation of this is to consider two neighbouring trajectories, one which satisfies the boundary condition (3.78) and one which satisfies $\boldsymbol{\lambda}(T + \Delta T) = \mathbf{c} + \Delta\mathbf{c}$.

Define $\mathbf{Y}(T + \Delta T) = \mathbf{Y}(T) + \Delta\mathbf{Y}$ to obtain

$$\mathbf{r}(\mathbf{c} + \Delta\mathbf{c}, T + \Delta T) = \mathbf{r}(\mathbf{c}, T) + \Delta\mathbf{Y} \tag{3.79}$$

From (3.74) follows $\Delta\mathbf{Y} = \left(\mathbf{A}(T)\mathbf{Y}(T) + \mathbf{B}_0(T)\mathbf{W}(T) - \mathbf{B}(T)\mathbf{R}^{-1}(T)\mathbf{B}^T(T)\boldsymbol{\lambda}(T)\right)\Delta T + o(\Delta T)$. The left hand side of equation (3.79) is expanded into a Taylor series about $\mathbf{c}$ and T. From (3.73) follows $\Delta\mathbf{c} = \left(-\mathbf{Q}(T)\mathbf{Y}(T) - \mathbf{A}^T(T)\boldsymbol{\lambda}(T)\right)\Delta T + o(\Delta T)$. In the limit as

$\Delta T \to 0$, eq. (3.79) then provides the invariant embedding equation

$$\frac{\partial \mathbf{r}}{\partial T} + \frac{\partial \mathbf{r}}{\partial \mathbf{c}}\left(-\mathbf{Q}(T)\mathbf{r}(\mathbf{c},T) - \mathbf{A}^T(T)\mathbf{c}\right) = \mathbf{A}(T)\mathbf{r}(\mathbf{c},T) + \mathbf{B}_0(T)\mathbf{W}(T) - \mathbf{B}(T)\mathbf{R}^{-1}(T)\mathbf{B}^T(T)\mathbf{c} \tag{3.80}$$

This is a non-linear partial differential equation for the function $\mathbf{r}(\mathbf{c},T)$. According to the definitions (3.77) and (3.78), its solution yields the unknown terminal condition $\mathbf{Y}(T)$ as a function of the duration of the control process, T, and the already known terminal condition on $\boldsymbol{\lambda}(T)$. Hence, if $\mathbf{r}(\mathbf{c},T)$ can be found by solving (3.80) with proper boundary conditions, the missing terminal condition $\mathbf{Y}(T)$ is obtained. With it, the original TPBVP (3.73) and (3.74) has been changed into an initial value problem, which can be solved by backward integration.

The invariant embedding equation (3.80) is difficult to solve exactly. Therefore, as an approximation we search for a perturbation series solution. According to (3.75) and (3.77) it is assumed that $\mathbf{c}$ is small, i.e. $|\mathbf{c}| \ll 1$. A perturbation solution in the parameter $\mathbf{c}$ is then set up. Ignoring terms of order higher than 1 in $\mathbf{c}$, the perturbation in $\mathbf{c}$ is governed by

$$\mathbf{r}(\mathbf{c},T) = \boldsymbol{\xi}(T) + \mathbf{S}(T)\mathbf{c} \tag{3.81}$$

where $\mathbf{S}(T)$ is an unknown matrix and $\boldsymbol{\xi}(T)$ is an unknown vector. The assumed solution (3.81) is substituted into the invariant embedding equation (3.80). Then equating terms of equal powers of $\mathbf{c}$ yield the following differential equations

$$\dot{\boldsymbol{\xi}}(t) = \left[\mathbf{A}(t) + \mathbf{S}(t)\mathbf{Q}(t)\right]\boldsymbol{\xi}(t) + \mathbf{B}_0(t)\mathbf{W}(t) \tag{3.82}$$

$$\dot{\mathbf{S}}(t) = \mathbf{A}(t)\mathbf{S}(t) + \mathbf{S}(t)\mathbf{A}^T(t) + \mathbf{S}(t)\mathbf{Q}(t)\mathbf{S}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t) \tag{3.83}$$

In (3.82) and (3.83) the variable terminal time is written as just t. The initial conditions for $\boldsymbol{\xi}(t)$ and $\mathbf{S}(t)$ are obtained by equating, cf. (3.78), $\mathbf{r}(\mathbf{c},0) = \boldsymbol{\xi}(0) + \mathbf{S}(0)\mathbf{c} = \mathbf{Y}(0) = \mathbf{Y}_0$. Thus, for arbitrary $\mathbf{c}$, the initial conditions are

$$\boldsymbol{\xi}(0) = \mathbf{Y}_0 \quad , \quad \mathbf{S}(0) = \mathbf{0} \tag{3.84}$$

Letting $\mathbf{c} = \mathbf{0}$, which is the correct value, it is seen from (3.78) and (3.81) that $\boldsymbol{\xi}(t) = \mathbf{Y}(t)$. By using this relationship in (3.82), we obtain the invariant embedding equation for the controlled state

$$\dot{\mathbf{Y}}(t) = \left[\mathbf{A}(t) + \mathbf{S}(t)\mathbf{Q}(t)\right]\mathbf{Y}(t) + \mathbf{B}_0(t)\mathbf{W}(t) \quad , \qquad \mathbf{Y}(0) = \mathbf{Y}_0 \tag{3.85}$$

Besides the inherent approximations due to the introduced perturbation solution, it is interesting and important to note that the state $\mathbf{Y}(t)$ calculated from (3.83) and (3.85) is not the optimal trajectory. Each state $\mathbf{Y}(t_1)$ calculated for a given time t_1 is the state which would result if t_1 were the terminal time, i.e. a suboptimal control is obtained.

The basic goal of the control design is a control law which determines the control force.

Since the invariant embedding equation (3.85) describes the controlled state the control force contained in the equation of motion (2.33) must be chosen in such a way that the solution of both equations are equal. A simple comparison of (2.33) and (3.85) reveals that this requirement is fulfilled if $\mathbf{B}(t)\mathbf{F}(t) = \mathbf{S}(t)\mathbf{Q}(t)\mathbf{Y}(t)$. This relationship only has a unique solution if $\mathbf{B}(t)$ is quadratic and non-singular, which is generally not the case.

Instead, the control force is obtained by choosing an algebraic form for the control law in which the coefficients will be determined. Assuming the control force is linear in the state vector, the control law can be written as (2.38) in which $\mathbf{G}_c(t)$ is an unknown $n_m \times n$ gain matrix. In this case n signifies the dimension of the state vector $\mathbf{Y}(t)$ and should not be confused with the dimension of the displacement vector $\mathbf{u}(t)$ in (2.22). n_m is the dimension of the control force vector. Inserting (2.38) into (2.33) yields the closed-loop equation

$$\dot{\mathbf{Y}}(t) = \left[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{G}_c(t)\right]\mathbf{Y}(t) + \mathbf{B}_0(t)\mathbf{W}(t) \tag{3.86}$$

The aim is to establish $\mathbf{G}_c(t)$ so that the solutions to the closed-loop equation (3.86) and the invariant embedding state equation (3.85) are approximately equal. For that purpose $\mathbf{G}_c(t)$ is determined in such a way that the eigenvalues of the system matrices are equal and the deviation between the eigenvectors are minimal.

The desired set of eigenvalues $\lambda_i(t), i = 1, 2, \ldots, n$ and the associated eigenvectors $\mathbf{\Psi}^{(i)}(t), i = 1, 2, \ldots, n$ are determined from the system matrix $[\mathbf{A}(t) + \mathbf{S}(t)\mathbf{Q}(t)]$ of the invariant embedding state equation (3.85).

The same set of eigenvalues $\lambda_i(t)$, $i = 1, 2, \ldots, n$ can be achieved for the closed-loop system matrix $[\mathbf{A}(t) - \mathbf{B}(t)\mathbf{G}_c(t)]$, if the fictitious time-invariant system at the arbitrary time $t = t_1$ described by the pair $\{\mathbf{A}(t_1), \mathbf{B}(t_1)\}$ is controllable, Brogan (1985). Notice, the concept of controllability is defined in Section 5.3.1. On this assumption a method of finding $\mathbf{G}_c(t)$ is given, which furthermore allows to interject judgement regarding desirability of the closed-loop eigenvectors, Brogan (1985).

If the closed-loop system matrix has the set of eigenvalues $\lambda_i(t)$, $i = 1, 2, \ldots, n$, then the eigenvectors $\mathbf{\Phi}^{(i)}$, $i = 1, 2, \ldots, n$ are determined as non-trivial solutions to the homogeneous equation $[\lambda_i\mathbf{I} - \mathbf{A} + \mathbf{B}\mathbf{G}_c]\mathbf{\Phi}^{(i)} = \mathbf{0}$. This is rearranged in the following way

$$\left[(\lambda_i\mathbf{I} - \mathbf{A}) \;\vdots\; \mathbf{B}\right] \begin{bmatrix} \mathbf{\Phi}^{(i)} \\ --- \\ \mathbf{G}_c\mathbf{\Phi}^{(i)} \end{bmatrix} = \mathbf{0} \tag{3.87}$$

The coefficient matrix in the above homogeneous equation is of dimension $n \times (n + n_m)$ and has rank n for any value of λ_i. Hence, the solution space of (3.87) consists of a linear combination of any n_m linear independent solution vectors. This set of independent solution vectors is assembled column-wise in an $(n + n_m) \times n_m$ dimensional matrix $\mathbf{U}(\lambda_i)$. According to (3.87) the matrix $\mathbf{U}$ is partitioned as

$$\mathbf{U}(\lambda_i) = \begin{bmatrix} \phi_1^{(i)} & \phi_2^{(i)} & \cdots & \phi_m^{(i)} \\ --- & --- & --- & --- \\ \mathbf{f}_1^{(i)} & \mathbf{f}_2^{(i)} & \cdots & \mathbf{f}_m^{(i)} \end{bmatrix} = \begin{bmatrix} \mathcal{P}(\lambda_i) \\ --- \\ \mathcal{F}(\lambda_i) \end{bmatrix} \tag{3.88}$$

where the substitution $\mathbf{f}_j^{(i)} = \mathbf{G}_c\phi_j^{(i)}$, $j = 1, 2, \ldots, n_m$ has been made. Being a solution to

(3.87), the eigenvector $\mathbf{\Phi}^{(i)}$ of the closed-loop system can be written as any linear combination of the basis vectors $\boldsymbol{\phi}_j$, $j = 1, 2, \ldots, n_m$, of the solution space, i.e.

$$\begin{aligned}\mathbf{\Phi}^{(i)} &= \boldsymbol{\phi}_1^{(i)} a_1 + \boldsymbol{\phi}_2^{(i)} a_2 + \cdots + \boldsymbol{\phi}_{n_m}^{(i)} a_{n_m} \\ &= \mathcal{P}(\lambda_i)\mathbf{a}(\lambda_i)\end{aligned} \tag{3.89}$$

where $\mathbf{a}(\lambda_i)$ is a vector of the free parameters a_j, $j = 1, 2, \ldots, n_m$. It is desired to determine $\mathbf{a}(\lambda_i)$ so that the deviation is to be minimum between $\mathbf{\Phi}^{(i)}$ and the associated eigenvector of the invariant embedding state equation, $\mathbf{\Psi}^{(i)}$, obtained from the eigenvalue problem $[\lambda_i \mathbf{I} - \mathbf{A} - \mathbf{SQ}]\mathbf{\Psi}^{(i)} = \mathbf{0}$ For that purpose the Euclidian norm of the error vector $\boldsymbol{\varepsilon} = \mathbf{\Psi}^{(i)} - \mathcal{P}(\lambda_i)\mathbf{a}(\lambda_i)$ is minimized leading to the least squares solution

$$\mathbf{a}(\lambda_i) = \left(\mathcal{P}^T(\lambda_i)\mathcal{P}(\lambda_i)\right)^{-1} \mathcal{P}^T(\lambda_i)\mathbf{\Psi}^{(i)} \tag{3.90}$$

When $\mathbf{a}(\lambda_i)$ is calculated from (3.90), the eigenvector $\mathbf{\Phi}^{(i)}$ is determined according to (3.89). Then, if (3.87) is still to be true the same linear combination of the columns in $\mathcal{F}(\lambda_i)$ must be selected, i.e.

$$\mathbf{G}_c \mathbf{\Phi}^{(i)} = \mathcal{F}(\lambda_i)\mathbf{a}(\lambda_i) \tag{3.91}$$

When the components of the solution vector to (3.87) are determined from (3.89) and (3.91), it is clear that $\mathbf{\Phi}^{(i)}$ is the eigenvector of the closed-loop system associated with λ_i. To obtain the desired eigenvalues and eigenvectors, $\mathbf{G}_c$ must then be selected to satisfy (3.91), but this equation is not sufficient. However, if an independent equation of this type is found for every λ_i, $i = 1, \ldots, n$, $\mathbf{G}_c$ can be determined from the composite set of equations $\mathbf{G}_c \left[\mathbf{\Phi}^{(1)}\ \mathbf{\Phi}^{(2)}\ \cdots\ \mathbf{\Phi}^{(n)}\right] = [\mathcal{F}(\lambda_1)\mathbf{a}(\lambda_1)\ \mathcal{F}(\lambda_2)\mathbf{a}(\lambda_2)\ \cdots\ \mathcal{F}(\lambda_n)\mathbf{a}(\lambda_n)]$, yielding

$$\mathbf{G}_c = \left[\mathcal{F}(\lambda_1)\mathbf{a}(\lambda_1)\quad \mathcal{F}(\lambda_2)\mathbf{a}(\lambda_2)\quad \cdots\quad \mathcal{F}(\lambda_n)\mathbf{a}(\lambda_n)\right]\left[\,\mathbf{\Phi}^{(1)}\quad \mathbf{\Phi}^{(2)}\quad \cdots\quad \mathbf{\Phi}^{(n)}\,\right]^{-1} \tag{3.92}$$

Substituting $\mathbf{G}_c = \mathbf{G}_c(t)$ into (2.38) a control law for the optimal non-linear control problem is obtained.

Implementation of the proposed control algorithm requires that all calculations are made on-line when the control force is to be determined. The necessary calculations comprise the solution of (3.83), the eigenvalue problem for the system matrix $[\mathbf{A}(t) + \mathbf{S}(t)\mathbf{Q}(t)]$ of the invariant embedding equation (3.85), and finally solution of (3.87) - (3.90) to obtain the feedback gain from (3.92). Equation (3.83) is solved by integrating forward in time, and from the solution $\mathbf{S}(t)$ at each time step the gain matrix $\mathbf{G}_c(t)$ is determined as described. If the structure is described by a linear time-invariant model and the weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ are time independent, $\mathbf{S}(t)$ establishes a stationary state in a very short period of time. In this case equation (3.83) can be simplified by setting $\dot{\mathbf{S}}(t) = \mathbf{0}$, and thus it becomes a purely algebraic equation yielding $\mathbf{S}$, and in the given circumstances $\mathbf{G}_c$ to be constant matrices.

3.3 Stochastic Optimal Control

When a civil engineering structure is subjected to external loadings that cannot be specified ahead of time, then deterministic cost functions of the type considered till now cannot be minimized by the choice of control forces. However, if the statistics of the external excitations are known, a cost function that is the expected value of the previous performance index can be minimized. This is called *stochastic optimal control.*

Solving the stochastic optimal control problem, two cases must be considered. In the first case of complete state information, the state vector $\mathbf{Y}(t)$ is known exactly at the time t, so feedback control is directly possible. In the second case, the problem of incomplete state information is addressed, where the state is measured in the presence of disturbances. Generally, stochastic quantities are indicated by capital letters and their sample values by small letters.

3.3.1 Structural Systems with Stochastic Excitations and Perfect Measurements

The general non-linear optimal control problem in Section 3.2 is considered, given the general performance index (2.52). However, the state vector and the control force is now modelled as stochastic processes $\{\mathbf{Y}(t),\ t \in [0,T]\}$ and $\{\mathbf{F}(t),\ t \in [0,T]\}$. For this reason the performance index is taken as the expectation

$$E\big[J[\mathbf{Y},\mathbf{F}]\big] = E\left[L_0(\mathbf{Y}(T),T) + \int_0^T L_1(\mathbf{Y}(t),\mathbf{F}(t),t)\,\mathrm{d}t\right] \tag{3.93}$$

The minimization of this expectation is constrained by the structural equation of motion (2.32). For this system the external excitation is assumed to be a non-stationary Gaussian white noise process $\{\mathbf{W}(t),\ t \in [0,T]\}$ with mean value function and covariance function described by the following

$$E[\mathbf{W}(t)] = \mathbf{0} \tag{3.94}$$

$$E\big[\mathbf{W}(t_1)\mathbf{W}^T(t_2)\big] = \mathbf{R}_{\mathbf{W}}(t_1)\delta(t_1 - t_2) \tag{3.95}$$

where $\mathbf{R}_{\mathbf{W}}(t)$ is a time-varying intensity function and $\delta(\cdot)$ is the Dirac delta function. The indicated formulation of the random external excitation is more general than it appears at first sight, because the state space model (2.32) in excess of the equations of motion may include a linear filter, where $\mathbf{W}(t)$ is the input and the output affects the equation of motion. For example, it is well known that an earthquake excitation can be modelled as a white noise process passed through an appropriate filter.

Due to the white noise excitation, $\{\mathbf{Y}(t),\ t \in [0,T]\}$, as given by, (2.32) becomes a Markov vector process in open as well as closed loop control. The drift vector and the diffusion matrix become, see Arnold (1974)

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}E\big[\mathbf{Y}(t+\Delta t)-\mathbf{Y}(t)\,\big|\,\mathbf{Y}(t)=\mathbf{y}\big]=\mathbf{G}(\mathbf{y},t)+\mathbf{B}(t)E\big[\mathbf{F}(t)\,\big|\,\mathbf{Y}(t)=\mathbf{y}\big] \tag{3.96}$$

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}E\big[\big(\mathbf{Y}(t+\Delta t)-\mathbf{Y}(t)\big)\big(\mathbf{Y}(t+\Delta t)-\mathbf{Y}(t)\big)^T\,\big|\,\mathbf{Y}(t)=\mathbf{y}\big]=\mathbf{B}_0(t)\mathbf{R}_{\mathbf{W}}(t)\mathbf{B}_0^T(t) \tag{3.97}$$

The problem of determining the control forces, which minimize the defined performance index (3.93), may in analogy with the corresponding deterministic problem be solved by using dynamic programming. Because of perfect measurements, it is assumed, that the observations $\mathbf{Y}(t)=\mathbf{y}$ are known without errors at the time t. Further, state observations prior to the time t are redundant, due to the Markov property. Hence, the expected performance index in (3.93) is replaced by the optimal cost function on condition that $\mathbf{Y}(t)=\mathbf{y}$

$$J^*(\mathbf{y},t)=\min_{\{\mathbf{F}(\tau),\tau\in[t,T]\}}E\big[J[\mathbf{Y},\mathbf{F},t]\,\big|\,\mathbf{Y}(t)=\mathbf{y}\big] \tag{3.98}$$

$$J[\mathbf{Y},\mathbf{F},t]=L_0(\mathbf{Y}(t),t)+\int_t^T L_1(\mathbf{Y}(\tau),\mathbf{F}(\tau),\tau)\,d\tau \tag{3.99}$$

Analogous to (3.20) the optimal cost function can be written

$$\begin{aligned}J^*(\mathbf{y},t)\;=\;\min_{\mathbf{F}(t)}\Big\{&E\big[J^*(\mathbf{Y}(t+\Delta t),t+\Delta t)\,\big|\,\mathbf{Y}(t)=\mathbf{y}\big]\\&+\;E\Big[\int_t^{t+\Delta t}L_1(\mathbf{Y}(\tau),\mathbf{F}(\tau),\tau)\,d\tau\,\Big|\,\mathbf{Y}(t)=\mathbf{y}\Big]\Big\}\end{aligned} \tag{3.100}$$

$J^*(\mathbf{Y}(t+\Delta t),t+\Delta t)$ is a stochastic variable, where the spatial variable $\mathbf{y}$ has been replaced by $\mathbf{Y}(t+\Delta t)$ in $J^*(\mathbf{y},t+\Delta t)$.

Due to (3.96), (3.97), the following adequate Taylor expansions to the first order in Δt are applied to the conditional expectation of this quantity.

$$\begin{aligned}&E\left[J^*(\mathbf{Y}(t+\Delta t),t+\Delta t)\mid\mathbf{Y}(t)=\mathbf{y}\right]=\\&\quad J^*(\mathbf{y},t)+\frac{\partial J^*(\mathbf{y},t)}{\partial t}\Delta t+\left(\frac{\partial J^*(\mathbf{y},t)}{\partial\mathbf{y}}\right)^T E\left[\dot{\mathbf{Y}}(t)\Delta t\,\middle|\,\mathbf{Y}(t)=\mathbf{y}\right]\\&\quad+\frac{1}{2}E\left[\dot{\mathbf{Y}}^T(t)\frac{\partial^2 J^*(\mathbf{y},t)}{\partial\mathbf{y}^2}\dot{\mathbf{Y}}(t)(\Delta t)^2\,\middle|\,\mathbf{Y}(t)=\mathbf{y}\right]+o(\Delta t)\\&\quad=J^*(\mathbf{y},t)+\frac{\partial J^*(\mathbf{y},t)}{\partial t}\Delta t+\left(\frac{\partial J^*(\mathbf{y},t)}{\partial\mathbf{y}}\right)^T\Big(\mathbf{G}(\mathbf{y},t)+\mathbf{B}(t)E[\mathbf{F}(t)\,|\,\mathbf{Y}(t)=\mathbf{y}]\Big)\Delta t\end{aligned}$$

$$+\frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 J^*(\mathbf{y},t)}{\partial \mathbf{y}^2}\mathbf{B}_0(t)\mathbf{R_W}(t)\mathbf{B}_0^T(t)\right)\Delta t + \mathbf{o}(\Delta t) \tag{3.101}$$

$\mathrm{Tr}(\cdot)$ is the trace of the indicated matrices, defined as $\mathrm{Tr}(\mathbf{AB}) = A_{ij}B_{ji}$.

For $\int_t^{t+\Delta t} L_1(\mathbf{Y}(\tau), \mathbf{F}(\tau), \tau)\,\mathrm{d}\tau$ the same Taylor expansion is applied as in (3.19). Inserting (3.101) into (3.100) the following partial differential equation is then derived for $\Delta t \to 0$

$$-\frac{\partial J^*(\mathbf{y},t)}{\partial t} = \min_{\mathbf{F}(t)}\left\{E\left[L_1(\mathbf{Y}(t),\mathbf{F}(t),t)\,|\,\mathbf{Y}(t)=\mathbf{y}\right] + \left(\frac{\partial J^*(\mathbf{y},t)}{\partial \mathbf{y}}\right)^T\right.$$

$$\left.\left(\mathbf{G}(\mathbf{y},t) + \mathbf{B}(t)E[\mathbf{F}(t)\,|\,\mathbf{Y}(t)=\mathbf{y}]\right) + \frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 J^*(\mathbf{y},t)}{\partial \mathbf{y}^2}\mathbf{B}_0(t)\mathbf{R}_W(t)\mathbf{B}_0^T(t)\right)\right\} \tag{3.102}$$

This is the stochastic H-J-B equation for the case of the completely known state $\mathbf{Y}(t)$. Again (3.102) is solved backwards in time. From (3.93), the starting value for evaluation of $J^*(\mathbf{y}(t),t)$ is $E[L_0(\mathbf{Y}(T),T)]$, which is $L_0(\mathbf{Y}(T),T)$ because $\mathbf{Y}(T)$ can be measured without error. Generally, the H-J-B equation is quite difficult to solve for analytic expressions for the optimal cost and control. A special case in which it can be solved easily, is represented in the following.

Linear Quadratic Control

Consider a linear system described by the equation of motion (2.33) where $\mathbf{W}(t)$ is a white Gaussian vector process noise. To suppress the vibrations of this system, it is desired to determine the stochastic control vector process $\{\mathbf{F}^*(t),\ t\in[0,T]\}$ on $[0,T]$ which minimizes the expected quadratic cost

$$E[J[\mathbf{Y},\mathbf{F}]] = E\left[\frac{1}{2}\mathbf{Y}^T(T)\mathbf{S}(T)\mathbf{Y}(T) + \frac{1}{2}\int_0^T \mathbf{Y}^T(t)\mathbf{Q}(t)\mathbf{Y}(t) + \mathbf{F}^T(t)\mathbf{R}(t)\mathbf{F}(t)\,\mathrm{d}t\right] \tag{3.103}$$

This is called the *linear quadratic Gaussian* (LQG) control problem. In the derivation of the solution to this optimization problem the conditioned control force is written $\mathbf{f} = E[\mathbf{F}(t)\,|\,\mathbf{Y}(t)=\mathbf{y}]$. On condition that the optimal control force $\mathbf{F}(t)$ is given by a linear closed-loop law of the same form as equation (2.38), it then follows that $E\left[\mathbf{F}^T(t)\mathbf{R}(t)\mathbf{F}(t)\,|\,\mathbf{Y}(t)=\mathbf{y}\right] = \mathbf{f}^T\mathbf{R}(t)\mathbf{f}$. According to these definitions the H-J-B equation (3.102) becomes

$$-\frac{\partial J^*(\mathbf{y},t)}{\partial t} = \min_{\mathbf{f}}\left\{\frac{1}{2}\mathbf{y}^T\mathbf{Q}(t)\mathbf{y} + \frac{1}{2}\mathbf{f}^T\mathbf{R}(t)\mathbf{f} + \left(\frac{\partial J^*(\mathbf{y},t)}{\partial \mathbf{y}}\right)^T\left(\mathbf{A}(t)\mathbf{y} + \mathbf{B}(t)\mathbf{f}\right)\right.$$

$$\left.+\frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2 J^*(\mathbf{y},t)}{\partial \mathbf{y}^2}\mathbf{B}_0(t)\mathbf{R}_W(t)\mathbf{B}_0^T(t)\right)\right\} \tag{3.104}$$

Equation (3.104) is subject to the terminal condition $J^*(\mathbf{y},T) = \frac{1}{2}E\left[\mathbf{Y}^T(T)\mathbf{S}(T)\mathbf{Y}(T)\right]$, $\forall\,\mathbf{y}\in\mathbf{R}^n$. Assuming that the control force is not constrained, the minimization can be performed by differentiating with respect to $\mathbf{f}$. This leads to (3.22) for the optimal control in terms of $\partial J^*(\mathbf{y},t)/\partial \mathbf{y}$.

The H-J-B equation (3.104) is solved by assuming that

$$J^*(\mathbf{y},t) = \frac{1}{2}\mathbf{y}^T\mathbf{S}(t)\mathbf{y} + \frac{1}{2}\mathrm{Tr}\Big(\int_t^T \mathbf{S}(\tau)\mathbf{B}_0(\tau)\mathbf{R}_W(\tau)\mathbf{B}_0^T(\tau)\,\mathrm{d}\tau\Big) \tag{3.105}$$

for some yet unknown deterministic matrix function $\mathbf{S}(t)$. Using (3.22) and (3.105) in (3.104) results in the Ricatti differential equation (3.24) for $\mathbf{S}(t)$. The optimal control forces are obtained by substituting (3.105) into the control law (3.22), which results in equation (3.28).

Hence, the optimal control law obtained as a solution to the stochastic LQG problem with complete state information is identical to the closed-loop control obtained from the LQ problem.

3.3.2 Structural Systems with Stochastic External Excitations and Imperfect Measurements

Closed-loop control strategies based on a state space formulation entails information that describes the actual state of the system. However, it is usually impossible to measure the entire state vector directly and the observations may furthermore be corrupted by noise. This is the problem of *incomplete state information.*

With uncertain or indirect measurements it is desirable to transmit the maximum amount of information about the state. At the same time it is desirable to find a control force, which responds least possible to the measurement errors. Therefore, the best control strategy involves optimal state estimation as well as optimal control.

In the beginning of this section we discuss the same stochastic optimization problem of a non-linear system as in Section 3.3.1 except that the measurements are noise-corrupted. The design problem consists of minimizing the expected cost function (3.93) with the dynamic constraints (2.32). The external excitation has the statistic properties given by (3.94) and (3.95).

The realized measurements at the present time t are contained in a vector $\mathbf{z}(t)$, and the set of measured data in the interval $[0,t]$ is given by $\{\mathbf{Z}(\tau),\ \tau \in [0,t]\}$.

A solution to the optimal control in the case of incomplete state information may also be obtained by means of dynamic programming. However, with imperfect knowledge of the state $\mathbf{Y}(t)$, all expectations should rather be conditioned on the available information $\{\mathbf{z}(\tau),\ \tau \in [0,t]\}$.

Using the total representation theorem of probability theory $E[X] = E[E[X\,|\,Y]]$, the optimal cost function on condition of the observations $\{\mathbf{z}(\tau),\ \tau \in [0,t]\}$ can be written

$$\begin{aligned} &\min_{\{\mathbf{F}(\tau),\,\tau\in]t,T]\}} E\big[J[\mathbf{Y},\mathbf{F},t]\,\big|\,\{\mathbf{z}(\tau),\ \tau\in[0,t]\}\big] \\ &= \min_{\{\mathbf{F}(\tau),\,\tau\in]t,T]\}} E\Big[E\big[J[\mathbf{Y},\mathbf{F},t]\,\big|\,\mathbf{Y}(t),\{\mathbf{z}(\tau),\ \tau\in[0,t]\}\big]\Big] \\ &= E\big[J^*(\mathbf{Y}(t),t)\,\big|\,\{\mathbf{z}(\tau),\ \tau\in[0,t]\}\big] \end{aligned} \tag{3.106}$$

where $J[\mathbf{Y},\mathbf{F},t]$ is given by (3.99) and $J^*\big(\mathbf{Y}(t),t\big)$ is given by (3.98), when $\mathbf{y}$ is replaced by the random state $\mathbf{Y}(t)$. This result follows, because the future expected performance index $E\big[J[\mathbf{Y},\mathbf{F},t]\,\big|\,\mathbf{Y}(t),\{\mathbf{z}(\tau),\ \tau\in[0,t]\}\big]$ is only affected by the measurements up to and including the time t to the extent, $\mathbf{Y}(t)$ can be predicted, i.e. the dependence on the measurements is implicitly through $\mathbf{Y}(t)$.

Analogous to (3.20) the optimal cost function on condition of $\{\mathbf{z}(\tau),\ \tau\in[0,t]\}$ can be written

$$
\begin{aligned}
&E\big[J^*(\mathbf{Y}(t),t)\,\big|\,\{\mathbf{z}(\tau),\ \tau\in[0,t]\}\big]\\
&=E\left[J^*(\mathbf{Y}(t+\Delta t),t+\Delta t)+\int_t^{t+\Delta t}L_1(\mathbf{Y}(\tau),\mathbf{F}(\tau),\tau)\,d\tau\,\middle|\,\{\mathbf{z}(\tau),\ \tau\in[0,t]\}\right]
\end{aligned}
\tag{3.107}
$$

Next, $J^*(\mathbf{Y}(t+\Delta t),t+\Delta t)$ can be expanded into a Taylor series to first order, given by (3.101), where $\mathbf{y}$ is replaced by $\mathbf{Y}(t)$. In the limit as $\Delta t\to 0$ the following version of the H-J-B-equation is then derived, Stengel (1986)

$$
\begin{aligned}
0=\min_{\mathbf{F}(t)}E\Bigg[&\frac{\partial J^*(\mathbf{Y}(t),t)}{\partial t}+L_1(\mathbf{Y}(t),\mathbf{F}(t),t)\\
&+\left(\frac{\partial J^*(\mathbf{Y}(t),t)}{\partial\mathbf{y}}\right)^T\Big(\mathbf{G}(\mathbf{Y}(t),t)+\mathbf{B}(t)E[\mathbf{F}(t)\,|\,\mathbf{Y}(t)]\Big)\\
&+\frac{1}{2}\mathrm{Tr}\left(\frac{\partial^2J^*(\mathbf{Y}(t),t)}{\partial\mathbf{y}^2}\mathbf{B}_0(t)\mathbf{R}_W(t)\mathbf{B}_0^T(t)\right)\,\Bigg|\,\{\mathbf{z}(\tau),\ \tau\in[0,t]\}\Bigg]
\end{aligned}
\tag{3.108}
$$

Notice, that $\mathbf{F}(t)$ is assumed to be a function of $\mathbf{Y}(t)$. Hence, $\mathbf{F}(t)$ in $L_1(\mathbf{Y}(t),\mathbf{F}(t),t)$ is only affected by the measurements through $\mathbf{Y}(t)$.

Equation (3.108) is solved backwards in time for the optimal cost $E\big[J^*(\mathbf{Y}(t),t)\,\big|\,\{\mathbf{z}(\tau),\ \tau\in[0,t]\}\big]$ with the boundary condition

$$
E\big[J^*(\mathbf{Y},T)\,\big|\,\{\mathbf{z}(\tau),\ \tau\in[0,T]\}\big]=E\big[L_0(\mathbf{Y},T)\,\big|\,\{\mathbf{z}(\tau),\ \tau\in[0,T]\}\big]
\tag{3.109}
$$

This yields a principal difficulty. Propagation of the expected optimal cost back from the terminal time to some intermediate time t is conditioned on $\{\mathbf{z}(\tau),\ \tau\in[0,T]\}$. However, the available set of measurements is $\{\mathbf{z}(\tau),\ \tau\in[0,t]\}$. To solve the H-J-B equation (3.108) it is then necessary, somehow, to predict the conditioning effect of the unknown and hence random future measurements $\{\mathbf{Z}(\tau),\ \tau\in]t,T]\}$ on the expectation of the future cost function $J^*(\mathbf{Y}(t),t)$. If this effect can only be approximated, then the stochastic control can only be a sub-optimum. Conversely, if the $J^*(\mathbf{Y}(t),t)$ is independent on $\{\mathbf{Z}(\tau),\ \tau\in]t,T]\}$, as is the case for the LQG problem described in the following, stochastic control can be optimum.

Linear Quadratic Control

Let the stochastic quadratic cost function be formulated as (3.103). The objective is to minimize this performance index by proper choice of $\{\mathbf{F}(\tau),\ \tau\in[0,T]\}$, subject to the linear equation of motion (2.33). The external excitation $\{\mathbf{W}(t),\ t\in[0,\infty[\}$ is a white random process with mean and covariance given by (3.94) and (3.95). The initial condition is random with mean and covariance expressed by

$$E[\mathbf{Y}(0)] = \mathbf{y}_0 \tag{3.110}$$

$$E\left[(\mathbf{Y}(0) - \mathbf{y}_0)(\mathbf{Y}(0) - \mathbf{y}_0)^T\right] = \mathbf{p}_0 \tag{3.111}$$

The observations $\mathbf{Z}(t)$ are assumed to be linearly related to the noise-corrupted state vector $\mathbf{Y}(t)$,

$$\mathbf{Z}(t) = \mathbf{H}(t)\mathbf{Y}(t) + \mathbf{V}(t) \tag{3.112}$$

where $\mathbf{H}(t)$ is a general time-varying transformation matrix and $\{\mathbf{V}(t),\, \tau \in [0,\infty[\}$, is a white measurement noise with

$$E[\mathbf{V}(t)] = \mathbf{0} \tag{3.113}$$

$$E\left[\mathbf{V}(t)\mathbf{V}^T(\tau)\right] = \mathbf{R}_\mathbf{V}(t)\delta(t-\tau) \tag{3.114}$$

In (3.114) $\mathbf{R}_\mathbf{V}(t)$ is a time-varying intensity function. It is assumed that $\mathbf{Y}(0)$, $\{\mathbf{W}(t),\, t \in [0,\infty[\}$ and $\{\mathbf{V}(t),\, t \in [0,\infty[\}$ are independent. As an illustration of a control system with an observation model given by (3.112), let the structural behaviour be described by a state space model with a state vector containing relative displacements and velocities, i.e $\mathbf{Y}^T = \left[\mathbf{v}^T\ \dot{\mathbf{v}}^T\right]$, c.f. (2.34). Then, if the measurements are performed with strain-gauges and the structure is linear, the vector $\mathbf{Z}(t)$ containing these measurements may by means of geometrical interpolation be expressed in terms of the displacements $\mathbf{v}(t)$ by a linear model, and consequently by (3.112). The importance and usefulness of representing the measurement errors by white Gaussian processes stem from the fact, that noise most often is due to superposition of a large number of small, independent, random effects and then according to the central limit theorem it is Gaussian. However, the proposed observation model is only meaningful in a mathematical sense and not physically, since $\mathbf{Z}(t)$ is represented by a sum of two terms, $\mathbf{HY}(t)$ and $\mathbf{V}(t)$ with a finite and infinite variance, respectively, and hence it becomes infinite.

Using the total representation theorem, equation (3.103) can be written

$$\begin{aligned} E[J[\mathbf{Y},\mathbf{F}]] = \frac{1}{2}E\Big\{ & E\left[\mathbf{Y}^T(T)\mathbf{S}(T)\mathbf{Y}(T) \,\middle|\, \{\mathbf{Z}(\tau),\, \tau \in [0,T]\}\right] \\ & + \int_0^T E\left[\mathbf{Y}^T(t)\mathbf{Q}(t)\mathbf{Y}(t) + \mathbf{F}^T(t)\mathbf{R}(t)\mathbf{F}(t) \,\middle|\, \{\mathbf{Z}(\tau),\, \tau \in [0,t]\}\right] dt \Big\} \end{aligned} \tag{3.115}$$

$\mathbf{F}(t)$ is only implicitly dependent on the measurements $\{\mathbf{Z}(\tau),\, \tau \in [0,t]\}$ through $\mathbf{Y}(t)$. Hence, $E\left[\mathbf{F}^T(t)\mathbf{R}(t)\mathbf{F}(t) \,\middle|\, \{\mathbf{Z}(\tau),\, \tau \in [0,t]\}\right] = E\left[\mathbf{F}^T(t)\mathbf{R}(t)\mathbf{F}(t) \,\middle|\, \mathbf{Y}(t)\right]$. To be able to calculate expected values of combined stochastic variables of $\mathbf{Y}(t)$ conditioned on $\{\mathbf{Z}(\tau),\, \tau \in [0,t]\}$, the conditional probability density function $f_{\mathbf{Y}(t)}(\mathbf{y} \,|\, \{\mathbf{Z}(\tau),\, \tau \in [0,t])$ is required.

