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To my Father
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The present thesis “Dynamic behaviour of Suction Caissons” has been prepared in connection with a PhD study carried out in the period February 2003 to September 2006 at the Department of Civil Engineering, Aalborg University, Denmark.

This thesis is divided into five numbered chapters; a list of references is situated after the last chapter. Four appendices associated with the main chapters are placed at the end of the report. The appendices are numbered with letters. Tables, equations and figures are indicated with consecutive numbers in each individual chapter or appendix. Cited references are marked as e.g. Veletsos and Wei (1971), with author specification and year of publication in the text.

The work presented in the thesis is new material, and are to published in a number of journal papers and technical reports. The work presented in Chapters 2 and 3 has been tentatively accepted for an international journal. The work presented in Chapter 4 is to be published as a journal paper. Appendix B, C and D are to published as technical reports.

Aalborg, December 4, 2006

Morten Liinggaard
Summary in English

Offshore wind energy is a promising source of energy in the near future, and is rapidly becoming competitive with other power generating technologies. The continuous improvement in wind turbine technology means that the wind turbines have increased tremendously in both size and performance during the last 25 years. In order to reduce the costs, the overall weight of the wind turbine components is minimized, which means that the wind turbine structures become more flexible and thus more sensitive to dynamic excitation. Since the first resonance frequency of the modern offshore wind turbines is close to the excitation frequencies of the rotor system, it is of utmost importance to be able to evaluate the resonance frequencies of the wind turbine structure accurately as the wind turbines increase in size. In order to achieve reliable responses of the wind turbine structure during working loads it is necessary to account for the possibilities of dynamic effects of the soil–structure interaction. The aim of this thesis is to evaluate the dynamic soil-structure interaction of foundations for offshore wind turbines, with the intention that the dynamic properties of the foundation can be properly included in a composite structure-foundation system. The work has been focused on one particular foundation type; the suction caisson.

The frequency dependent stiffness (impedance) of the suction caisson has been investigated by means of a three-dimensional coupled Boundary Element/Finite Element model, where the soil is simplified as a homogenous linear viscoelastic material. The dynamic stiffness of the suction caisson is expressed in terms of dimensionless frequency-dependent coefficients corresponding to the different degrees of freedom. Comparisons with known analytical and numerical solutions indicate that the static and dynamic behaviour of the foundation are predicted accurately with the applied model. The analysis has been carried out for different combinations of the skirt length, soil stiffness and the Poisson’s ratio of the subsoil. Subsequently, the high-frequency impedance has been determined for the use in lumped-parameter models of wind turbine foundations.

The requirement for real-time computations in commercial software packages for performance and loading analysis of wind turbines, do not conform with the use of e.g. a three-dimensional coupled Boundary Element/Finite Element Method, where the foundation stiffness is evaluated in the frequency domain. For that reason, the dynamic stiffness (impedance) for each degree of freedom have been formulated into lumped-parameter models with frequency independent coefficients, suitable for implementation in standard dynamic finite element schemes. The lumped-parameter models have been used to simulate the soil–structure interaction within a numerical finite element model of a Vestas V90 3.0 MW offshore wind turbine with a suction caisson foundation. The simulations of the soil–structure interaction by means of lumped-parameter model approximations of the impedance have shown that the concept is useful for use in applications where the performance of the wind turbine are to be analysed.

Experimental modal analyses have been carried out with the intention of estimating the natural frequencies of an existing Vestas 3.0 MW offshore wind turbine. The experimental modal analysis of the wind turbine makes use of "Output-only modal identification" which is utilized when the modal properties are identified from measured responses only. The experimental modal analyses have shown that the approach is a useful tool to estimate the response of the wind turbine.
Summary in Danish

Offshore vindenergi er en lovende energiressource i den nærmeste fremtid, og bliver hurtigt konkurrencedygtig med andre energiproducerende teknologier. Den fortsatte forbedring af vindmølleteknologien har betydet, at vindmøllerne er vokset enormt i både størrelse og ydelse indenfor de sidste 25 år. For at reducere omkostningerne er vægten vindmøllekomponenterne minimieret, hvilket betyder at vindmøllekonstruktionen bliver mere fleksibel, og dermed mere følsom overfor dynamiske påvirkninger. På grund af, at den første resonansfrekvens for en moderne offshore vindmølle ligger tæt op af omlovsfrekvenserne for rotorsystemet, er det yderst vigtigt at være i stand til at evaluere møllens resonansfrequenser præcist i takt med at vindmøllerne bliver større. For at opnå et pålideligt respons af vindmøllekonstruktionen under anvendelsesstillstande er det nødvendigt at tage højde for mulige dynamiske effekter, forårsaget af jord–struktur interaktionen. Målet med denne afhandling er at undersøge den dynamiske jord–struktur interaktion af fundamenter til offshore vindmøller, med den hensigt, at de dynamiske egenskaber af fundamenterne kan inkluderes i et sammensat konstruktion–fundament system. Arbejdet har været fokuseret på en bestemt fundamenttype; sugebøttefundamentet.