However, in this case, where the external loadings and the measurement errors are Gaussian and the system equations (2.33) and (3.112) are linear, $f_{\mathbf{Y}(t)}(\mathbf{y}\,|\,\{\mathbf{Z}(\tau),\ \tau\in[0,t]\})$ becomes Gaussian too. Hence, to calculate the expected cost it is sufficient to know the following conditioned first and second moments of $\mathbf{Y}(t)$

$$E\big[\mathbf{Y}(t)\,\big|\,\{\mathbf{Z}(\tau),\ \tau\in[0,t]\}\big]=\hat{\mathbf{Y}}(t) \tag{3.116}$$

$$E\Big[\Big(\mathbf{Y}(t)-\hat{\mathbf{Y}}(t)\Big)\Big(\mathbf{Y}(t)-\hat{\mathbf{Y}}(t)\Big)^T\,\Big|\,\{\mathbf{Z}(\tau),\ \tau\in[0,t]\}\Big]=\mathbf{P}(t) \tag{3.117}$$

The random vector $\hat{\mathbf{Y}}(t)$ and the random matrix $\mathbf{P}(t)$ are, respectively, the regressions of $\mathbf{Y}(t)$ and $\big(\mathbf{Y}(t)-\hat{\mathbf{Y}}(t)\big)\big(\mathbf{Y}(t)-\hat{\mathbf{Y}}(t)\big)^T$ on the measurements $\{\mathbf{Z}(\tau),\ \tau\in[0,t]\}$. By replacing each term in (3.115) by its trace, interchanging the expectation and trace operations, and using (3.116),(3.117) the performance index can be written as

$$\begin{aligned}E[J[\mathbf{Y},\mathbf{F}]]&=\frac{1}{2}E\Big[\mathrm{Tr}\Big(\mathbf{S}(T)\Big\{\mathbf{P}(T)+\hat{\mathbf{Y}}(T)\hat{\mathbf{Y}}^T(T)\Big\}\Big)\\&\quad+\mathrm{Tr}\Big(\int_0^T\mathbf{Q}(t)\Big\{\mathbf{P}(t)+\hat{\mathbf{Y}}(t)\hat{\mathbf{Y}}^T(t)\Big\}\,dt\Big)+\int_0^T E\big[\mathbf{F}^T(t)\mathbf{R}(t)\mathbf{F}(t)\,|\,\mathbf{Y}(t)\big]\,dt\Big]\\&=J_C+J_E\end{aligned} \tag{3.118}$$

where J_C is the *certainty-equivalent performance index*

$$\begin{aligned}J_C&=\frac{1}{2}E\Big[\mathrm{Tr}\Big(\mathbf{S}(T)\hat{\mathbf{Y}}(T)\hat{\mathbf{Y}}^T(T)\Big)\\&\quad+\mathrm{Tr}\Big(\int_0^T\mathbf{Q}(t)\hat{\mathbf{Y}}(t)\hat{\mathbf{Y}}^T(t)dt\Big)+\int_0^T E\big[\mathbf{F}^T(t)\mathbf{R}(t)\mathbf{F}(t)\,|\,\mathbf{Y}(t)\big]\,dt\Big]\end{aligned} \tag{3.119}$$

and J_E is the cost due to estimation error

$$J_E=\frac{1}{2}E\Big[\mathrm{Tr}\Big(\mathbf{S}(T)\mathbf{P}(T)\Big)+\mathrm{Tr}\Big(\int_0^T\mathbf{Q}(t)\mathbf{P}(t)\,dt\Big)\Big] \tag{3.120}$$

The conditional covariance matrix $\mathbf{P}(t)$ in (3.120) must be determined from the theory of state estimation. Within this subject the concern is to estimate the state of a system on the basis of measurements which may be indirect and uncertain. The estimate $\hat{\mathbf{Y}}(t)$ of the state vector may be determined from different criteria depending on the particular problem. A common technique is to determine an optimal estimate in the sense that it minimizes the expectation of the square of the error between the actual state, $\mathbf{Y}(t)$, and the corresponding estimate, $\hat{\mathbf{Y}}(t)$. This has come to be called the *minimum covariance of error estimate* or simply minimum-variance estimator. The estimation techniques of interest here are those determining conditional mean and variance estimates as defined in (3.116) and (3.117). However, one may quite easily show that the state estimate determined from the minimum-variance criterion is the conditional mean estimate.

Equation (3.117) defines the conditional covariance of the state estimate error. According to the total representation theorem the unconditioned covariance matrix is then given as

$$\mathbf{p}(t) = E\Big[\big(\mathbf{Y}(t) - \hat{\mathbf{Y}}(t)\big)\big(\mathbf{Y}(t) - \hat{\mathbf{Y}}(t)\big)^T\Big] = E\big[\mathbf{P}(t)\big] \tag{3.121}$$

The optimal state estimate is obtained by minimizing the norm of $\mathbf{p}(t)$ given by the Kalman filter equations, see Appendix A. Then the error covariance matrix $\mathbf{p}(t)$ develops according to the differential equation,

$$\dot{\mathbf{p}}(t) = \mathbf{A}(t)\mathbf{p}(t) + \mathbf{p}(t)\mathbf{A}^T(t) + \mathbf{B}_0(t)\mathbf{R}_W(t)\mathbf{B}_0^T(t) - \mathbf{p}(t)\mathbf{H}^T(t)\mathbf{R}_{\mathbf{V}}^{-1}(t)\mathbf{H}(t)\mathbf{p}(t) \quad ,$$
$$\mathbf{p}(0) = \mathbf{p}_0 \tag{3.122}$$

and the state estimate update is determined from

$$\dot{\hat{\mathbf{Y}}}(t) = \mathbf{A}(t)\hat{\mathbf{Y}}(t) + \mathbf{B}(t)\mathbf{F}(t) + \mathbf{K}_f(t)\Big(\mathbf{Z}(t) - \mathbf{H}(t)\hat{\mathbf{Y}}(t)\Big) \quad , \qquad \hat{\mathbf{Y}}(0) = \mathbf{y}_0 \tag{3.123}$$

where $\mathbf{K}_f(t)$ is a Kalman filter gain defined as

$$\mathbf{K}_f(t) = \mathbf{p}(t)\mathbf{H}^T(t)\mathbf{R}_{\mathbf{V}}^{-1}(t) \tag{3.124}$$

It appears from (3.122) that $\mathbf{p}(t)$ is independent of the statistics of the measurements $\{\mathbf{Z}(\tau),\ \tau \in [0,t]\}$ and the state vectors $\{\mathbf{Y}(\tau),\ \tau \in [0,t]\}$ beyond the noise covariance matrices $\mathbf{R_V}(t)$ and $\mathbf{R_W}(t)$. Especially, $\mathbf{p}(t)$ is independent of the statistics of previous control forces $\{\mathbf{F}(\tau),\ \tau \in [0,t]\}$. This implies that J_E in (3.120) is unaffected by $\mathbf{F}(t)$, and therefore, the control forces $\{\mathbf{F}(\tau),\ \tau \in [0,T]\}$ that minimize J_C also minimize J. The optimal control is then obtained by minimizing the certainty-equivalent cost J_C subject to the dynamic constraint (3.123). Since the last term in (3.123) is a zero-mean white noise process this is an optimization problem identical to the linear stochastic control problem in Section 3.3.1 with $\hat{\mathbf{Y}}(t)$ replacing $\mathbf{Y}(t)$. Thus, the optimal control law takes the same form as (3.28), i.e.

$$\mathbf{F}(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T\mathbf{S}(t)\hat{\mathbf{Y}}(t) \tag{3.125}$$

where $\mathbf{S}(t)$ is the Ricatti matrix obtained from (3.24). This is a quite important result known as the *separation principle.* It can be stated as follows. The control law derived from the stochastic optimal control problem with incomplete state information is independent of the optimal estimation algorithm, and vice versa. This is clear: The estimation algorithm (3.122)-(3.124) does not contain the control law (3.125), and vice versa. Furthermore, the results satisfy the *certainty-equivalence property.* If a stochastic optimal control problem possesses this property, then the control function is the same as the deterministic optimal control function. The latter is satisfied because the closed-loop control law (3.28) obtained as a solution to the deterministic LQ problem takes the same form as (3.28).

The importance of the separation and certainty-equivalence property is that the LQG regulator design can be accomplished in two separate stages: The Kalman filter design and the control feedback design.

3.4 Conclusions

Optimal control strategies for active vibration suppression of linear elastic structures may be formulated on the basis of either a distributed or discrete parameter model. For the distributed parameter system, it is difficult to determine a continuous optimal feedback control law which optimizes the system response according to a specified quadratic criterion. It is seen to be so, because the control is a function of both time and spatial coordinates. Even when one is able to determine this optimal feedback control law, one faces the difficulty of implementing it, because of the demand of distributed sensors and actuators.

To circumvent this problem, the main emphasis has been placed on the design of control algorithms for discretized structures controlled by a finite number of actuators. For this class of systems it has been shown how an optimal open-closed loop control law can be determined by minimizing a quadratic performance index. This solution is also applicable when the control design is accomplished in modal space, and it becomes especially simple when the optimal modal control force is going to be determined independently for each mode. The latter strategy, called independent modal space control, requires that the number of controlled modes must be equal to the number of actuators in order to synthesize the actual control forces from the modal control forces.

The designed optimal control law is a "backward-in-time" solution. This name is used, because a set of differential equations involving the external loadings has to be solved backwards from the final time, before the control can be evaluated. Unfortunately, many environmental loads are not known a priori, and therefore, it is not possible to determine the optimal control forces.

A common approach is to disregard the open-loop term and only use closed-loop control. This solution is optimal if there are no external loadings. However, the same closed-loop control is shown to be optimal, if the external excitations can be modelled by a Gaussian white noise process, and the optimality conditions are formulated so as to minimize the expected cost of the previous quadratic performance index. Moreover, the certainty-equivalence property of the linear quadratic solution has been proved, which states that the deterministic closed-loop control function is also optimal in the case of stochastic control with incomplete state information.

To circumvent the assumptions of external loadings known ahead in time or given as "white" noise, the optimal control problem is solved by using the technique of invariant embedding. This method leads to a "forward-in-time" solution for the control forces. The developed control algorithm can be generalized to include non-linear structural systems, and its feasibility is studied in Chapter 5, where the general optimal control problem of non-linear structures is treated.

Chapter 4

Implementation of Active Control

Implementation of active control possesses some practical problems which may be taken into account by application of the control schemes developed in the preceding. Besides the real practical problems of, for example, how to transmit the desired control forces via one or more actuators, other important problems arise from a practical point of view, because the control design has been based on idealized system descriptions under ideal conditions. A consequence of using models with a finite number of degrees-of-freedom to describe a real structure and application of discrete actuators and sensors is the so-called spillover effects, (Section 4.1). Another important aspect is the discrete time nature of a control algorithm to be executed by a digital computer, (Section 4.2). This phenomenon is treated along with the problem arising from an inevitable time delay between the measured feedback information and execution of the associated feedback force.

4.1 Spillover Effects

A civil engineering structure is by nature a continuum whose dynamic behaviour generally may be described by a continuously distributed parameter system. Especially for large structures, application of optimal control is in such cases encountered with complicated numerical problems, cf. Section 3.1. Because of these difficulties, a model reduction procedure is generally carried out, whereby the distributed parameter system is reduced to a model with a finite number of degrees-of-freedom. Subsequently, the control parameters are calculated on the basis of this discretized system. One approach is to discretize the structure in space by expanding the distributed dependent variable into a finite series of eigenfunctions. This technique was demonstrated in Section 3.2.1 in relation to the optimal control problem. Other conceivable approaches are the finite element method, the finite difference method, or the boundary element method. The latter approach was used by O'Donoghue and Atluri (1986) in connection with optimal control of large space structures.

Control spillover means that the vibrations in modes, which are not controlled by the control algorithm, are excited by the control force actuators. Vibrations in uncontrolled modes will then always be present, no matter how much the controlled modes are damped. Actually, the spillover vibrations will increase if the control forces are increased. *Observation spillover* means that the motion of the controlled modes cannot be reconstructed, because the displacement and velocity measurements from the sensors are influenced by the motion in the uncontrolled modes, Balas (1978).

In the analysis of spillover effects it is customary to consider a certain number of modes, which may be determined from the equations of motion describing either a distributed-parameter system or a discretized system. These modes are referred to as *modelled.* In some cases, it may not be feasible to control all the modelled modes, and then the modelled modes are further divided into two categories, *controlled* and *residual* modes.

When a control algorithm, designed for a discretized continuum is synthesized and applied to a real structure inevitable errors such as control and observation spillover are introduced, which may degrade the structural performance seriously, Soong and Chang (1982). The effect of control spillover has been explored in a qualitative manner by Meirovitch and Baruh (1982) in connection with independent modal space control of distributed-parameter systems. Both the influence of control and observation spillover has been studied by Chung et al. (1989) when only some of the modes associated with a discretized system are controlled.

In the following the two phenomena - control and observation spillover - are elaborated in connection with the control of a distributed parameter system.

4.1.1 Control Spillover

The following derivations assume implicitly that the modelled modes are readily obtained as eigensolutions of a distributed-parameter system, as described in section 3.2.1. Further, let the modal equations of motion representing the controlled and residual modes, respectively, be internally decoupled. Then the equations of motion for the modal coordinates $\mathbf{q}_c$ and $\mathbf{q}_r$ associated with the controlled and residual modes are given, respectively, by (3.42) and the following differential equations

$$\ddot{\mathbf{q}}_r + \mathbf{\Xi}_r\dot{\mathbf{q}}_r + \mathbf{\Lambda}_r\mathbf{q}_r = \mathbf{w}_r(t) + \mathbf{f}_r(t) \tag{4.1}$$

In (4.1) the following vectors are introduced

$$\begin{aligned} \mathbf{q}_r &= \begin{bmatrix} q_{n_c+1} & q_{n_c+2} & \cdots & q_{n_c+n_r} \end{bmatrix}^T \\ \mathbf{w}_r &= \begin{bmatrix} w_{n_c+1} & w_{n_c+2} & \cdots & w_{n_c+n_r} \end{bmatrix}^T \\ \mathbf{f}_r &= \begin{bmatrix} f_{n_c+1} & f_{n_c+2} & \cdots & f_{n_c+n_r} \end{bmatrix}^T \end{aligned} \tag{4.2}$$

where n_c and n_r are the number of controlled and residual modes, respectively. The components of the matrices $\mathbf{\Xi}_r$ and $\mathbf{\Lambda}_r$ are obtained from equations (3.44) and (3.45) by replacing the index c with r, and by increasing the indices on the right-hand side by n_c. By analogy with the specification of the modal control forces $\mathbf{f}_c$ given by (3.47) and (3.46), the modal control forces $\mathbf{f}_r$ associated with the residual modes can be expressed as

$$\mathbf{f}_r(t) = \mathbf{b}_r\mathbf{F}(t) \tag{4.3}$$

The components of the matrix $\mathbf{b}_r$ in (4.3) are obtained from the definition of the modal control forces (3.41) by analogy with $\mathbf{b}_c$, cf. (3.46). The physical control forces $\mathbf{F}(t)$ are assumed to be given in closed-loop form expressed in terms of the modal coordinates as

$$\mathbf{F}(t) = -\mathbf{g}_{c1}\mathbf{q}_c(t) - \mathbf{g}_{c2}\dot{\mathbf{q}}_c(t) \tag{4.4}$$

The feedback gains $\mathbf{g}_{c1}$ and $\mathbf{g}_{c2}$ in (4.4) could for instance be determined on the basis of the optimal control formalism, cf. (3.50). The action of the feedback forces is considered by substituting (4.3) and (4.4) into (3.42) and (4.1)

$$\ddot{\mathbf{q}}_c + (\mathbf{\Xi}_c + \mathbf{b}_c\mathbf{g}_{c2})\,\dot{\mathbf{q}}_c + (\mathbf{\Lambda}_c + \mathbf{b}_c\mathbf{g}_{c1})\,\mathbf{q}_c = \mathbf{w}_c(t) \tag{4.5}$$

$$\ddot{\mathbf{q}}_r + \mathbf{\Xi}_r\dot{\mathbf{q}}_r + \mathbf{\Lambda}_r\mathbf{q}_r = \mathbf{w}_r(t) - \mathbf{b}_r\left(\mathbf{g}_{c1}\mathbf{q}_c + \mathbf{g}_{c2}\dot{\mathbf{q}}_c\right) \tag{4.6}$$

Hereby, it is found that the resulting modal damping and stiffness are modified for the controlled modes, while there are no changes in the dynamic properties of the residual modes. On the other hand, equation (4.6) implies that the feedback control forces represented on the right hand side excite the residual modes. This phenomenon is known as *control spillover*. Theoretically, control spillover can be eliminated if the controllers are implemented in such a way that the control forces are applied at the nodal points of all residual mode shapes. Then $\mathbf{b}_r$ is equal to a zero matrix.

4.1.2 Observation Spillover

Implementation of the control law (4.4) requires the knowledge of the modal displacement $\mathbf{q}_c(t)$ and modal velocity $\dot{\mathbf{q}}_c(t)$ at all times. In this section the problem of estimating $q_{c,m}$ and $\dot{q}_{c,m}$, $m = 1, 2, \ldots, n_c$ from discrete measurements is considered. It is assumed that there are n_s sensors capable of measuring displacements and velocities at discrete points. Z_α designates the measured displacement of the point with the referential coordinates $\mathbf{x}_\alpha^T = \begin{bmatrix} x_{\alpha1} & x_{\alpha2} & x_{\alpha3} \end{bmatrix}$ in a direction given by the unit vector $\mathbf{e}_\alpha^T = \begin{bmatrix} e_{\alpha1} & e_{\alpha2} & e_{\alpha3} \end{bmatrix}$. Since the displacement field $\mathbf{u}(\mathbf{x}, t)$ is given by (3.38), the measured displacements can be expressed as

$$Z_\alpha(t) = \mathbf{e}_\alpha^T\mathbf{U}^{(0)}(\mathbf{x}_\alpha, t) + \sum_{m=1}^{\infty}\mathbf{e}_\alpha^T\mathbf{\Phi}^{(m)}(\mathbf{x}_\alpha)q_m(t) \quad , \quad \alpha = 1, \ldots, n_s \tag{4.7}$$

A vector of measured displacements $\mathbf{Z} = \begin{bmatrix} Z_1 & Z_2 & \cdots & Z_{n_s} \end{bmatrix}$ is introduced. Equation (4.7) can then be written in terms of $\mathbf{q}_c(t)$ and $\mathbf{q}_r(t)$ as

$$\mathbf{Z}(t) = \mathbf{Z}^{(0)}(t) + \mathbf{H}_c\mathbf{q}_c(t) + \mathbf{H}_r\mathbf{q}_r(t) + \Delta\mathbf{Z}(t) \tag{4.8}$$

where the components of $\mathbf{Z}^{(0)}(t)$, $\mathbf{H}_c$, $\mathbf{H}_r$ and $\Delta\mathbf{Z}(t)$ are given by

$$\mathbf{Z}^{(0)T}(t) = \begin{bmatrix} \mathbf{e}_1^T\mathbf{U}^{(0)}(\mathbf{x}_1, t) & \cdots & \mathbf{e}_{n_s}^T\mathbf{U}^{(0)}(\mathbf{x}_{n_s}, t) \end{bmatrix} \tag{4.9}$$

$$\mathbf{H}_c = \begin{bmatrix} \mathbf{e}_1^T\mathbf{\Phi}^{(1)}(\mathbf{x}_1) & \cdots & \mathbf{e}_1^T\mathbf{\Phi}^{(n_c)}(\mathbf{x}_1) \\ \vdots & \ddots & \vdots \\ \mathbf{e}_{n_s}^T\mathbf{\Phi}^{(1)}(\mathbf{x}_{n_s}) & \cdots & \mathbf{e}_{n_s}^T\mathbf{\Phi}^{(n_c)}(\mathbf{x}_{n_s}) \end{bmatrix} \tag{4.10}$$

$$\mathbf{H}_r = \begin{bmatrix} \mathbf{e}_1^T \mathbf{\Phi}^{(n_c+1)}(\mathbf{x}_1) & \cdots & \mathbf{e}_1^T \mathbf{\Phi}^{(n_c+n_r)}(\mathbf{x}_1) \\ \vdots & \ddots & \vdots \\ \mathbf{e}_{n_s}^T \mathbf{\Phi}^{(n_c+1)}(\mathbf{x}_{n_s}) & \cdots & \mathbf{e}_{n_s}^T \mathbf{\Phi}^{(n_c+n_r)}(\mathbf{x}_{n_s}) \end{bmatrix} \tag{4.11}$$

$$\Delta Z_\alpha(t) = \sum_{m=n_c+n_r+1}^{\infty} \mathbf{e}_\alpha^T \mathbf{\Phi}^{(m)}(\mathbf{x}_\alpha) q_m(t) \quad , \quad \alpha = 1, 2, \ldots, n_s \tag{4.12}$$

In order to reconstruct the controlled modal coordinates it is assumed, that only the controlled modes contribute to $\mathbf{Z}(t)$, corresponding to $\mathbf{q}_r(t) = \mathbf{0}$ and $\Delta\mathbf{Z}(t) = \mathbf{0}$. Furthermore, it is required that $\mathbf{H}_c$ is quadratic, which is satisfied if the number of sensors are equal the number controlled modes, i.e. $n_s = n_c$, and is non-singular. On these assumptions equation (4.8) yields

$$\mathbf{q}_c(t) \simeq \mathbf{H}_c^{-1} \left(\mathbf{Z}(t) - \mathbf{Z}^{(0)}(t)\right) \tag{4.13}$$

In the same manner the modal velocities $\dot{\mathbf{q}}_c(t)$ can be determined from the measured velocities $\dot{\mathbf{Z}}(t), \dot{\mathbf{Z}}^{(0)}(t)$. This gives

$$\dot{\mathbf{q}}_c(t) \simeq \mathbf{H}_c^{-1} \left(\dot{\mathbf{Z}}(t) - \dot{\mathbf{Z}}^{(0)}(t)\right) \tag{4.14}$$

The feedback forces based on the approximate modal displacements $\mathbf{q}_c(t)$ and modal velocities $\dot{\mathbf{q}}_c(t)$ given by (4.13) and (4.14) are obtained by substituting into (4.4). Using (4.8), the result can be written

$$\mathbf{F}(t) = -\mathbf{g}_{c1}\mathbf{H}_c^{-1}\Big(\mathbf{H}_c\mathbf{q}_c + \mathbf{H}_r\mathbf{q}_r + \Delta\mathbf{Z}\Big) - \mathbf{g}_{c2}\mathbf{H}_c^{-1}\Big(\mathbf{H}_c\dot{\mathbf{q}}_c + \mathbf{H}_r\dot{\mathbf{q}}_r + \Delta\mathbf{Z}\Big) \tag{4.15}$$

Inserting (4.15) into (3.42) and (4.1), provides

$$\begin{aligned} &\ddot{\mathbf{q}}_c + (\mathbf{\Xi}_c + \mathbf{b}_c\mathbf{g}_{c2})\,\dot{\mathbf{q}}_c + (\mathbf{\Lambda}_c + \mathbf{b}_c\mathbf{g}_{c1})\,\mathbf{q}_c = \\ &\qquad \mathbf{w}_c(t) - \mathbf{b}_c\Big(\mathbf{g}_{c1}\mathbf{H}_c^{-1}\Big[\mathbf{H}_r\mathbf{q}_r + \Delta\mathbf{Z}\Big] + \mathbf{g}_{c2}\mathbf{H}_c^{-1}\Big[\mathbf{H}_r\dot{\mathbf{q}}_r + \Delta\dot{\mathbf{Z}}\Big]\Big) \end{aligned} \tag{4.16}$$

$$\begin{aligned} &\ddot{\mathbf{q}}_r + (\mathbf{\Xi}_r + \mathbf{b}_r\mathbf{g}_{c2}\mathbf{H}_c^{-1}\mathbf{H}_r)\,\dot{\mathbf{q}}_r + (\mathbf{\Lambda}_r + \mathbf{b}_r\mathbf{g}_{c1}\mathbf{H}_c^{-1}\mathbf{H}_r)\,\mathbf{q}_r = \\ &\qquad \mathbf{w}_r(t) - \mathbf{b}_r\Big(\mathbf{g}_{c1}\Big[\mathbf{q}_c + \mathbf{H}_c^{-1}\Delta\mathbf{Z}\Big] + \mathbf{g}_{c2}\Big[\dot{\mathbf{q}}_c + \mathbf{H}_c^{-1}\Delta\dot{\mathbf{Z}}\Big]\Big) \end{aligned} \tag{4.17}$$

It appears from (4.17) that the dynamic properties of the modal coordinates associated with the residual modes are no longer the same as in (4.6). They are now influenced by the feedback gains and the eigenproperties of the controlled modes through $\mathbf{H}_c$. This phenomenon is referred to as *observation spillover.* A consequence of observation spillover may be, that the resulting damping or stiffness matrix of (4.16) become negative definite, rendering the control system unstable. As in the case of control spillover, observation spillover can also be eliminated if the sensors are mounted at the nodal points of the residual mode shapes.

4.2 Discrete Time Control

Implementation of a control algorithm in real-time requires on-line calculations in the control-loop. Nowadays, due to flexibility, reliability and speed, there is a trend to apply digital computers to these on-line calculations instead of analog technology. As a consequence, the measurements of the structural response are digitized by analog-digital (A/D) converters and control forces are applied in the form of piecewise step functions through the use of digital-analog (D/A) converters. Fig. 4.1 shows a basic scheme of a closed-loop computer control system.

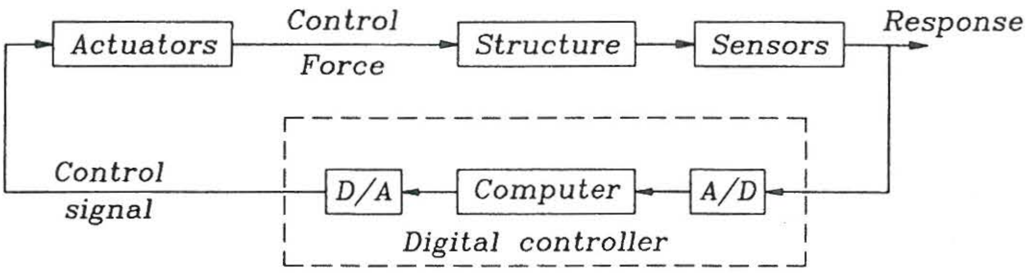

Figure 4.1: Closed-loop on-line computer control scheme.

The control algorithms developed in Chapter 3 imply continuous functions for the measurements and the control forces. As mentioned, these functions have a discrete-time nature for the digital controller and therefore, it is necessary to reformulate the already developed continuous-time control algorithms.

In treating the continuous-time systems, the assumptions have been made that all operations in the control-loop as shown in fig. 4.1 can be performed instantaneously. In reality, however, time is consumed in processing measured information, in performing on-line computation, and in executing the control forces required. This time delay causes unsynchronized application of the control forces, which, besides affecting the control effectiveness, may cause instability of the system.

The importance of time-delay compensation in structural control has been demonstrated in the laboratory by Chung et al. (1988,1989). In these papers a compensation method based on a continuous-time formulation has been proposed, which include modification of the control gain by performing phase shift of the measured state variables in the modal domain. Different control algorithms with discrete-time nature have also been proposed. A method proposed by Rodellar et al. (1987a) computes the control force at the "present" time that produce a desired structural response over a certain prediction horizon. This predictive control algorithm has also been modified to compensate for time-delay, Rodellar et al. (1987b). Other algorithms based on the optimal control formalism with discrete-time nature and with time-delay compensation have been proposed by Chung et al. (1987), Pu and Hsu (1988).

With the time-delay in mind, a discrete-time control algorithm is here formulated on the basis of the latter papers. Firstly, an optimal control algorithm is developed in the case of deterministic external loadings and a completely measurable state vector. Secondly, the corresponding stochastic problem with incomplete state information is considered.

4.2.1 Linear Quadratic Control

The discretized linear structural system in consideration is represented in state space form. Assuming a time delay denoted by Δt in the control forces the equation of motion can be written as, cf. (2.33)

$$\dot{\mathbf{Y}}(t) = \mathbf{A}(t)\mathbf{Y}(t) + \mathbf{B}_0(t)\mathbf{W}(t) + \mathbf{B}(t)\mathbf{F}(t - \Delta t) \quad , \qquad \mathbf{Y}(0) = \mathbf{Y}_0 \tag{4.18}$$

Suppose the state vector $\mathbf{Y}(t)$ is sampled with a period h for on-line calculation. Let the sampling instants be represented by the set $\{t_k = kh,\ k = 0, 1, \ldots\}$. The sampled state is thus represented by the signal $\{\mathbf{Y}(kh),\ k = 0, 1, \ldots\}$.

A common situation in computer control is that the D/A converter is so constructed, that it holds the analog signal constant until a new conversion is commanded. Since the control signal is discontinuous, it is necessary to specify its behaviour at the discontinuities. The convention that the signal is continuous from the right is adopted. By choosing the sampling instants, t_k, as the time instants when the control changes, the control force between two consecutive sampling instants is thus written as $\mathbf{F}(t) = \mathbf{F}(kh)$, $t \in [kh, kh+h[$. The external excitation is assumed to be constant between two consecutive sampling instants, and it is specified with the same notation as the control forces, i.e. $\mathbf{W}(t) = \mathbf{W}(kh)$, $t \in [kh, kh+h[$.

The time-delay is assumed to be a multiple of the sampling period, i.e. $\Delta t = dh$, where d is an integer. Given the state at the sampling instant t_k the state at the next sampling instant is then obtained by solving (4.18). This yields,

$$\begin{aligned} \mathbf{Y}(kh+h) &= \mathbf{\Theta}(kh+h, kh)\mathbf{Y}(kh) + \mathbf{\Gamma}_0(kh+h, kh)\mathbf{W}(kh) \\ &\quad + \mathbf{\Gamma}(kh+h, kh)\mathbf{F}(kh-dh) \quad , \qquad \mathbf{Y}(0) = \mathbf{Y}_0 \end{aligned} \tag{4.19}$$

In the above equation $\mathbf{\Theta}(\cdot,\cdot)$ is the fundamental matrix of (4.18) satisfying

$$\frac{\partial}{\partial t}\mathbf{\Theta}(t,\tau) = \mathbf{A}(t)\mathbf{\Theta}(t,\tau) \quad , \quad t > \tau \quad , \quad \mathbf{\Theta}(\tau,\tau) = \mathbf{I} \tag{4.20}$$

Further, the following matrices are defined in (4.19)

$$\mathbf{\Gamma}_0(t,s) = \int_s^t \mathbf{\Theta}(t,\tau)\mathbf{B}_0(\tau)\,\mathrm{d}\tau \tag{4.21}$$

$$\mathbf{\Gamma}(t,s) = \int_s^t \mathbf{\Theta}(t,\tau)\mathbf{B}(\tau)\,\mathrm{d}\tau \tag{4.22}$$

The difference equation (4.19) describes the discrete-time system to be controlled.

To suppress the structural vibrations over a time interval $[dh, Nh]$, the control forces $\mathbf{F}(kh)$ will be selected to minimize the quadratic performance index

$$J[\mathbf{Y},\mathbf{F}] = \frac{1}{2}\mathbf{Y}^T(Nh)\mathbf{S}(Nh)\mathbf{Y}(Nh)$$

$$+\frac{1}{2}\sum_{k=d}^{N-1}\left[\mathbf{Y}^T(kh)\mathbf{Q}(kh)\mathbf{Y}(kh)+\mathbf{F}^T(kh-dh)\mathbf{R}(kh-dh)\mathbf{F}(kh-dh)\right] \quad (4.23)$$

in which $\mathbf{S}(Nh)$ and $\mathbf{Q}(Nh)$ are symmetric positive semi-definite weighting matrices, and $\mathbf{R}(kh)$ is a symmetric positive definite weighting matrix. $J[\mathbf{Y},\mathbf{F}]$ here specifies functional dependence on the discrete functions $\mathbf{Y}(kh)$ and $\mathbf{F}(kh)$.

The objective is to determine the control sequence $\{\mathbf{F}^*(kh),\ k=0,1,\ldots,N-d-1\}$ to minimize (4.23) subject to the dynamic constraint (4.19). To solve this discrete-time optimal control problem the dynamic programming approach will be used. Hence, define $J^*(\mathbf{Y}(kh),kh)$ as the minimum cost of the process, starting at t_k. Obviously, for $k=N$

$$J^*(\mathbf{Y}(Nh),Nh)=\frac{1}{2}\mathbf{Y}^T(Nh)\mathbf{S}(Nh)\mathbf{Y}(Nh) \quad (4.24)$$

$J^*(\mathbf{Y}(kh),kh)$, $k=d,d+1,\ldots,N$ is determined recursively backwards from the terminal time t_N. Thus, the optimal cost from time t_k on is equal to

$$\begin{aligned}J^*(\mathbf{Y}(kh),kh)=\min_{\mathbf{F}(kh-dh)}&\left\{\frac{1}{2}\mathbf{Y}^T(kh)\mathbf{Q}(kh)\mathbf{Y}(kh)\right.\\&\left.+\frac{1}{2}\mathbf{F}^T(kh-dh)\mathbf{R}(kh-dh)\mathbf{F}(kh-dh)+J^*(\mathbf{Y}(kh+h),kh+h)\right\}\\k=d,d+1,\ldots,N-1&\end{aligned} \quad (4.25)$$

This non-linear difference equation has the boundary condition given by (4.24). It is solved by assuming a solution of the form

$$J^*(\mathbf{y},kh)=\frac{1}{2}\mathbf{y}^T\mathbf{S}(kh)\mathbf{y}+\mathbf{y}^T\mathbf{T}(kh)+V(kh) \quad , \qquad k=d,d+1,\ldots,N \quad (4.26)$$

where $\mathbf{S}$ is a symmetric positive semi-definite $n\times n$ matrix, $\mathbf{T}$ is an $n\times 1$ vector, and V is a scalar. They will be selected so as to force (4.26) to satisfy (4.25).

First (4.26) is inserted into the right hand side of (4.25), and next (4.19) is used to eliminate $\mathbf{Y}(kh+h)$. Since there are no restrictions on $\mathbf{F}(kh-dh)$, the minimizing $\mathbf{F}(kh-dh)$ is found by setting the gradient with respect to $\mathbf{F}(kh-dh)$ equal to zero. After some calculations this yields

$$\begin{aligned}\mathbf{F}^*(kh-dh)&=-\mathbf{D}(kh)\boldsymbol{\Gamma}^T(kh+h,kh)\Big[\mathbf{S}(kh+h)\boldsymbol{\Theta}(kh+h,kh)\mathbf{Y}(kh)\\&\qquad+\mathbf{S}(kh+h)\boldsymbol{\Gamma}_0(kh+h,kh)\mathbf{W}(kh)+\mathbf{T}(kh+h)\Big]\end{aligned} \quad (4.27)$$

where

$$\mathbf{D}(kh)=\left[\mathbf{R}(kh-dh)+\boldsymbol{\Gamma}^T(kh+h,kh)\mathbf{S}(kh+h)\boldsymbol{\Gamma}(kh+h,kh)\right]^{-1} \quad (4.28)$$

Substituting (4.26)-(4.28) into (4.25) and utilizing (4.19), the assumed form of $J^*(\mathbf{y},kh)$ can be forced to be a solution for all $\mathbf{y}$ by requiring that the quadratic terms, the linear terms, and the terms not involving $\mathbf{y}$ all balance individually. This requires

$$\mathbf{S}(kh) = \mathbf{Q}(kh) + \boldsymbol{\Theta}^T(kh+h,kh)\left[\mathbf{S}(kh+h) - \mathbf{S}(kh+h)\boldsymbol{\Gamma}(kh+h,kh)\mathbf{D}(kh)\boldsymbol{\Gamma}^T(kh+h,kh)\mathbf{S}(kh+h)\right]\boldsymbol{\Theta}(kh+h,kh) \quad (4.29)$$

$$\begin{aligned}\mathbf{T}(kh) = \boldsymbol{\Theta}^T(kh+h,kh)\Big[&\mathbf{T}(kh+h) + \mathbf{S}(kh+h)\boldsymbol{\Gamma}_0(kh+h,kh)\mathbf{W}(kh)\\ &- \mathbf{S}(kh+h)\boldsymbol{\Gamma}(kh+h,kh)\mathbf{D}(kh)\boldsymbol{\Gamma}^T(kh+h,kh)\mathbf{S}(kh+h)\boldsymbol{\Gamma}_0(kh+h,kh)\mathbf{W}(kh)\\ &- \mathbf{S}(kh+h)\boldsymbol{\Gamma}(kh+h,kh)\mathbf{D}(kh)\boldsymbol{\Gamma}^T(kh+h,kh)\mathbf{T}(kh+h)\Big]\end{aligned} \quad (4.30)$$

$$\begin{aligned}V(kh) = \frac{1}{2}&\mathbf{W}^T(kh)\boldsymbol{\Gamma}_0^T(kh+h,kh)\Big[\mathbf{S}(kh+h)\\ &-\mathbf{S}(kh+h)\boldsymbol{\Gamma}(kh+h,kh)\mathbf{D}(kh)\boldsymbol{\Gamma}^T(kh+h,kh)\mathbf{S}(kh+h)\Big]\boldsymbol{\Gamma}_0(kh+h,kh)\mathbf{W}(kh)\\ &- \mathbf{W}^T(kh)\Big[\boldsymbol{\Gamma}_0^T(kh+h,kh)\mathbf{S}(kh+h)\boldsymbol{\Gamma}(kh+h,kh)\mathbf{D}(kh)\boldsymbol{\Gamma}^T(kh+h,kh)\\ &- \boldsymbol{\Gamma}_0^T(kh+h,kh)\Big]\mathbf{T}(kh+h)\\ &+ \frac{1}{2}\mathbf{T}(kh+h)\boldsymbol{\Gamma}(kh+h,kh)\mathbf{D}(kh)\boldsymbol{\Gamma}^T(kh+h,kh)\mathbf{T}(kh+h) + V(kh+h)\end{aligned} \quad (4.31)$$

The boundary conditions are $\mathbf{S}(Nh) = \mathbf{S}_N$, $\mathbf{T}(Nh) = \mathbf{0}$, and $V(Nh) = 0$. Equation (4.29) is the discrete-time matrix Ricatti equation.