Kravene til realtidsberegninger i kommercielle lastberegningssystemer til vindmøller stemmer ikke overens med brugen af f.eks. kobled element/randelementmetode modeller, hvor fundamentsstivheden er evaluert i frekvensdomænet. På grund af dette er den dynamiske stivhed (impedans) for hver frihedsgrad formuleret ved hjælp af lumped-parameter modeller med frekvensafhængige koefficienter, anvendelige for implementering i standard elementmetodeformuleringer. Lumped-parameter modellerne er anvendt til at simulere jord–struktur interaktionen i forbindelse med en elementmetodeformulering af en Vestas V90 3.0 MW offshore vindmølle med et sugebøttefundament. Simuleringerne af jord–struktur interaktionen ved hjælp af lumped-parameter model approksimationen af impedansen viser at konceptet er anvendeligt i forbindelse med applikationer, hvor anvendelsesstillstanden af en vindmølle skal analyseres.


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Chapter 1
Introduction

Offshore wind energy is a promising source of energy in the near future, and the plans for expanding the capacity in the European seas will increase the percentage of total electrical consumption produced by wind power. The offshore wind power generation by wind turbines is briefly introduced in this chapter. Subsequently, the various foundation types for offshore wind turbines are discussed. Afterwards, the need for further research and within foundation of offshore wind turbines and the research aims of the project are presented. At the end of the chapter an outline of the thesis is given.

1.1 Offshore wind turbines

Wind energy has proven to become a significant and powerful global energy resource. The international developments in wind energy technology has been governed by the European market for the last 25 years and the wind energy technologies have been well established in most western European countries. In 2004 Germany achieved 7 % (16,629 MW), Spain 6.5 % (8,263 MW), and Denmark 20 % (3,117 MW) of the total national electrical consumption from wind energy resources, according to the Global Wind Energy Council (www.gwec.net). The general aim of the Renewable Energy Policies for most countries with significant wind energy potential is to increase the percentage of total electrical consumption produced by wind power. The wind power has until recently been based on onshore wind turbines, but the newly developed megawatt sized wind turbines and new knowledge about offshore wind conditions are improving the economics of offshore wind power. Hence, offshore wind energy is rapidly becoming competitive with other power generating technologies. The continuous improvement in wind turbine technology means that the wind turbines have increased tremendously in both size and performance during the last 25 years. The general output of the wind turbines is improved by lager rotors and more powerful generators. In order to reduce the costs, the overall weight of the wind turbine components is minimized, which means that the wind turbine structures become more flexible and thus more sensitive to dynamic excitation.
1.2 Foundations for offshore wind turbines

The actual selection of type of foundation for a wind turbine or wind turbine park is governed by several factors. The choice of the particular design usually depends on following factors (Feld 2004):

♦ Soil condition
♦ Water depth
♦ Possible erosion
♦ Size and type of wind turbine
♦ Environmental conditions (wave height, current, ice, etc.)
♦ Economics and politics!

The recent major offshore wind farm projects in Europe have been dominated by two types of foundation solutions: the gravitational foundation and the monopile. The monopile solution has been used at Horns Rev, Samso, North Hoyle and Kentish Flats, whereas the offshore projects at Nysted and Middelgrunden are based on gravitational foundations. In future projects at increasing water depths and/or with greater wind turbines tripod foundations or jackets may become practicable. Moreover, a five-year research and development project at Aalborg University has proven the novel principle of the suction caisson to be feasible in suitable soil conditions in water depth from near shore to approximately 40 meters. The concepts for offshore wind turbine foundations at relatively shallow waters (0 - 50 m) are sketched in Figure 1.1. Each concept is presented in the following subsections. Floating offshore wind turbines are not included, see e.g. Bulder et al. (2003) and Hydro (2005) for details.
Figure 1.1: Concepts for offshore wind turbine foundations at relatively shallow waters. From left to right: Gravity based, suction caisson, monopile and tripod foundation.
1.2.1 Gravity based foundations

The gravity foundation is designed to carry the load from the wind turbine by compression load on the seabed, and thus avoiding tensile forces between the base of the foundation and the seabed. The dead load of the gravity based foundation has to be sufficient to prevent the foundation from uplifting, tilting and sliding. On the other hand the dead load must not create subsoil failure. This mass distribution of the gravity based foundation is balanced by the diameter and the height of the foundation in order to ensure the overall stability. The gravity load is secured by either massive concrete foundations or caisson type foundations (steel or concrete) where additional ballast is added by means of sand, concrete or rock material. A caisson type concrete foundation is illustrated in Figure 1.2. Some short comments on the concept:

- Possible to float out - reduces cost of crane vessel
- Fabrication/material costs are low (when using concrete)
- Not competitive at larger water depth
- Vulnerable to the presence of soft soil deposits below the foundation base
- Vulnerable to erosion and scour
- Sea bed preparation is necessary prior to installation (reduced by using steel skirts)
- Ice-cone usually integrated in design

Figure 1.2: Gravity based foundation.