In order to determine the optimal control force sequence $\{\mathbf{F}^*(kh),\ k = 0, 1, \ldots, N-d-1\}$ given by (4.27), equation (4.29) is solved first, and next (4.30). Actually, (4.31) never needs to be solved if the only interest is finding the optimal control. $V(kh)$ is only needed if $J^*(\mathbf{y}, kh)$ has to be calculated. Solving (4.30) backwards in time requires an a priori knowledge of the external loadings $\mathbf{W}(kh)$, which is generally not possible for excitations encountered in structural engineering. Therefore, the control law will be formulated by assuming $\mathbf{W}(kh) = \mathbf{0}$, $k = d, d+1, \ldots, N-1$, corresponding an autonomous system. Then (4.30) is a homogeneous equation with zero initial conditions $\mathbf{T}(Nh) = \mathbf{0}$, so $\mathbf{T}(kh) = \mathbf{0}$, $k = d, d+1, \ldots, N-1$. Additionally, $V(kh) = 0$, $k = d, d+1, \ldots, N-1$. Given these assumptions the control signal at the time t_k can be written as

$$\mathbf{F}^*(kh) = -\mathbf{G}_c(kh+dh)\mathbf{Y}(kh+dh) \quad (4.32)$$

where $\mathbf{G}_c(kh)$ is a feedback gain sequence defined as

$$\mathbf{G}_c(kh) = \mathbf{D}(kh)\boldsymbol{\Gamma}^T(kh+h,kh)\mathbf{S}(kh+h)\boldsymbol{\Theta}(kh+h,kh) \quad (4.33)$$

To compute the feedback control force $\mathbf{F}(kh)$ at the time t_k it appears from (4.32) that knowledge of the state $\mathbf{Y}(kh + dh)$ at the future time t_{k+d} is needed. This is obtained by utilizing the state equation to predict $\mathbf{Y}(kh + dh)$ from the measured state $\mathbf{Y}(kh)$. By applying (4.19) repeatedly, where $\mathbf{F}(kh)$ is given by (4.32), the state $\mathbf{Y}(kh + dh)$ can be expressed as

$$\begin{aligned}\mathbf{Y}(kh + dh) &= \mathbf{\Theta}(kh + dh, kh)\mathbf{Y}(kh) \\ &+ \sum_{m=0}^{d-1} \mathbf{\Theta}(kh + dh, kh + mh + h) \Big(\mathbf{\Gamma}_0(kh + mh + h, kh + mh)\mathbf{W}(kh + mh) \\ &+ \mathbf{\Gamma}(kh + mh + h, kh + mh)\mathbf{F}(kh - dh + mh)\Big) \end{aligned} \tag{4.34}$$

To predict $\mathbf{Y}(kh+dh)$ from $\mathbf{Y}(kh)$ according to (4.34) it is required that the external loading $\mathbf{W}(kh+mh)$, $m = 0, 1, \ldots, d-1$ is known a priori. This is usually not the case, and so, the states must be reconstructed by neglecting the term with $\mathbf{W}(kh)$ in (4.34).

Steady-State Feedback Gains

A time-invariant system is considered, in which the system matrices $\mathbf{A}$, $\mathbf{B}_0$ and $\mathbf{B}$ are constant. Hereby, the fundamental matrix $\mathbf{\Theta}(t, \tau)$ becomes

$$\mathbf{\Theta}(t, \tau) = e^{\mathbf{A}(t-\tau)} \quad , \qquad t \geq \tau \tag{4.35}$$

where $e^{\mathbf{A}t}$ is given by (2.44). The following matrices are defined

$$\mathbf{A}_d = \mathbf{\Theta}(kh + h, kh) = \mathbf{\Theta}(h, 0) \tag{4.36}$$

$$\mathbf{B}_{d0} = \mathbf{\Gamma}_0(kh + h, kh) = \mathbf{\Gamma}_0(h, 0) \tag{4.37}$$

$$\mathbf{B}_d = \mathbf{\Gamma}(kh + h, kh) = \mathbf{\Gamma}(h, 0) \tag{4.38}$$

According to the definitions (4.36)-(4.38) the difference equation (4.19) for the structural behaviour can then be written as

$$\mathbf{Y}(kh + h) = \mathbf{A}_d\mathbf{Y}(kh) + \mathbf{B}_{d0}\mathbf{W}(kh) + \mathbf{B}_d\mathbf{F}(kh - dh) \tag{4.39}$$

Furthermore, the weighting matrices $\mathbf{Q}$ and $\mathbf{R}$ are assumed to be time-invariant. Then, if the operating time of the digital control system is longer than the duration of the external excitations, it is often justifiable to use only the steady-state constant feedback gains. This is because $\mathbf{S}(kh)$, obtained by solving (4.29) backwards in time, reached constant values in a few time steps from the terminal time. When the sequence $\mathbf{S}$ has approached a constant value, we have $\mathbf{S}(kh - h) = \mathbf{S}(kh)$. The steady-state solution of the discrete-time Ricatti difference equation (4.29) is thus determined from the system of non-linear equations

$$\mathbf{S} = \mathbf{A}_d^T \left(\mathbf{S} - \mathbf{S}\mathbf{B}_d \left(\mathbf{R} + \mathbf{B}_d^T \mathbf{S} \mathbf{B}_d \right)^{-1} \mathbf{B}_d^T \mathbf{S} \right) \mathbf{A}_d + \mathbf{Q} \tag{4.40}$$

Under the given assumptions of time-independent system and weighting matrices constant feedback gains can then be determined by applying the solution of (4.40) in (4.33). Since the steady-state gain matrix $\mathbf{G}_c$ is time-invariant and independent of the state of the structure, it can be precomputed and stored in the computer before the recursive control law (4.32) is implemented.

4.2.2 Linear Quadratic Gaussian Regulator

The control algorithm developed in the preceding section was based on the assumption that the entire state vector can be measured without disturbances. Furthermore, implementation of the optimal control law implied a priori knowledge of the external excitation. Following, a control algorithm based on more realistic conditions is presented. For further reference, see Stengel (1986) and Sage and White (1977).

For the structure in consideration the discrete-time equation of motion is represented by (4.19). The external excitation $\{\mathbf{W}(kh),\ k = 0, 1, \ldots, N-1\}$ is modelled by a non-stationary white Gaussian random sequence independent of the initial condition $\mathbf{Y}_0$, where the first and second order moments are given by

$$E[\mathbf{W}(kh)] = \mathbf{0} \tag{4.41}$$

$$E\left[\mathbf{W}(kh)\mathbf{W}^T(lh)\right] = \begin{cases} \mathbf{0} & , \quad k \neq l \\ \frac{1}{h}\mathbf{R}_\mathbf{W}(kh) & , \quad k = l \end{cases} \tag{4.42}$$

$\mathbf{R}_\mathbf{W}(kh)$ is the covariance matrix of an equivalent continuous time Gaussian white noise process, as specified by (3.94) and (3.95). The initial condition on the state $\mathbf{Y}(kh)$ at the time $t = 0$ is a Gaussian random variable with mean $\mathbf{y}_0$ and covariance matrix $\mathbf{p}_0$, see (3.110) and (3.111).

The measurements $\mathbf{Z}(kh)$ of the structural movements at stage kh are assumed to be linearly related to the state by

$$\mathbf{Z}(kh) = \mathbf{H}(kh)\mathbf{Y}(kh) + \mathbf{V}(kh) \quad , \qquad k = 0, 1, \ldots N-1 \tag{4.43}$$

In the above equation $\mathbf{H}$ is a time-varying transformation matrix of dimension $n_s \times n$, and $\{\mathbf{V}(kh),\ k = 0, 1, \ldots, N-1\}$ is a white Gaussian random sequence independent of $\{\mathbf{W}(kh),\ k = 0, 1, \ldots, N-1\}$ and the initial conditions $\mathbf{Y}_0$, where the first and second moments are given by

$$E[\mathbf{V}(kh)] = \mathbf{0} \tag{4.44}$$

$$E\left[\mathbf{V}(kh)\mathbf{V}^T(lh)\right] = \begin{cases} \mathbf{0} & , \quad k \neq l \\ \frac{1}{h}\mathbf{R_V}(kh) & , \quad k = l \end{cases} \tag{4.45}$$

where $\mathbf{R_V}(kh)$ is the covariance matrix of an equivalent continuous time Gaussian white noise process, as specified by (3.113) and (3.114). The objective of the control design is to find the control forces $\{\mathbf{F}(kh),\ k = 0, 1, \ldots, N-d-1\}$ which minimize the expected cost

$$J = \frac{1}{2}E\left[\mathbf{Y}^T(Nh)\mathbf{S}(Nh)\mathbf{Y}(Nh) + \sum_{k=d}^{N-1}\left(\mathbf{Y}^T(kh)\mathbf{Q}(kh)\mathbf{Y}(kh) + \mathbf{F}^T(kh-dh)\mathbf{R}(kh-dh)\mathbf{F}(kh-dh)\right)\right] \tag{4.46}$$

subject to the dynamic constraint (4.19). This is the discrete-time linear quadratic Gaussian (LQG) problem.

The result will not be derived here. Derivations would be in much the same way of the continuous-time case that has already been developed. Not surprisingly, it would show that the control algorithm is composed of a state estimator followed by a deterministic closed-loop LQ regulator. Thus, the separation and certainty-equivalence principle also holds for discrete-time systems.

According to the certainty-equivalence principle the optimal control is a feedback of the optimal state estimate $\hat{\mathbf{Y}}$ and not $\mathbf{Y}$. As a consequence of the time delay between the measuring of the state and the application of the associated feedback force, the control force becomes a feedback of the state estimate $\hat{\mathbf{Y}}(kh+dh)$. The available information to determine this estimate is the measurements $\{\mathbf{Z}(jh),\ j = 0, 1, \ldots, k\}$ up to and including the time t_k. The predicted estimate based on this information set is denoted by $\hat{\mathbf{Y}}(kh+dh|kh)$.

The optimal control law is then obtained by replacing $\mathbf{Y}(kh+dh)$ in (4.32) by $\hat{\mathbf{Y}}(kh+dh|kh)$. This yields

$$\mathbf{F}^*(kh) = -\mathbf{G}_c(kh+dh)\hat{\mathbf{Y}}(kh+dh|kh) \tag{4.47}$$

The feedback gain $\mathbf{G}_c(kh)$ in (4.47) is the same as that determined in connection with the associated deterministic optimal control problem, cf. (4.33). From the derivation of the optimal control law, it follows that the estimate $\hat{\mathbf{Y}}(kh+dh|kh)$ has to be taken as the conditional mean $E\big[\mathbf{Y}(kh+dh)|\{\mathbf{Z}(jh),\ j = 0, 1, \ldots, k\}\big]$. By analogy with the associated continuous-time estimation problem, this estimate is equivalent to the minimum covariance of error estimate. Hence, the predicted estimate may be determined from the discrete-time version of the Kalman filter equations, cf. Appendix A.

Given the state estimate $\hat{\mathbf{Y}}(kh-h) = \hat{\mathbf{Y}}(kh-h|kh-h)$ at the previous time instant, t_{k-1}, the equation of motion is first used to propagate this estimate to the present sampling instant, t_k, without regard to measurements. This gives

$$\hat{\mathbf{Y}}(kh|kh-h) = \mathbf{\Theta}(kh, kh-h)\hat{\mathbf{Y}}(kh-h) + \mathbf{\Gamma}(kh, kh-h)\mathbf{F}(kh-dh-h) \quad ,$$
$$\hat{\mathbf{Y}}(0|-1) = \mathbf{Y}_0 \tag{4.48}$$

Next, the state estimate update $\hat{\mathbf{Y}}(kh)$ is determined from

$$\hat{\mathbf{Y}}(kh) = \mathbf{Y}(kh|kh-h) + \mathbf{K}_f(kh)\left[\mathbf{Z}(kh) - \mathbf{H}(kh)\hat{\mathbf{Y}}(kh|kh-h)\right] \tag{4.49}$$

in which $\mathbf{K}_f$ is the filter gain. The criterion for choosing $\mathbf{K}_f$ is to minimize the norm of the estimation error matrix $\mathbf{p}(kh)$ defined as

$$\mathbf{p}(kh) = E\left[\left(\mathbf{Y}(kh) - \hat{\mathbf{Y}}(kh|kh-h)\right)\left(\mathbf{Y}(kh) - \hat{\mathbf{Y}}(kh|kh-h)\right)^T\right] \tag{4.50}$$

Then, see (A.25)

$$\mathbf{K}_f(kh) = \mathbf{H}^T(kh)\left[\frac{1}{h}\mathbf{R}_\mathbf{V}(kh) + \mathbf{H}(kh)\mathbf{p}(kh)\mathbf{H}^T(kh)\right]^{-1} \tag{4.51}$$

where the error covariance matrix propagates according to

$$\begin{aligned} \mathbf{p}(kh+h) = & -\mathbf{\Theta}(kh+h,kh)\mathbf{p}(kh)\mathbf{H}^T(kh)\left[\frac{1}{h}\mathbf{R}_\mathbf{V}(kh) + \mathbf{H}(kh)\mathbf{p}(kh)\mathbf{H}^T(kh)\right]^{-1} \\ & \mathbf{H}(kh)\mathbf{p}(kh)\mathbf{\Theta}^T(kh+h,kh) + \mathbf{\Theta}(kh+h,kh)\mathbf{p}(kh)\mathbf{\Theta}^T(kh+h,kh) \\ & + \frac{1}{h}\mathbf{\Gamma}_0(kh+h,kh)\mathbf{R}_\mathbf{W}(kh)\mathbf{\Gamma}_0^T(kh+h,kh) \end{aligned} \tag{4.52}$$

The initial conditions for (4.52) are $\mathbf{p}(0) = \mathbf{p}_0$, where $\mathbf{p}_0$ is defined in (3.111). The filter equations, which are sequential in nature, are processed on-line. However, the filter gain and covariance computations are unaffected by the control forces, and therefore, equations (4.51), (4.52) can be solved beforehand, if the structural dynamics are known.

When the "filtered" estimate $\hat{\mathbf{Y}}(kh)$ is determined from the measurements up to and including the time t_k, the predicted estimate $\hat{\mathbf{Y}}(kh+dh|kh)$ is finally obtained by applying (4.48) repeatedly. This gives

$$\begin{aligned} \hat{\mathbf{Y}}(kh+dh|kh) = & \mathbf{\Theta}(kh+dh,kh)\hat{\mathbf{Y}}(kh|kh) \\ & + \sum_{m=0}^{d-1}\mathbf{\Theta}(kh+dh,kh+mk+h)\mathbf{\Gamma}(kh+mh+h,kh+mh)\mathbf{F}(kh-dh+mh) \end{aligned} \tag{4.53}$$

Implementation of a closed-loop control system based on the predicted estimate (4.53) may cause some difficulties in practice. To utilize the filter equations it is assumed that exact descriptions of system dynamics and noise statistics are known. This is often not the case. When some information is not precise the effect of uncertainties for instance in connection with the structural dynamics may be investigated by sensitivity analysis.

4.3 Conclusions

Some of the most important practical problems to be taken into account in a control scheme development has been treated in this chapter. The undesired spillover effects arise when a feedback control law based on a discretized model is used to reduce vibrations of a real structure. This problem has received a great deal of attention by researchers, and different methods of spillover compensation have been proposed. However, a thorough investigation has been omitted. Future research of this basic practical problem will among others be about demonstrating the practicability of the proposed compensation methods. Such experimental studies must be performed with large multi-degree-of-freedom structural models and experimental tests of that magnitude are beyond the scope of this study.

In the second part of this chapter, an optimal control scheme is developed with regard to the discrete time nature in application of a control algorithm, and the time delay, due to the fact that all operations in a control-loop cannot be performed instantaneously. This control scheme forms the basis of a series of experimental tests, which were carried out in a laboratory and are described in Chapter 6.

Chapter 5

Optimal Control of Nonlinear Structural Systems

Active control has traditionally been applied to linear elastic structures and there is little information on its extension to nonlinear structures. However, the rationale for such an extension is simple. Under severe environmental loads, large flexible structures tend to exhibit a nonlinear behaviour as the deformation increases. If these nonlinear deformations are excessive, extensive repairs or even demolition of the structure may be necessary. If, on the other hand, the structure is allowed to enter the nonlinear range, but we subsequently intervene with an active control mechanism to prevent the kind of deformations that would render the structure unserviceable, then an additional factor of safety is introduced and a lighter design is possible.

The feasibility in controlling general nonlinear structures by means of pulse generators has been explored by Masri et al. (1982) and Miller et al. (1988). In a work done by Reinhorn et al. (1987) an active pulse control approach for reducing the response of structures undergoing inelastic deformations has been developed. The proposed control strategies in these papers fall into the category of bounded state control. In Abdel-Rohman and Nayfeh (1987b) the nonlinear oscillations of a hinged-hinged single-span bridge are controlled by a combined active and passive control mechanism. Here, a nonlinearity of geometrical nature has been eliminated and the resonance effect has been reduced.

Concerning active vibration control of nonlinear structures there is a lack of research within the design of control schemes based on the optimal control formalism. The basic concept of optimal control of nonlinear systems is not new. It has been the subject of electrical and control engineering for the last 3 decades. However, within these areas only eigenvibrations from given initial conditions are usually to be controlled. External loadings do not appear in the equations of motion. Against it, external dynamic loads are the major sources inducing vibrations that should be controlled considering civil engineering structures. As a result, active control of nonlinear civil engineering structures presents a unique problem to be solved.

In this chapter emphasis is placed on optimal control with respect to a quadratic performance index. Using this criterion two suboptimal control algorithms are presented (Section 5.1). Next, the control design based on a stochastic description of the external loading is treated, and in this connection the problem of measurement uncertainties is addressed (Section 5.2). The utility of the proposed control strategies is demonstrated by considering a hysteretic one-degree-of-freedom oscillator subjected to a white Gaussian noise (Section 5.3). Until now the dynamic parameters of the structural system have been assumed to be known. Next, the problem of incorporating an identification scheme which will sequentially update information of unknown system dynamics is addressed (Section 5.4).

5.1 Optimal Control

The concern of this section is the optimal control of non-linear time-varying structures with respect to a quadratic performance index. It is assumed that the structural system is represented by the non-linear state space model (2.32). With the general performance index (2.52) the necessary conditions for optimality have been given in Section 2.3.2, see equations (3.68)-(3.70), and the optimal control law was found to satisfy (3.72). In the special case of a quadratic performance index (2.53) it is easily verified that the optimal control law still satisfies (3.72) in the same form. By eliminating the optimal control force $\mathbf{F}(t)$ from the equation of motion (2.32) and the adjoint state equation (3.69), it is found that the optimal state $\mathbf{Y}(t)$ and the adjoint state vector of Lagrange multipliers $\boldsymbol{\lambda}(t)$ must satisfy

$$\dot{\mathbf{Y}}(t) = \mathbf{G}(\mathbf{Y},t) + \mathbf{B}_0(t)\mathbf{W}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\boldsymbol{\lambda}(t) \quad , \qquad \mathbf{Y}(0) = \mathbf{Y}_0 \tag{5.1}$$

$$\dot{\boldsymbol{\lambda}}(t) = -\mathbf{Q}(t)\mathbf{Y}(t) - \frac{\partial \mathbf{G}^T(\mathbf{Y},t)}{\partial \mathbf{Y}}\boldsymbol{\lambda}(t) \quad , \qquad \boldsymbol{\lambda}(T) = \mathbf{0} \tag{5.2}$$

The terminal value in (5.2) implies that the weighting matrix for the terminal state is taken as zero, i.e. $\mathbf{S}(T) = \mathbf{0}$, cf. (3.69).

The optimal control forces $\mathbf{F}^*(t)$ depend linearly on the the co-state vector $\boldsymbol{\lambda}(t)$, see (3.72). However, in general it is impossible to find a closed-form solution of the nonlinear two-point boundary-value problem (5.1) and (5.2) to obtain $\boldsymbol{\lambda}(t)$ and with it, the control forces. Neither, would it in general be possible to determine a numerical solution and derive a precalculated control since the external loading history $\mathbf{W}(t)$ is not known a priori. Then in order to obtain a closed-loop controller which may be able to function in an on-line fashion for vibration suppression of real structures, there is no alternative except to use some suboptimal control algorithm.

Within the area of electrical and control engineering a number of methods have been proposed to determine approximate optimal control laws. Essentially, these take the form of a truncated series expansion of the control law and the optimal value function $J^*(\mathbf{Y},t)$ satisfying the H-J-B equation.

Some techniques, Lukes (1969), Willemstein (1977), Yoshida and Loparo (1989), consist of representing the optimal feedback law as Taylor series either in the state or co-state vector. Suboptimal control laws proposed by Baldwin and Sims Williams (1969) and Nishikawa et al. (1971) are, respectively, provided by a truncated perturbation series expansion for the optimal value function and the control law. Perturbation methods normally require, that a dominant linear term in the drift vector $\mathbf{G}(\mathbf{Y},t)$ and a dominant quadratic term in the performance index can be identified. Finally, Pearson (1962) has made a control algorithm by linearizing the equations of motion about their present state and then solving the optimal control problem for this instantaneous linear system.

In this chapter design of control algorithms for structural systems with a hysteretic restoring force is dealt with. For these systems the drift vector $\mathbf{G}(\mathbf{Y},t)$ is a highly non-linear and non-analytical function of the state variables, for which no dominant linear term can be identified. Consequently, perturbation analysis is unsuitable in this case. Since, the proposed suboptimal control algorithms based on global Taylor series expansions require that the nonlinearities are analytical, these algorithms cannot be applied either. Furthermore, out of these methods only the one proposed by Baldwin and Sims Williams (1969) forms the basis of an on-line controller, as desired. Some other control algorithms have been proposed for optimal control of non-linear systems, Durbeck (1965), Friedland (1966), Garrad et al. (1967), Burghart (1968), Eller and Aggarwal (1968), Chin and Pengilley (1970). However, none of these techniques are applicable for on-line control of structures for which the drift vector is non-analytical. The only control algorithm which is directly usable for non-analytical systems and which is formulated to operate on-line, is Pearson's equivalent linearization method. This method is presented briefly in the following section.

5.1.1 Pearson's Equivalent Linearization Method

The basis of this approach is the adoption of a linearized time and state dependent model of the nonlinear equations of motion (2.32). Hence, it is assumed that the equation of motion in the interval $]t,T]$ can be represented by the linearized equations of motion

$$\begin{aligned}\dot{\mathbf{Y}}(\tau) &= \mathbf{G}(\mathbf{y}(t),\tau)+\mathbf{A}(t,\tau)\big(\mathbf{Y}(\tau)-\mathbf{y}(t)\big)+\mathbf{B}_0(\tau)\mathbf{W}(\tau)+\mathbf{B}(\tau)\mathbf{F}(\tau) \quad , \quad \tau\in\,]t,T] \\ \mathbf{Y}(t) &= \mathbf{y}(t)\end{aligned} \tag{5.3}$$

where

$$\mathbf{A}(t,\tau)=\left.\frac{\partial\mathbf{G}(\mathbf{Y}(t),\tau)}{\partial\mathbf{Y}}\right|_{\mathbf{Y}(t)=\mathbf{y}(t)} \tag{5.4}$$

It is assumed, that the state $\mathbf{Y}(t)=\mathbf{y}(t)$ at the time t can be measured perfectly. Notice, that even for hysteretic systems, the gradient (5.4) exists almost everywhere in the state space.

By considering the present time t as fixed, an optimal control law is derived for (5.3) to minimize the performance index

$$J[\mathbf{Y},\mathbf{F},t]=\frac{1}{2}\int_t^T\left(\mathbf{Y}^T(\tau)\mathbf{Q}(t)\mathbf{Y}(\tau)+\mathbf{F}^T(\tau)\mathbf{R}(t)\mathbf{F}(\tau)\right)\,d\tau \tag{5.5}$$

This controller approximates the optimal control for the non-linear structure at the time t. This problem was presented in Section 3.2. Using a closed-loop approximation, corresponding to ignoring the equivalent loading $\mathbf{G}(\mathbf{y}(t),\tau)-\mathbf{A}(t,\tau)\mathbf{y}(t)+\mathbf{B}(\tau)\mathbf{W}(\tau)$, this derivation lead to the control law (3.28). The obtained control law involves the steady-state solution of the algebraic Ricatti equation (3.30), which is solved at each instant of time t with the current system matrix $\mathbf{A}(t)=\mathbf{A}(t,t)$ defined by (5.4).

The described suboptimal control law is thus a linear feedback with a time-varying feedback gain, which is determined at each instant of time by minimizing a quadratic performance index subjected to the equivalent linearized equations of motion.

5.1.2 Invariant Embedding

At the beginning of Section 5.1 it was explained that solving the optimal quadratic control problem for non-linear structures by using calculus of variations leads to a non-linear TPBVP. In what follows an approximate solution based on the invariant embedding technique is presented for this TPBVP. Furthermore, a closed-loop controller is designed. The derivations are analogous to those given in Section 3.2.3 for the linear quadratic control problem.

The original TPBVP of (5.1) and (5.2) is imbedded into a class of more general single-point boundary-value problems by letting the terminal boundary condition on $\boldsymbol{\lambda}(t)$ take a general value $\mathbf{c}$ rather than $\mathbf{0}$. Furthermore, let $\mathbf{r}(\mathbf{c}, T)$ denote the missing final condition on $\mathbf{Y}(t)$ at the terminal time T. Then (3.80) is replaced by

$$\frac{\partial \mathbf{r}}{\partial T} + \frac{\partial \mathbf{r}}{\partial \mathbf{c}}\left(-\mathbf{Q}(T)\mathbf{r} - \frac{\partial \mathbf{G}^T(\mathbf{r},T)}{\partial \mathbf{r}}\mathbf{c}\right) = \mathbf{G}(\mathbf{r},T) + \mathbf{B}_0(T)\mathbf{W}(T) - \mathbf{B}(T)\mathbf{R}^{-1}(T)\mathbf{B}^T(T)\mathbf{c} \tag{5.6}$$

The perturbation solution in the parameter $\mathbf{c}$ as given by (3.81) is next applied. Concurrently, truncated Taylor series expansions are introduced in the form $\mathbf{G}(\mathbf{r},t) = \mathbf{G}(\boldsymbol{\xi},t) + \frac{\partial \mathbf{G}}{\partial \boldsymbol{\xi}}\mathbf{S}(t)\mathbf{c}$ and $\frac{\partial \mathbf{G}}{\partial \mathbf{r}} = \frac{\partial \mathbf{G}}{\partial \boldsymbol{\xi}}$. Subsequently, by analogy with the derivation of the invariant embedding equations (3.83) and (3.85), the assumed solution provides the following equations

$$\dot{\mathbf{Y}}(t) = \mathbf{G}(\mathbf{Y}(t),t) + \mathbf{S}(t)\mathbf{Q}(t)\mathbf{Y}(t) + \mathbf{B}_0(t)\mathbf{W}(t) \quad , \qquad \mathbf{Y}(0) = \mathbf{Y}_0 \tag{5.7}$$

$$\dot{\mathbf{S}}(t) = \frac{\partial \mathbf{G}(\mathbf{Y}(t),t)}{\partial \mathbf{Y}}\mathbf{S}(t) + \mathbf{S}(t)\frac{\partial \mathbf{G}^T(\mathbf{Y}(t),t)}{\partial \mathbf{Y}} + \mathbf{S}(t)\mathbf{Q}(t)\mathbf{S}(t) - \mathbf{B}(t)\mathbf{R}^{-1}(t)\mathbf{B}^T(t) \quad ,$$
$$\mathbf{S}(0) = \mathbf{0} \tag{5.8}$$

The differential equation (5.7) describes, in combination with (5.8), the controlled state, obtained by application of the invariant embedding technique. However, the objective of the control design is to develop a control law which determines the control force. To achieve the control law, the same technique is employed as for the associated linear control problem in Section 3.2.3.

The basic idea is to choose an algebraic form for the control law in which the coefficients will be determined. As for the linear structural system the control law is here assumed to be linear in the state vector as formulated by (2.38), where $\mathbf{G}_c(t)$ is an unknown feedback matrix to be determined. In the development of $\mathbf{G}_c(t)$, it is assumed that at each instant of time t the nonlinear structural system may be modeled by a linear system of the form (5.3), which was introduced in connection with Pearson's method. Insertion of the assumed control law (2.38) then yields

$$\dot{\mathbf{Y}}(\tau) = \left[\mathbf{A}(t) + \mathbf{B}(t)\mathbf{G}_c(t)\right]\mathbf{Y}(\tau) + \mathbf{G}(\mathbf{y}(t),t) - \mathbf{A}(t)\mathbf{y}(t) + \mathbf{B}_0(t)\mathbf{W}(\tau) \quad , \quad \tau \in]t,T]$$

$$\mathbf{Y}(t) = \mathbf{y}(t) \tag{5.9}$$

Correspondingly, a linear approach is adopted for the invariant embedding equation (5.7) of the form

$$\dot{\mathbf{Y}}(\tau) = [\mathbf{A}(t) + \mathbf{S}(t)\mathbf{Q}(t)]\mathbf{Y}(\tau) + \mathbf{G}(\mathbf{y}(t), t) - \mathbf{A}(t)\mathbf{y}(t) + \mathbf{B}_0(t)\mathbf{W}(\tau) \quad , \quad \tau \in]t, T]$$
$$\mathbf{Y}(t) = \mathbf{y}(t) \tag{5.10}$$

The matrix $\mathbf{A}(t) = \mathbf{A}(t,t)$ of the linear system equations (5.9) and (5.10) is given by (5.4) for $t = \tau$. The associated initial state $\mathbf{Y}(t) = \mathbf{y}(t)$ is assumed to be available at the present time t as measurements. The gain selection is then made at each instant of time t from the requirement that the eigenvalues of the system matrices $(\mathbf{A}(t) - \mathbf{B}(t)\mathbf{G}_c(t))$ and $(\mathbf{A}(t) + \mathbf{S}(t)\mathbf{Q}(t))$ must be equal and the deviation between the corresponding eigenvectors must be minimal. A method to determine $\mathbf{G}_c(t)$ from this requirement was represented in Section 3.2.3.

The proposed suboptimal control algorithm forms the basis of an on-line controller. During the control, $\mathbf{S}(t)$ is determined by forward integration of (5.8), in which, the partial derivatives are calculated from the present state $\mathbf{Y}(t) = \mathbf{y}(t)$. From the solution $\mathbf{S}(t)$ at the present time t, the feedback gain $\mathbf{G}_c(t)$ is determined as described, and finally substituted into the control law (2.38) to obtain the control force.

5.2 Stochastic Optimal Control

As in the design of control algorithms for linear structural systems, the design criteria for non-linear structures may be formulated according to a stochastic control formalism. In this connection the external excitation is represented by a stochastic process. Then the equation of motion constitutes a stochastic matrix differential equation and the control force is determined so that the expected value of the performance index defined in the deterministic formulation is minimized.

5.2.1 Perfect Measurements

The stochastic optimization of non-linear structures is considered in Section 3.3.1 for the case of perfect measurements. By using the dynamic programming approach the H-J-B equation for the optimal cost function $J^*(\mathbf{y}, t)$ was set up, cf. (3.102). However, this equation is generally extremely difficult to solve for analytic expressions for the optimal cost and control. Therefore, the control algorithms based on the stochastic optimal control formalism may be suboptimal like the algorithms based on the deterministic formulation.

A common approach in the literature is to divide the solution into nominal and perturbational parts, where the former represents an off-line (prior) solution and the latter represents an on-line (real-time) solution, see e.g. Stengel (1986). For that purpose the external loading is considered to be composed of a deterministic mean value and a random part, where the former is known a priori and the latter is assumed to be small and additive. Then the resulting suboptimal solution suggests that the optimal controlled state history could well be approximated by the sum of a deterministic non-linear control problem, computed off-line with the deterministic external loading, plus a stochastic neighbouring-optimal response. The neighbouring-optimal solution is based on a linear perturbation from the nominal response, and involves the solution of a stochastic linear quadratic control problem. The developments are summarized in the following.

The deterministic part of the external loading and the corresponding state is denoted $\mathbf{w}_0(t)$ and $\mathbf{y}_0(t)$, respectively. The nominal-optimal state, $\mathbf{y}_0^*(t)$, is calculated using a numerical method to solve the non-linear TPBVP of (5.1) and (5.2) with $\mathbf{w}_0(t)$, $\mathbf{y}_0(t)$ replacing $\mathbf{W}(t)$, $\mathbf{Y}(t)$. The associated nominal-optimal control force is denoted $\mathbf{f}_0^*(t)$. Notice, that $\mathbf{y}_0(t)$ and $\mathbf{f}_0^*(t)$ cannot be interpreted as the mean-value functions of the optimal solution to the stochastic control problem (5.1) and (5.2), unless these are linear.

An approximate solution for the incremental state $\Delta\mathbf{Y}(t) = \mathbf{Y}(t) - \mathbf{y}_0^*(t)$ due to the stochastic part of the external loading is obtained by linearizing the equation of motion around the nominal state,

$$\Delta\dot{\mathbf{Y}}(t) = \mathbf{A}(t)\Delta\mathbf{Y}(t) + \mathbf{B}_0(t)\Delta\mathbf{W}(t) + \mathbf{B}(t)\Delta\mathbf{F}(t) \quad , \qquad \Delta\mathbf{Y}(0) = \mathbf{0} \tag{5.11}$$

where $\Delta\mathbf{W}(t) = \mathbf{W}(t) - \mathbf{w}_0(t)$, $\Delta\mathbf{F}(t) = \mathbf{F}(t) - \mathbf{f}_0^*(t)$ and

$$\mathbf{A}(t) = \left.\frac{\partial\mathbf{G}(\mathbf{Y}(t),t)}{\partial\mathbf{Y}}\right|_{\mathbf{Y}(t)=\mathbf{y}_0^*(t)} \tag{5.12}$$

The additive random loading $\{\Delta\mathbf{W}(t),\, t \in [0,T]\}$ is assumed to be a non-stationary zero-mean Gaussian white noise process.

The associated performance index is obtained from an expansion of the expected quadratic cost (3.103) about $\mathbf{y}_0^*(t)$ and $\mathbf{f}_0^*(t)$ retaining terms up to second order. By collecting terms of zero order, first order and second order in $\Delta\mathbf{Y}(T)$, $\Delta\mathbf{F}(T)$, $\Delta\mathbf{Y}(t)$ and $\Delta\mathbf{F}(t)$ this yields

$$\begin{aligned} &E\big[J[\mathbf{y}_0^*,\mathbf{f}_0^*,\Delta\mathbf{Y},\Delta\mathbf{F}]\big] = \\ &\quad E\Big[\frac{1}{2}\mathbf{y}_0^{*T}(T)\mathbf{S}(T)\mathbf{y}_0^*(T) + \frac{1}{2}\int_0^T \left(\mathbf{y}_0^{*T}(t)\mathbf{Q}(t)\mathbf{y}_0^*(t) + \mathbf{f}_0^{*T}(t)\mathbf{R}(t)\mathbf{f}_0^*(t)\right) dt \\ &\quad + \mathbf{y}_0^{*T}(T)\mathbf{S}(T)\Delta\mathbf{Y}(T) + \int_0^T \left(\mathbf{y}_0^{*T}(t)\mathbf{Q}(t)\Delta\mathbf{Y}(t) + \mathbf{f}_0^{*T}(t)\mathbf{R}(t)\Delta\mathbf{F}(t)\right) dt \\ &\quad + \frac{1}{2}\Delta\mathbf{Y}^T(T)\mathbf{S}(T)\Delta\mathbf{Y}(T) + \frac{1}{2}\int_0^T \left(\Delta\mathbf{Y}^T(t)\mathbf{Q}(t)\Delta\mathbf{Y}(t) + \Delta\mathbf{F}^T(t)\mathbf{R}(t)\Delta\mathbf{F}(t)\right) dt\Big] \end{aligned} \tag{5.13}$$

Then, it is seen from (3.66),(5.11) that first order terms cancel, if $\mathbf{y}_0^*(t)$, $\mathbf{f}_0^*(t)$ is the optimal control due to the loading $\mathbf{w}_0(t)$. Hence, the optimal control of the incremental state vector is obtained by minimizing the performance index

$$E\big[J[\Delta\mathbf{Y},\Delta\mathbf{F}]\big] = E\Big[\frac{1}{2}\Delta\mathbf{Y}^T(T)\mathbf{S}(T)\Delta\mathbf{Y}(T)$$

$$+\frac{1}{2}\int_0^T \left(\Delta\mathbf{Y}^T(t)\mathbf{Q}(t)\Delta\mathbf{Y}(t)+\Delta\mathbf{F}^T(t)\mathbf{R}(t)\Delta\mathbf{F}(t)\right)\mathrm{d}t\Bigg] \tag{5.14}$$

The optimal control, $\Delta\mathbf{F}^*(t)$, minimizing (5.14) subject to (5.11) is given by

$$\Delta\mathbf{F}^*(t) = -\mathbf{R}^{-1}(t)\mathbf{B}^T(t)\mathbf{S}(t)\Delta\mathbf{Y}^*(t) \tag{5.15}$$

where $\mathbf{S}(t)$ is the solution of the matrix Ricatti equation (3.24). The approximated total optimal control $\mathbf{F}^*(t)$ is thus given by

$$\begin{aligned}\mathbf{F}^*(t) &= \mathbf{f}_0^*(t)+\Delta\mathbf{F}^*(t)\\ &= \mathbf{f}_0^*(t)-\mathbf{R}^{-1}\mathbf{B}^T(t)\mathbf{S}(t)\left(\mathbf{Y}(t)-\mathbf{y}_0^*(t)\right)\end{aligned} \tag{5.16}$$

In real-time control the perfect measurements $\mathbf{Y}(t) = \mathbf{y}(t)$ at the time t are used for the state vector in (5.16).

The method is based on an assumption, that the random response $\Delta\mathbf{Y}(t)$ is relatively small compared to the deterministic response $\mathbf{y}_0(t)$ caused by the mean value $\mathbf{w}_0(t)$ of the external excitation, rendering the global linearization (5.11), (5.12) of the nonlinear drift vector valid. Examples of this is a fast train passing a flexible bridge, where the stochastic loading is due to surface irregularities, and most wind gust loadings on buildings. Notice, that the mean wind loading $\mathbf{w}_0$ and the mean wind response $\mathbf{y}_0$ are normally modelled as constants or quasi-static quantities, implying that $\mathbf{f}_0(t) \equiv \mathbf{0}$.

5.2.2 Imperfect Measurements

The suboptimal control algorithms represented in the preceding sections are formulated as closed-loop controllers, where the control force can be modified in some way by feedback information that describes the actual state of the structural system. Obviously, feedback entails measurements, and these may be uncertain or indirect. If so, it is necessary to estimate the state history that is most likely to have caused the results of the measurements.

In the general case of stochastic optimal control based on imperfect measurements, the control and estimation strategies must be designed concurrently, i.e. one depends on each other. Using the dynamic programming approach to solve the stochastic optimization problem leads to the H-J-B equation for the optimal cost $J^*(\mathbf{Y}(t),t)$. Given a non-linear structural model written in state space form by (2.32) and a general performance index (2.52), the H-J-B equation is given by (3.108).

However, solving this equation backwards in time from the final condition (3.109) to some intermediate time t requires that the conditioning effect of the future measurements $\{\mathbf{Z}(\tau), \tau \in]t,T]\}$ on the future cost function is known. For the LQG regulator which possesses the separation property as described in Section 3.3.2, the conditioning effect is completely predictable, and hence stochastic control can be optimum. For non-linear structures the future conditioning effect can only be approximated, and then the stochastic control can only be suboptimal.

In the remainder of this section the stochastic optimal control problem based on a quadratic performance index is considered. The objective is to minimize the quadratic performance index (3.103) subject to the non-linear equation of motion (2.32) with an external excitation described by a Gaussian white noise. The state is observed through a noisy measurement described by the non-linear model

$$\mathbf{Z}(t) = \mathbf{C}(\mathbf{Y}(t), t) + \mathbf{V}(t) \tag{5.17}$$

where the measurement noise $\{\mathbf{V}(t),\ t \in [0, \infty[\}$ is assumed to be a non-stationary Gaussian white noise process with the first and second moments (3.113) and (3.114), respectively. For this problem the performance index can be rewritten by analogy with (3.118). By this reformulation the conditional expectations $\hat{\mathbf{Y}}(t)$ and $\mathbf{P}(t)$ have been defined, representing respectively, a state estimate and a state-estimate-error covariance, cf. (3.116) and (3.117). It is now desired to develop recursive equations determining present values for these estimates.

State estimation is considerably more difficult when the structural system is nonlinear. Gaussian external loadings and measurement errors cause non-Gaussian response, and hence calculation of the defined conditional mean estimate requires a predetermined estimate of the non-Gaussian probability density function $f_{\mathbf{Y}(t)}(\mathbf{y}|\{\mathbf{Z}(\tau),\ \tau \in [0, t]\})$. Based on the theory of continuous Markov vector processes Minai and Suzuki (1987) have derived differential forms for this conditional probability density function given observations during a finite time interval. These differential forms are called fundamental equations for stochastic estimates, and considered as an extension of the Fokker-Planck-Kolmogorov equation to stochastic estimate problems. From these fundamental equations, differential forms of the conditional moment equations are obtained. However, as might be suspected, solution of these equations will not often be a simple task.

Instead the state estimate, $\hat{\mathbf{Y}}(t)$ will be derived from the criterion of minimum covariance of the error in filtering, which leads to the extended Kalman filter. To derive these filter equations, the non-linear equations of motion (2.32) and the nonlinear observation equations (5.17) are expanded in Taylor series about an assumed optimum state $\mathbf{Y}(t) = \hat{\mathbf{Y}}(t)$. Retaining terms up to first order, this yields the following differential equation for the future states

$$\dot{\mathbf{Y}}(\tau) = \mathbf{G}(\hat{\mathbf{Y}}(t), \tau) + \mathbf{A}(t, \tau)\left(\mathbf{Y}(t) - \hat{\mathbf{Y}}(t)\right) + \mathbf{B}_0(\tau)\mathbf{W}(\tau) + \mathbf{B}(\tau)\mathbf{F}(\tau) \quad ,$$
$$\tau \in]t, T] \tag{5.18}$$

$$\mathbf{Z}(\tau) = \mathbf{C}(\hat{\mathbf{Y}}(t), \tau) + \mathbf{H}(t, \tau)\left(\mathbf{Y}(\tau) - \hat{\mathbf{Y}}(t)\right) + \mathbf{V}(\tau) \quad , \quad \tau \in]t, T] \tag{5.19}$$

where

$$\mathbf{A}(t, \tau) = \left.\frac{\partial \mathbf{G}(\mathbf{Y}(t), \tau)}{\partial \mathbf{Y}}\right|_{\mathbf{Y}(t) = \hat{\mathbf{Y}}(t)} \quad , \quad \mathbf{H}(t, \tau) = \left.\frac{\partial \mathbf{C}(\mathbf{Y}(t), \tau)}{\partial \mathbf{Y}}\right|_{\mathbf{Y}(t) = \hat{\mathbf{Y}}(t)} \tag{5.20}$$

The derivations leading to the optimum linear filter may now be repeated, cf. Appendix A. In this derivation it will be assumed that the estimate $\hat{\mathbf{Y}}(t)$ in (5.18) and (5.19) is a known quantity. Thus, state updates are determined from the original nonlinear equations using a linear function of the filter residual

$$\dot{\hat{\mathbf{Y}}}(t) = \mathbf{G}(\hat{\mathbf{Y}},t) + \mathbf{B}(t)\mathbf{F}(t) + \mathbf{K}_f(t)\Big(\mathbf{Z}(t) - \mathbf{C}(\hat{\mathbf{Y}},t)\Big) \quad , \qquad \hat{\mathbf{Y}}(0) = \mathbf{y}_0 \tag{5.21}$$

where the filter gain $\mathbf{K}_f(t)$ is given by (3.124). The error covariance matrix $\mathbf{p}(t)$ fulfils (3.122), where $\mathbf{A}(t,t)$ and $\mathbf{H}(t,t)$ as given by (5.20) are inserted for $\mathbf{A}(t)$ and $\mathbf{H}(t)$. In this connection equation (3.122) is an approximate expression for the error covariance based on the linearized equations of motion. Consequently, the extended Kalman filter is suboptimal. The partial derivative matrices given by (5.20) are evaluated at current values of the variable $\hat{\mathbf{Y}}$ and therefore, they cannot be precomputed. Consequently, the gain matrix also depends on the state estimate, and all three filter equations (5.21), (3.122) and (3.124) must be calculated in real time.

Although the extended Kalman filter equations are suboptimal, it becomes apparent that the control and estimation strategies depend on each other. The propagation of the estimation error covariance $\mathbf{p}(t)$ described by (3.122) depends on $\hat{\mathbf{Y}}(t)$ through $\mathbf{A}(t,t)$, and the state estimate $\hat{\mathbf{Y}}(t)$ is affected by the control force $\mathbf{F}(t)$ through (5.21). Hence, the control force affects the quality of the state estimates. Conversely, the feedback control force is dependent on the state estimate. This intercoupling of estimation and control, two roles that are possibly conflicting, is often called the *dual effect*.

In the dual control problem the control forces are determined with the aim of improving state estimation, while providing a minimum cost, i.e. both terms in the split up version of the performance index (3.118) are minimized. Recalling that deterministic optimal control of nonlinear structural systems requires iteration and that optimal nonlinear estimation is so difficult that only suboptimum solutions are normally practical, it is not surprising that the dual control of nonlinear structures has similar limitations.

An approximate control algorithm may be designed using the separation property of the LQG-regulator, where the control and estimation strategies can be developed independently. This approximate design technique is based on the assumption that the optimal performance index J can be determined by minimizing the two terms J_E and J_C in (3.118) independently. The cost due to the estimation error J_E is minimized by using the extended Kalman filter to estimate the state vector. The control forces are then obtained by minimizing the cost J_C subject to the dynamic constraint (5.21). Since the last term in (5.21) is a stochastic process this is an optimization problem identical to the non-linear stochastic problem in Section 5.2.1 with $\hat{\mathbf{Y}}(t)$ replacing $\mathbf{Y}(t)$. Thus the suboptimal control law takes the form

$$\mathbf{F}(t) = -\mathbf{G}_c(t)\hat{\mathbf{Y}}(t) \tag{5.22}$$

where the gain matrix $\mathbf{G}_c$ is determined according to the equivalent linearization technique in Section 5.1.1 or the invariant embedding technique in Section 5.1.2. Since $\mathbf{G}(t)$ is state dependent in both cases, its present value is determined from the estimate $\hat{\mathbf{Y}}(t)$.

5.3 Example

The above control algorithms are validated by applying them to digital simulated data generated from a hysteretic one-degree-of-freedom oscillator subjected to a white Gaussian noise. For an oscillator with viscous damping and a hysteretic restoring force the equation of motion may be written in the form

$$m\ddot{u} + c\dot{u} + g(\{u(\tau),\ \tau \in [0,t]\}) = f(t) + \nu W(t) \tag{5.23}$$

where m and c are, respectively, the mass and the viscous damping coefficient, $f(t)$ is the control force, $\{W(t),\ \tau \in [0,\infty[\}$ is a unit Gaussian white noise, ν is a constant intensity factor, and $g(\{u(\tau),\ \tau \in [0,t]\})$ represents the hysteretic restoring term which is a functional of the preceding deformation history $\{u(\tau),\ \tau \in [0,t]\}$. Generally g can be decomposed into a linear, non-hysteretic term and a hysteretic component. Thus

$$g = \alpha k u + (1-\alpha) k z_1 \tag{5.24}$$

In the above equation $z_1(t)$ is the hysteretic component of the restoring force, α is a non-dimensional factor measuring the relative contribution of the non-hysteretic term, and k is a linear stiffness coefficient. Here $z_1(t)$ will be described by the Bouc-Wen model, Bouc (1967), Wen (1976), given in differential form as

$$\dot{z}_1 = -\gamma_1 |\dot{u}|\, z_1 |z_1|^{n_B-1} - \beta_1 \dot{u}\, |z_1|^{n_B} + A_1 \dot{u} \tag{5.25}$$

In (5.25) γ_1, β_1, n_B and A_1 are "loop parameters", which control the shape of the hysteresis loop. The number of parameters in (5.27) and (5.25) may be reduced by one, by introducing the following new parameters

$$\begin{aligned} &z = z_1(1-\alpha)k \quad , \quad A_B = A_1(1-\alpha)k \\ &\gamma = \gamma_1\big[(1-\alpha)k\big]^{1-n_B} \quad , \quad \beta = \beta_1\big[(1-\alpha)k\big]^{1-n_B} \\ &k_B = \alpha k \end{aligned} \tag{5.26}$$

Equations (5.24) and (5.25) then become

$$g = k_B u + z \tag{5.27}$$

$$\dot{z} = -\gamma |\dot{u}|\, z\, |z|^{n_B-1} - \beta \dot{u}\, |z|^{n_B} + A_B \dot{u} \tag{5.28}$$

The total structural model represented by (5.23), (5.27) and (5.28) is written in state space form as

$$\dot{\mathbf{Y}}(t) = \mathbf{A}\big(\mathbf{Y}(t)\big)\mathbf{Y}(t) + \mathbf{B}f(t) + \mathbf{B}_0 W(t) \tag{5.29}$$

where

$$\mathbf{Y} = \begin{bmatrix} u \\ \dot{u} \\ z \end{bmatrix} \quad , \quad \mathbf{B} = \begin{bmatrix} 0 \\ 1/m \\ 0 \end{bmatrix} \quad , \quad \mathbf{B}_0 = \begin{bmatrix} 0 \\ \nu/m \\ 0 \end{bmatrix}$$

$$\mathbf{A}(\mathbf{Y}(t)) = \begin{bmatrix} 0 & 1 & 0 \\ -k_B/m & -c/m & -1/m \\ 0 & -\gamma \text{sgn}(\dot{u})z|z|^{n_B-1} - \beta|z|^{n_B} + A_B & 0 \end{bmatrix} \tag{5.30}$$

The control force is obtained by minimizing the quadratic performance index (2.53) in which $\mathbf{Q}$ is a 3×3 matrix chosen as

$$\mathbf{Q} = \begin{bmatrix} \omega_0^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where $\omega_0 = \sqrt{k/m}$ is the angular eigenfrequency for the corresponding linear oscillator. Since, there is only one control force $\mathbf{R}$ becomes a scalar designated r. The weighting matrix for the terminal state is chosen to be zero, i.e. $\mathbf{S}(T) = \mathbf{0}$.

5.3.1 Equivalent Linearization

The basis of the equivalent linearization technique is the adoption of a linear model at the present time t in the form (5.3). For the defined problem the equivalent system matrix in the linearized model is simply chosen as $\mathbf{A}(t) = \mathbf{A}(\mathbf{Y}(t))$, i.e $\mathbf{A}(t)$ is given by (5.30), rather than using the gradient (5.4). This corresponds to replacing tangential differentials of the drift vector with secantial differentials. This has been done in order to circumvent the problem of the lack of differentiability of the state vector in the equilibrium state $\mathbf{Y}(t) = \mathbf{0}$. Then the suboptimal control force is determined by minimizing a quadratic performance index subject to the instantaneous time-invariant linear model. The solution to this problem is specified in terms of the stationary solution of the Ricatti equation. However, a unique solution to this equation only exists if the fictitious time-invariant linear system described by the pair $\{\mathbf{A}(t), \mathbf{B}\}$ is completely controllable, see e.g. Levis (1986a). This condition is not satisfied for this example as described in the following.

Consider the linear eigenvalueproblem associated with the linear eigenvibrations of (5.3), $(\lambda_i \mathbf{I} - \mathbf{A}(t))\, \mathbf{\Phi}^{(i)}(t) = \mathbf{0}$. Notice, that $\lambda_i(t)$ and $\mathbf{\Phi}^{(i)}(t)$ are time-dependent through the dependence of $\mathbf{A}(t)$ and the state $\mathbf{Y}(t)$ from which the state equations are linearized. Further, formulate the adjoint eigenvalueproblem $\left(\lambda_i \mathbf{I} - \mathbf{A}^T(t)\right) \mathbf{\Psi}^{(i)}(t) = \mathbf{0}$, which yields the same eigenvalues. The eigenvectors $\mathbf{\Phi}^{(j)}(t)$ and $\mathbf{\Psi}^{(i)}(t)$ are orthogonal and can be normalized so as to satisfy

$$\mathbf{\Psi}^{(i)T}(t)\mathbf{\Phi}^{(j)}(t) = \begin{cases} 0 & , \quad i \neq j \\ 1 & , \quad i = j \end{cases} \quad , \qquad i, j = 1, 2, 3 \tag{5.31}$$

Define the matrices

$$\begin{aligned} \mathbf{\Lambda}(t) &= \text{diag}\left(\lambda_1(t) \quad \lambda_2(t) \quad \lambda_3(t) \right) \\ \mathbf{\Phi}(t) &= \left[\mathbf{\Phi}^{(1)}(t) \quad \mathbf{\Phi}^{(2)}(t) \quad \mathbf{\Phi}^{(3)}(t) \right] \\ \mathbf{\Psi}(t) &= \left[\mathbf{\Psi}^{(1)}(t) \quad \mathbf{\Psi}^{(2)}(t) \quad \mathbf{\Psi}^{(3)}(t) \right] \end{aligned} \tag{5.32}$$

Since the eigenvectors are linearly independent, the solution $\mathbf{Y}(\tau)$, $t \in [t, T]$ of the linearized stationary model (5.3), can be expressed as

$$\mathbf{Y}(\tau) = \mathbf{\Phi}(t)\mathbf{q}(\tau) \tag{5.33}$$

where $\mathbf{q}(\tau)$ is a 3-dimensional vector of modal coordinates. Substituting (5.33) into (5.29) and multiplying by $\mathbf{\Psi}^T(t)$ from the left yield

$$\dot{\mathbf{q}}(\tau) = \mathbf{\Lambda}(t)\mathbf{q}(\tau) + \mathbf{\Psi}^T(t)\mathbf{B}f(\tau) + \mathbf{\Psi}^T(t)\mathbf{B}_0W(\tau) \tag{5.34}$$

For the linearized hysteretic system one of the eigenvalues in $\mathbf{\Lambda}$ is zero and the component of $\mathbf{\Psi}^T\mathbf{B}$ in the model equation of motion associated with this eigenvalue is zero. Then the corresponding mode is unaffected by the control force, and hence the equivalent linear system is not completely *controllable.*

In the following the eigenvalues λ_i are ordered so that the first $n_c = 2$ modal equations in (5.34) are those, which are affected by the control force. Next, the optimal control problem is modified according to the assumption, that the state vector can be expressed in terms of the modal coordinates associated with these controllable modes,

$$\mathbf{Y}(\tau) = \mathbf{\Phi}_c(t)\mathbf{q}_c(\tau) \tag{5.35}$$

where $\mathbf{q}_c = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T$ and $\mathbf{\Phi}_c(t) = \begin{bmatrix} \mathbf{\Phi}^{(1)}(t) & \mathbf{\Phi}^{(2)}(t) \end{bmatrix}$. Conversely, $\mathbf{q}_c(\tau)$ can be expressed in terms of $\mathbf{Y}(\tau)$ by multiplying (5.33) by $\mathbf{\Psi}_c(t) = \begin{bmatrix} \mathbf{\Psi}^{(1)}(t) & \mathbf{\Psi}^{(2)}(t) \end{bmatrix}$ from the left and using the orthogonality properties (5.31), yielding

$$\mathbf{q}_c(\tau) = \mathbf{\Psi}_c^T(t)\mathbf{Y}(\tau) \tag{5.36}$$

The equations of motion for the controllable modes are then obtained by substituting (5.35) into (5.29) and by multiplying with $\mathbf{\Psi}_c^T(t)$ from the left, leading to

$$\dot{\mathbf{q}}_c(\tau) = \mathbf{\Lambda}_c(t)\mathbf{q}_c(\tau) + \mathbf{\Psi}_c^T(t)\mathbf{B}f(\tau) + \mathbf{\Psi}_c^T(t)\mathbf{B}_0W(\tau) \tag{5.37}$$

where $\mathbf{\Lambda}_c(t) = \text{diag}\begin{pmatrix} \lambda_1(t) & \lambda_2(t) \end{pmatrix}$. Next, substitute (5.35) into the performance index (5.5). With $\mathbf{S}(T) = \mathbf{0}$ and the definition of a modified weighting matrix $\mathbf{Q}_c(t) = \mathbf{\Phi}_c^T(t)\mathbf{Q}(t)\mathbf{\Phi}_c(t)$, this gives

$$J[\mathbf{q}_c, f, t] = \frac{1}{2}\int_t^T \left(\mathbf{q}_c^T(\tau)\mathbf{Q}_c(t)\mathbf{q}_c(\tau) + f^2(\tau)r(t)\right) d\tau \tag{5.38}$$

The control force is then determined by minimizing the above modified performance index subject to the dynamic constraint (5.37). By using closed-loop control, this yields

$$f(\tau) = -\frac{1}{r(t)}\mathbf{B}^T\mathbf{\Psi}_c(t)\mathbf{S}_c(t)\mathbf{q}_c(\tau) \tag{5.39}$$

where $\mathbf{S}_c(t)$ is the solution of the algebraic Ricatti equation

$$\mathbf{S}_c(t)\mathbf{\Lambda}_c(t) + \mathbf{\Lambda}_c^T(t)\mathbf{S}_c(t) - \frac{1}{r(t)}\mathbf{S}_c(t)\mathbf{\Psi}_c^T(t)\mathbf{B}\mathbf{B}^T\mathbf{\Psi}_c(t)\mathbf{S}_c(t) + \mathbf{Q}_c(t) = \mathbf{0} \tag{5.40}$$

The control force at the present time t is finally obtained by substituting (5.36) into (5.39),

$$f(t) = -\frac{1}{r(t)}\mathbf{B}^T\mathbf{\Psi}_c(t)\mathbf{S}_c(t)\mathbf{\Psi}_c^T(t)\mathbf{Y}(t) \tag{5.41}$$

5.3.2 Invariant Embedding

By application of the invariant embedding technique the feedback gain is derived from the solution $\mathbf{S}(t)$ of the Ricatti equation (5.8), which involves the partial derivative $\frac{\partial \mathbf{G}(\mathbf{Y}(t),t)}{\partial \mathbf{Y}}$, which for the considered hysteretic system exists everywhere, except in the equilibrium state for $\mathbf{Y} = \mathbf{0}$. On the basis of the solution $\mathbf{S}(t)$, the feedback gain is determined from the requirement that the eigenvalues of the equivalent linearized invariant embedding equation and the linearized equation of motion should be equal. For this example the matrix $\mathbf{A}(t)$ of the equivalent linear systems constructed as (5.9) and (5.10), are chosen to be equal to $\mathbf{A}\big(\mathbf{Y}(t)\big)$ given by (5.30).

When the present time t is considered as fixed, it was found above that the fictitious time-invariant linear system described by the pair $\{\mathbf{A}(t), \mathbf{B}\}$ is not completely controllable. Consequently, it is not possible to achieve any set of closed-loop eigenvalues by applying a state feedback matrix $\mathbf{G}_c(t)$, because the eigenvalues of $\mathbf{A}(t)$ associated with the non-controllable eigenmodes will remain unchanged in the closed-loop system.

In this example the eigenvalue associated with the non-controllable eigenmode is zero. Hence, the desired set of closed-loop eigenvalues is specified by $\{\lambda_1(t), \lambda_2(t), 0\}$. One of the eigenvalues of the closed-loop matrix $[\mathbf{A}(t) + \mathbf{B}\mathbf{G}_c(t)]$ is always zero no matter how $\mathbf{G}_c(t)$ is selected, i.e. the remaining eigenvalues $\lambda_1(t)$ and $\lambda_2(t)$ must be obtained by selecting the 3 parameters of $\mathbf{G}_c(t)$. $\mathbf{G}_c$ must satisfy $\mathbf{G}_c\left[\mathbf{\Phi}^{(1)}\ \mathbf{\Phi}^{(2)}\right] = [\mathcal{F}(\lambda_1)a(\lambda_1)\ \mathcal{F}(\lambda_2)a(\lambda_2)]$, cf. (3.92), which has no unique solution. Here, the feedback coefficient multiplied by the hysteretic component $z(t)$ is selected to be zero, i.e. $G_{c,3} = 0$, and hence the remaining feedback coefficients must satisfy

$$\left[\begin{array}{cc} G_{c,1} & G_{c,2} \end{array}\right] = \left[\begin{array}{cc} \mathcal{F}(\lambda_1)a(\lambda_1) & \mathcal{F}(\lambda_2)a(\lambda_2) \end{array}\right] \left[\begin{array}{cc} \Phi_1^{(1)} & \Phi_1^{(2)} \\ \Phi_2^{(1)} & \Phi_2^{(2)} \end{array}\right]^{-1} \tag{5.42}$$

5.3.3 Simulation Results

The simulation results were generated by numerically integrating equation (5.29), using the fourth-order Runge-Kutta algorithm with automatic step size, i.e. it takes larger steps where the solution is more slowly changing, cf. PC-MATLAB (1989). Two series of simulation were carried out using each of the two aforementioned control algorithms. By application of the invariant embedding technique equations, (5.8) was integrated simultaneously in order to calculate the control force.

For convenience, all the parameters and variables in the state space model (5.29) are taken to be non-dimensional. The normalization has been made in such a way, that the stationary variance of the deflection u and the velocity $\dot{u}$ is equal to 1 for the corresponding linear oscillator ($\alpha = 1$), when the oscillator is subjected to a unit intensity Gaussian white noise, i.e $E[W(t_1)W(t_2)] = \delta(t_1 - t_2)$. This condition requires that $\omega_0 = 1$ and $\nu = 2m\sqrt{\zeta}$. Consequently, the time t has been made non-dimensional with respect to $1/\omega_0$.

The hysteresis parameters γ, β, A_B and n_B which control the shape and scale of the hysteresis loops are selected in proportion to the introduced normalisation. To see the variation of the hysteretic component z with u, integration of (5.25) is performed. Depending on the conditions of $\dot{u}$ and z being positive or negative this yields

$$\frac{dz}{du} = A_B \pm (\gamma + \beta) z^{n_B} \tag{5.43}$$

which can be integrated in closed-form. Provided $(\gamma + \beta) > 0$ the hysteresis loop converges towards the horizontal asymptotes $-z_{sup}$ and z_{sup} for which $\frac{dz}{du} = 0$. Then according to (5.43)

$$z_{sup} = \sup z(t) = \left(\frac{A_B}{\gamma + \beta} \right)^{1/n_B} \tag{5.44}$$

Conversely, $z(t)$ is unbounded for $(\gamma + \beta) < 0$.

z_{sup} is normalised in proportion to the stationary standard deviation of the corresponding linear oscillator. Hence, this parameter becomes a measure of the excitation level. Small values indicate heavy excitation and vice versa. In the following z_{sup} is selected to be equal the prescribed standard deviation for the corresponding linear oscillator, i.e. $z_{sup} = 1$, indicating a heavy non-linear response. The particular parameter values are $\gamma = 0.5$, $\beta = 0.5$, $A_B = 1.0$ and $n_B = 5$. The remaining structural parameters are selected as $m = 1$, $\zeta = 0.1$ and $\alpha = 0.1$.

The white noise excitation is approximated with a realization of a broad-banded zero-mean Gaussian process. Define a set of equidistant time instants $\{t_k = k\Delta t, k = 1, 2, \ldots\}$ with intervals Δt, and let $\{W(t_k), k = 1, 2, \ldots\}$ be a sequence of mutually independent stochastic variables, all identically distributed $N(0, \sigma_W^2)$. Then the external excitation $W(t)$, $t \in]t_k, t_{k+1}[$ is determined by linear interpolation between $W(t_k)$ and $W(t_{k+1})$.

The autospectral density function of this process is given by, see Clough and Penzien (1974)

$$S_{WW}(\omega) = \frac{6 - 8\cos(\omega\Delta t) + 2\cos(2\omega\Delta t)}{(\omega\Delta t)^4} S_0 \tag{5.45}$$

in which S_0 is given by

$$S_0 = \frac{\sigma_W^2 \Delta t}{2\pi} \tag{5.46}$$

It is evident as $\Delta t \to 0$, that this simulated process becomes a Gaussian white noise of intensity S_0 over the entire frequency range. In order, that S_0 should represent the autospectral density of a unit intensity white noise process with autospectral density $S_{WW} = \frac{1}{2\pi}$, it follows that $\sigma_W^2 = 1/(\Delta t)$. From the requirement that $S_{WW}(\omega)$ is not allowed to deviate more than lit is found that

$$\frac{\Delta t}{T_0} < \frac{1}{51.2} \tag{5.47}$$

where $T_0 = 2\pi/\omega_0$ is the eigenperiod for the corresponding linear undamped system. According to this requirement generation is then performed with the time step $\Delta t = T_0/50$. The uncontrolled structural response, corresponding to the selected parameters and for a given realization of the external excitation, is first determined. The associated hysteresis loops for z and the total restoring force g are shown in fig. 5.1. From this, it is evident that the restoring force characteristics are rather non-linear at the selected level of excitation.

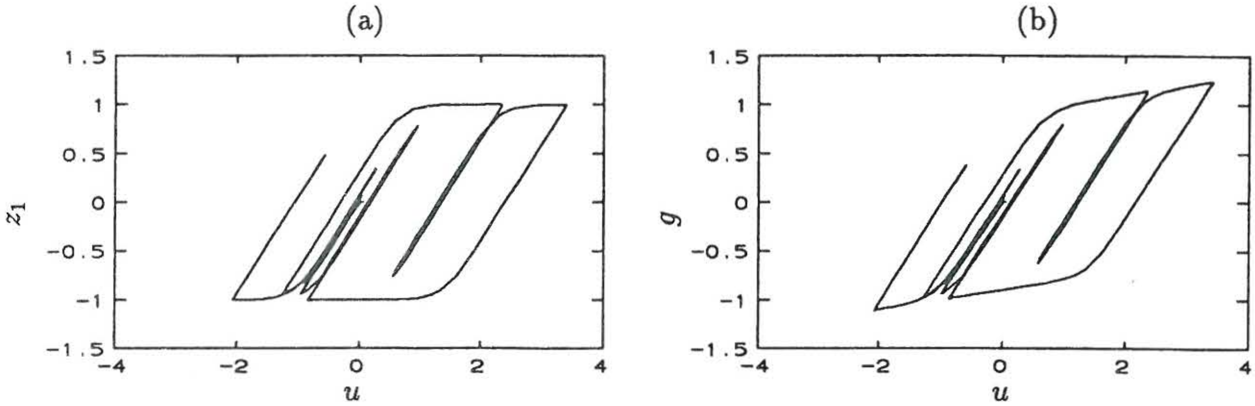

Figure 5.1: Variation of the hysteretic restoring force with displacement u for $f = 0$. (a) Hysteretic restoring force component, z_1. (b) Total hysteretic force, g.

Subsequently, the structural response is simulated for the same external excitation, applying different time-series of the control force determined from the equivalent linearization technique in Section 5.3.1 and the invariant embedding technique in Section 5.3.2. With a weighting parameter of $r = 10$ the controlled displacements based on each of the two control algorithms are compared to the uncontrolled level in fig. 5.2. The applied control forces are shown in fig. 5.3. It appears, that the vibration level has been reduced considerably, and that the time-series for the controlled displacements are very much alike. If a greater reduction of the structural response is desired the control force should be given less weight in the objective function (smaller r), which results in larger control forces.

In order to assess the control efficiency of the applied control strategies, numerical integration has been carried out to determine the performance index for the simulated data. By using this measure the invariant embedding algorithm ($J = 5.19$) gives a better result than the equivalent linearization algorithm ($J = 5.92$). The control procedure has been repeated for different values of r using the same external excitation. Comparing the performance indices associated with the applied control strategies, see fig. 5.4, the invariant embedding technique seems generally to be more effective than the equivalent linearization technique. The tendency is the same for other realizations of the external excitation.

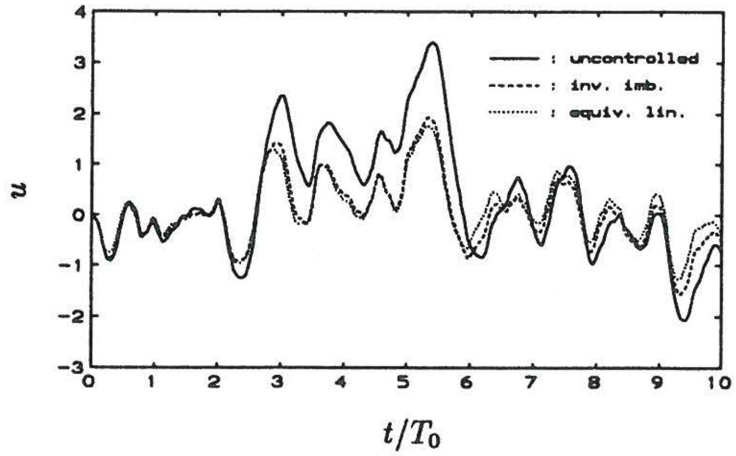

Figure 5.2: Comparison of uncontrolled and controlled displacements, $r = 10$.

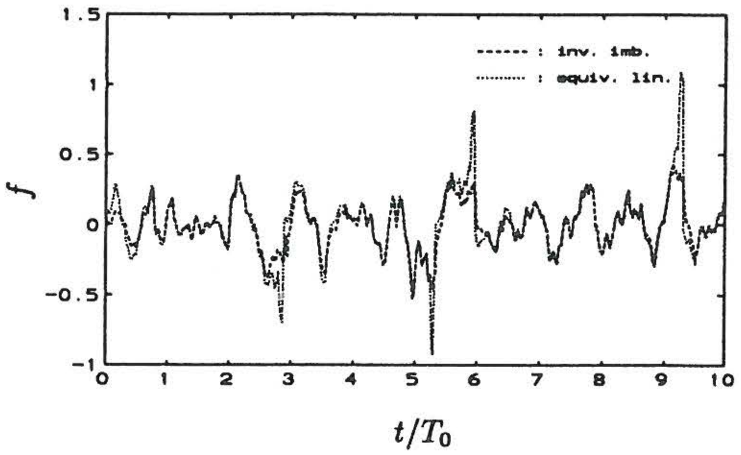

Figure 5.3: Control forces, $r = 10$.

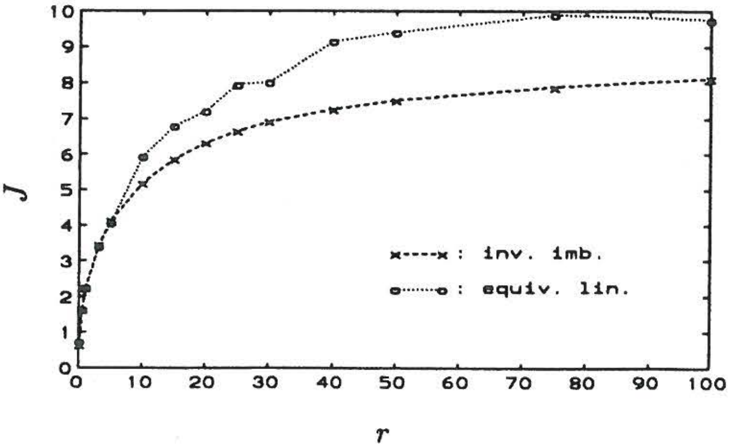

Figure 5.4: Comparison of performance indices for different values of r

For small values of r (< 10) the minimum value of the performance indices are almost equal. When the control systems are operating in this region the control forces become large and a substantial reduction in response is obtained. This means that the hysteretic component of the restoring force is negligible and thus, the response becomes linear. And for linear systems the feedback gains for the two control schemes converge quickly towards the same values, so that the deviation in the performance is insignificant.

In practice the weighting coefficient r must be selected according to the available actuator and an acceptable maximum vibration level. If the actuator is able to generate large control forces, it is of course possible to reduce the vibrations significantly. However, energy dissipation due to hysteretic effect will limit the structural response and therefore, the exploitation effect of the control force in proportion to the attained vibration suppression could be increased, if a vibration level with some hysteretic characteristics is acceptable.

5.4 Adaptive Control

Control algorithms designed from the criterion of minimizing a performance index are based on the availability of appropriate models for the structural dynamics. In structural control the models are mainly developed from a theoretical approach, which makes use of physical laws that govern the structural system (e.g. Newton's law), as described in Section 2.1. However, implementation of a control algorithm based on an assumed structural model provides a knowledge of the model parameters, such as mass, stiffness and damping. A common procedure is to determine these parameters from measurements of the structural behaviour by means of a system identification scheme and use these estimates in the control design. However, under heavy excitation the stiffness and damping parameters of the structural system may change due to local or global damage. For such systems more complicated so-called *adaptive control algorithms* have been developed. These control schemes estimate the structural parameters and perform vibration control on-line.

A survey of adaptive control theory and its application has been presented by Åström (1983). In a paper by Yao (1979) there is a literature review in which the problem of system identification in conjunction with on-line control has received some attention. Some of the principles of adaptive control of single-input and single-output dynamic systems are outlined in Åström and Wittenmark (1971), Şafak (1989a,1989b).

Addressing the adaptive optimal control problem for non-linear structural systems the equation of motion is assumed to be written in state space form as

$$\dot{\mathbf{Y}}(t) = \mathbf{G}(\boldsymbol{\theta}(t), \mathbf{Y}(t), t)\mathbf{Y}(t) + \mathbf{B}_0(\boldsymbol{\theta}(t), t)\mathbf{W}(t) + \mathbf{B}(\boldsymbol{\theta}(t), t)\mathbf{F}(t) \tag{5.48}$$

where $\boldsymbol{\theta}$ is an n_p-dimensional vector of the unknown parameters. The associated performance index J is formulated as in the case of known structural parameters, cf. the general function (2.52) or the quadratic function (2.53). The aim of adaptive control is to find the optimal control forces minimizing the performance index, together with the optimal estimation of the unknown parameters, $\boldsymbol{\theta}(t)$.

If the external excitation $\mathbf{W}(t)$ is described by a stochastic model the objective is to minimize the expected value $E[J]$ subject to the stochastic constraints (5.48). In what follows the stochastic description is applied. To perform the estimation of the parameters, measurements of the structural response are needed. Let the measurements $\mathbf{Z}(t)$ be a non-linear function of the state $\mathbf{Y}(t)$ and affected by the random noise $\mathbf{V}(t)$, cf. (5.17).

The problem of finding a control, which minimizes the expected cost function, is difficult to solve. On the assumption that a solution exists, a functional equation (a H-J-B equation) can be derived for the optimal cost function $J^*(\mathbf{Y}(t), t)$ using dynamic programming. The H-J-B equation can be solved analytically only in very simple cases, e.g. in relation to the LQG regulator problem with known structural parameters. When the structural parameters are unknown the H-J-B equation associated with the LQG problem becomes very complicated and cannot be solved analytically. The adaptive LQG problem has been investigated by Morimoto (1990), Casiello and Loparo (1989), Caines and Chen (1985), Rishel (1986), Hijab (1983,1986). This is an extremely difficult problem, where there is no closed-form solution for the control forces. Hence, for the defined adaptive control problem there is often no alternative except to use some suboptimal control.

By analogy with the stochastic optimal regulator based on imperfect measurements, the adaptive optimal regulator can be thought of as composed of two parts: a nonlinear estimator and a feedback regulator, Åström (1987). The estimator generates the conditional probability of the state and parameters on condition of the measurements, $f_{\mathbf{Y}\boldsymbol{\theta}}(t)(\mathbf{y}, \boldsymbol{\theta} | \{\mathbf{Z}(\tau),\ \tau \in [0, t]\})$. In principle, there is no distinction between the parameters and the other state variables for adaptive schemes derived from the stochastic control theory. The feedback regulator is a nonlinear functional, which maps this distribution into control forces. This functional can be computed off-line. The distribution $f_{\mathbf{Y}\boldsymbol{\theta}}(t)(\mathbf{y}, \boldsymbol{\theta} | \{\mathbf{Z}(\tau),\ \tau \in [0, t]\})$ must, however, be updated on-line, which in general requires the solution of a complicated nonlinear filtering problem.