Morten Liingaard
1.2 Foundations for offshore wind turbines

1.2.2 Monopile Foundation

The monopile foundation consists of a large diameter monopile installed in the seabed. The monopile is a welded steel pile with the same diameter as the lower section of the wind turbine tower. However, the diameter may vary depending on the type of connection between the pile and tower and the stiffness requirements of the foundation. In Figure 1.3 the monopile is shown with a transition piece. The load from the wind turbine is transferred to the surrounding soil by lateral earth pressure on the monopile. The length of the pile is governed by the required lateral resistance of the surrounding soil. The design depth (depth below seabed) of the monopile is typically between 20 - 30 m depending on water depth, wind turbine size/load and soil conditions. The monopile is installed by hydraulic hammer (driven pile), vibratory hammer (vibrated pile) or may be drilled into the seabed. Short comments on the concept:

♦ Simple fabrication with welded steel pile with either welded flange or grouted flange connection between tower and pile.

♦ The wind turbine structure with monopile becomes flexible for large water depths (25 m). Results in decrease of natural frequency of the structure.

♦ No preparations of the seabed are necessary; however, scour protection may be required when the concept is used at sandy locations.

♦ Requires heavy duty piling/drilling equipment (problem of sufficiently heavy hammers for diameters greater than 5-6 m).

♦ Foundation type is not suitable for locations with many large boulders in the seabed.

Figure 1.3: Monopile foundation.


1.2.3 Tripod Foundation

The tripod concept originates in the numerous steel tripod foundations for the offshore oil and gas industry. The tripod is a welded steel structure with a centre column supported by three piles in separate pile sleeves. The tripod concept is sketched in Figure 1.4. The tripod can be supported by either vertical or inclined piles. The design criterion for the piles in a tripod construction is usually the axial bearing capacity, contrary to the monopile. The main design parameters for the tripod structure are the height of the welded joint and the base radius (offset of pile sleeves). Short comments on the concept:

- Suitable for larger water depths.
- Minimum of preparations are required at the site before installation.
- Known technology from oil & gas industry.

The tripod in Figure 1.4 is a traditional steel structure with piles. There are several alternative solutions to this concept. Suction caissons may be used instead of piles, and the main structure may be constructed as a jacket construction (Tjelta 1995) or as a concrete section, with the possibility of ballasting the foundation, see e.g. Vølund (2005).
1.2 Foundations for offshore wind turbines

1.2.4 Suction caissons

Recent research and development projects (Byrne 2000; Feld 2001; Houlsby et al. 2005) have shown that suction caissons (see Figure 1.5) may be used as offshore wind turbine foundations in suitable soil conditions. Suction caissons (also denoted as bucket foundations or skirted foundations) have previously been used as anchors and foundations for several offshore platforms. Here, the suction caissons are mainly subject to vertical and horizontal loads. On the other hand, when suction caissons are applied as monopod foundations for wind turbines, they must be able to sustain a significant overturning moment. At greater water depths the monopod solution may become uneconomical and a foundation concept with three or four smaller suction caissons may become appropriate (see the previous subsection). The overturning moment is then stabilized by the opposing vertical reactions of the suction caissons, see (Houlsby et al. 2005; Senders 2005). The suction caisson is installed by using suction as the driving force and does not require heavy installation equipment. Lowering the pressure in the cavity between the foundation and the soil surface causes a water flow to be generated, which again causes the effective stresses to be reduced around the tip of the skirt. Hence, the penetration resistance is reduced. A fully operational 3.0 MW offshore wind turbine was installed on a prototype of the suction caisson foundation at the test field in Frederikshavn, Denmark in late 2002. The project is described in details in Ibsen et al. (2005). Short comments on the concept:

♦ Hybrid of pile and gravity based foundation.
♦ Fabrication/material costs comparable to that of the monopile concept.
♦ Non proven technology for large water depths.
♦ Decommissioning is a relatively simple process.
1.3 Need for further research

A modern offshore wind turbine (1.5 to 2 MW) is typically installed with a variable speed system so the rotational speed of the rotor varies from, for example, 10 - 20 RPM. This means that the excitation frequency of the rotor system varies. The first excitation frequency interval then becomes 0.17 - 0.33 Hz (for 10 - 20 RPM) and is referred to as the $1\Omega$ frequency interval. The second excitation frequency interval corresponds to the rotor blade frequency that depends on the number of blades. For a three-bladed wind turbine the $3\Omega$ frequency interval is equal to 0.5 - 1.0 Hz (for 10 - 20 RPM). Since the first resonance frequency $\omega_1$ of the modern offshore wind turbines is placed between $1\Omega$ and $3\Omega$, it is of outmost importance to be able to evaluate the resonance frequencies of the wind turbine structure accurately as the wind turbines increase in size.