A suboptimal control scheme introduced here belongs to the category of so-called *self-tuning* controllers, which are developed according to the described solution strategy, Åström and Wittenmark (1973), Åström et al. (1977). This is designed for control systems with unknown but constant or slowly varying structural parameters. The regulator is accomplished by designing a control law as if the parameters were known exactly and by introducing a recursive estimator. The true parameter values are then concurrently replaced by their estimated values in the control law.

Comparing the self-tuning regulator with the optimal regulator some principal differences are ascertained. In the self-tuning regulator, the states are separated into two groups, the ordinary state variables of the underlying constant parameter model and the parameters which are assumed to vary slowly. In the optimal stochastic regulator there is no distinction between the parameters and the other state variables. The design calculations in the self-tuning regulators are made as if the parameters were known exactly. There are no attempts to modify the control law when the estimates are uncertain. In the optimal stochastic regulator, there is a feedback from the conditional distribution of the parameters and states which take full account of uncertainties. The comparison indicates that it may be useful to add parameter uncertainties to the self-tuning regulator. However, the regulator in many cases has the desirable self-tuning property, i.e. the controller converges to the one that could be designed if the structural model was known a priori. In the following, a self-tuning controller is proposed.

5.4.1 Self-Tuning Regulator

There are many possible self-tuning regulators depending on the control design and parameter estimation techniques used. Two methods to generate a sequential control law were proposed by Sage (1966a,1966b). However, these methods are based on the assumption that the structural system is stationary over subintervals. The type of regulator considered employs a recursive estimator set up from the extended Kalman filter and a suboptimal control design based on the invariant embedding technique.

The extended Kalman filter provides a basis for sequential estimation of the unknown structural parameters that is straightforward. The applied technique is analog to algorithms proposed by Detchmendy and Shridar (1966), Desai and Lalwani (1969), and it has among others been investigated for parametric identification of hysteretic systems by Distefano and Rath (1975), Roberts and Sadeghi (1990) in the case without control forces. Further, Seibold and Fritzen (1991) have applied the extended Kalman filter to recursive identification of the parameters in a nonlinear model representing a simple rotor. The basic idea is to augment the physical system of equations (5.48) by a differential equation modelling the unknown parameters. Assuming they are constant the augmented system is governed by a state equation of the following form

$$\dot{\mathbf{Y}}_A(t) = \mathbf{G}_A(\mathbf{Y}_A(t), \mathbf{F}(t), t) + \mathbf{B}_{0A}(\mathbf{Y}_A(t), t)\mathbf{W}(t) \quad , \qquad \mathbf{Y}_A(0) = \hat{\mathbf{Y}}_{A0} \tag{5.49}$$

where

$$\mathbf{Y}_A(t) = \begin{bmatrix} \mathbf{Y}(t) \\ \boldsymbol{\theta}(t) \end{bmatrix} \quad , \quad \mathbf{G}_A(\mathbf{Y}_A(t), t) = \begin{bmatrix} \mathbf{G}(\mathbf{Y}_A(t), t) + \mathbf{B}(\mathbf{Y}_A(t), t)\mathbf{F}(t) \\ \mathbf{0} \end{bmatrix} \quad ,$$

$$\mathbf{B}_{0A}(\mathbf{Y}_A(t), t) = \begin{bmatrix} \mathbf{B}_0(\mathbf{Y}_A(t), t) \\ \mathbf{0} \end{bmatrix} \tag{5.50}$$

Notice, unlike the preceding state space formulations that the matrices $\mathbf{B}$ and $\mathbf{B}_0$ are now dependent on the state vector $\mathbf{Y}_A(t)$ due to their dependence on $\boldsymbol{\theta}(t)$ The initial values $\hat{\mathbf{Y}}_A(0)$ contain the initial values $\mathbf{Y}(0)$ of the state vector and the initial guess $\hat{\boldsymbol{\theta}}(0)$ of the system parameters. From (5.49) it may appear that $\hat{\boldsymbol{\theta}}(t)$ will always be $\hat{\boldsymbol{\theta}}(0)$. This is because (5.49) deals with the dynamics of the system with exact state variables and exact system parameters $\boldsymbol{\theta}(t)$. In reality these parameters must be estimated based on measurements. Equation (5.49) is then replaced by an updating differential equation specifying the development of the state variables estimates.

The measurements are assumed to be the same as in the case of known structural parameters, cf. (5.17). Thus, the observation equation in terms of the augmented state vector becomes

$$\mathbf{Z}(t) = \mathbf{C}_A(\mathbf{Y}_A(t), t) + \mathbf{V}(t) \tag{5.51}$$

in which $\mathbf{C}_A(\mathbf{Y}_A(t), t) = \mathbf{C}(\mathbf{Y}(t), t)$. The statistics of the external excitation and the measurement noise is defined as before, cf. (3.94),(3.95) and (3.113),(3.114). The recursive estimator for the augmented system follows Section 5.2.2 directly. The state estimate is found by integrating the nonlinear differential equation that models structure and parameter dynamics and that is driven by measurement residuals,

$$\dot{\hat{\mathbf{Y}}}_A(t) = \mathbf{G}_A(\hat{\mathbf{Y}}_A(t), \mathbf{F}(t), t) + \mathbf{K}_f(t)\Big(\mathbf{Z}(t) - \mathbf{C}_A(\hat{\mathbf{Y}}_A(t), t)\Big) \quad , \quad \hat{\mathbf{Y}}_A(0) = \mathbf{Y}_{A0} \quad (5.52)$$

The filter gain $\mathbf{K}_f(t)$ is defined as

$$\mathbf{K}_f(t) = \mathbf{p}(t)\mathbf{H}_A^T(t)\mathbf{R}_V^{-1}(t) \quad (5.53)$$

and the covariance update is determined from

$$\begin{aligned}\dot{\mathbf{p}}(t) &= \mathbf{A}_A(t)\mathbf{p}(t) + \mathbf{p}(t)\mathbf{A}_A^T(t) + \mathbf{B}_{0A}(\mathbf{Y}_A(t), t)\mathbf{R}_W(t)\mathbf{B}_{0A}^T(\mathbf{Y}_A(t), t) \\ &\quad - \mathbf{p}(t)\mathbf{H}_A^T(t)\mathbf{R}_V^{-1}(t)\mathbf{H}_A(t)\mathbf{p}(t) \quad , \qquad \mathbf{p}(0) = \mathbf{p}_0 \end{aligned} \quad (5.54)$$

where

$$\mathbf{H}_A(t) = \left.\frac{\partial \mathbf{C}_A(\mathbf{Y}_A(t), t)}{\partial \mathbf{Y}_A}\right|_{\mathbf{Y}_A(t) = \hat{\mathbf{Y}}_A(t)} \quad , \quad \mathbf{A}_A(t) = \left.\frac{\partial \mathbf{G}_A(\mathbf{Y}_A(t), t)}{\partial \mathbf{Y}_A(t)}\right|_{\mathbf{Y}_A(t) = \hat{\mathbf{Y}}_A(t)} \quad (5.55)$$

The initial conditions for $\mathbf{p}(t)$ reflecting the initial uncertainty in the initial values and the a priori estimate $\hat{\boldsymbol{\theta}}$ of the structural parameters will usually be unknown and may therefore be chosen in a more-or-less arbitrary manner. In principle, the larger the norm of $\mathbf{p}(0) = \mathbf{p}_0$, the greater the rate of convergence. However, for very large values of the matrix elements, the estimation scheme can be unstable. It is convenient to set $\mathbf{p}_0 = \lambda_0 \mathbf{I}$, where $\mathbf{I}$ is a unit matrix of dimension $(n + n_p) \times (n + n_p)$ and λ_0 is a number to be selected by the designer.

The control law generated from the estimated states and parameters is here based on the suboptimal control solution derived from the invariant embedding technique. In the case of known structural parameters a linear feedback in the state estimate was proposed, cf. (5.22), where the feedback gain was determined from the solution $\mathbf{S}(t)$ of (5.8). The same control law is utilized for the self-tuning regulator except that the structural parameters are concurrently replaced by their present estimate in the equations, which are used to determine the feedback coefficients.

5.4.2 Numerical Example

The proposed self-tuning regulator based on the invariant embedding control scheme and the extended Kalman filter is validated by application to the hysteretic one-degree-of freedom oscillator utilized in Section 5.3. Thus, the equation of motion written in state space form is given by (5.29). In this example a reduced estimation problem is considered where the mass $m = 1$ and the hysteretic parameter $n_A = 5$ are assumed to be known. Hence, the vector of unknown structural parameters becomes $\boldsymbol{\theta}^T = [c \; k_B \; \gamma \; \beta \; A_B]^T$. The state and parameter estimation is performed from measurements made on the displacement u and the velocity $\dot{u}$ as described by the observation model (5.51). The intensity of the measurement noise was selected to be 10% of the intensity of the external loading.

The simulation results were generated using the same values for the structural parameters, and the same realization for the external excitation as in the example in Section

5.3. The controlled state $\mathbf{Y}(t)$ was determined by numerically integrating the equation of motion (5.29) given the initial condition $\mathbf{Y}^T = [0\ 0\ 0]^T$. Recursive state and parameter estimates were obtained by simultaneously integrating (5.52) from the initial condition $\hat{\mathbf{Y}}_A^T(0) = [\hat{\mathbf{Y}}^T(0)\,;\, \hat{\boldsymbol{\theta}}^T(0)] = [0\ 0\ 0\,;\ 0.1\ 0.15\ 0.38\ 0.38\ 0.45]$. The initial values for the unknown structural parameters deviate 50% from their true value. Finally, equation (5.54) was integrated from the initial condition $\mathbf{p}(0) = \lambda_0 \mathbf{I}$ where λ_0 was selected as 0.1.

Given the described conditions the obtained estimates of the state variables are shown in fig. 5.5. The estimates of the directly measured variables u and $\dot{u}$ are seen to nearly match their true values. The hysteretic restoring force z converge to its true value within the time corresponding to 3 eigenperiods. Within the same period of time the structural parameters k_B and A_B approach their true values. The convergence of the parameter γ is relatively good, but it is slowly. However, the remaining parameters c and β differ a good deal from their true values.

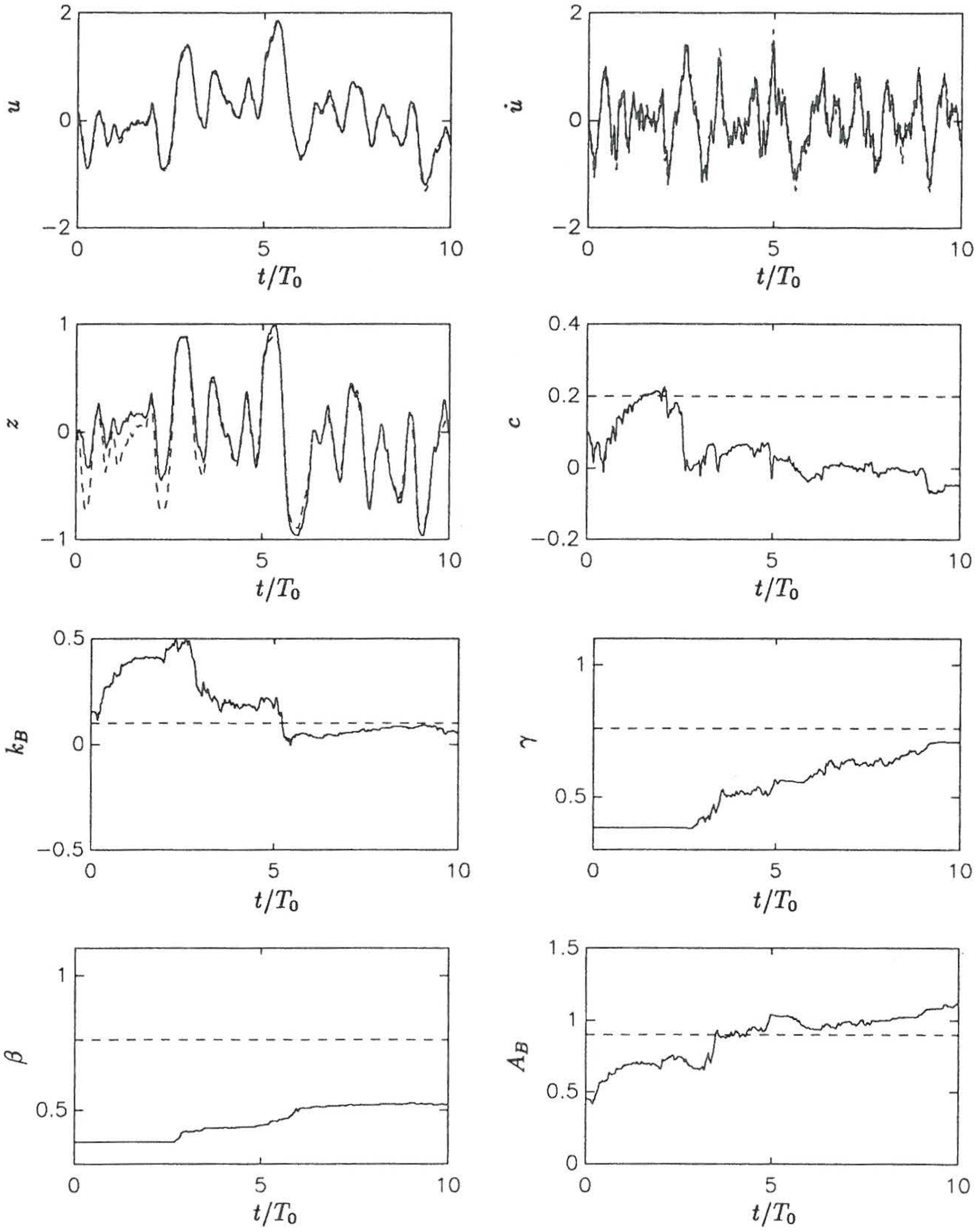

Figure 5.5: Evolution of estimated state variables of the augmented model during self-tuning control, $r = 10$. The dashed lines indicate the true values.

Although some of the parameter estimates are inaccurate the reduction in control efficiency is limited compared to the case of known parameters, which appears from a comparison of the displacements as seen in fig. 5.6.

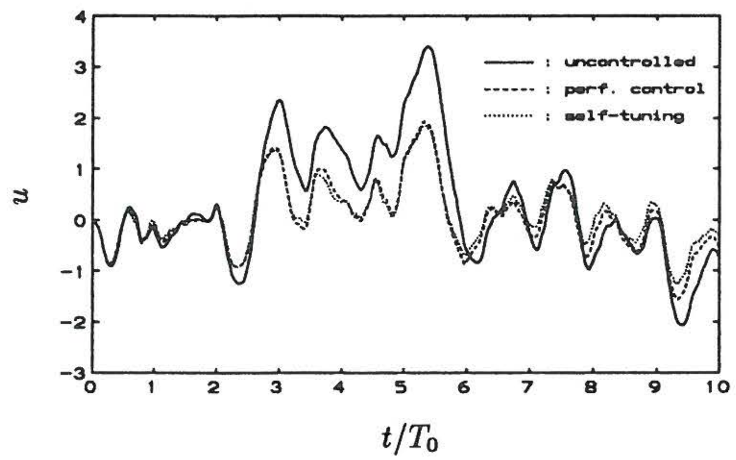

Figure 5.6: Controlled displacement obtained from a perfectly known structural model and a sequentially estimated model, respectively, compared to the uncontrolled response.

The degradation for the performance due to uncertainty in the estimated structural parameters may also be illustrated by comparing the function values for the associated performance index. As seen from the comparison in fig. 5.7 of the performance index for different values of r the uncertainty is seen to have moderate effect. Hence it is concluded that the extended Kalman filter yields sufficiently accurate parameter estimates and that the proposed self-tuning controller is applicable to vibration suppression of hysteretic systems.

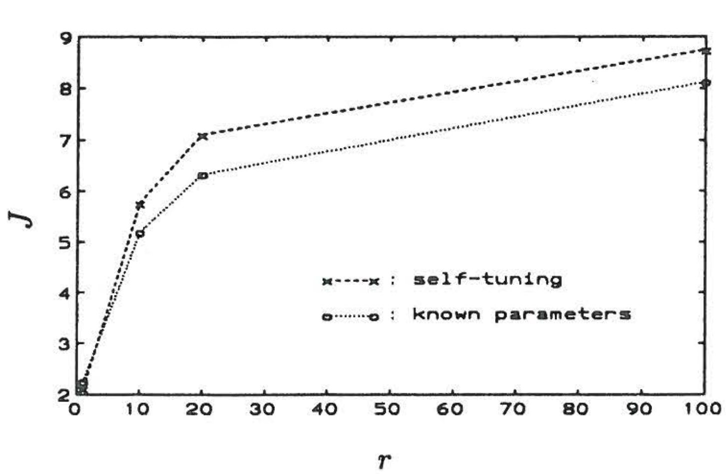

Figure 5.7: Comparison of values for the performance index in case of known and unknown structural parameters, respectively.

5.5 Conclusions

Design of optimal control strategies for nonlinear structures has been dealt with in this chapter. The applicability of a linear feedback controller has been emphasized, which is developed by use of the invariant embedding technique. By application to a numerical example the proposed control algorithm has been compared to Pearson's method of equivalent linearization. For the given example the invariant embedding method leads to the best results according to the defined performance index.

The last section represents an adaptive controller which concatenates the invariant embedding control scheme and the extended Kalman filter for state and parameter estimation. Although the parameter estimates obtained from the sequential identification scheme are not perfect the resulting controller seems to operate satisfactorily.

Either of the considered control algorithms are suitable for on-line control of typical flexible structures subjected to arbitrary dynamic environments. However, by implementation in real-time control one has to take into account that a certain amount of computational effort is required for calculations involved in determining the control law. Consequently, a discretization in time of the applied control strategy must be accomplished in such a way, that the control force only has to be changed at discrete time instants.

If a reduction in calculation time is desired for either of the considered control strategies, e.g. by control of large structures with many degrees-of-freedom, then the calculations required to determine the feedback gains may easily be reduced. Using the invariant embedding method a considerable reduction may be obtained, if the feedback coefficients are determined from the requirement that just a certain number of eigenvalues of the linearized equation of motion and of the invariant embedding equation are identically equal. Correspondingly, the calculation time will be reduced by Pearson's method if just a certain number of modes associated with the linearized model are controlled.

Chapter 6

Experimental Study

Experiments of active control were carried out in the laboratory in order to study the possible application of active control to structures under seismic excitation. A control scheme based on the discrete-time LQG regulator described in Section 4.2 was employed to reduce the structural response under base motion generated by a seismic simulator.

Most of the previous work done in active structural control has been analytical and numerical, assuming ideal conditions on which active control is implemented. The feasibility of the LQG-regulator used in these tests has been validated by applying it to simulated data, see e.g. Pu and Hsu (1988) and Suhardjo and Spencer (1990). However, within the framework of structural control a demand for experimental evidence of the applicability on more realistic conditions led to this study.

An important question to be treated and verified through experimental studies is the effect of the inevitable uncertainties in parametric identification on the performance of the active control system. Yao (1987) has examined some types of uncertainties and discussed their effect on the reliability of the structural performance. A sensitivity study has been conducted by Yang and Akbarpour (1990) to investigate this effect by using a simple simulation approach. However, this numerical study neglects the modelling error for a real structure and assume that the entire state vector is available without noise. These errors are present in experimental studies.

Experimental studies performed by Chung et al. (1988,1989), Carotti and Lio (1991) have demonstrated the feasibility of the active tendon control system to suppress vibrations of seismic frame buildings. In this experimental test an active, tuned mass damper (TMD) is implemented. Numerical investigations of the active mass damper system for reduction of adverse effects of earthquake and wind have been made, using different control schemes showing varying degrees of efficiency, see e.g Hrovat et al. (1983), Yang and Samali (1983a) and Abdel-Rohman (1987a).

A detailed description of the experimental setup is given in Section 6.1, and a formulation of a mathematical model describing the structural behaviour is given in Section 6.2. In Section 6.3 the identification techniques used to determine the dynamic parameters are presented. The control algorithm for this specific structural system is set up in Section 6.4 and the results are shown in Section 6.5.

6.1 Experimental Setup

The experimental setup used in this study consists of a cantilever model on a seismic simulator and an active mass damper implemented at the top of the model. The earthquake simulator used to shake the model is controlled by a microcomputer. Based on measurements of the forced vibrations the same computer is used to calculate and generate control signals for the active control device. The model structure and a diagram of the experimental setup are shown in fig. 6.1.

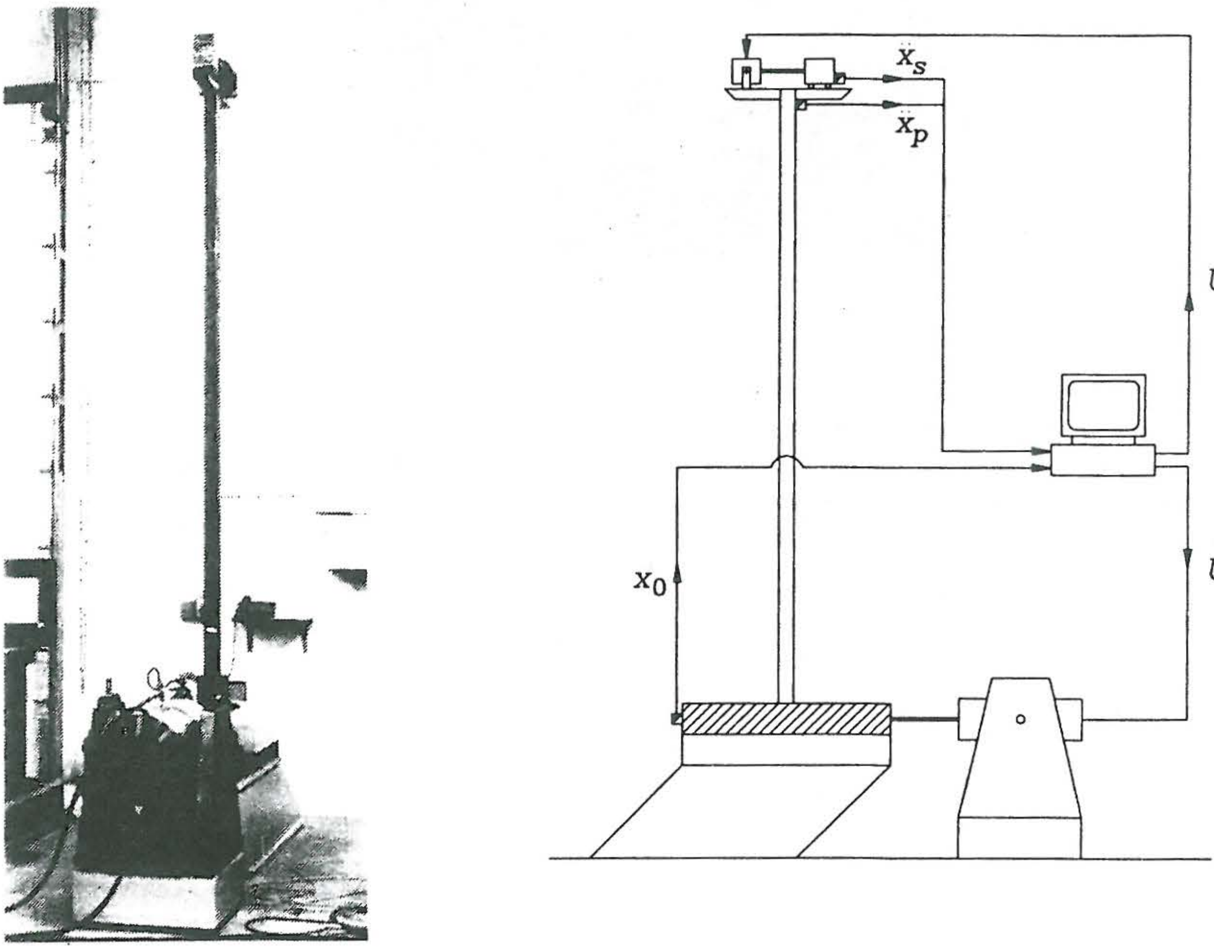

Figure 6.1: Model structure and diagram of control system.

The model structure is simply a 3 m high box profile (70mm × 70mm × 4mm) - called the pile. The pile is bolted to a plate of steel which is in turn mounted on two vertically placed sheets, one at each end of the plate. This foundation represents a movable base. The base movements are created by a vibration exciter, Brüel & Kjær, type 4818, using a signal generated by the computer. By computer generated signals, any kind of base movements can be produced.

At the top of the pile, an active mass damper is mounted to generate the desired control force. It consists of a vibration exciter (called the actuator), Brüel & Kjær, type 4809, and a movable secondary mass. The maximum force amplitude in harmonic excitations of this actuator is 44.5 N. The actuator and the secondary mass are mounted on a console on either side of the pile to avoid considerable excentric loading. The actuator is controlled by a drive voltage generated by the computer.

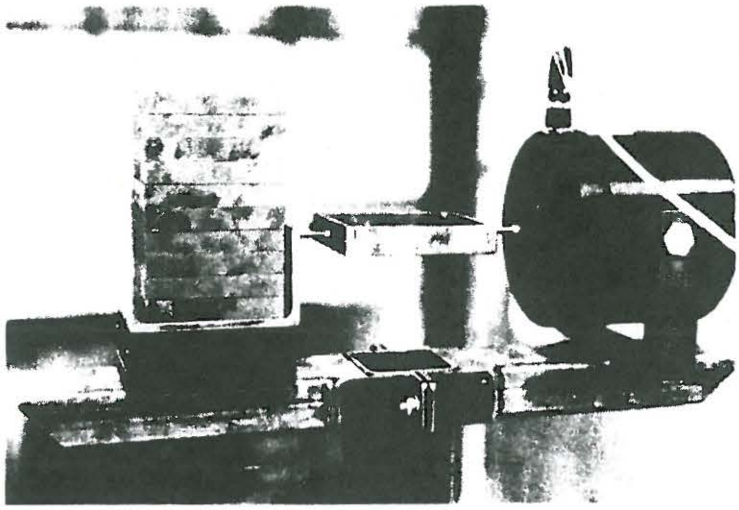

Figure 6.2: Active mass damper.

The secondary mass is mounted on a roller bearing nearly without friction and rigidly connected with the moving element of the actuator. The maximum displacement of the moving element is 8 mm peak to peak, which is less than the maximum displacement of the roller bearing. The secondary mass is an assembly of small masses, each of approximately 1 kg, to make the total secondary mass variable.

Accelerometers are mounted on the model structure to provide feedback information about the structural behaviour. One accelerometer is mounted on the steel plate which is part of the base, one is mounted at the top of the pile, and one is mounted on the secondary mass.

The on-line calculations of the control algorithm are carried out by an ARIANE 386-PC microcomputer. The microprocessor is operated by MS-DOS 5 operating system and real-time processed Turbo Pascal 5.0 commands. The acquired data are transferred to the microprocessor and the computed data are transmitted to the actuator and base exciter, using an analog and digital I/O board, DT2821.

6.2 Modelling

A compound model for the experimental setup is obtained by making a mathematical description of the behaviour of each component and afterwards synthesizing each model to a joint model.

The vibration exciter used to generate the control force operates like a loudspeaker, where the movement is produced by a current passing through a coil in a magnetic field. The vibration level of the moving element and with it the secondary mass is controlled by regulating the drive voltage $U_s(t)$. The generated force $f(t)$ will be modelled by

$$f(t) = \gamma_s U_s(t) \tag{6.1}$$

where γ_s is a constant depending on the amplifier and the exciter. The vibration level of the exciter has to be controlled in the interval about 2-10 Hz. In this region experimental tests have shown that the moving mass and its suspension may be represented by a spring-dashpot-mass system.

The vibration exciter used to generate the base motion operates in the same way as the actuator. But, unlike the actuator, it is not necessary to have a precise model describing the behaviour of the base, because the only requirement for the movement is, that the spectral density of the base acceleration $\ddot{x}_0(t)$ becomes broad-banded. Since the absolute value of the frequency response function between the drive voltage $U_0(t)$ and $\ddot{x}_0(t)$ is flat, this is simply obtained by generating $U_0(t)$ as a realization of a white noise. In order to control the vibration level the following approximate model is formulated,

$$\ddot{x}_0(t) = \gamma_0 U_0(t) \tag{6.2}$$

where γ_0 is a constant depending on the exciter and the amplifier.

The model structure is a continuum with a distributed mass and a concentrated mass representing the fixed part of the actuator. This system will be modelled by a one-degree-of-freedom system corresponding to the first mode. The vibrations of the higher order modes are negligible compared with the first mode.

In the description of the interaction between the three components - vibration exciter, pile and actuator - it is assumed, that the base motion is not affected by the movements of the pile and actuator. To make this assumption acceptable, the vibration exciter is designed in such a way, that the mass is about 10 times bigger than the mass of the remaining part of the model structure. Hence the inertia forces which affect the latter part and are transmitted to the base, become small compared to the driving force used to accelerate the base.

The compound experimental setup may according to the assumptions introduced be represented by the structural model shown in fig. 6.3. The eigenfuction in the first mode has been normalized so its argument at the position of the mass damper is equal to 1. The corresponding modal coordinate x_p can then physically be interpreted as the horizontal displacement of the pile at the position of the mass damper. m_p, c_p and k_p are modal mass, modal damping coefficient and modal spring stiffness corresponding to this normalization. The mass damper is modelled as a single-degree-of-freedom linear elastic, linear viscous damped system with mass m_s, damping coefficient c_s, and spring stiffness k_s. The horizontal displacement of the mass m_s is termed x_s. The internal electro-magnetic force between the pile and the mass-damper is designated $f(t)$

For the system of fig. 6.3, it is straightforward to derive the following system equations

$$m_p\ddot{x}_p + (c_p + c_s)\dot{x}_p + (k_p + k_s)x_p = k_p x_0 + c_p\dot{x}_0 + k_s x_s + c_s\dot{x}_s + f(t) \tag{6.3}$$

$$m_s\ddot{x}_s + c_s\dot{x}_s + k_s x_s = k_s x_p + c_s\dot{x}_p - f(t) \tag{6.4}$$

It is the strategy to suppress the movements of the pile relative to the the base. Hence, introduce the displacement of the pile at the position of the mass damper relative to the base as $v_p = x_p - x_0$ and the displacement of the mass damper relative to the pile as

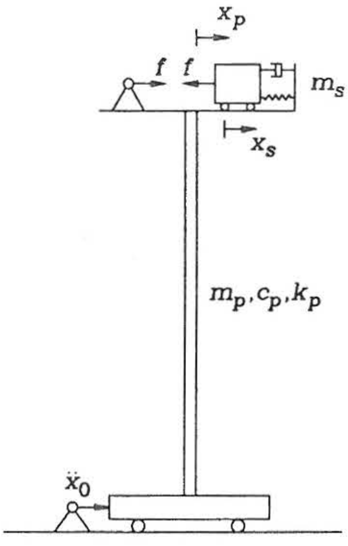

Figure 6.3: Structural model of experimental setup.

$v_s = x_s - x_p$. Use of these quantities in (6.3) and (6.4) provides

$$\mathbf{M}\ddot{\mathbf{v}} + \mathbf{C}\dot{\mathbf{v}} + \mathbf{K}\mathbf{v} = \mathbf{b}_0\ddot{x}_0(t) + \mathbf{b}f(t) \tag{6.5}$$

where

$$\mathbf{v} = \begin{bmatrix} v_p \\ v_s \end{bmatrix} \quad , \quad \mathbf{M} = \begin{bmatrix} m_p & 0 \\ m_s & m_s \end{bmatrix} \quad , \quad \mathbf{C} = \begin{bmatrix} c_p & -c_s \\ 0 & c_s \end{bmatrix} \quad , \quad \mathbf{K} = \begin{bmatrix} k_p & -k_s \\ 0 & k_s \end{bmatrix}$$

$$\mathbf{b}_0 = \begin{bmatrix} -m_p \\ -m_s \end{bmatrix} \quad , \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The system matrices **M**, **C** and **K** in (6.5) and the loading parameters γ_s and γ_0 in (6.1) and (6.2) are not known a priori, and consequently, they are going to be estimated.

The control scheme will be based on the following state space representation of the equation of motion (6.5) written in terms of (6.1),

$$\dot{\mathbf{Y}}(t) = \mathbf{A}\mathbf{Y}(t) + \mathbf{B}_0\ddot{x}_0(t) + \mathbf{B}U_s(t) \tag{6.6}$$

In equation (6.6), $\mathbf{Y}(t)$ and $\mathbf{A}$ take the same form as in (2.34), and

$$\mathbf{B}_0 = \begin{bmatrix} 0 & 0 & (\mathbf{M}^{-1}\mathbf{b}_0)^T \end{bmatrix}^T \quad , \quad \mathbf{B} = \begin{bmatrix} 0 & 0 & \gamma_s(\mathbf{M}^{-1}\mathbf{b})^T \end{bmatrix}^T \tag{6.7}$$

6.3 System Identification

System identification deals with the problem of building mathematical models of dynamic systems, based on observed system data. The area has an established collection of basic techniques, and some of these will be applied to estimate the unknown parameters in the selected mathematical model for the structure.

The basic idea behind the different identification techniques is to select the unknown parameters so that the equations of motion fit a set of observed input-output data as well as possible. The equations of motion describe the relationship between continuous time input and output signals, but the observed data are here obtained by digital sampling at discrete equidistant instants of time. Therefore, the differential equations describing the controlled system will be represented by discrete-time approximate models, in which the unknown parameters will be estimated. Afterwards, a connection is established between the parameters in the discrete-time model and the continuous-time model.

The system identification is carried out from tests with a single-input and a single-output. The input signal used to excite the model in a particular test is assumed to be known and is designated $U(t)$. Hence, in the case of a single-input the equations of motion (6.5) are expressed in state space form as

$$\dot{\mathbf{Y}}(t) = \mathbf{A}\mathbf{Y}(t) + \mathbf{B}_U U(t) \tag{6.8}$$

in which $\mathbf{Y}(t)$ and $\mathbf{A}$ take the same form as in (2.34). The vector $\mathbf{B}_U$ is dependent on the applied input signal, which may either be a drive voltage or the base acceleration. The observed output data for the system identification are accelerations. To identify the parameters from a particular test the relative accelerations at one of the two degrees-of-freedom is used, designated $\dot{Z}_p(t)$ and $\dot{Z}_s(t)$, respectively. The associated directly measured absolute accelerations are designated $\dot{Z}_{p,a}(t)$ and $\dot{Z}_{s,a}(t)$. Hence according to the defined relative movements, we have $\dot{Z}_s(t) = \dot{Z}_{s,a}(t) - \dot{Z}_{p,a}(t)$ and $\dot{Z}_p(t) = \dot{Z}_{p,a}(t) - \ddot{x}_0(t)$, where $\ddot{x}_0(t)$ is the measured base accelerations. Let $\dot{Z}(t)$ represent either of the two defined relative accelerations. This signal can be expressed in terms of the state vector by the general observation model

$$\dot{Z}(t) = \mathbf{H}\dot{\mathbf{Y}}(t) \tag{6.9}$$

where the time-invariant transformation matrix $\mathbf{H} = [0\ 0\ 1\ 0]$ for $\dot{Z}(t) = \dot{Z}_p(t)$ and $\mathbf{H} = [0\ 0\ 0\ 1]$ for $\dot{Z}(t) = \dot{Z}_s(t)$. In the following generalized formulation $\mathbf{Y}(t)$ is assumed to be an $n-$dimensional vector.

6.3.1 Discrete-time Model for Input-Output Data

The input signals contained in $U(t)$ and the measured output signals given by $\dot{Z}(t)$ are in principle discrete-time functions, which are specified at equidistant time instants $t_k = kh$, where h is the sampling period. The signals are specified with the convention defined in section 4.2.1. In what follows the objective is to derive a difference equation describing the relationship between the input and output signals. For that purpose equation (6.8) is solved to obtain the future state $\mathbf{Y}(kh+h)$ given $\mathbf{Y}(kh)$. Hereby, we have

$$\mathbf{Y}(kh+h) = \mathbf{A}_d\mathbf{Y}(kh) + \mathbf{B}_{dU}U(kh) \tag{6.10}$$

where $\mathbf{A}_d$ is given by (4.36) and $\mathbf{B}_{dU} = \int_0^h e^{\mathbf{A}t}\mathbf{B}_U\,dt = \mathbf{A}^{-1}(\mathbf{A}_d - \mathbf{I})\mathbf{B}_U$. The measured response $\dot{Z}(kh+jh)$ at the future sampling instant t_{k+j} can be expressed in terms of the state vector by substituting (6.8) into (6.9),

$$\dot{Z}(kh+jh) = \mathbf{HAY}(kh+jh) + \mathbf{HB}_U U(kh+jh) \tag{6.11}$$

Next, by applying (6.10) repeatedly in (6.11), the future measurements can be formulated in terms of $\mathbf{Y}(kh)$. After some straightforward rewriting, this yields

$$\begin{aligned}\dot{Z}(kh+jh) &= \mathbf{HAA}_d^j\mathbf{Y}(kh) + \sum_{i=1}^{j}\mathbf{HA}_d^{j-i}(\mathbf{A}_d-\mathbf{I})\mathbf{B}_U U(kh-h+ih)\\ &\quad + \mathbf{HB}_U U(kh+jh)\end{aligned} \tag{6.12}$$

In (6.12) it has been used, that $\mathbf{A}_d^{j-1}\mathbf{A} = \mathbf{AA}_d^{j-1}$, which is satisfied because the considered matrices have the same eigenvectors.

Introduce the characteristic equation $\det(\mu\mathbf{I}-\mathbf{A}_d)=0$. When the determinant is expanded, it yields the characteristic polynomial

$$\det(\mu\mathbf{I}-\mathbf{A}_d) = \alpha_n\mu^n + \alpha_{n-1}\mu^{n-1} + \cdots + \alpha_1\mu + \alpha_0 = \Delta_d(\mu) \tag{6.13}$$

where $\alpha_n = 1$. Multiplying each $\dot{Z}(kh+jh)$ in (6.12) by α_j for $j=0,1,\ldots,n$, and adding all the equations together, the following is obtained

$$\begin{aligned}&\dot{Z}(kh+nh) + \alpha_{n-1}\dot{Z}(kh+nh-h) + \cdots + \alpha_1\dot{Z}(kh+h) + \alpha_0\dot{Z}(kh) =\\ &\mathbf{HA}\left(\mathbf{A}_d^n + \alpha_{n-1}\mathbf{A}_d^{n-1} + \cdots + \alpha_1\mathbf{A}_d + \alpha_0\mathbf{I}\right)\mathbf{Y}(kh)\\ &+\beta_n U(kh+nh) + \beta_{n-1}U(kh+nh-h) + \cdots + \beta_0 U(kh)\end{aligned} \tag{6.14}$$

in which β_j, $j=0,1,\ldots,n$ have been introduced. From the derivation it follows that

$$\beta_n = \mathbf{HB}_U \tag{6.15}$$

$$\beta_j = \sum_{i=j+1}^{n}\alpha_i\mathbf{HA}_d^{i-j-1}(\mathbf{A}_d-\mathbf{I})\mathbf{B}_U + \alpha_j\mathbf{HB}_U \quad , \quad j=0,1,\ldots,n-1 \tag{6.16}$$

In the second line of equation (6.14) the terms inside the brackets represent the characteristic matrix polynomial $\Delta_d(\mathbf{A}_d)$. According to the Cayley-Hamilton theorem every matrix satisfies its own characteristic equation, i.e $\Delta_d(\mathbf{A}_d) = \mathbf{0}$. Hence, the second line in (6.14) is cancelled, and hereby equation (6.14) represents the desired discrete-time relationship between the input and the output data.

To facilitate the formulation of equation (6.14) the polynomial $A_\alpha(q)$ is introduced as

$$A_\alpha(q) = 1 + \alpha_{n-1}q^{-1} + \cdots + \alpha_1 q^{-n+1} + \alpha_0 q^{-n} \tag{6.17}$$

where q is the shift operator

$$q^{-j}\dot{Z}(kh) = \dot{Z}(kh - jh) \tag{6.18}$$

In addition, the polynomial $B_\beta(q)$ is introduced as

$$B_\beta(q) = \beta_n + \beta_{n-1}q^{-1} + \cdots + \beta_1 q^{-n+1} + \beta_0 q^{-n} \tag{6.19}$$

Utilizing the defined expressions (6.17) - (6.19), equation (6.14) can be written as

$$A_\alpha(q)\dot{Z}(kh) = B_\beta(q)U(kh) \tag{6.20}$$

According to the difference equation (6.20), the structural output $\dot{Z}(kh)$ can be exactly calculated once the input $U(kh)$ is known. This is unrealistic. In most cases such a linear model cannot describe the structural behaviour exactly. Furthermore, the sensors that measure the output are usually subjected to noise. Within this linear framework it is assumed that the effects of such uncertainties can be lumped into an additive term $e(kh)$ at the output,

$$A_\alpha(q)\dot{Z}(kh) = B_\beta(q)U(kh) + e(kh) \tag{6.21}$$

When the disturbance $\{e(kh),\ k = 1, 2, \ldots\}$ is a white noise sequence, equation (6.21) represents a so-called ARX model, where AR refers to the autoregressive part $A_\alpha(q)\dot{Z}(kh)$ and X to the extra input $B_\beta(q)U(kh)$, cf. Ljung (1987). Subsequently, the coefficients in $A_\alpha(q)$ and $B_\beta(q)$ are determined by means of an estimation procedure.

Define a vector $\boldsymbol{\theta}$ containing the unknown parameters to be determined as

$$\boldsymbol{\theta} = \begin{bmatrix} \alpha_{n-1} & \alpha_{n-2} & \cdots & \alpha_0 & \beta_n & \beta_{n-1} & \cdots & \beta_0 \end{bmatrix}^T \tag{6.22}$$

and the vector

$$\varphi(kh) = \begin{bmatrix} -\dot{Z}(kh-h) & \cdots & -\dot{Z}(kh-nh) & U(kh) & \cdots & U(kh-nh) \end{bmatrix}^T \tag{6.23}$$

Then equation (6.21) can be rewritten as

$$\dot{Z}(kh) = \varphi^T(kh)\boldsymbol{\theta} + e(kh) \tag{6.24}$$

The model (6.24) predict the measurement at the time kh on the basis of information provided by other observed variables $\varphi(kh)$. Therefore, $e(kh)$ is designated the prediction error. Having observed a set of input data $\{U(kh),\ k = 1, 2, \ldots, N\}$ and output data $\{\dot{Z}(kh), k = 1, 2, \ldots, N\}$ estimates of $\boldsymbol{\theta}$ are determined by minimizing the square of the prediction errors. Hence, the criterion function to be minimized is

$$J_N(\boldsymbol{\theta}) = \sum_{k=1}^{N} \left(\dot{Z}(kh) - \varphi^T(kh)\boldsymbol{\theta}\right)^2 \tag{6.25}$$

This is the *least squares estimate*. In order to find the estimate $\hat{\boldsymbol{\theta}}$ which minimizes (6.25), define the N-dimensional column vector

$$\dot{\mathbf{Z}}_N = \left[\ \dot{Z}(h) \quad \dot{Z}(2h) \quad \cdots \quad \dot{Z}(Nh)\ \right]^T \tag{6.26}$$

and the $N \times (2n+1)$ matrix

$$\mathbf{\Phi}_N = \left[\ \varphi(h) \quad \varphi(2h) \quad \cdots \quad \varphi(Nh)\ \right]^T \tag{6.27}$$

Then the criterion (6.25) can be written in the form

$$J_N(\boldsymbol{\theta}) = \left(\dot{\mathbf{Z}}_N - \mathbf{\Phi}_N\boldsymbol{\theta}\right)^T \left(\dot{\mathbf{Z}}_N - \mathbf{\Phi}_N\boldsymbol{\theta}\right) \tag{6.28}$$

Minimizing $J_N(\boldsymbol{\theta})$ yields the following estimate, see e.g. Ljung (1987)

$$\hat{\boldsymbol{\theta}} = \left[\mathbf{\Phi}_N^T\mathbf{\Phi}_N\right]^{-1} \mathbf{\Phi}_N^T\dot{\mathbf{Z}}_N \tag{6.29}$$

6.3.2 Expressions for the Dynamic Parameters

Given estimates of the parameters in the ARX model (6.21) it is desired to determine the structural parameters in the continuous time model (6.5).

The estimated AR-parameters α_i, $i = 0, 1, \ldots, n-1$ correspond to the coefficients in the characteristic polynomial of $\mathbf{A}_d$, cf. (6.13). Hence, the eigenvalues μ_i of $\mathbf{A}_d$ can be found by solving the associated characteristic equation $\Delta_d(\mu) = 0$.

The eigenvalues λ_i of $\mathbf{A}$ are determined from the characteristic equation $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$. Expanding the determinant the following characteristic polynomial is obtained

$$\det(\lambda\mathbf{I} - \mathbf{A}) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = \Delta(\lambda) \tag{6.30}$$

According to (2.44) it follows, that $\mathbf{A}_d = e^{\mathbf{A}h}$ and $\mathbf{A}$ have the same eigenvectors, and that $\mathbf{A}_d$ has the eigenvalues $\mu_i = \exp(\lambda_i h)$. When μ_i is determined, λ_i is then simply given by

$$\lambda_i = \frac{\ln(\mu_i)}{h} \tag{6.31}$$

The invariants a_i, $i = 0, 1, \ldots, n-1$ in (6.30) can be set up by inserting the estimated values, λ_i, repeatedly, into the characteristic equation $\Delta(\lambda) = 0$. This yields,

$$\begin{bmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ 1 & \lambda_2 & \cdots & \lambda_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} -\lambda_1^n \\ -\lambda_2^n \\ \vdots \\ -\lambda_n^n \end{bmatrix} \tag{6.32}$$

By solving (6.32) an estimate of the invariants in the characteristic equation (6.30) is obtained, designated $\hat{a}_i$, $i = 0, 1, \ldots, n-1$. These coefficients may also be expressed in terms of the still unknown structural parameters by expanding the determinant $\det(\lambda\mathbf{I} - \mathbf{A})$. In the experimental setup in Section 6.2, $n = 2$ and the unknown structural parameters are $m_p, m_s, c_p, c_s, k_p, k_s$, see (6.5). Matrix $\mathbf{A}$ is given by (2.34), (6.5). Hence, the following system of nonlinear equations is obtained for the identification of the structural parameters

$$a_i(m_p, m_s, c_p, c_s, k_p, k_s) = \hat{a}_i \quad , \qquad i = 0, 1, 2, 3 \tag{6.33}$$

In (6.33) there are 4 equations, but there are 6 unknown structural parameters. A general method to determine the additional 2 unknowns will not be given here. Instead, the additional 2 equations required to estimate the 6 structural parameters according to algorithm described above are simply obtained by performing an additional test with the pile alone without the secondary mass, called test I.

Test I

For test I the secondary mass is removed, and hence the equation of motion is given by

$$m_p\ddot{v}_p + c_p\dot{v}_p + k_p v_p = -m_p\ddot{x}_0 \tag{6.34}$$

In this test the base is driven by a band-limited white-noise voltage. Equation (6.34) is written in state space form as (6.8) with $\mathbf{Y} = [v_p \; \dot{v}_p]^T$. The associated ARX model is formulated, and the AR-parameters are estimated using the base acceleration as input, i.e $U(t) = \ddot{x}_0(t)$, and the relative velocity of the pile at the position of the mass damper as output. The observation model (6.9) then takes the form $\dot{Z}(t) = [0 \; 1]\,\dot{\mathbf{Y}}(t)$. Next, estimates of the coefficients in the associated characteristic polynomial (6.30) are determined from (6.31) and (6.32), designated $\hat{a}_{p1}$, $\hat{a}_{p2}$. Equations for these coefficients in terms of m_p, c_p and k_p are set up by formulating the characteristic polynomial (6.30) from the state space model of (6.34). By analogy with (6.33), this leads to the following equations

$$\frac{c_p}{m_p} = \hat{a}_{p1} \tag{6.35}$$

$$\frac{k_p}{m_p} = \hat{a}_{p0} \tag{6.36}$$

Test II

In test II the total model was considered, but the base was fixed, i.e. $\ddot{x}_0(t) = 0$. The input used to excite the model was a band-limited white-noise voltage driving the actuator, and the measured structural output was the acceleration $\ddot{x}_p(t)$, i.e $U(t) = U_s(t)$ and according to the formulated state space model $\dot{Z}(t) = [0 \; 0 \; 1 \; 0]\,\mathbf{Y}(t)$, cf. (6.9). Based on the observed input-output data the coefficients $\hat{a}_0, \hat{a}_1, \hat{a}_2, \hat{a}_3$ are estimated as described above. Mathematical expressions for these parameters written in terms of the unknown structural parameters are developed from the equations of motion (6.3) and (6.4). This leads to

$$\frac{m_s(c_p + c_s) + m_p c_s}{m_p m_s} = \hat{a}_3 \tag{6.37}$$

$$\frac{m_s(k_p + k_s) + m_p k_s + c_p c_s}{m_p m_s} = \hat{a}_2 \tag{6.38}$$

$$\frac{k_s c_p + k_p c_s}{m_p m_s} = \hat{a}_1 \tag{6.39}$$

$$\frac{k_p k_s}{m_p m_s} = \hat{a}_0 \tag{6.40}$$

Equations (6.35)-(6.40) constitute a system of 6 non-linear equations for the unknown structural parameters. The solution is obtained by means of the MATLAB algorithm *fsolve*, which is based on the Gauss-Newton method to find roots in non-linear equations, see PC-MATLAB (1989). The results are shown in table 6.1.

m_p	c_p	k_p	m_s	c_s	k_s
kg	Ns/m	10^3N/m	kg	Ns/m	10^3N/m
22.6	57.6	16.1	11.7	40.4	8.85

Table 6.1: Estimated structural parameters

The eigenvalues used to estimate the structural parameters may also be expressed in terms of angular eigenfrequencies and damping ratios according to the definition (2.37). The dynamic parameters associated with the pile (ω_p, ζ_p), the actuator (ω_s, ζ_s) and the combined system $(\omega_1, \omega_2, \zeta_1, \zeta_2)$ are shown in table 6.2.

$\frac{\omega_p}{2\pi}$	$\frac{\omega_s}{2\pi}$	$\frac{\omega_1}{2\pi}$	$\frac{\omega_2}{2\pi}$	ζ_p	ζ_s	ζ_1	ζ_2
Hz	Hz	Hz	Hz	%	%	%	%
4.25	4.38	2.96	6.17	0.76	6.3	3.7	8.3

Table 6.2: Estimated dynamic parameters

As seen from table 6.2, the mass damper is tuned to work as a passive vibration absorber against vibrations in the first mode of the structure $(\omega_p \sim \omega_s)$. Only vibration frequencies different from ω_p are basically controlled actively. Further, it should be noted that the secondary mass m_s is 50% of the primary mass m_p. For prototypes of tuned mass dampers this ratio will typically be 1% - 5%. However, in this experimental study the practicable maximum displacement of the secondary mass relative to the pile is small and following, its maximum acceleration. Therefore, to increase its inertia force and with it the damping effect, the secondary mass has been increased.

6.3.3 Loading parameters

The level of the control force and the base acceleration is controlled by the computer according to the relations (6.1) and (6.2). In each of these two equations the loading parameters γ_s and γ_0 are contained, respectively. These still unknown parameters are estimated in the following.

For that reason, the input signal $U(t)$ used in each of the two tests carried out to estimate γ_s and γ_0, respectively, is a drive voltage. The loading parameters are in each of the two tests contained in the vector $\mathbf{B}_U$, and are related to the estimated ARX parameters as described by (6.15) and (6.16). Consider the problem of estimating γ_s from test II, where $\mathbf{B}_U = \mathbf{B}$, given in (6.7). For test II $n = 4$, and hence (6.15) and (6.16) represent 5 equations. However, in these equations the only unknown is γ_s and consequently, there is no unique solution for γ_s. This is also the fact for γ_0, which may be determined from test I. Because of this problem, the loading parameters are estimated on the basis of a spectral analysis.

By using a spectral analysis, transfer functions for the observed input-output data are estimated directly, without using a parametric model. Next the loading parameters are selected so that the transfer function based on the equations of motion fit the estimated transfer function as well as possible.

An estimate of a transfer function associated with a single-input $U(t)$ and a single-output $\dot{Z}(t)$ is obtained from

$$\hat{H}_{\dot{Z}U}(\omega) = \frac{\hat{\Phi}_{\dot{Z}U}(\omega)}{\hat{\Phi}_{UU}(\omega)} \tag{6.41}$$

where $\hat{\Phi}_{\dot{Z}U}(\omega)$ and $\hat{\Phi}_{UU}(\omega)$ are estimated cross-spectral and auto-spectral density functions. The above estimate is determined for $\frac{N}{2} - 1$ equidistant frequencies, $\omega_k = 2\pi k/(Nh)$, $k = 1, 2, \ldots, \frac{N}{2} - 1$. Here, the estimate is calculated by means of the MATLAB algorithm *spectrum*, see PC-MATLAB (1989), which implements the *Welch method.* For further information, see e.g. Oppenheim and Schafer (1975).

Next, the corresponding frequency response function based on the equation of motion (6.8) and the observation model (6.9) is set up. For that purpose the Fourier transform is introduced,

$$\mathbf{Y}(\omega) = \int_{-\infty}^{\infty} \mathbf{Y}(t) e^{-i\omega t} \mathrm{d}t \tag{6.42}$$

where $\mathbf{Y}(\omega)$ is the Fourier transform of the function $\mathbf{Y}(t)$. By determining the Fourier transform of (6.8) and (6.9), the frequency response function $H_{\dot{Z}U}(\omega)$ is obtained as

$$\begin{aligned} H_{\dot{Z}U}(\omega) &= \frac{\dot{Z}(\omega)}{U(\omega)} \\ &= (i\omega)\mathbf{H}[i\omega\mathbf{I} - \mathbf{A}]^{-1}\mathbf{B}_U \end{aligned} \tag{6.43}$$

It is convenient to define $\Delta(i\omega) = \det(i\omega\mathbf{I} - \mathbf{A})$ corresponding to the characteristic polynomial $\Delta(\lambda)$ in (6.30). Then $[i\omega\mathbf{I} - \mathbf{A}]^{-1} = \mathrm{Adj}[i\omega\mathbf{I} - \mathbf{A}]/\Delta(i\omega)$, where $\mathrm{Adj}(\cdot)$ is the transposed of a matrix formed by cofactors of its argument. The frequency response function $H_{\dot{Z}U}(\omega)$ can then be rewritten as

$$H_{\dot{Z}U}(\omega) = \frac{(i\omega)\mathbf{H}\mathrm{Adj}[i\omega\mathbf{I} - \mathbf{A}]\mathbf{B}_U}{\Delta(i\omega)} \tag{6.44}$$

In the model based frequency response function (6.44) the unknown loading parameter is contained in $\mathbf{B}_U$ for the particular test. Consider the problem of estimating γ_s from test II. In that case, $H = [0\ 0\ 1\ 0]$, $\mathbf{A}$ takes the same form as in (2.34), and $\mathbf{B}_U = \mathbf{B}$ as given by (6.7). According to these specifications the numerator in (6.44) can be expressed as a product of the loading parameter γ_s and a polynomial $P_{\dot{Z}U}(\omega)$ of order $n = 4$, i.e.

$$\begin{aligned}(i\omega)\mathbf{H}\mathrm{Adj}[i\omega\mathbf{I} - \mathbf{A}]\mathbf{B}_U &= \gamma_s P_{\dot{Z}U}(\omega) \\ &= \gamma_s\big(b_3(i\omega)^3 + b_2(i\omega)^2 + b_1 i\omega + b_0\big)i\omega\end{aligned} \tag{6.45}$$

in which the coefficients b_m, $m = 0, 1, 2, 3$ follow from this rewriting. The model based frequency response function (6.44) then takes the form

$$H_{\dot{Z}U}(\omega) = \frac{P_{\dot{Z}U}(\omega)\gamma_s}{\Delta(i\omega)} \tag{6.46}$$

Let the polynomials in the numerator and the denominator of (6.46) be expressed in terms of the already estimated structural parameters, using the designation $\hat{P}_{\dot{Z}U}(\omega)$ and $\hat{\Delta}(i\omega)$. Then, given the non-parametric estimate $\hat{H}_{\dot{Z}U}(\omega)$, the loading parameter is determined by minimizing

$$J_H = \sum_{k=1}^{\frac{N}{2}-1} \left|\hat{H}_{\dot{Z}U}(\omega_k)\hat{\Delta}(i\omega_k) - \gamma_s\hat{P}_{\dot{Z}U}(\omega_k)\right|^2 \tag{6.47}$$

For test I the model based frequency response function can be written in the same form as (6.46), and the cost function to be minimized as (6.47) with γ_0 replacing γ_s. In the present tests $\dot{Z}(t) = \ddot{v}_p(t)$. The loading parameter γ_0 is determined from the frequency response function estimate $\hat{H}_{\ddot{v}_pU_0}(\omega)$. Based on the equation of motion (6.34) this function expressed in terms of the estimated dynamic parameters becomes

$$H_{\ddot{v}_pU_0}(\omega) = \frac{\frac{1}{m_p}\omega^2\gamma_0}{-\omega^2 + \hat{a}_{p1}i\omega + \hat{a}_{p0}} \tag{6.48}$$

For test II the applied frequency response function is

$$H_{\ddot{v}_pU_s}(\omega) = \frac{\frac{1}{m_p}\omega^4\gamma_s}{\omega^4 - \hat{a}_3 i\omega - \hat{a}_2\omega^2 + \hat{a}_1 i\omega + \hat{a}_0} \tag{6.49}$$

By performing the minimization of J_H in (6.47) associated with each of the two tests, the estimates shown in table 6.3 are obtained. Besides being dependent on the exciters the loading parameters are also dependent on the amplifiers, since the drive voltages used to estimate the parameters are the input voltages for the amplifiers. Consequently, the two parameters are not comparable, and do not directly express the required voltage to generate the base acceleration and the control force, respectively.

γ_0	γ_s
N/(V· kg)	N/V
0.467	0.022

Table 6.3: Estimated loading parameters

The directly estimated frequency response functions as given by (6.41) and the corresponding one based on the estimated structural and loading parameters as given by (6.48) and (6.48) are compared in fig. 6.4 and fig. 6.5.

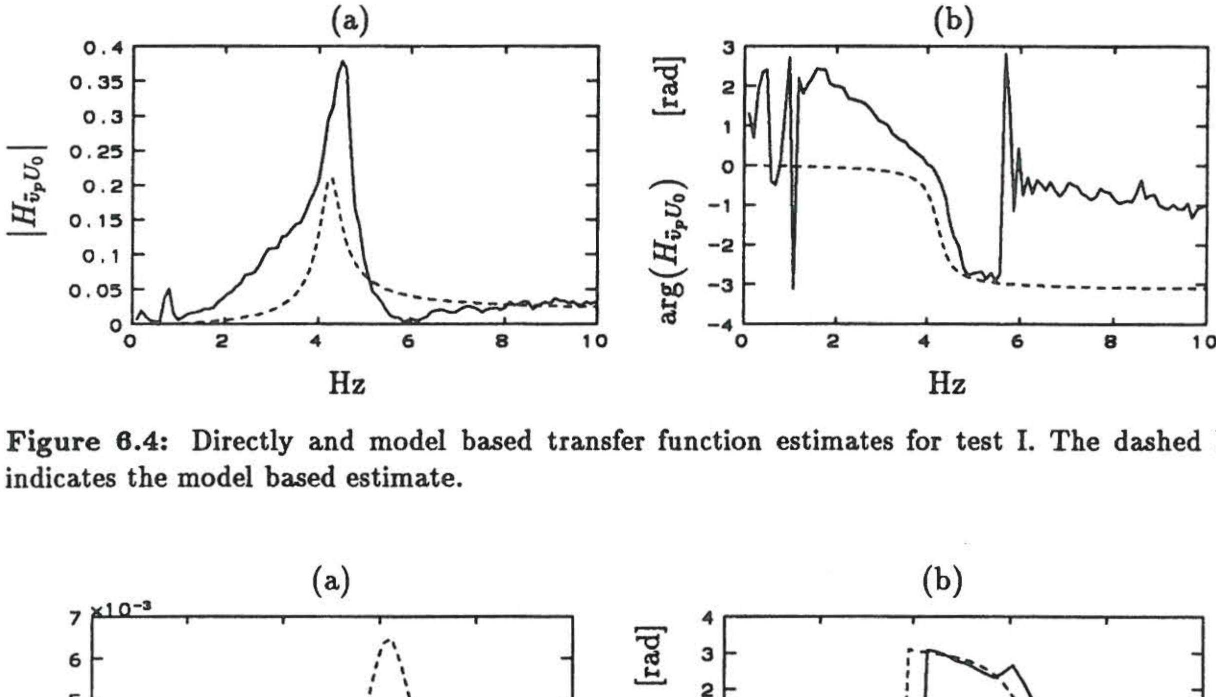

Figure 6.4: Directly and model based transfer function estimates for test I. The dashed line indicates the model based estimate.

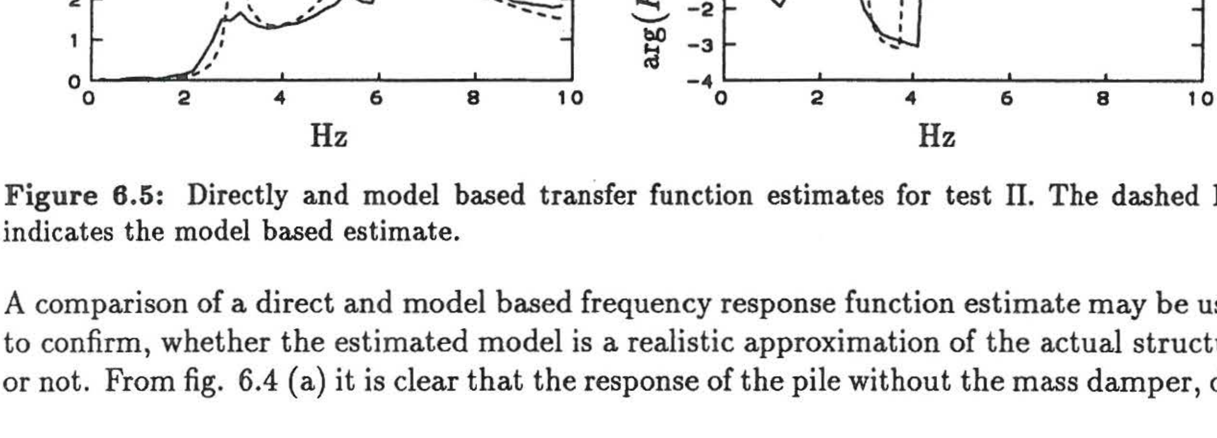

Figure 6.5: Directly and model based transfer function estimates for test II. The dashed line indicates the model based estimate.

A comparison of a direct and model based frequency response function estimate may be used to confirm, whether the estimated model is a realistic approximation of the actual structure or not. From fig. 6.4 (a) it is clear that the response of the pile without the mass damper, due to a broad-banded "seismic voltage" $U_0(t)$, is dominated by the fundamental frequency of the pile as described by the model. However, the phase shift between U_0 and the response of the pile cannot be described by the applied model. This deviation is due to uncertainties in the simplified model (6.2) for the relation between $U_0(t)$ and $\ddot{x}_0(t)$. However, as mentioned in connection with the set up of this model, the aim is to control the absolute value of the base acceleration with the drive voltage. Hence, it is most important that there is a reasonable agreement between the absolute value of the model based transfer function estimate and the absolute value of the corresponding non-parametric estimate, and less important for the phase. By considering the complete model, the results in fig. 6.5 seem to confirm that the model is able to describe the structural behaviour caused by the control voltage, $U_s(t)$.

6.4 Control Design

In order to formulate the control law, with the discrete-time nature of the digital control procedure in mind, the equation of motion (6.6) is approximated by a difference equation. By using the same approximation procedure as in section 4.2.1 the relation between two consecutive sampling instants with the time difference h can be written as

$$\mathbf{Y}(kh+h) = \mathbf{A}_d\mathbf{Y}(kh) + \mathbf{B}_{d0}\ddot{x}_0(kh) + \mathbf{B}_d U_s(kh) \tag{6.50}$$

in which $\mathbf{A}_d$, $\mathbf{B}_{d0}$, and $\mathbf{B}_d$ are given, respectively, by (4.36)-(4.38).

In this experimental study the sampled structural response $\mathbf{Z}(kh)$ is assumed to be linearly related to the state $\mathbf{Y}(kh)$ and affected by an additive noise $\mathbf{V}(kh)$ as described by (4.43). The state vector contains the relative velocity and displacement of the secondary mass and of the pile at the mass damper, cf. (6.5). However, the response at these places is measured by means of accelerometers, designated $\dot{Z}_{p,a}$ and $\dot{Z}_{s,a}$, where the index a indicates that it is the absolute acceleration. In analogy with the continuous-time definitions introduced in connection with the observation model (6.9), let $\dot{Z}_s(kh) = \dot{Z}_{s,a}(kh) - \dot{Z}_{p,a}(kh)$ and $\dot{Z}_p(kh) = \dot{Z}_{p,a}(kh) - \ddot{x}_0(kh)$, where $\ddot{x}_0(kh)$ is the measured base acceleration. Then the "measured relative velocities", $Z_s(kh), Z_p(kh)$, are obtained by numerical integration of $\dot{Z}_s(kh), \dot{Z}_p(kh)$. A temporary value $Z_{s,T}(kh)$ is first determined by using the trapezoidal rule,

$$Z_{s,T}(kh) = \left(\frac{1}{2}\dot{Z}_s(0) + \frac{1}{2}\dot{Z}_s(kh) + \sum_{i=1}^{k-1}\dot{Z}_s(ih)\right)h \tag{6.51}$$

Since the mean value of the base excitation is zero and the structural model is approximately linear, the mean value of $Z_s(kh)$ has to be zero. To satisfy this requirement the velocity is determined as the output of the following moving average filter of order N_0,

$$Z_s(kh) = Z_{s,T}(kh) - \frac{1}{N_0}\sum_{i=k+1-N_0}^{k} Z_{s,T}(kh) \tag{6.52}$$

The "measured velocity" $Z_p(t)$ is determined similarly. Hence, $\mathbf{Z}(kh)$ and $\mathbf{H}$ in the observation model (4.43) are given by

$$\mathbf{Z}(kh) = \begin{bmatrix} Z_p(kh) \\ Z_s(kh) \end{bmatrix} \quad , \quad \mathbf{H} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6.53}$$

The additive noise $\{\mathbf{V}(kh),\ k = 1, 2, \ldots\}$ represents deviations between the integrated signals $\mathbf{Z}(kh)$ and the true velocities $\dot{\mathbf{v}}(kh)$. This signal and the base acceleration $\{\ddot{x}_0(kh),\ k = 1, 2, \ldots\}$ are modelled as mutually independent sequences of zero-mean, independent Gaussian random variables with covariances given by

$$E\left[\mathbf{V}(kh)\mathbf{V}^T(lh)\right] = \begin{cases} \mathbf{0} & , \quad k \neq l \\ \sigma_V^2 \mathbf{I} & , \quad k = l \end{cases} \tag{6.54}$$

$$E\left[\ddot{x}_0(kh)\ddot{x}_0(lh)\right] = \begin{cases} 0 & , \quad k \neq l \\ \sigma_0^2 & , \quad k = l \end{cases} \tag{6.55}$$

The standard deviation σ_0 in (6.55) is estimated from measurements of $\ddot{x}_0$. It is difficult to estimate σ_V in (6.54) and therefore, it is just considered as a weighting coefficient which is chosen by the designer. For smaller σ_V, the measurements are perceived as more reliable.

The control law for the driving voltage $U_s(kh)$ is found by minimizing a quadratic performance index of the same form as (4.46). Here, the time-delay dh between the measured structural response and the application of the corrective force must be specified. For this experimental test the time-delay is only due to on-line calculations which have to be performed within each sampling interval, i.e. $d = 1$. Concerning the weighting matrices in (4.46), the matrix $\mathbf{S}(Nh)$ weighting the relative importance attached to the final state is taken as zero. The remaining weighting matrices are chosen to be constant. Hence, the performance index for this problem becomes

$$J = \frac{1}{2} E\left[\sum_{k=1}^{N-1} \mathbf{Y}^T(kh)\mathbf{Q}\mathbf{Y}(kh) + rU_s^2(kh - h)\right] \tag{6.56}$$

The optimal closed-loop control voltage $U_s^*(kh)$ that minimizes (6.56) is given by the feedback law (4.47) with $U_s^*(kh)$ replacing $\mathbf{F}^*(kh)$. The feedback gain $\mathbf{G}_c(kh)$ is time-varying for the optimal control problem, but as an approximation a time-invariant gain is used based on a solution to the steady-state part of the Ricatti equation (4.40). Then, the control law is given by

$$U_s(kh) = -\mathbf{G}_c\hat{\mathbf{Y}}(kh + h|kh) \tag{6.57}$$

where $\mathbf{G}_c$ is given according to (4.28) and (4.33) as

$$\mathbf{G}_c = \left(r + \mathbf{B}_d^T\mathbf{S}\mathbf{B}_d\right)^{-1}\mathbf{B}_d^T\mathbf{S}\mathbf{A}_d \tag{6.58}$$

The predicted estimate $\hat{\mathbf{Y}}(kh + h|kh)$ in (6.57) is determined from the Kalman filter equations (4.48)-(4.52). The filter gain contained in this estimation algorithm may be found by computing (4.51)-(4.52) recursively, which can be done off-line. However, the convergence properties of the filter gain $\mathbf{K}_f(kh)$ determined hereby, are the same as for the feedback gain $\mathbf{G}_c$, i.e. that all the components of $\mathbf{K}_f$ in a short period of time reach a stationary value. Therefore, to simplify the estimation procedure the steady-state solution for the filter gain is used for the entire control period.

The recursive equation for the one-step ahead predicted estimate $\hat{\mathbf{Y}}(kh+h|kh)$ can be written in the form, cf. Appendix A

$$\begin{aligned}\hat{\mathbf{Y}}(kh+h|kh) &= \mathbf{A}_d\hat{\mathbf{Y}}(kh|kh-h) + \mathbf{B}_d U_s(kh-h) \\ &\quad + \mathbf{A}_d\mathbf{L}_f\Big(\mathbf{Z}(kh) - \mathbf{H}\hat{\mathbf{Y}}(kh|kh-h)\Big) \qquad (6.59)\end{aligned}$$

The filter gain $\mathbf{L}_f$ in (6.59) is related to $\mathbf{K}_f$ in the way given by (A.25), and it takes the form, cf. (A.21)

$$\mathbf{L}_f = \mathbf{A}_d\mathbf{H}^T\left(\sigma_V^2\mathbf{I} + \mathbf{H}\mathbf{p}\mathbf{H}^T\right)^{-1} \qquad (6.60)$$

The stationary solution for the covariance of the estimation error, $\mathbf{p}$, is determined from (A.22). With $\mathbf{p}(kh+h) = \mathbf{p}(kh) = \mathbf{p}$, this yields

$$\mathbf{p} = \mathbf{A}_d\mathbf{p}\mathbf{A}_d^T - \mathbf{A}_d\mathbf{p}\mathbf{H}^T\left(\mathbf{H}\mathbf{p}\mathbf{H}^T + \sigma_V^2\mathbf{I}\right)^{-1}\mathbf{H}\mathbf{p}\mathbf{A}_d^T + \sigma_0^2\mathbf{B}_{d0}\mathbf{B}_{d0}^T \qquad (6.61)$$

Equation (6.61) is of the same form as the steady-state Ricatti equation (4.40) of the optimal control problem . Hence, the computer program available to solve this equation can also be used to solve (6.61) for $\mathbf{p}$.

6.5 Results

The effect of using active vibration control was examined by using the control algorithm described in Section 6.4 to suppress the relative vibrations caused by a banded white noise excitation. Furthermore, the passive damping effect of the mass damper was examined.

To see the passive damping effect the frequency response function from the base acceleration $\ddot{x}_0(t)$ to the relative acceleration of the pile was estimated and compared for two different tests. In the first test the secondary mass was removed, and in the second test it was implemented. Modulus of the two frequency response functions are depicted in fig. 6.6. From this it is evident that the structural response caused by components of the base excitation in the interval 3.5 - 5 Hz can be reduced considerably by means of a passive mass damper. But the disadvantage of passive vibration control is revealed, too. In the interval around the two eigenfrequencies of the combined system, 2.96 Hz and 6.17 Hz, the vibration level will be increased.

The actively controlled test was performed on the basis of the system parameters listed in Table 6.4. The weighting coefficients in $\mathbf{Q}$ were selected so that the first term in the performance index (6.54) represents the mechanical energy associated with the pile alone

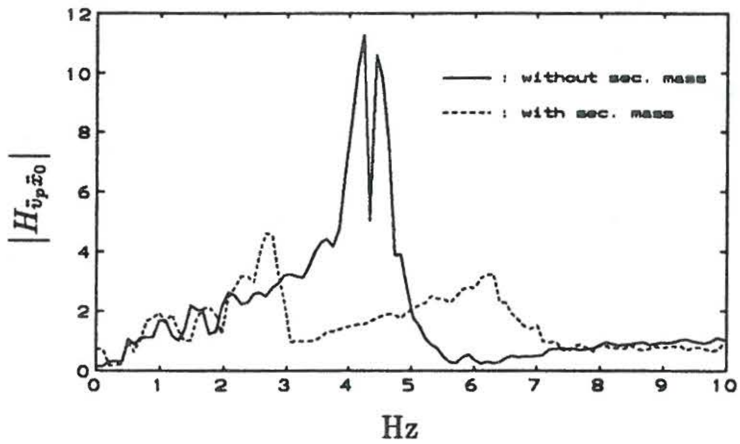

Figure 6.6: Comparison of directly transfer function estimates for the model structure without and with the secondary mass.

Parameter	Quantity
Control weighting parameters, $\mathbf{Q}$	$\begin{bmatrix} 712 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
r	5×10^{-9}
Control gain, $\mathbf{G}_c$	$\begin{bmatrix} 1.41 & 0.695 & -0.052 & -0.073 \end{bmatrix} \times 10^5$
Transformation matrix, $\mathbf{H}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Filter parameters, σ_0	0.33 V
σ_V	1×10^{-3} m/s
Filter gain, $\mathbf{L}_f$	$\begin{bmatrix} -0.0128 & 0.0099 \\ -0.0109 & -0.0302 \\ 0.9936 & -0.0020 \\ -0.0029 & 0.9744 \end{bmatrix}$
Filter order, N_0	17
Sampling period, h	0.02 s

Table 6.4: System parameters of controlled system.

at each sampling instant, i.e. $Q(1,1) = \omega_p^2$ and $Q(3,3) = 1$. The weighting coefficient r associated with the control force was selected from information on performance limits on the actuator. Concerning the filter parameters, σ_0 was estimated directly from the knowledge of the excitation, whereas σ_V was chosen more or less randomly from comparisons of the measured and estimated signals. The measured signals were considered as reliable, and therefore σ_V was selected to obtain a good agreement between the two signals. As expected the agreement was better for smaller values of σ_V. On the other hand, due to inevitable measurement errors it should not be taken as zero. Both the relative velocity of the pile and the secondary mass were measured for the state estimation. If only one of the velocities associated with the two-degree-of-freedom system was measured, an improvement of the passive mass damper could not be obtained.

The effect of the actively controlled mass damper is illustrated in fig. 6.7, where the one-sided auto-spectral density function $\Phi_{\dot{v}_p}(\omega)$ for the relative velocity $\dot{v}_p$ is depicted for the actively and passively controlled mass damper, respectively. A reduction of the energy level is obtained at the lowest eigenfrequency of the combined system, i.e 2.97 Hz. Around the second eigenfrequency, 6.17 Hz, there is only a small difference in the two tests. The spectral peak at 9 Hz is due to an enhanced energy level in the base excitation. This frequency fits the eigenfrequency of the vibration exciter. The associated standard deviations, which corresponds to the area below the curve of the power spectral density functions, are 0.013 by passive control and 0.011 by active control. This is a reduction of 15 %.