The resonance frequencies (or natural frequencies) are usually to be evaluated at early stages of the design procedure, and for that reason it is crucial to have accurate numerical formulations of the entire wind turbine structure. At present, the soil–structure interaction between the wind turbine foundation and the surrounding soil is taken into account, for example by elongation of structural beam elements to simulate reduction of the resonance frequencies. Another approach is to use static soil springs for describing the soil–structure interface. However, these simple methods do not account for any dynamic reaction of the surrounding soil, i.e. the damping and inertial effects are not dealt with.

Preliminary studies have shown that the stiffness of suction caissons varies significantly in certain intervals of excitation frequencies (Liingaard et al. 2005). The same pattern may be observed for surface foundations vibrating on layered soil (Andersen and Clausen 2005). In order to achieve reliable responses of the wind turbine structure during working loads it is necessary to account for the possibilities of dynamic effects of the soil–structure interaction. The dynamic effects of the foundation interacting with the soil can be explained by the fact that the observed stiffness of the foundation depends on the excitation frequency of an applied load, i.e the foundation stiffness is frequency dependent.

1.4 Research aims

The purpose of this thesis is to evaluate the frequency dependent stiffness of suction caisson foundations for offshore wind turbines, with the intention that the dynamic properties of the foundation can be properly included in a composite structure–foundation system. For comparison, experimental evaluations of the resonance frequencies of an existing offshore wind turbine have been utilized. The research aims are categorized as:

- Evaluation the frequency dependent stiffness of suction caisson foundations
- Experimental estimation of resonance frequencies
- Formulation and implementation of lumped-parameter models that accounts for frequency dependent behaviour of the soil–structure interface
1.4.1 Evaluation of the frequency dependent stiffness of suction caissons

This part concerns the dynamic soil–structure interaction of steel suction caissons applied as foundations for offshore wind turbines. The soil is simplified as a homogenous linear viscoelastic material and the dynamic stiffness of the suction caisson is expressed in terms of dimensionless frequency-dependent coefficients corresponding to the different degrees of freedom. The dynamic stiffness coefficients for the skirted foundation are evaluated by means of a three-dimensional coupled boundary element/finite element model.

1.4.2 Experimental estimation of resonance frequencies

Experimental modal analyses have been carried out with the intention of estimating the natural frequencies of an existing Vestas 3.0 MW offshore wind turbine. The experimental modal analysis of the wind turbine makes use of "Output-only modal identification" which is utilized when the modal properties are identified from measured responses only.

1.4.3 Formulation and implementation of lumped-parameter models

In order to meet the requirements of real-time calculations and analysis in time domain, lumped-parameter models are particularly useful. A lumped-parameter model represents the frequency dependent soil-structure interaction of a massless foundation placed on or embedded into an unbounded soil domain. A key feature is that the models consist of real frequency-independent coefficients in a certain arrangement, which can be formulated into stiffness, damping and/or mass matrices. Thus, the lumped-parameter model can be incorporated into standard dynamic finite element programs.

1.5 Thesis outline

The thesis contains five chapters and four associated appendices, related to the topics described in Section 1.4.

Chapters 2 and 3 contain the evaluation of the frequency dependent stiffness of suction caissons. The method of analysis is explained in details in the first part of Chapter 2. In the main part of Chapter 2, the frequency dependent stiffness of the vertical degree of freedom is examined. The vertical frequency dependent stiffness is evaluated by a three-dimensional coupled boundary element/finite element model, and parameter studies for different combinations of the skirt length, Poisson’s ratio and the ratio between soil stiffness and skirt stiffness have been performed. Moreover, the behaviour at high frequencies is investigated with the intention of applications for lumped-parameter models.

Chapter 3 concerns torsional and coupled sliding/rocking vibrations. The dynamic stiffness components are evaluated by a three-dimensional coupled boundary element/finite element model. The torsional frequency dependent stiffness has been determined for different combinations of the skirt length, and the frequency dependent stiffness of the coupled sliding/rocking motion has been determined for different combinations of
the skirt length and Poisson’s ratio. Again, the high frequency behaviour is investigated with the intention of applications for lumped-parameter models.

Chapter 4 concerns the formulation and implementation of lumped-parameter models. The main purpose of the chapter is to investigate the natural frequencies of the Vestas 3.0 MW offshore wind turbine with experimental and numerical methods. The experimental estimation of the natural frequencies is carried out by means of experimental modal analysis of the structure. The experimental procedure is presented in the first part of Chapter 4. In the second part of Chapter 4, the natural frequencies are evaluated by a finite element model, where the soil-structure interaction is modelled by two types of foundation models. In the first approach, the soil–structure interaction is modelled by static springs, and in the second approach, the frequency dependent behaviour of the structure-foundation system is taken into consideration by applying lumped-parameter models.

Chapter 5 contains the main conclusion of the thesis, and directions for future work are given, based on the findings in this thesis.

Appendix A describes the closed-form solution to the vertical dynamic stiffness of the infinite cylinder. The solution is used in Chapter 2.

Appendix B concerns the basic theory and principles for experimental modal analysis. The sections within the appendix are: Output-only modal analysis software, general digital analysis, basics of structural dynamics and modal analysis and system identification.