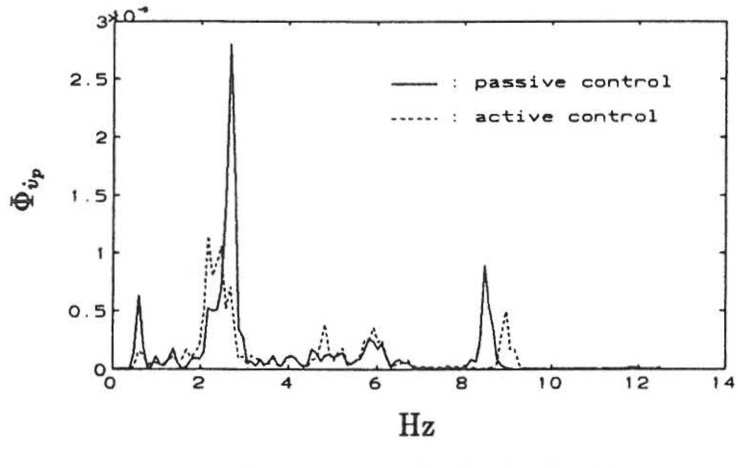

Figure 6.7: One-sided auto-spectral densities for $\dot{v}_p$ obtained from experimental tests with passive and active control.

A comparison between the displacements for the uncontrolled and controlled tests shows the same tendencies as those emphasized in connection with the velocities. The relatively modest reduction is due to the fact, that the response of the present test structure is dominated by the first mode, and the passive mass damper is optimally tuned against these vibrations.

Next, it is examined whether the control efficiency is sensitive to the applied state estimation algorithm or not. This sensitivity analysis is performed from simulated data generated from the equations of motion for the structural model. First, the control procedure is simulated with the assumption that the entire state vector is available, i.e. the control force is a feedback of the true state. In the simulation the same seismic excitation is used as in the experimental test. Next, the control force is obtained as a feedback of the estimated state, which is determined from the simulated structural accelerations according to the algorithm used in the experimental test. In the calculation of the "measured" velocity there is not introduced any measurement noise in excess of the errors introduced by the numerical integration.

The one-sided auto-spectral density functions for the two simulations show a good agreement, see fig. 6.8. Hence, the control efficiency seems not to be sensitive to the applied state estimator, if the mathematical model describes the structural behaviour exactly.

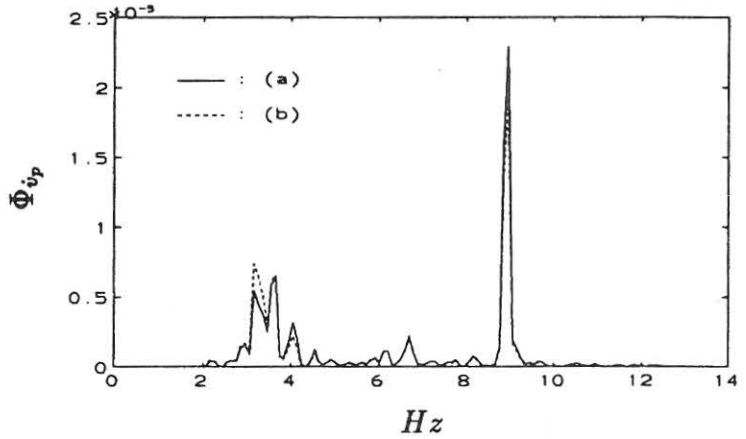

Figure 6.8: One-sided auto-spectral densities for $\dot{v}_p$ with active control obtained from simulated data. (a) Perfectly known state. (b) Estimated state.

By comparison of the spectral density functions in fig. 6.8 and the corresponding one given by the dashed line in fig. 6.7, there is seen to be some discrepancy, which of course is due to uncertainties in the mathematical model. The standard deviation $\sigma_{\dot{v}_p}$ obtained by the numerical simulation of the passive and active control system is 0.017 and 0.011, respectively. This is a reduction of $\Delta\sigma_{\dot{v}_p} = 35\%$ obtained by a control force with a standard deviation $\sigma_{U_s} = 0.49$ V. The corresponding values for the experimental test are $\Delta\sigma_{\dot{v}_p} = 13\%$ and $\sigma_{U_s} = 0.67$ V. Using these results as a measure of the control efficiency, it is evident that the model uncertainty has an effect. During the experimental test these uncertainties affect the control algorithm through the state estimation, which is based on the Kalman filter. State estimates determined from uncertain equations of motion will of course be defective and hence, the feedback control force is less effective than expected according to numerical simulations. In this test, the model uncertainties implied, that the velocity associated with each degree-of-freedom had to be measured in order to obtain satisfactory state estimates and with it, to improve the tuned mass damper by controlling it actively.

6.6 Conclusions

Active structural control by a LQG control algorithm has been carried out experimentally using active mass damper control. Implementation of the control algorithm requires knowledge of a structural model and of its parameters, such as damping and stiffness. Therefore, a mathematical model describing the structural behaviour was set up with regard to the interaction between the structure and the control device.

The structural parameters were determined from the estimated parameters in two different ARX-models representing measured input-output data. A connection between these parameters was developed by setting up the characteristic equation for the poles of the structural system, both based on of the continuous time model and on the ARX-model. However, it was not possible to make an unambiguous determination of the loading parameters from the estimated ARX-parameters. For that reason the loading parameters were determined by fitting the frequency response function for the continuous time model to a directly estimated transfer function obtained from a spectral analysis method.

Comparison of the direct and model based transfer function estimates showed small discrepancies. However, due to uncertainties in the mathematical model, it was not possible to make adequate estimates of the state vector from one sensor. Only, when the velocity associated with each of the two degrees-of-freedom was measured, an improvement of the passive mass damper could be obtained by active control. This indicates as expected, that the state estimates may be improved by employing more sensors, and that a closed-loop algorithm based on state feedback is sensitive to uncertainties in the state estimate. Hence, it is evident that an analysis of the efficiency of a given control algorithm ought to include a sensitivity study with respect to these effects, either by numerical studies or by experimental tests.

The experimental tests carried out in this study have shown some prospects of applying active mass dampers in structures under earthquake excitation. In spite of model uncertainties, it was possible to improve the properties of the passive mass damper by controlling it actively according to a closed-loop control scheme in which the feedback gains were determined from the optimal control formalism and the state provided for feedback were estimated via the Kalman filter.

Chapter 7

Conclusions

Active vibration control of civil engineering structures has been studied. In the work carried out main emphasis has been attached to the design of optimal control schemes, where the objective is to minimize the vibrations of a structure by using "acceptable" levels of control forces. According to this formalism a new control algorithm has been developed on the basis of the invariant embedding technique. This control strategy is applicable both to vibration control of nonlinear structural systems and to adaptive control of structures with unknown dynamic parameters. Furthermore, a laboratory test has been carried out to study the prospects of an active mass damper for vibration suppression of structures under earthquake excitation. A survey of the complete study and the obtained results is presented in Section 7.1.

On the basis of the simulation results obtained here and by other researchers it appears that active structural control is applicable as a concept of vibration abatement of civil engineering structures. Likely, the experimental study carried out here and laboratory tests carried out by other researchers give promising results for this concept. But a general application of active control for vibration suppression of civil engineering structures will not be prominent before some key problems are solved.

Civil engineering structures are complex systems, but there is a demand for simple control systems to improve reliability and reduce the cost, typically tantamount to control schemes based on greatly simplified models of the structural behaviour. Hence, problems caused by modelling errors, control and observation spillover arise. Furthermore, active control of large civil engineering structures requires the ability to generate and apply large control forces over sustained periods of time to the structure. This problem requires among others a hardware development. Finally, problems exist with respect to acceptance by the civil engineering and construction profession, especially when structural safety is to rely upon an active control system.

The solution of these key problems provides a continuation of the theoretical research within the field of active structural control and concurrently with this, more laboratory test must be performed using larger multi-degree-of-freedom structural models. Driven by the evident advantages of active structural control to ensure safety and comfort these problems will probably be solved in the future. Hence, for the next generation of buildings active structural control may be an alternative to traditional static methods of conservative design and become an integral part of the building system.

7.1 Summary of the Thesis

Chapter 2. Analysis and design of control schemes for active structural control generally imply an analytical model which describes the structural behaviour under environmental loads and the applied control forces. Such a model has been set up from a theoretical approach, which makes use of Cauchy's law of motion and the mechanical properties of the structure (e.g. mass, stiffness, damping, etc.). This continuous model is based on the assumption of distributed continuous control forces and the fact that civil engineering structures in nature are distributed parameter systems. However, a control design based on continuous systems is generally quite complicated and therefore, a discretized model is introduced in which the control forces are discrete elements. In preparation for the following analysis the equations of motion for both the distributed and discrete parameter system are written in state space form.

Given the equations of motion for a structure, it is in principle possible to design a control system on ideal conditions according to a specified performance criterion for the structural response. Some of the most common categories of performance specifications applied in structural control are presented. Using the pole assignment method the control is determined in such a way that the eigenvalues of a closed-loop system take a set of values prescribed by the designer. The purpose of another method, bounded state control, is to maintain a set of structural response variables within an allowable region. Finally, there is a broad category of optimal control algorithms concerned with developing control systems which are the best possible with respect to a standard, a so-called performance index. In the case of a time-dependent performance index, which is quadratic in the present states and the present control forces, the resulting control scheme has been presented, known as instantaneous optimal control. In contradiction, a classical optimal control algorithm is derived by minimizing the structural response with a minimum of control forces over a specified period of time. This study has mainly been about the derivation of control algorithms according to the latter performance specification.

Chapter 3. Optimal control of a structure described by a distributed parameter model possesses some complicated numerical problems. Further, even if it is possible to determine a control law, its practicability is enclosed by a requirement of distributed measurements of the structural response. These problems are pointed out. Hence, both to simplify the control design and to improve the applicability of the derived control schemes, a general method is to discretize the model equations in space.

Given a distributed model, one approach is to expand the distributed dependent variables into a finite set of eigenfunctions and then control this discretized system. This technique becomes of particular interest when the corresponding modal equations are decoupled. Then, the modal control forces may be determined separately according to a prescribed performance specification for the particular mode and next, the physical control forces are synthesized, called independent modal space control. Besides this method, optimal control design has been treated for discretized systems in which the dependent variables describe the displacement at discrete points. The optimality conditions to be satisfied in this case have been formulated both by means of calculus of variations and dynamic programming. Use of the former method leads to a two-point boundary-value problem for the controlled state and the co-state, while the latter approach results in the Hamilton-Jacobi-Bellman equation for the minimum cost function. In the case of special interest, where the performance index is quadratic, it has been demonstrated that the two types of equations yield the same control law, called the LQ-regulator.

The optimal control design possesses a principal difficulty, because a determination of the optimal control force requires that the external loading is known a priori. Hence, it is called a backwards-in-time solution. Unfortunately, this is usually not the case by control of civil engineering structures subjected to environmental loads. Instead, one may use an optimal closed-loop control law, which is obtained by ignoring the external loading in the derivation.

However, another approximate solution is derived in this thesis on the basis of the invariant embedding technique. The basic idea of this method is to change the original TPBVP into a class of more general initial value problems. Hereby, a set of differential equations for the controlled state has been derived, and they can be solved by integrating forward in time from specified initial conditions. However, an explicit expression for the control force is not obtained by using the invariant embedding technique and therefore, a new control algorithm has been developed in order to find the control forces. For that purpose, the idea of the pole assignment method has been utilized by letting the control force be determined by closed-loop feedback. The control gain matrix is then selected in such a way, that the eigenvalues of the system matrix associated with the equation of motion and with the invariant embedding equation for the controlled state, are equal. This solution for the control gain matrix is not unique and therefore, it is furthermore required that the deviation between the corresponding eigenvectors must be minimal. The developed control algorithm has been generalized to include non-linear structural systems, and its feasibility has been explored by optimal control of a hysteretic system to be described later.

Chapter 4. The formulated control algorithms in Chapter 3 are based on idealized structural models on ideal conditions. Hence, implementation of such a control scheme give rise to practical considerations. One type of problem due to ideal system descriptions, called spillover, is described in Chapter 4. It is common to distinguish between control and observation spillover. The former phenomenon designates the fact that a state feedback force developed from a discrete structural model will inevitably affect the uncontrolled modes by application to real structures. Since this action is more-or-less random, the vibration level of the uncontrolled modes may be increased due to the control forces. Observation spillover means that the motion of the controlled modes cannot be reconstructed, because the measurements of the structural response are influenced by the motion in the uncontrolled modes, This phenomenon may lead to instability.

Another important consideration in real-time control implementation is the discrete-time nature of a control scheme when it is processed by a digital computer. In preparation for accomplishment of an experimental test the LQ-regulator described in Chapter 3 is reformulated from an assumption of known variation of the structural dynamics and the external loading between consecutive time instants. This reformulation also takes into account the time delay between a measured information and the execution of the associated control force.

Chapter 5. When a large civil engineering structure is subjected to severe external loadings, the deformation may increase, even by application of active vibration control, to a level where the behaviour becomes nonlinear. To include this effect the proposed control algorithm based on the invariant embedding technique is extended to encompass nonlinear structures. The developed control scheme is validated by application to simulated data generated from a single-degree-of-freedom hysteretic system subjected to a white Gaussian noise. A comparison is made with another control algorithm, which has also been derived from the criterion of minimizing a quadratic performance index, called Pearson's equivalent linearization technique. According to the simulated data the algorithm developed from the invariant embedding technique showed the best properties.

Under heavy external excitation the dynamic parameters such as stiffness and damping may change due to local or global damage. If the structural parameters and their variation are unknown, more complicated, so-called adaptive control algorithms have been developed. These control schemes estimate the structural parameters and perform control on-line, simultaneously. A self-tuning controller has been developed, where unknown but constant or slowly varying structural parameters are estimated while the control is performed. In this algorithm the control forces are calculated from the invariant embedding control scheme by substituting the unknown parameters by their present estimate. The self-tuning controller has been documented by simulation results generated from a hysteretic structural model.

Chapter 6. An experimental test has been carried out in the laboratory in order to study the possible application of active control to structures under seismic excitation. The experimental setup consist of a cantilever model on a seismic simulator and an active mass damper implemented at the top of the model. The mass damper is tuned to reduce the vibrations at the fundamental frequency of the model structure. A feedback control law is implemented where the feedback gains were determined according to the optimal control formalism and the state vector was estimated from the Kalman filter. The basis for this control algorithm is a mathematical model for the structural behaviour. Such a model is set up from a theoretical approach, and the included parameters are estimated from a system identification scheme. Here, the parameters in an equivalent discrete-time model are first identified and then the corresponding parameters of the continuous-time model were derived. On the basis of the estimated model it was possible to improve the passive mass damper by controlling it actively according to the above-mentioned control algorithm.

Finally, the report contains an appendix of optimal filtering and prediction.

Bibliography

[1] Abdel-Rohman, M. and Leipholz, H.H. (1978a) : "Active Control of Flexible Structures," *Journal of the Structural Division,* Vol. 104, No. ST8, pp. 1251-1266, 1978.

[2] Abdel-Rohman, M. and Leipholz, H.H. (1978b) : "Structural Control by Pole Assignment Method," *Journal of the Engineering Mechanics Division,* Vol. 104, No. EM5, pp. 1159-1175, 1978.

[3] Abdel-Rohman, M. and Leipholz, H.H.E. (1979) : "General Approach to Active Structural Control," *Journal of the Engineering Mechanics Division,* Vol. 105, No. EM6, pp. 1007-1023, 1979.

[4] Abdel-Rohman, M., Quintana, V.H. and Leipholz, H.H. (1980a) : "Optimal Control of Civil Engineering Structures," *Journal of the Engineering Mechanics Division,* Vol. 106, No. EM1, pp. 57-73, 1980.

[5] Abdel-Rohman, M. and Leipholz, H.H. (1980b) : "Automatic Active Control of Structures," *Journal of the Structural Division,* Vol. 106, No. ST3, pp. 663-677, 1980.

[6] Abdel-Rohman, M. and Leipholz, H.H. (1981) : "Stochastic Control of Structures," *Journal of the Structural Division,* Vol. 107, No. ST7, pp. 1313-1325, 1981.

[7] Abdel-Rohman, M. (1982) : "Active Control of Large Structures," *Journal of the Engineering Mechanics Division,* Vol. 108, No. EM5, pp. 719-730, 1982.

[8] Abdel-Rohman, M. and Leipholz, H.H. (1983) : "Active control of tall buildings," *Journal of Structural Engineering,* Vol. 109, No. 3, pp. 628-645, 1983.

[9] Abdel-Rohman, M. and Leipholz, H.H.E. (1984a) : "Optimal Feedback Control of Elastic, Distributed Parameter Structures," *Computers & Structures,* Vol. 19, No. 5/6, pp. 801-805, 1984.

[10] Abdel-Rohman, M. (1987a) : "Feasibility of Active Control of Tall Buildings against Wind," *Journal of Structural Engineering,* Vol. 113, No. 2, pp. 349-362, 1987.

[11] Abdel-Rohman, M. and Nayfeh, A.H. (1987b) : "Active Control of Nonlinear Oscillations in Bridges," *Journal of Engineering Mechanics,* Vol. 113, No. 3, pp. 335-348, 1987.

[12] Abdel-Rohman, M. (1987c) : "Design of Active Damping in Flexible Structures," *Proc. ASME Vibrations Conference,* Boston, MA, 1987.

[13] Al'brekht, E.G. (1961) : "On the Optimal Stabilization of Nonlinear Systems," *PMM J. Appl. Math. and Mechanics,* Vol. 25, No. 5, pp.1254-1266, 1961.

[14] Arnold, L. (1974) : *Stochastic Differential Equations - Theory and Applications,* John Wiley and Sons, New York, 1974.

[15] Baba, S., Ninoniya, K. and Hayashi, Y. (1989) : "Active Optimal Control of Structure Using Optimal Observer," *Journal of Engineering Mechanics,* Vol. 115, No. 11, 1989.

[16] Baldwin, J.F. and Sims Williams, J.H. (1969) : "The Use of a Method of Perturbations in the Synthesis of Closed-Loop Control Laws for Non-Linear Systems," *Automatica,* Vol. 5, pp. 357-367, 1969.

[17] Bailey, T. and Hubbard, Jr.J.E. (1985) : "Distributed Piezoelectric-Polymer Active Vibration Control of a Cantilever Beam," *Journal of Guidance and Control,* Vol. 8, No. 5, pp. 605-611, 1985.

[18] Balas, M (1978) : "Active Control of Flexible Systems," *Journal of Optimization Theory and Applications,* Vol. 25, No. 3, pp. 415-436, 1978.

[19] Baruh, H. (1987) : "A recursive Pole Placement Method for Large Flexible Structures," *Proc. ASME Vibrations Conference,* Boston, MA, 1987.

[20] Baz, A. and Poh, S. (1988) : "Performance of an Active Control System with Piezoelectric Actuators," *Journal of Sound and Vibration,* Vol. 126, No. 2, pp. 327-343, 1988.

[21] Baz, A. and Poh, S. (1990) : "Experimental Implementation of the Modified Independent Modal Space Control Method," *Journal of Sound and Vibration,* Vol. 139, No. 1, pp. 133-149, 1990.

[22] Bellman, R., Kalaba, R. and Wing, G.M. (1960) : "Invariant Imbedding and the Reduction of Two-Point Boundary Value Problems to Initial Value Problems," *Proceedings of the National Academy of Sciences,* Vol. 46, No. 12, pp. 1646-1649, 1960.

[23] Bellman, R. and Kalaba, R. (1961) : "On the Fundamental Equations of Invariant Imbedding, I," *Proceedings of the National Academy of Sciences,* Vol. 47, No. 3, pp. 336-338, 1961.

[24] Bellman, R. and Kalaba, R. (1963) : "A Note on Hamilton's Equations and Invariant Imbedding," *Quarterly of Applied Mathematics,* Vol. 21, No. 2, pp. 166-168, 1963.

[25] Bellman, R., Kagiwada, H.H. and Kalaba, R. (1967) : "Invariant Imbedding and the Numerical Integration of Boundary-Value Problems for Unstable Linear Systems of Ordinary Differential Equations," *Communications of the ACM,* Vol. 10, No. 2, pp. 100-102, 1967.

[26] Bendsøe, M.P., Olhoff, N. and Taylor, J.E. (1987) : "On the Design of Structure and Controls for Optimal Performance of Actively Controlled Flexible Structures," *Mechanics of Structures and Machines,* Vol. 15, No. 3, pp. 265-295, 1987.

[27] Bouc, R. (1967) : "Forced Vibrations of a Mechanical System with Hysteresis," *Abstract Proc. of 4th Conf. on Non-Linear Oscillations, Prague, Czechoslovakia,* 1967

[28] Bourdache-Siguerdidjane, H. and Fliess, M. (1987) : "Optimal Feedback Control of Non-linear Systems," *Automatica,* Vol. 23, No. 3, pp. 365-372, 1987.

[29] Brogan, W.L. (1985) : *Modern Control Theory,* Prentice-Hall, Inc., 1985.

[30] Bryson, E.A. Jr. (1975) : *Applied Optimal Control,* John Wiley & Sons, Inc., 1975.

[31] Burghart, J.H. (1969) : "A Technique for Suboptimal Feedback Control of Nonlinear Systems," *IEEE Transactions on Automatic Control,* pp. 530-533, 1969.

[32] Caines, P.E. and Chen, H.F. (1985) : "Optimal Adaptive LQG Control for Systems with Finite State Parameters," *IEEE Transactions on Automatic Control,* Vol. AC-30, No. 2, pp. 185-189, 1985.

[33] Carotti, A. and Lio, G. (1991) : "Experimental Active Control: Bench Tests on Controller Units," *Engineering Structures,* Vol. 13, pp. 242-252, 1991.

[34] Casiello, F. and Loparo, K.A. (1989) : "Optimal Control of Unknown Parameter Systems," *IEEE Transactions on Automatic Control,* Vol. 34, No. 10, pp. 1092-1094, 1989.

[35] Chang, J.C.H., Min I.J. and Soong, T.T. (1980a) : "Optimal Controller Placement in Modal Control of Complex Systems," *Journal of Mathematical Analysis and Applications,* Vol. 75, pp. 340-358, 1980.

[36] Chang, J.C.H. and Soong, T.T. (1980b) : "Structural Control Using Active Tuned Mass Dampers," *Journal of the Engineering Mechanics Division,* Vol. 106, No. EM6, pp. 1091-1098, 1980.

[37] Cheng, F.Y. and Pantelides, C.P. (1987) : "Optimal Active Control of Wind Structures Using Instantaneous Algorithm," *Proc. ASME Vibrations Conference,* Boston, MA, 1987.

[38] Chin, S.M. and Pengilley, C.J. (1970) : "The Application of Harmonic Linearization to Optimal Control Problems," *International Journal of Control,* Vol. 12, No. 6, pp. 999-1007, 1970.

[39] Chung, L.L. and Soong, T.T. (1987) : "Practical Considerations in Discrete-Time Structural Control," *Proc. ASME Vibrations Conference,* Boston, MA, 1987.

[40] Chung, L.L., Reinhorn, A.M. and Soong, T.T. (1988) : "Experiments on Active Control of Seismic Structures," *Journal of Engineering Mechanics,* ASCE, Vol. 114, No. 2, pp. 241-256, 1988.

[41] Chung, L.L., Soong, T.T. and Reinhorn, A.M. (1989) : "Experimental Study of Active Control for MDOF Seismic Structures," *Journal of Engineering Mechanics,* ASCE, Vol. 115, No. 8, pp. 1609-1627, 1989.

[42] Clough, R.W. and Penzien J. (1982) : *Dynamics of Structures,* McGraw-Hill, Inc., 1982.

[43] Desai, R.C. and Lalwani,C.S. (1969) : "System Estimation by Invariant Imbedding with Unknown Inputs," *International Journal of Control,* Vol. 12, No. 5, pp. 909-911, 1969.

[44] Desanghere, G. and Vansevenant. E. (1991) : "An Adaptive Tuned Vibration Absorber," *Proceedings of the 9th International Modal Analysis Conference - April 1991, Firenze, Italy*

[45] Detchmendy, D.M. and Shridar, R. (1966) : "Sequential Estimation of States and Parameters in Noisy Nonlinear Dynamical Systems," *Journal of Basic Engineering,* ASME, pp. 362-368, 1966.

[46] Distefano, N. and Rath, A. (1975) : "Sequential Identification of Hysteretic and Viscous Models in Structural Seismic Dynamics," *Computer Methods in Applied and Engineering,* Vol. 6, pp. 219-232, 1975.

[47] O'Donoghue, P.E. and Atluri, S.N. (1986) : "Control of Dynamic Response of a Continuum Model of a Large Space Structure," *Computers & Structures,* Vol. 23, No. 2, pp. 199-209, 1986.

[48] Durbeck, R.C. (1965) : "An Approximation Technique for Suboptimal Control," *IEEE Transactions on Automatic Control,* Vol. AC-10, pp. 144-149, 1965.

[49] Ebrahimi, N.D. (1988) : "Open-loop Vibration Control of Elastic Structural Elements," *Earthquake Engineering and Structural Dynamics,* Vol. 16, pp. 433-442, 1988.

[50] Ebrahimi, N.D. (1989a) : "Vibration Suppression of Structures," *Earthquake Engineering and Structural Dynamics,* Vol. 18, pp. 1-10, 1989.

[51] Ebrahimi, N.D. (1989b) : "Optimal Vibration Abatement of Mechanical Systems," *Earthquake Engineering and Structural Dynamics,* Vol. 18, pp. 507-516, 1989.

[52] Eisenberg, B.R. and Sage, P.S. (1966) : "Closed Loop Optimization of Fixed Configuration Systems," *International Journal of Control,* Vol. 3, No. 3, pp. 183-194, 1966.

[53] Eller, D.H. and Aggarwal, J.K. (1968) : "Sub-optimal Control of Non-linear Single Input Systems," *International Journal of Control,* Vol. 8, No. 2, pp. 113-121, 1968.

[54] Friedland, B. (1966) : "A Technique of Quasi-Optimum Control," *Journal of Basic Engineering,* ASME, pp. 437-443, 1966.

[55] Garrard, W.L., McClamroch, N.H. and Clarck, L.G. (1967) : "An Approach to Sub-optimal Feedback Control of Non-Linear Systems," *International Journal of Control,* Vol. 5, No. 5, pp. 425-435, 1967.

[56] Grandhi, R.V. (1989) : "Structural and Control Optimization of Space Structures," *Computers & Structures,* Vol. 31, No. 2, pp. 139-150, 1989.

[57] Hijab, O.B. (1983) : "The Adaptive LQG Problem - Part I," *IEEE Transactions on Automatic Control,* Vol. AC-28, No. 2, pp. 171-178, 1983.

[58] Hijab, O.B. (1986) : "The Adaptive LQG Problem, II," *Stochastics,* Vol. 16, pp. 25-49, 1986.

[59] Hrovat, D., Barak, P. and Rabins, M. (1983) : "Semi-Active versus Passive or Active Tuned Mass Dampers for Structural Control," *Journal of Structural Engineering,* Vol. 109, No. 3, pp. 691-705, 1983.

[60] Kagiwada, H.H. and Kalaba, R.E. (1968) : "Derivation and Validation of an Initial- Value Method for Certain Nonlinear Two-Point Boundary-Value Problems," *Journal of Optimization Theory and Applications,* Vol. 2, No. 6, pp. 378-385, 1968.

[61] Kalaba, R.E. and Spingarn, K. (1977) : "Numerical Solution of a Nonlinear Two-Point Boundary Value Problem by an Imbedding Method," *Nonlinear Analysis, Theory, Methods & Applications,* Vol. 1, No. 2, pp. 129-133, 1977.

[62] King, J.-Y., Lin, R.C. and Lee, H.-T (1991) : "Active Vibration Control of a Beam with Accelerometer and Vibrator," *Proceeding of the 9th International Modal Analysis Conference - April 1991, Firenze, Italy,* pp. 765-768.

[63] Lewis, Frank. L. (1986a) : *Optimal Control,* John Wiley & Sons, Inc., 1986.

[64] Lewis, Frank. L. (1986b) : *Optimal Estimation with an Introduction to Stochastic Control Theory,* John Wiley & Sons, Inc., New York, 1986.

[65] Ljung, L. (1987) : *System Identification - Theory for the User,* Prentice-Hall, Inc., New Jersey, 1987.

[66] Lukes, D.L. (1969) : "Optimal Regulation of Nonlinear Dynamical Systems," *SIAM J. Control,* Vol. 7, No. 1, pp. 75-100, 1969.

[67] Luzzato, E. and Jean, M. (1983) : "Mechanical Analysis of Active Vibration Damping in Continuous Structures," *Journal of Sound and Vibration,* Vol. 86, No. 4, pp. 455-473, 1983.

[68] Martin, C.R. and Soong, T.T. (1976) : "Modal Control of Multistorey Structures," *Journal of the Engineering Mechanics Division,* Vol. 102, No. EM4, pp. 613-623, 1976.

[69] Masri, S.F., Bekey, G.A. and Caughey, T.K. (1982) : "On-Line Control of Nonlinear Flexible Structures," *Journal of Applied Mechanics,* Vol. 49, pp. 877-884, 1982.

[70] McNamara, Roberts J. (1977) : "Tuned Mass Dampers for Buildings," *Journal of the Structural Division,* Vol. 103, No. ST9, pp. 1785-1798, 1977.

[71] Meirovitch L. and Baruh, H. (1982) : "Control of Self-adjoint Distributed-parameter System," *Journal of Guidance and Control,* Vol. 5, No. 1, pp. 60-66, 1982.

[72] Meirovitch, L., Baruh, H. and Öz, H. (1983a) : "A Comparison of Control Techniques for Large Flexible Systems," *Journal of Guidance and Control,* Vol. 6, No. 4, pp. 302-310, 1983.

[73] Meirovitch, L. and Silverberg, L.M. (1983b) : "Control of Structures Subjected to Seismic Excitation," *Journal of Engineering Mechanics,* Vol. 109, No. 3, pp. 604-618, 1983.

[74] Meirovitch, L., Baruh, H., Montgomery, R.C. and Williams J.P. (1984b) : "Nonlinear Natural Control of an Experimental Beam," *Journal of Guidance and Control,* Vol. 7, No. 4, July-August 1984.

[75] Meirovitch, L. and Ghosh, D. (1987) : "Control of Flutter in Bridges," *Journal of Engineering Mechanics,* Vol. 113, No. 5, pp. 720-736, May 1987.

[76] Miller, R.K., Masri, S.F., Dehghanyar, T.J. and Caughey, T.K. (1988) : "Active Vibration Control of Large Civil Structures," *Journal of Engineering Mechanics,* ASCE, Vol. 114, No. 9, pp. 1542-1570, Sept., 1988.

[77] Minai, R: and Suzuki, Y. (1987) : "Stochastic Estimates of Nonlinear Dynamics," *U.S.-Japan Joint Seminar on Stochastic Approaches in Earthquake Engineering,* Florida Atlantic University, Boca Raton, Florida, May 6 and 7, pp. 1-27, 1987.

[78] Molinari, B.P. (1974) : "Expressions for Control and Loss in Least-Squares Tracking and Smoothing," *IEEE Transactions on Automatic Control,* Vol. AC-19, pp. 146-147, April 1974.

[79] Morimoto, H. (1990) : "Adaptive LQG Regulator via the Separation Principle," *IEEE Transactions on Active Control,* Vol. 35, No. 1, pp. 85-88, 1990.

[80] Nishikawa, Y., Sannomiya, N. and Itakura, H. (1971) : "A Method for Suboptimal Design of Nonlinear Feedback Systems," *Automatica,* Vol. 7, pp. 703-712, 1971.

[81] Oppenheim, A.V. and Schafer, R.W. (1975) : *Digital Signal Processing,* Prentice-Hall, 1975.

[82] Pantelides, C.P. (1991) : "Control of Seismic Response of Turbomachine Foundations," *Earthquake Engineering and Structural Dynamics,* Vol. 20, pp. 839-848, 1991.

[83] PC-MATLAB User's Guide (1989), The MathWorks, Inc., South Natick, MA 01760, USA.

[84] Pearson, J.D. (1962) : "Approximation Methods in Optimal Control," *Journal of Electronics and Control,* Vol. 13, 1962.

[85] Pu, J.-P. and Hsu, D.-S. (1988) : "Optimal control of tall buildings," *Journal of Engineering Mechanics,* Vol. 114, No. 6, pp. 973-989, 1988.

[86] Pu, J.-P. and Kelly, J.M. (1991) : "Active Control and Seismic Isolation," *Journal of Engineering Mechanics,* Vol. 117, No. 10, pp. 2221-2236.

[87] Reinhorn, A.M. and Manolis, G.D. (1985) : "Current State of Knowledge on Structural Control," *Shock and Vibration Digest,* Vol. 17, pp. 35-41, 1985.

[88] Reinhorn, A.M., Manolis, G.D. and Wen, C.Y. (1987) : "Active Control of Inelastic Structures," *Journal of Engineering Mechanics,* Vol. 113, No. 3, pp. 315-333, March 1987.

[89] Reinhorn, A.M. and Manolis, G.D. (1989) : "Recent Advances in Structural Control," *Shock and Vibration Digest,* Vol. 21, pp. 3-8, Jan. 1989.

[90] Rishel, R. (1986a) : "An Exact Formula for a Linear Quadratic Adaptive Stochastic Optimal Control Law," *SIAM Journal on Control and Optimization,* Vol. 24, No. 4, pp. 667-674, 1986.

[91] Rodellar, J., Chung, L.L., Soong, T.T. and Reinhorn, A.M. (1987a) : "Experimental Digital Predictive Control," *Proc. ASME Vibrations Conference,* Boston, MA, 1987.

[92] Rodellar, J., Barbat, A.H. and Martín-Sánchez, J.M. (1987b) : "Predictive Control of Structures," *Journal of Engineering Mechanics,* Vol. 113, No. 6, June 1987.

[93] Roorda, J. (1975) : "Tendon Control in Tall Structures," *Journal of the Structural Division,* Vol. 101, No. ST3, pp. 505-521.

[94] Roberts, J.B. and Sadeghi, A.H. (1990) : "Sequential parametric identification and response of hysteretic oscillators with random excitation," *Structural Safety,* Vol. 8, pp. 45-68, 1990.

[95] Sadek, I.S. and Adali, S. (1984) : "Control of the Dynamic Response of a Damped Membrane by Distributed Forces," *Journal of Sound and Vibration,* Vol. 96, No. 3, pp. 391-406, 1984.

[96] Sadek, I.S., Adali, S., Sloss, J.M. and Bruch, J.C. Jr. (1987) : "Optimal Distributed Control of Continuous Beam with Damping," *Journal of Sound and Vibration,* Vol. 117, No. 2, pp. 207-218, 1987.

[97] Sadek, I.S., Bruch, J.C. Jr., Sloss, J.M. and Adali, S. (1988) : "Structural Control of a Variable Cross-section Beam by Distributed Forces," *Mechanics of Structures and Machines,* Vol. 16, No. 3, pp. 313-333, 1988-89.

[98] Sae-Ung, S. and Yao, J.T.P. (1978) : "Action Control of Building Structures," *Journal of the Engineering Mechanics Division,* Vol. 104, No. EM2, pp. 335-350, 1978.

[99] Şafak, E. (1989a) : " Adaptive Modeling, Identification, and Control of Dynamic Structural Systems. I: Theory," *Journal of Engineering Mechanics,* Vol. 115, No. 11, pp. 2386-2405, 1989.

[100] Şafak, E. (1989b) : " Adaptive Modeling, Identification, and Control of Dynamic Structural Systems. II: Applications," *Journal of Engineering Mechanics,* Vol. 115, No. 11, pp. 2406-2426, 1989.

[101] Sage, A.P. and Ellis, T.W. (1966a): "Sequential Suboptimal Adaptive Control of Nonlinear Systems," *Proceedings of the International Electronics Conference,* No. 22, pp. 692-697, 1966.

[102] Sage, A.P. and Eisenberg, B.R. (1966b): "Suboptimal Adaptive Control of a Nonlinear Plant," *IEEE Transactions on Automatic Control,* pp. 621-623, July, 1966.

[103] Sage, A.P. and White, C.C. (1977) : *Optimum Systems Control,* second edition, Prentice Hall, Inc., 1977.

[104] Samali, B., Yang, J.N. and Liu, S.C. (1985) : "Active Control of Seismic-excited Buildings," *Journal of Structural Engineering,* Vol. 111, No. 10, pp. 2165-2180, 1985.

[105] Schäfer, B.E. and Holzach, H. (1985) : "Experimental Research on Flexible Beam Modal Control," *Journal of Guidance and Control,* Vol. 8, No. 5, pp. 597-604, Sept.-Oct. 1985.

[106] Seibold, S., Fritzen, C.-P. (1991) : "Recursive Identification of Nonlinear Systems by means of the Extended Kalman Filter," *Proceeding of the 9th International Modal Analysis Conference - April 1991, Firenze, Italy,* pp. 802-809.

[107] Sinha, A. (1988) : "Optimal Vibration Control of Flexible Structures for Specified Modal Decay Rates," *Journal of Sound and Vibration,* Vol. 123, No. 1, pp. 185-188, 1988.

[108] Sloss, J.M., Adali, S., Sadek, I.S. and Bruch, J.C. Jr. (1988) : "Displacement Feedback Control of Beams under Moving Loads," *Journal of Sound and Vibration,* Vol. 123, No. 3, pp. 457-464, 1988.

[109] Soong, T.T. and Skinner, G.T. (1981) : "Experimental Study of Active Structural Control," *Journal of Engineering Mechanics Division,* Vol. 107, No. EM6, pp.1057-1067, 1981.

[110] Soong,T.T. and Chang, J.C.H. (1982) : "Active Vibration Control of Large Flexible Structures," *Shock and Vibration Bulletin,* Vol. 52, pp. 47-54, 1982.

[111] Soong,T.T. (1988) : "State-of-the-Art Review: Active Structural Control in Civil Engineering," *Engineering Structures,* Vol. 10, pp. 74-84, April 1988.

[112] Soong, T.T. (1990) : *Active Structural Control: Theory and Practice,* Longman, London, and Wiley, New York, 1990.

[113] Spencer, A.J.M (1980) : *Continuum Mechanics,* Longman Inc., New York, 1980.