Appendix C explains the steps of establishing a lumped-parameter model. Following sections are included in this appendix: Static and dynamic formulation, Simple lumped-parameter models and Advanced lumped-parameter models. The content of the appendix concerns Chapter 4 and Appendix D.

Appendix D describes the lumped-parameter models for a suction caisson with a ratio between skirt length and foundation diameter equal to 1/2, embedded into an viscoelastic soil. The models are presented for three different values of the shear modulus of the subsoil. Subsequently, the assembly of the dynamic stiffness matrix for the foundation is considered, and the solution for obtaining the steady state response, when using lumped-parameter models is given. The content of the appendix concerns Chapter 4.
Chapter 2
Dynamic stiffness of suction caissons—vertical vibrations

The dynamic response of offshore wind turbines are affected by the properties of the foundation and the subsoil. The purpose of this chapter is to evaluate the dynamic soil–structure interaction of suction caissons for offshore wind turbines. The investigation is limited to a determination of the vertical dynamic stiffness of suction caissons. The soil surrounding the foundation is homogenous with linear viscoelastic properties. The dynamic stiffness of the suction caisson is expressed by dimensionless frequency-dependent dynamic stiffness coefficients corresponding to the vertical degree of freedom. The dynamic stiffness coefficients for the foundations are evaluated by means of a dynamic three-dimensional coupled Boundary Element/Finite Element model. Comparisons are made with known analytical and numerical solutions in order to evaluate the static and dynamic behaviour of the Boundary Element/Finite Element model. The vertical frequency dependent stiffness has been determined for different combinations of the skirt length, Poisson’s ratio and the ratio between soil stiffness and skirt stiffness. Finally the dynamic behaviour at high frequencies is investigated.

2.1 Introduction

Wind turbines have increased tremendously in both size and performance during the last 25 years. The general output of the wind turbines is improved by larger rotors and more powerful generators. In order to reduce the costs, the overall weight of the wind turbine components is minimized, which means that the wind turbine structures become more flexible and thus more sensitive to dynamic excitation at low frequencies. The foundation principles for the recent major offshore wind farm projects in Europe have been dominated by two types of foundation solutions: the gravitational foundation and the monopile. Recent research and development projects Houltsby, Ibsen, and Byrne (2005) have shown that suction caissons (see Figure 2.1) may be used as offshore wind turbine foundations in suitable soil conditions and water depths up to approximately 40 meters. Suction caissons (also denoted as bucket foundations or skirted foundations) have previously been used as anchors and foundations for several offshore platforms. Here, the suction caissons are mainly subject to vertical and horizontal loads. On the other hand, when suction caissons are applied as monopod foundations for wind turbines, they must be able to sustain a significant overturning moment. At greater water depths the monopod solution may become uneconomical and a foundation concept with three
or four smaller suction caissons may become appropriate. The overturning moment is then stabilized by the opposing vertical reactions of the suction caissons, see (Houlsby et al. 2005; Senders 2005). The suction caisson is installed by using suction as the driving force and does not require heavy installation equipment. Lowering the pressure in the cavity between the foundation and the soil surface causes a water flow to be generated, which again causes the effective stresses to be reduced around the tip of the skirt. Hence, the penetration resistance is reduced. A fully operational 3.0 MW offshore wind turbine was installed on a prototype of the suction caisson foundation at the test field in Frederikshavn, Denmark in late 2002. The project is described in details in Ibsen et al. (2005).

The purpose of this chapter is to evaluate the vertical impedance of suction caisson foundations for offshore wind turbines, with the intention that the dynamic properties of the foundation can be properly included in a composite structure–foundation system. The frequency dependent dynamic stiffness is evaluated by means of a dynamic three-dimensional coupled Boundary Element/Finite Element (BE/FE) program BEASTS by Andersen and Jones (2001a). Initially, the solution methods for analysing soil–structure interaction are briefly introduced in Section 2.2. Afterwards, the method applied in this chapter, i.e. the coupled BE/FE model, is described in Section 2.3. The definitions of static and dynamic stiffness for the suction caisson are presented in Section 2.4. Prior to the analysis, two benchmark tests are shown in Section 2.5. The results obtained by analysing the vertical dynamic stiffness of suction caissons are presented in Section 6 and the findings are discussed in Section 2.7. The main conclusions of the chapter are given in Section 2.8. In this chapter the impedance is equal to the dynamic stiffness of the foundation, i.e. the impedance contains both a real and an imaginary part.

### 2.2 Analysis methods for dynamic soil-structure interaction

The classical methods for analysing vibrations of foundations are based on analytical solutions for massless circular foundations resting on an elastic half-space. The classical solutions by Reissner, Quinlan and Sung were obtained by integration of Lamb's solution for a vibrating point load on a half-space (Richart et al. 1970; Das 1993). The mixed boundary value problems with prescribed conditions under the foundation and zero traction at the remaining free surface were investigated by Veletsos and Wei (1971) and Luco and Westmann (1971). The integral equations of the mixed boundary value problems were evaluated and tabulated for a number of excitation frequencies. A closed-form solution has been presented by Krenk and Schmidt (1981). Whereas analytical and semi-analytical solutions may be formulated for surface footings with a simple geometries, numerical analysis is required in the case of flexible embedded foundations with complex geometry. Thus, in the present analysis of suction caissons for offshore wind turbines, a coupled boundary element/finite element model is applied. The Finite Element Method (FEM) is very useful for the analysis of structure with local inhomogeneities and complex geometries. However, only a finite region can be discretized. Hence, at the artificial boundaries of the unbounded domain, e.g. soil, transmitting boundary conditions must be applied as suggested by Higdon (1990), Higdon (1992) and (Krenk 2002). Numerous
2.3 Boundary element/finite element formulation

The dynamic stiffness of the suction caissons is evaluated by means of the dynamic three-dimensional coupled Boundary Element Method/Finite Element Method program BEASTS by Andersen and Jones (2001a). The boundary element part of BEASTS is an extension of the theory presented by Domínguez (1993), which has been modified to account for open domains and to allow a coupling with finite elements, see Andersen and Jones (2001b) for details.

2.3.1 Boundary element formulation

Let $\mathbf{x}$ define a point in the three-dimensional Cartesian space and let $\omega$ denote the cyclic frequency. The governing equation of motion for a three-dimensional body $\Omega$ in the
frequency domain is then given by
\[
\frac{\partial \sigma_{ij}(x, \omega)}{\partial x_j} + \rho B_i(x, \omega) + \omega^2 \rho U_i(x, \omega) = 0, \quad x \in \Omega, \quad (2.1)
\]
where summation is carried out over repeated indices. \( U_i(x, \omega) \) \((i=1,2,3)\) and \( \sigma_{ij}(x, \omega) \) \((j=1,2,3)\) are the complex amplitudes of the displacement field and the stresses, respectively. The latter may be computed from the displacements by the constitutive relation. Further, \( \rho B_i(x, \omega) \) are the body forces. The boundary conditions on the surface \( \Gamma \) of the body \( \Omega \) are:
\[
\begin{align*}
U_i(x, \omega) &= \hat{U}_i(x, \omega) \quad \text{for} \quad x \in \Gamma_U, \\
P_i(x, \omega) &= \hat{P}_i(x, \omega) \quad \text{for} \quad x \in \Gamma_P,
\end{align*}
\]
where the displacement amplitude \( \hat{U}_i(x, \omega) \) is given on one part of the boundary, \( \Gamma_U \), and the surface traction \( \hat{P}_i(x, \omega) = \sigma_{ij}(x, \omega) n_j(x) \) is given on the remaining part of the boundary, \( \Gamma_P \). Here \( n_j(x) \) are the components of the outward unit normal to the surface. To obtain the boundary element formulation of Equation (2.1), a second state \( U^*_il(x, \omega; \xi) \) is identified as the fundamental solution to the equation of motion
\[
\frac{\partial \sigma^*_{ijl}(x, \omega; \xi)}{\partial x_j} + \rho \delta(x - \xi) \delta_{il} + \omega^2 \rho U^*_il(x, \omega; \xi) = 0,
\]
where \( \delta(x - \xi) \) is the Dirac delta function in vector form and \( \delta_{il} \) is the Kronecker delta. It should be noted that the Green’s function \( U^*_il(x, \omega; \xi) \) represents a \( 3 \times 3 \) matrix, i.e. there are three displacement components at the receiver point \( x \) for each direction \( l \) of the load applied at the source point \( \xi \). In three dimensions, \( U^*_il(x, \omega; \xi) \) has a singularity of the order \( 1/r \), whereas the corresponding stress field \( \sigma^*_{ijl}(x, \omega; \xi) \) has a singularity of the order \( 1/r^2 \).

The fundamental solution is based on wave propagation in the full space and therefore only represents body waves emanating from the source, i.e. dilatation and shear waves with phase velocities \( c_P \) and \( c_S \), respectively. The velocities \( c_P \) and \( c_S \) are given as
\[
c_P = \sqrt{\frac{\lambda + 2G}{\rho}}, \quad c_S = \sqrt{\frac{G}{\rho}}, \quad (2.4)
\]
where \( \lambda \) and \( G \) are the Lamé constants of the material, and \( \rho \) is the mass density. The Lamé constants \( \lambda \) and \( G \) can be written in terms of Young’s modulus \( E \) and Poisson’s ratio \( \nu \) by the following relations:
\[
G = \frac{E}{2(1 + \nu)}; \quad \lambda = \frac{\nu E}{(1 + \nu)(1 - 2\nu)}, \quad (2.5)
\]
Material damping is introduced by a complex Young’s modulus \( E^* \), resulting in complex Lamé constants. The complex Young’s modulus \( E^* \) is given by
\[
E^* = E (1 + i\eta), \quad (2.6)
\]
where \( \eta \) is the loss factor of the material and \( i = \sqrt{-1} \) is the imaginary unit. Note that the loss factor is assumed to be constant for all frequencies, i.e., hysteretic damping is assumed.