[114] Spencer, B.F. and Suhardjo, J. (1990) : "Recent Theoretical Developments in Active Structural Control," *Proc. Structures Congress,* pp. 129-132,

[115] Stengel, R.F. (1986) : *Stochastic Optimal Control - Theory and Application,* Wiley, New York, 1986.

[116] Suhardjo, J. and Spencer, B.F. Jr. (1990) : "Feedback-Feedforward Control of Structures under Seismic Excitation," *Structural Safety,* Vol. 8, No 1-4, pp. 69-89, 1990.

[117] Thesbjerg, L. (1990) : "Active Vibration Control of Civil Engineering Structures under Earthquake Excitation," *Fracture & Dynamics,* No. 25, ISSN 0902-7513 R9027, 1990.

[118] Tzafestas, S.G. and Stavroulakis, P. (1983) : "Recent Advances in the Study of Distributed Parameter Systems," *Journal of Franklin Institute,* Vol. 315, No. 5/6, pp. 285-305, May/June 1983.

[119] Udwadia, F.E. and Tabaie, S. (1981a) : "Pulse Control of Single Degree-of-Freedom System," *Journal of Engineering Mechanics Division,* Vol. 107, No. EM6, pp. 997-1009, 1981.

[120] Udwadia, F.E. and Tabaie, S. (1981b) : "Pulse Control of Structural and Mechanical Systems," *Journal of Engineering Mechanics Division,* Vol. 107, No. EM6, pp. 1011-1028, 1981.

[121] Wen, Y.-K. (1976) : "Method for Random Vibration of Hysteretic Systems," *Journal of the Engineering Mechanics,* Vol. 102, No. EM2, pp. 249-263, 1976.

[122] Willemstein, A.P. (1977) : "Optimal Regulation of Nonlinear Dynamical Systems on a Finite Interval," *SIAM Journal on Control and Optimization,* Vol. 15, No. 6, pp. 1050-1069, 1977.

[123] Wong, H.L. and Luco, J.E. (1991) : "Structural Control Including Soil-Structure Interaction Effects," *Journal of Engineering Mechanics,* Vol. 117, No. 10, pp. 2237-2250, 1991.

[124] Yang, J.-N. (1975) : "Application of Optimal Control Theory to Civil Engineering Structures," *Journal of the Engineering Mechanics Division,* Vol. 101, No. EM6, pp. 819-838, 1975.

[125] Yang, J.-N. and Giannopulos, F. (1978) : "Active Tendon Control of Structures," *Journal of the Engineering Mechanics Division,* Vol. 104, No. EM3, pp. 551-568, 1978.

[126] Yang, J.-N. and Mingchien, J.L. (1982) : "Optimal Critical-Mode Control of Building under Seismic Load," *Journal of the Engineering Mechanics Division,* Vol. 108, No. EM6, pp. 1167-1185, 1982.

[127] Yang, J.N. and Samali, B. (1983a) : "Control of Tall Buildings in Along-Wind Motion," *Journal of Structural Engineering,* Vol. 109, No. 1, pp. 50-68, January 1983.

[128] Yang, J.N. and Lin, M.J. (1983b) : "Building Critical-Mode Control: Nonstationary Earthquake," *Journal of Engineering Mechanics,* Vol. 109, No. 6, pp. 1375-1389, 1983.

[129] Yang, J.N., Akbarpour, A. and Ghaemmaghami, P. (1987) : "New Optimal Control Algorithms for Structural Control," *Journal of Engineering Mechanics,* ASCE, Vol. 113, No. 9, pp. 1369-1386, Sept., 1987.

[130] Yang, J.N. and Soong, T.T. (1988) : "Recent Advances in Active Control of Civil Engineering Structures," *J. Prob. Eng. Mech,* Vol. 3, pp. 179-188, 1988.

[131] Yang, J.N. and Akbarpour, A. (1990) : "Effect of System Uncertainty on Control of Seismic-Excited Buildings," *Journal of Engineering Mechanics,* Vol. 116, No. 2, 1990.

[132] Yao. J.T.P. (1972) : "Concept of Structural Control," *Journal of the Structural Division,* Vol. ST7, pp. 1567-1574, 1972.

[133] Yao. J.T.P. (1980) : "Identification and Control of Structural Damage," *Solid Mechanics Archives,* Vol. 5, pp. 325-345, 1980.

[134] Yao. J.T.P. (1987) : "Uncertainties in Structural Control," *Proc. ASME Vibrations Conference,* Boston, MA, pp. 197-200, 1987.

[135] Yoshida, T. and Loparo, K.A. (1989) : "Quadratic Regulatory Theory for Analytic Non-linear Systems with Additive Controls," *Automatica*, Vol. 25, No. 4, pp. 531-544, 1989.

[136] Åström, K.J. and Wittenmark, B. (1984) : *Computer Controlled Systems; Theory and Design*, Prentice-Hall, Inc., 1984.

[137] Åström, K.J. and Wittenmark, B. (1971a) : "Problems of Identification and Control," *Journal of Mathematical Analysis and Applications*, Vol. 34, pp. 91-113, 1971.

[138] Åström, K.J. and Wittenmark, B. (1973) : "On Self Tuning Regulators," *Automatica*, Vol. 9, pp. 185-199, 1973.

[139] Åström, K.J., Borisson, U., Ljung, L. and Wittenmark, B. (1977) : "Theory and Applications of Self-Tuning Regulators," *Automatica*, Vol. 13, pp. 457-476, 1977.

[140] Åström, K.J. (1983) : "Theory and Applications of Adaptive Control - A Survey," *Automatica*, Vol. 19, No. 5, pp. 471-486, 1983.

[141] Åström, K.J. (1987) : "Adaptive Feedback Control," *Proceedings of the IEEE*, Vol. 75, No. 2, pp. 185-217, 1987.

Appendix A

Optimal Filtering and Prediction

Consider the development of recursive algorithms for estimation of the individual variables in a state space model, when the measurements are indirect and contain random errors. The recursive process of determining the most likely values of the state variables given a set of measurements up to the present time is called *filtering* or *prediction*, depending on whether present or future values of the state variables are found. The resulting dynamic system representing the recursive estimator is called a filter regardless of the filtering or prediction problem is solved. The development in this appendix is mainly based on Sage and White (1977), Bryson (1975), Åström (1984), and Stengel (1986).

Suppose that the intention is to estimate the state vector $\mathbf{Y}(t)$ of a linear time-varying structural system described in state space form by

$$\dot{\mathbf{Y}}(t) = \mathbf{A}(t)\mathbf{Y}(t) + \mathbf{B}_0(t)\mathbf{W}(t) + \mathbf{B}(t)\mathbf{F}(t) \quad , \qquad \mathbf{Y}(0) = \mathbf{Y}_0 \tag{A.1}$$

where $\mathbf{F}(t)$ and $\mathbf{W}(t)$ represent known control forces and unknown random loadings, respectively. $\mathbf{Y}_0$ is the initial values, which are assumed to be normally distributed random variables. $\{\mathbf{W}(t),\, t \in [0,\infty[\}$ is a Gaussian white noise process independent of $\mathbf{Y}_0$. The first and second order moments of the external excitation and the initial state vector are

$$E[\mathbf{W}(t)] = \mathbf{0} \tag{A.2}$$

$$E\big[\mathbf{W}(t_1)\mathbf{W}^T(t_2)\big] = \mathbf{R}_{\mathbf{W}}(t_1)\delta(t_1 - t_2) \quad , \qquad \forall\,(t_1,t_2) \in [0,\infty[\times[0,\infty[\tag{A.3}$$

$$E[\mathbf{Y}(0)] = \mathbf{y}_0 \tag{A.4}$$

$$E\left[(\mathbf{Y}(0) - \mathbf{y}_0)(\mathbf{Y}(0) - \mathbf{y}_0)^T\right] = \mathbf{p}_0 \tag{A.5}$$

A noise-corrupted linear observation of the state vector

$$\mathbf{Z}(t) = \mathbf{H}(t)\mathbf{Y}(t) + \mathbf{V}(t) \tag{A.6}$$

is available for processing. The measurement disturbance $\{\mathbf{V}(t),\ t \in [0,\infty[\}$ is a Gaussian white noise process independent of $\{\mathbf{Y}(t),\ t \in [0,\infty[\}$ and $\{\mathbf{W}(t),\ t \in [0,\infty[\}$ with first and second order moments

$$E[\mathbf{V}(t)] = \mathbf{0} \tag{A.7}$$

$$E\big[\mathbf{V}(t_1)\mathbf{V}^T(t_2)\big] = \mathbf{R_V}(t_1)\delta(t_1 - t_2) \quad , \qquad \forall (t_1, t_2) \in [0,\infty[\times [0,\infty[\tag{A.8}$$

Discrete-Time Optimum Filter

An optimum filter for the discrete-time equivalent of the defined model is first derived. In the sampled version of the continuous-time model, the time-dependent functions in (A.1)-(A.8) are specified at a set of discrete time instants $\{t_k,\ k = 0, 1, 2, \ldots\}$ which are assumed to be equally spaced in time with a period h, i.e. $t_k = kh$. The transformation matrix $\mathbf{H}(t)$ in (A.6) is considered as constant in the interval $[kh, kh + h[$, and $\mathbf{F}(t)$, $\mathbf{W}(t)$ and $\mathbf{V}(t)$ are represented by stochastic variables $\mathbf{F}(kh)$, $\mathbf{W}(k)$ and $\mathbf{V}(k)$ in the same interval. Equations (A.1) and (A.6) can then be represented by the following difference equation

$$\mathbf{Y}(kh + h) = \mathbf{\Theta}(kh + h, kh)\mathbf{Y}(kh) + \mathbf{\Gamma}_0(kh + h, kh)\mathbf{W}(k) + \mathbf{\Gamma}(kh + h, kh)\mathbf{F}(kh) \tag{A.9}$$

$$\mathbf{Z}(kh) = \mathbf{H}(kh)\mathbf{Y}(kh) + \mathbf{V}(k) \tag{A.10}$$

where the matrices $\mathbf{\Theta}(\cdot,\cdot)$, $\mathbf{\Gamma}_0(\cdot,\cdot)$ and $\mathbf{\Gamma}(\cdot,\cdot)$ may be defined by (4.20), (4.21) and (4.22). $\{\mathbf{W}(k),\ k = 0, 1, \ldots\}$ and $\{\mathbf{V}(k),\ k = 0, 1, \ldots\}$ are assumed to be mutually independent Gaussian zero mean stochastic sequences with

$$E\left[\mathbf{W}(k)\mathbf{W}^T(l)\right] = \begin{cases} \mathbf{0} & , \quad k \neq l \\ \frac{1}{h}\mathbf{R_W}(kh) & , \quad k = l \end{cases} \tag{A.11}$$

$$E\left[\mathbf{V}(k)\mathbf{V}^T(l)\right] = \begin{cases} \mathbf{0} & , \quad k \neq l \\ \frac{1}{h}\mathbf{R_V}(kh) & , \quad k = l \end{cases} \tag{A.12}$$

To simplify the notation in the following, let $\mathbf{Y}(k) = \mathbf{Y}(kh)$, $\mathbf{Z}(k) = \mathbf{Z}(kh)$, $\mathbf{\Theta}(k+1, k) = \mathbf{\Theta}(kh + h, kh)$, $\mathbf{\Gamma}_0(k+1, k) = \mathbf{\Gamma}_0(kh + h, kh)$, $\mathbf{\Gamma}(k+1, k) = \mathbf{\Gamma}(kh + h, kh)$ and $\mathbf{H}(k) = \mathbf{H}(kh)$. Formulating the one-step ahead prediction problem, the objective is to determine the regression of first kind $\hat{\mathbf{Y}}(k+1|k)$ of $\mathbf{Y}(k+1)$ on the set of measurements $\{\mathbf{Z}(j),\ j = 0, 1, \ldots, k\}$. The notation $\hat{\mathbf{Y}}(k+1|k)$ is used to indicate, that it is an estimate based on measurements from the time interval $[0, t_k]$. In the development it is required that the function $\hat{\mathbf{Y}}(k+1|k)$ must have the following characteristics: It is linearly dependent on the past observations $\{\mathbf{Z}(j),\ j = 0, 1, \ldots, k\}$, i.e. the regression is assumed to be linear, it is unbiased in that $E\big[\hat{\mathbf{Y}}(k+1|k)\big] = E\,[\mathbf{Y}(k+1)]$, and it minimizes a suitable norm of the covariance matrix $E\big[\Delta\mathbf{Y}(k+1|k)\Delta\mathbf{Y}^T(k+1|k)\big]$, where

$$\Delta\mathbf{Y}(k+1|k) = \mathbf{Y}(k+1) - \hat{\mathbf{Y}}(k+1|k) \tag{A.13}$$

is called the one-stage prediction estimation error. The dynamic system for $\hat{\mathbf{Y}}(k+1|k)$ satisfying these requirements is called a *linear minimum error variance one-step ahead filter.*

Let the estimator have the form

$$\begin{aligned}\hat{\mathbf{Y}}(k+1|k) &= \mathbf{\Theta}(k+1,k)\hat{\mathbf{Y}}(k|k-1) + \mathbf{\Gamma}(k+1,k)\mathbf{F}(k) \\ &\quad + \mathbf{L}_f(k)\Big[\mathbf{Z}(k) - \mathbf{H}(k)\hat{\mathbf{Y}}(k|k-1)\Big]\end{aligned} \tag{A.14}$$

in which $\mathbf{L}_f(k)$ is an unknown filter gain to be determined. Substituting (A.9) and (A.14) into (A.13) the governing equation for the prediction error becomes

$$\begin{aligned}\Delta\mathbf{Y}(k+1|k) &= \big[\mathbf{\Theta}(k+1,k) - \mathbf{L}_f(k)\mathbf{H}(k)\big]\Delta\mathbf{Y}(k|k-1) + \mathbf{\Gamma}_0(k+1,k)\mathbf{W}(k) \\ &\quad - \mathbf{L}_f(k)\mathbf{V}(k)\end{aligned} \tag{A.15}$$

The filter gain $\mathbf{L}_f(k)$ in (A.15) is optimally selected to minimize the error variance, which is denoted by $\mathbf{p}(k)$

$$\mathbf{p}(k) = E\Big[\big(\Delta\mathbf{Y}(k|k-1) - E\left[\Delta\mathbf{Y}(k|k-1)\right]\big)\big(\Delta\mathbf{Y}(k|k-1) - E\left[\Delta\mathbf{Y}(k|k-1)\right]\big)^T\Big] \tag{A.16}$$

The mean value of $\Delta\mathbf{Y}(k|k-1)$ is obtained by taking expectation of (A.15). According to (A.2) and (A.7), this yields

$$E[\Delta\mathbf{Y}(k+1|k)] = \Big(\mathbf{\Theta}(k+1,k) - \mathbf{L}_f(k)\mathbf{H}(k)\Big)E[\Delta\mathbf{Y}(k|k-1)] \tag{A.17}$$

Assuming $E\big[\hat{\mathbf{Y}}(0|-1)\big] = \mathbf{y}_0$, it follows from (A.13) that $E\big[\Delta\mathbf{Y}(0|-1)\big] = \mathbf{0}$. Consequently, $E\big[\Delta\mathbf{Y}(k+1|k)\big] = \mathbf{0}$ for all $k \geq 0$ and that is, $\hat{\mathbf{Y}}(k+1|k)$ is unbiased. Utilizing equation (A.15) the error variance defined by (A.16) now gives

$$\begin{aligned}\mathbf{p}(k+1) &= \Big(\mathbf{\Theta}(k+1,k) - \mathbf{L}_f(k)\mathbf{H}(k)\Big)\mathbf{p}(k)\Big(\mathbf{\Theta}(k+1,k) - \mathbf{L}_f(k)\mathbf{H}(k)\Big)^T \\ &\quad + \frac{1}{h}\mathbf{\Gamma}_0(k+1,k)\mathbf{R}_\mathbf{W}(k)\mathbf{\Gamma}_0^T(k+1,k) + \frac{1}{h}\mathbf{L}_f(k)\mathbf{R}_\mathbf{V}(k)\mathbf{L}_f^T(k)\end{aligned} \tag{A.18}$$

since $\Delta\mathbf{Y}(k|k-1)$, $\mathbf{W}(k)$ and $\mathbf{V}(k)$ are uncorrelated. From (A.18) it follows that if $\mathbf{p}(k)$ is positive semi-definite, then $\mathbf{p}(k+1)$ is also positive semi-definite. Then the criterion of minimizing the error variance is satisfied by minimizing the scalar $\boldsymbol{\alpha}^T\mathbf{p}(k+1)\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is an arbitrary vector. It is assumed that the optimal gain matrix, $\mathbf{L}_f$, is known up to the time t_{k-1}. From (A.18)

$$\boldsymbol{\alpha}^T\mathbf{p}(k+1)\boldsymbol{\alpha} =$$

$$\boldsymbol{\alpha}^T\Big(\mathbf{\Theta}(k+1,k)\mathbf{p}(k)\mathbf{\Theta}^T(k+1,k) + \frac{1}{h}\mathbf{\Gamma}_0(k+1,k)\mathbf{R}_\mathbf{W}(k)\mathbf{\Gamma}_0^T(k+1,k)$$

$$- \mathbf{L}_f(k)\mathbf{H}(k)\mathbf{p}(k)\mathbf{\Theta}^T(k+1,k) - \mathbf{\Theta}(k+1,k)\mathbf{p}(k)\mathbf{H}^T(k)\mathbf{L}_f^T(k)$$

$$+ \mathbf{L}_f(k)\left[\frac{1}{h}\mathbf{R_V}(k) + \mathbf{H}(k)\mathbf{p}(k)\mathbf{H}^T(k)\right]\mathbf{L}_f^T(k)\Big)\boldsymbol{\alpha} \tag{A.19}$$

Equation (A.19) is rewritten by adding and subtracting equal terms to the right hand side as follows

$$\boldsymbol{\alpha}^T\mathbf{p}(k+1)\boldsymbol{\alpha} =$$

$$\boldsymbol{\alpha}^T\left\{\mathbf{\Theta}\mathbf{p}(k)\mathbf{\Theta}^T + \frac{1}{h}\mathbf{\Gamma}_0\mathbf{R_W}\mathbf{\Gamma}_0^T - \mathbf{\Theta}\mathbf{p}(k)\mathbf{H}^T\left(\frac{1}{h}\mathbf{R_V} + \mathbf{H}\mathbf{p}(k)\mathbf{H}^T\right)^{-1}\mathbf{H}\mathbf{p}(k)\mathbf{\Theta}^T\right\}\boldsymbol{\alpha}$$

$$+ \boldsymbol{\alpha}^T\left\{\left[\mathbf{L}_f - \mathbf{\Theta}\mathbf{p}(k)\mathbf{H}^T\left(\frac{1}{h}\mathbf{R_V} + \mathbf{H}\mathbf{p}(k)\mathbf{H}^T\right)^{-1}\right]\left[\frac{1}{h}\mathbf{R_V} + \mathbf{H}\mathbf{p}(k)\mathbf{H}^T\right]\right.$$

$$\left.\left[\mathbf{L}_f - \mathbf{\Theta}\mathbf{p}(k)\mathbf{H}^T\left(\frac{1}{h}\mathbf{R_V} + \mathbf{H}\mathbf{p}(k)\mathbf{H}^T\right)^{-1}\right]^T\right\}\boldsymbol{\alpha} \tag{A.20}$$

The scalar to be minimized in (A.20) has two terms, where the first part is independent of $\mathbf{L}_f$. The second part is non-negative because the matrix $(\frac{1}{h}\mathbf{R_V} + \mathbf{H}\mathbf{p}(k)\mathbf{H}^T)$ is positive definite. The minimum is thus obtained if $\mathbf{L}_f$ is chosen such that the second part of (A.20) is zero. Then

$$\mathbf{L}_f(k) = \mathbf{\Theta}(k+1,k)\mathbf{H}^T(k)\left(\frac{1}{h}\mathbf{R_V}(k) + \mathbf{H}(k)\mathbf{p}(k)\mathbf{H}^T(k)\right)^{-1} \tag{A.21}$$

$$\mathbf{p}(k+1) = \mathbf{\Theta}(k+1,k)\mathbf{p}(k)\mathbf{\Theta}^T(k+1,k) + \frac{1}{h}\mathbf{\Gamma}_0(k+1,k)\mathbf{R_W}(k)\mathbf{\Gamma}_0^T(k+1,k)$$

$$- \mathbf{\Theta}(k+1,k)\mathbf{p}(k)\mathbf{H}^T(k)\left(\frac{1}{h}\mathbf{R_V}(k) + \mathbf{H}(k)\mathbf{p}(k)\mathbf{H}^T(k)\right)^{-1}\mathbf{H}(k)\mathbf{p}(k)\mathbf{\Theta}^T(k+1,k) \quad ,$$

$$\mathbf{p}(0) = \mathbf{p}_0 \tag{A.22}$$

The reconstruction defined by (A.14), (A.21), and (A.22) is called the *Kalman filter*. Equation (A.14) represents a recursive description of the linear minimum error variance single-stage predictor which requires the Kalman gain $\mathbf{L}_f$ given by (A.21). Calculation of the Kalman gain requires knowledge of the minimum error variance represented in the recursive equation (A.22).

Next, it is desired to determine the filter estimate $\hat{\mathbf{Y}}(k|k) = \hat{\mathbf{Y}}(k)$ to the state $\mathbf{Y}(k)$ based on the observations $\{\mathbf{Z}(j),\ j = 0,1,\ldots,k\}$. Having in mind that the estimates are unbiased, they are related according to the difference equation

$$\hat{\mathbf{Y}}(k+1|k) = \mathbf{\Theta}(k+1,k)\hat{\mathbf{Y}}(k) + \mathbf{\Gamma}(k+1,k)\mathbf{F}(k) \tag{A.23}$$

Substituting (A.23) into (A.14) and premultiplying both sides by $\mathbf{\Theta}^{-1}(k+1,k)$, yield

$$\hat{\mathbf{Y}}(k) = \hat{\mathbf{Y}}(k|k-1) + \mathbf{K}_f(k)\Big[\mathbf{Z}(k) - \mathbf{H}(k)\hat{\mathbf{Y}}(k|k-1)\Big] \quad , \qquad \hat{\mathbf{Y}}(0) = \mathbf{y}_0 \tag{A.24}$$

where

$$\begin{aligned}\mathbf{K}_f(k) &= \mathbf{\Theta}(k+1,k)^{-1}\mathbf{L}_f(k) \\ &= \mathbf{H}^T(k)\left(\frac{1}{h}\mathbf{R}_\mathbf{V}(k) + \mathbf{H}(k)\mathbf{p}(k)\mathbf{H}^T(k)\right)^{-1}\end{aligned} \tag{A.25}$$

The last line in equation (A.23) is obtained by insertion of (A.21). Having determined the filtered estimate $\hat{\mathbf{Y}}(k)$, a prediction of arbitrary time steps is simply obtained by applying (A.23) repeatedly.

Continuous-Time Optimum Filter

The continuous-time version of the Kalman filter will be derived from the discrete case. For that purpose the following simple approximation of the equations of motion (A.1) is introduced

$$\mathbf{Y}(t+h) = [\mathbf{I} + \mathbf{A}(t)h]\mathbf{Y}(t) + \mathbf{B}_0(t)\mathbf{W}(t)h + \mathbf{B}(t)\mathbf{F}(t)h + \mathbf{o}(h) \tag{A.26}$$

In order that the approximation (A.26) correspond to the applied difference equation for the state (A.9), it is seen that

$$\mathbf{\Theta}(k+1,k) = [\mathbf{I} + \mathbf{A}(t)h]_{t=kh} \tag{A.27}$$

$$\mathbf{\Gamma}(k+1,k) = \mathbf{B}(t)h\big|_{t=kh} \tag{A.28}$$

$$\mathbf{\Gamma}_0(k+1,k) = \mathbf{B}_0(t)h\big|_{t=kh} \tag{A.29}$$

The continuous-time equivalent of the covariance update in (A.22) is developed by inserting (A.27) and (A.29) into this equation and next dividing by h. After rearranging

$$\begin{aligned}\frac{\mathbf{p}(kh+h) - \mathbf{p}(kh)}{h} &= \mathbf{A}(kh)\mathbf{p}(kh) + \mathbf{p}(kh)\mathbf{A}^T(kh) + \mathbf{A}(kh)\mathbf{p}(kh)\mathbf{A}^T(kh)h \\ &\quad + \mathbf{B}_0(kh)\left[\mathbf{R}_\mathbf{W}(kh)\right]\mathbf{B}_0(kh) - \left[\mathbf{I} + \mathbf{A}(kh)h\right]\mathbf{p}(kh)\mathbf{H}^T(kh) \\ &\quad \left[\mathbf{R}_\mathbf{V}(kh) + \mathbf{H}(kh)\mathbf{p}(kh)\mathbf{H}^T(kh)h\right]^{-1}\mathbf{H}(kh)\mathbf{p}(kh)\left[\mathbf{I} + \mathbf{A}(kh)h\right]^T\end{aligned} \tag{A.30}$$

In the limit as h goes to zero (A.30) approaches the differential equation

$$\dot{\mathbf{p}}(t) = \mathbf{A}(t)\mathbf{p}(t) + \mathbf{p}(t)\mathbf{A}^T(t) + \mathbf{B}_0(t)\mathbf{R}_\mathbf{W}(t)\mathbf{B}_0^T(t) - \mathbf{p}(t)\mathbf{H}^T(t)\mathbf{R}_\mathbf{V}^{-1}(t)\mathbf{H}(t)\mathbf{p}(t) \quad ,$$

$$\mathbf{p}(0) = \mathbf{p}_0 \tag{A.31}$$

The state estimation equation is derived similarly. Substituting (A.21), (A.27) and (A.28) into (A.14) and dividing by h yield

$$\frac{\hat{\mathbf{Y}}(kh+h|kh) - \hat{\mathbf{Y}}(kh|kh-h)}{h} =$$
$$\mathbf{A}(kh)\hat{\mathbf{Y}}(kh|kh-h) + \mathbf{B}(kh)\mathbf{F}(kh) + \Big[\mathbf{I} + \mathbf{A}(kh)h\Big]\,\mathbf{p}(kh)\mathbf{H}^T(kh)$$
$$\Big[\mathbf{R_V}(kh) + \mathbf{H}(kh)\mathbf{p}(kh)\mathbf{H}^T(kh)h\Big]^{-1}\Big[\mathbf{Z}(kh) - \mathbf{H}(kh)\hat{\mathbf{Y}}(kh|kh-h)\Big] \tag{A.32}$$

In the limit as h goes to zero $[\hat{\mathbf{Y}}(kh+h|kh) - \hat{\mathbf{Y}}(kh|kh-h)]/h$ becomes $\mathrm{d}\hat{\mathbf{Y}}(t)/\mathrm{d}t$, and the state estimate is seen to fulfil the stochastic differential equation

$$\dot{\hat{\mathbf{Y}}}(t) = \mathbf{A}(t)\hat{\mathbf{Y}}(t) + \mathbf{B}(t)\mathbf{F}(t) + \mathbf{K}_f(t)\Big[\mathbf{Z}(t) - \mathbf{H}(t)\hat{\mathbf{Y}}(t)\Big] \quad , \qquad \hat{\mathbf{Y}}(0) = \mathbf{y}_0 \tag{A.33}$$

where the continuous optimal filter gain $\mathbf{K}_f(t)$ is given as

$$\mathbf{K}_f(t) = \mathbf{p}(t)\mathbf{H}^T(t)\mathbf{R}_\mathbf{V}^{-1}(t) \tag{A.34}$$

In the limit as $h \to 0$, $\Delta\mathbf{Y}(kh+h|kh) = \mathbf{Y}(kh+h) - \hat{\mathbf{Y}}(kh+h|kh)$ will approach the error of the state estimation $\mathbf{Y}(kh) - \hat{\mathbf{Y}}(kh)$. The covariance matrix of $(\mathbf{Y}(kh) - \hat{\mathbf{Y}}(kh))$ on condition of the set of measurements $\{\mathbf{Z}(jh),\ j = 0, 1, \ldots, k\}$ is $\mathbf{P}(kh)$, see (3.117). The unconditioned covariance matrix is then $E\big[\mathbf{P}(kh)\big]$, according to the representation theorem. In the limit as $h \to 0$, we then have

$$\mathbf{p}(t) = E\big[\mathbf{P}(t)\big] \tag{A.35}$$

This completes the derivation of the continuous-time version of the Kalman filter. The linear unbiased state estimator is given by (A.33), where the optimally selected gain to minimize the error variance is given (A.34). The associated covariance of the estimation error is propagating as specified by the matrix differential equation (A.31).

Appendix B

Resumé in Danish

Formålet med den foreliggende afhandling har været at konstruere styringsalgoritmer til aktiv svingningsdæmpning af bygningskonstruktioner samt at eftervise deres anvendelighed, hvilket er opnået gennem numerisk simulering og ved udførelse af laboratorieforsøg. Udgangspunktet for dette arbejde er kravet om effektive dæmpningssystemer, som skal tilvejebringe en nødvendig sikkerhed og/eller komfort for bærende konstruktioner, når de udsættes for dynamiske påvirkninger som jordskælv, vindstød eller bølgelaster.

Design af styringssystemer baseres generelt på et sæt bevægelsesligninger, som beskriver responset af konstruktionen under påvirkning af såvel ydre laster som kontrolkræfter. Der formuleres et sådan sæt ligninger ud fra en referencebeskrivelse af Cauchys bevægelsesligning og ved indførelse af lineære fysiske og geometriske betingelser. Denne kontinuerte model med et uendeligt antal frihedsgrader inkluderer det generelle tilfælde, hvor kontrolkræfterne er kontinuert fordelte laster. Det er imidlertid temmelig kompliceret at designe et styringssystem ud fra en kontinuer model, og derfor indføres en diskret model med et endeligt antal frihedsgrader. I denne model antages, at kontrolkræfterne er koncentrerede laster, hvilket ud fra en praktisk betragtning er mere realistisk end antagelsen om kontinuerligt fordelte kontrollaster. Med henblik på den efterfølgende analyse opstilles de to typer bevægelsesligninger på tilstandsform svarende til sæt af 1. ordens differentialligninger.

Styringssystemer til aktiv svingningsdæmpning kan designes ud fra forskellige kriterier om hvordan og i hvilken grad bevægelsen ønskes dæmpet. De mest almindeligt benyttede dæmpningskriterier inden for aktiv svingningsdæmpning af bygningskonstruktioner omtales. Én metode benytter en feedback styring, som er lineært propertional med tilstandsvektoren, hvor feedback matricen er valgt således, at egenværdierne (polerne) til systemmatricen for tilstandsligningerne antager foreskrevne værdier. En anden kategori af styringssystemer benytter pulsbelastninger, som udløses, når flytningerne i visse punkter overskrider en fastsat grænseværdi. Endelig er der en bred kategori af optimale kontrolalgoritmer, som bestemmer styrekraften ved at minimere et såkaldt tabsindeks. Der er udviklet en speciel type af styringssystemer, der minimerer en tidsafhængig tabsfunktion, men i den klassiske formulering designes styringssystemet således, at konstruktionens respons minimeres med anvendelse af de mindst mulige kontrolkræfter over en tidsperiode svarende til varigheden af den ydre belastning. I forbindelse med dette studium har opgaven været at designe styresystemer i forhold til det sidstnævnte kriterium.

Optimal kontrol af konstruktioner beskrevet ved kontinuerte modeller giver almindeligvis anledning til meget komplicerede numeriske problemer. Selv om det i et tilfælde skulle være muligt at bestemme en feedback kontrollov vil den i praksis ikke kunne implementeres, idet den forudsætter, at bevægelsen måles kontinuert fordelt over hele konstruktionen. Disse problemer er beskrevet. For at simplificere beregningerne i forbindelse med formuleringen af en kontrollov samt for at forenkle implementeringen foretages der en diskretisering af de kontinuerte bevægelsesligninger.

For et givet diskretiseret system er betingelsesligningerne for et optimalt kontrolsystem opstillet hhv. ved hjælp af variationsregning og dynamisk programmering. Den førstnævnte metode fører til et to-punkts randværdiproblem for den kontrollerede og den adjungerede tilstand, mens den sidstnævnte metode fører til Hamilton-Jacobi-Bellman ligningen for den minimale tabsfunktion. I det specielle tilfælde, hvor tabsindekset er kvadratisk i tilstandsvektoren og kontrolkræfterne, er det vist, at de to typer af ligninger fører til den samme kontrollov, kaldet en LQ-regulator.

Optimalt design af et styringssystem til aktiv svingningsdæmpning af bygningskonstruktioner er principielt ikke muligt, idet det forudsætter, at den ydre lastpåvirkning er kendt a priori. Dette er almindeligvis ikke tilfældet for den type laster, som giver anledning til store uønskede svingninger. I stedet benyttes i mange tilfælde en feedback regulering, som kun er optimal for en fri svingning uden nogen ydre lastpåvirkning.

Som et nyt forslag til løsning af det optimale styringsproblem for konstruktioner under påvirkning af ydre laster er der opstillet en kontrollov ud fra invariant embedding teknikken. Ideen i denne metode er, at det oprindelige to-punkts randværdiproblem omformuleres til et mere generelt begyndelsesværdiproblem. Ved dernæst at benytte en pertubationsanalyse til løsning af dette problem er der udledt et sæt differentialligninger for den kontrollerede tilstand af konstruktionen. Disse ligninger kan løses ved integration fremad i tiden ud fra en givet begyndelsestilstand og kræver således ikke et forhåndskendskab til den ydre lastpåvirkning. Den udledte løsning giver imidlertid ikke et eksplicit udtryk for kontrolkraften. Som forslag til en kontrollov for styrekraften er der benyttet en feedback regulering, hvor feedback matricen er fastsat ud fra et krav om, at egenværdierne til systemmatricerne i tilstandsligningerne for den kontrollerede tilstand givet henholdsvis ved bevægelsesligningen og den invariante imbedding ligning skal være ens. Denne løsning for feedback matricen er ikke entydig, og derfor kræves det ydermere, at afvigelsen mellem de tilhørende egenvektorer er minimal. Kontrolalgoritmen kan udvides til også et omfatte ikke lineære systemer, som er behandlet senere i rapporten, og anvendeligheden af metoden er undersøgt i denne forbindelse.

De opstillede kontrolalgoritmer er baseret på idealiserede strukturelle modeller. Ved implementering af én af disse kontrolalgoritmer vil der således i praksis kunne opstå problemer pga. afvigelser mellem modellen og den virkelige konstruktion. Én type af problemer, kaldet spillover, er beskrevet i kapitel 4. Der skelnes mellem to typer af spillover, nemlig kontrol- og observations-spillover. Kontrol spillover betegner den uundgåelige påvirkning fra styrekraften af de egensvingningsformer, som ikke er modelleret i de benyttede bevægelsesligninger. Da denne effekt er mere eller mindre tilfældig, kan dette fænomen bevirke, at svingningerne af de ukontrollerede egensvingningsformer forstærkes. Observations-spillover refererer til det fænomen, at de kontrollerede egensvingningsformer ikke kan identificeres eksakt, pga. at målingerne af det strukturelle respons er influeret af de ukontrollerede egensvingningsformer. Dette fænomen kan føre til instabilitet af kontrolsystemet.

I det ideelle tilfælde er det forudsat, at styrekraften kan ændres kontinuerligt, samt at processerne i en kontrolløkke kan udføres momentant. I praksis kan disse forudsætninger ikke opfyldes. Ved anvendelse af en digital computer til on-line beregninger og regulering af kontrolkraften kan styringsprocessen kun korrigeres til diskrete tidspunkter. Under hensyntagen til denne tidsdiskrete natur samt tidsforsinkelsen i styringsprocessen er der lavet en omformulering af LQ-regulatoren.

Under ekstreme belastninger, som fører til store deformationer, vil den dynamiske opførsel af bygningskonstruktioner ofte være ulineær, også ved anvendelse af aktiv svingningsdæmpning. For at tage højde for disse ulineariteter er den foreslåede kontrolalgoritme, der er udledt ud fra invariant imbedding, udvidet til også at omfatte ulineære strukturelle systemer. Dens anvendelighed er dokumenteret ved simulering af et hysteresesystem med én frihedsgrad under påvirkning af Gaussisk hvid støj. Der er lavet en sammenligning med en anden algoritme, Pearsons ækvivalente lineariseringsmetode, som også er udledt ud fra et kriterium om at minimere et tabsindeks. Bedømt ud fra de simulerede data fremgår det, at kontrolalgoritmen baseret på den invariante imbedding metode er mest effektiv.

I tilfælde af ekstreme dynamiske belastninger vil de dynamiske karakteristika kunne ændre sig pga. lokale eller globale ødelæggelser. Aktiv svingningsdæmpning som korrigerer for disse ændringer benævnes adaptiv styring. En adaptiv kontrolalgoritme estimerer således de strukturelle parametre, samtidig med at kontrolkræfterne reguleres. Et forslag til et adaptivt styringssystem er udarbejdet, hvor ukendte strukturelle parametre, som enten er konstante eller varierer langsomt, bliver estimeret løbende under kontrolafviklingen. Kontrolkraften beregnes ud fra den invariante imbedding algoritme ved at benytte de øjeblikkelige estimater til bestemmelse af feedback koefficienterne. Effektiviteten af denne kontrolalgoritme er ligeledes eftervist ved simulering af et hysteresesystem.

Muligheden for at anvende aktiv svingningsdæmpning for bygninger udsat for et jordskælv er blevet undersøgt gennem et laboratorieforsøg. Forsøgsmodellen består af en fast indspændt søjle placeret på en jordskælvssimulator, samt en aktiv massedæmper monteret i toppen af søjlen. Massedæmperen er tunet til at reducere svingningerne ved den laveste egenfrekvens. Den modificerede LQ-regulator, der er udledt i kapitel 4, benyttes til regulering af massedæmperen. Implementeringen af denne kontrolalgoritme err baseret på en valgt matematisk model, hvor de indgående parametre er bestemt ved systemidentifikation. På basis af den benyttede kontrolalgoritme er det muligt at forøge effektiviteten af massedæmperen ved at styre denne aktivt. Effektivitetsforøgelsen som følge af den aktive dæmpning er dog begrænset sammenlignet med den passive dæmpning, hvilket skyldes, at dæmperen er optimalt tunet, og konstruktionens respons er domineret af den første egensvingning.

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