The fundamental solution is applied as a weight function in the weak formulation of the equation of motion (2.1) for the physical field and vice versa. After some manipulations, and disregarding body forces in the interior of the domain, Somigliana’s identity is derived:

\[
C_{il}(x) U_l(x, \omega) + \int_{\Gamma} P^*_{il}(x, \omega; \xi) U_l(\xi, \omega) \, d\Gamma \xi = \int_{\Gamma} U^*_{il}(x, \omega; \xi) P_l(\xi, \omega) \, d\Gamma \xi \tag{2.7}
\]

Here \( P^*_{il}(x, \omega; \xi) \) is the surface traction related to the Green’s function \( U^*_{il}(x, \omega; \xi) \). \( C_{il}(x) \) is a doubly indexed scalar that only depends on the geometry of the surface \( \Gamma \). In particular, \( C_{il}(x) = 1/2 \delta_{il} \) on a smooth part of the boundary \( \Gamma \) and \( C_{il}(x) = \delta_{il} \) inside the body \( \Omega \). A detailed derivation of (2.7) and properties of \( C_{il}(x) \) are given in (Andersen 2002; Domínguez 1993).

In order to evaluate the boundary integral equations in (2.7) for a point \( x \) on the boundary, the surface is discretized into a finite number of boundary elements. The boundary integral equation can then be solved numerically for any point \( x \) on the boundary. The boundary can be discretized by different types of elements with varying order of integration. In the present study, quadrilateral elements with quadratic interpolation are employed, due to the fact that nine-noded boundary elements are superior in performance and convergence compared to elements with constant or linear interpolation (Andersen 2002).

To obtain the BE formulation, the state variable fields on the boundary are discretized. \( U_j(x) \) and \( P_j(x) \) be the vectors storing the displacements and tractions at the \( N_j \) nodes in element \( j \). The displacement and traction fields over the element surface \( \Gamma_j \) then become

\[
U(x, \omega) = \Phi_j(x) U_j(\omega), \quad P(x, \omega) = \Phi_j(x) P_j(\omega), \tag{2.8}
\]

where \( \Phi_j(x) \) is a matrix storing the interpolation, or shape, functions for the element. This allows the unknown values of the state variables to be taken outside the integrals in Equation (2.7). Finally, the three-row matrices originating from Equation (2.7) for each of the observation points may be assembled into a single matrix equation for the entire BE domain,

\[
H(\omega) U(\omega) = G(\omega) P(\omega). \tag{2.9}
\]

Component \((i, k)\) of the matrices \( H(\omega) \) and \( G(\omega) \) stores the influence from degree-of-freedom \( k \) to degree-of-freedom \( i \) for the traction and the displacement, respectively, i.e., the integral terms on the left- and right-hand side of Equation (2.7). The geometric constants \( C_{il}(x) \) are absorbed into the diagonal of \( H(\omega) \).

### 2.3.2 Coupling of FE and BE regions

The finite element (FE) region of the model is formulated by the equation of motion in the frequency domain (Andersen and Jones 2002):

\[
(-M \omega^2 + iC + K) U = K_{FE} U = F, \tag{2.10}
\]
where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices, respectively. $\mathbf{U}$ contains the nodal displacements and $\mathbf{F}$ the nodal forces. Hysteretic material damping is assumed, i.e. $\mathbf{C} = \eta \mathbf{K}$. Hence, the damping term is independent of the circular frequency $\omega$.

In the subsequent analysis, the foundation consists of relatively thin structures (skirt) and the use of boundary elements in this region is inappropriate due to the singularities of the Green’s functions. In these regions finite elements are used. In order to couple a BE domain formulated in terms of surface tractions with an FE region with loads applied in terms of nodal forces, a transformation matrix $\mathbf{T}$ is defined, such that $\mathbf{F} = \mathbf{T}\mathbf{P}$. Here $\mathbf{F}$ is the vector of nodal forces equivalent to the tractions $\mathbf{P}$ applied on the surface of the domain. The transformation matrix only depends on the spatial interpolation functions, i.e. the shape functions, for the elements along the interaction boundary. Hence, $\mathbf{T}$ may be determined once and for all and applied in all analyses with a given model geometry. Subsequently, for each frequency the matrix

$$\mathbf{T}\mathbf{G}^{-1}\mathbf{H} = \mathbf{K}_\text{BE}$$

defines an equivalent dynamic stiffness matrix for the boundary element domain. This operation turns the BE domain into a macro finite element. It should be noted that $\mathbf{K}_\text{BE}$ is a fully populated and asymmetrical matrix, as opposed to $\mathbf{K}_\text{FE}$ which is a sparsely populated, banded and symmetric matrix. For detailed discussion regarding the coupling between a BE and FE regions, see Andersen and Jones (2001b).

### 2.4 Static and dynamic stiffness formulation

A generalized massless axisymmetric foundation with a rigid base has six degrees of freedom: one vertical, two horizontal, two rocking and one torsional. The six degrees of freedom and the corresponding forces and moments are shown in Figure C.1. For a harmonic excitation with the cyclic frequency $\omega$, the dynamic stiffness matrix $\mathbf{S}$ is related to the vector of forces and moments $\mathbf{R}$ and the vector of displacements and rotations $\mathbf{U}$ as follows:

$$\mathbf{R} = \mathbf{SU}$$

![Figure 2.2: Degrees of freedom for a rigid surface footing: (a) displacements and rotations, and (b) forces and moments.](image-url)
The component form of Equation (C.3) can be written as:

\[
\begin{bmatrix}
\frac{V}{G_s R^2} \\
\frac{H_1}{G_s R^2} \\
\frac{H_2}{G_s R^2} \\
\frac{T}{G_s R^3} \\
\frac{M_1}{G_s R^3} \\
\frac{M_2}{G_s R^3}
\end{bmatrix}
= \begin{bmatrix}
S_{VV} & 0 & 0 & 0 & 0 & 0 \\
0 & S_{HH} & 0 & 0 & 0 & -S_{MH} \\
0 & 0 & S_{HH} & 0 & S_{MH} & 0 \\
0 & 0 & 0 & S_{TT} & 0 & 0 \\
0 & 0 & S_{MH} & 0 & S_{MM} & 0 \\
0 & -S_{MH} & 0 & 0 & 0 & S_{MM}
\end{bmatrix}
\begin{bmatrix}
W/R \\
U_1/R \\
U_2/R \\
\theta_T \\
\theta_{M1} \\
\theta_{M2}
\end{bmatrix}
\] (2.13)

where \( R \) is the radius of the foundation and \( G_s \) is the shear modulus of the soil. The components in \( S \) are functions of the cyclic frequency \( \omega \) and Poisson’s ratio of the soil \( \nu_s \). The nonzero terms in \( S \) can be written as:

\[
S_{ij}(a_0) = K^0_{ij}[k_{ij}(a_0) + i a_0 c_{ij}(a_0)], \quad (i, j = H, M, T, V),
\] (2.14)

where \( K^0_{ij} \) is the static value of \( ij \)th stiffness component, whereas \( k_{ij} \) and \( c_{ij} \) are the dynamic stiffness and damping coefficients, respectively. Furthermore, \( a_0 = \omega R/c_S \) is the dimensionless frequency where \( c_S \) is the shear wave velocity of the soil. The real part of Equation (2.14) is related to the stiffness and inertia properties of the soil–structure system, whereas the imaginary part describes the damping of the system. For a soil without material dissipation, \( c_{ij} \) reflects the geometric damping, i.e. the radiation of waves into the subsoil.

In some situations it is useful to examine the magnitude and phase angle of Equation (2.14) in addition to the real and imaginary parts of the dynamic stiffness. The magnitude (complex modulus) and the phase angle \( \phi_{ij} \) of \( S_{ij} \) are given by

\[
|S_{ij}| = K^0_{ij} \sqrt{(k_{ij})^2 + (a_0 c_{ij})^2}, \quad \phi_{ij} = \arctan \left( \frac{a_0 c_{ij}}{k_{ij}} \right).
\] (2.15)

### 2.5 Benchmark tests

The coupled BE/FE model of the suction caisson has been tested and compared with known analytical and numerical results. The first comparison concerns the capability of determining the static stiffness of the suction caisson by the BE/FE formulation. In the second comparison the BE/FE model of the suction caisson has been used to reproduce the vertical dynamic stiffness of a surface foundation by setting the skirt properties equal to the properties of the surrounding soil.

#### 2.5.1 Verification of the vertical static stiffness

The vertical static stiffness \( K^0_{VV} \), corresponds to the stiffness of the soil-foundation system without any inertial or material dissipation effects. The vertical static stiffness coefficient has been determined by means of a static finite element analysis in ABAQUS (Abaqus 2003). These static results have been used as convergence criteria for the element mesh size in the subsequent boundary element analyses of the dynamic stiffness. The reason for using the static stiffness as convergence criteria is that the shape of the impedance (location of peaks as function of frequency) converges with a relatively coarse mesh,
compared to the actual magnitude of the impedance. Surprisingly, it turns out that the magnitude of the impedance is the critical convergence parameter. The static stiffness from the FE/BE models are estimated for a very low excitation frequency, $a_0 = 0.01$, where the inertial effects are negligible.

**ABAQUS model**

The static three-dimensional ABAQUS model of the suction caisson consists of a foundation and near-field soil domain modelled by second order finite elements and a far field soil domain modeled by infinite elements. The skirt of the suction caisson is flexible, considering the fact that the skirt thickness is small compared to the height or diameter of the foundation. The lid is assumed to be rigid. The lid is modelled as a solid finite element section with a thickness of one meter and the same material properties as the skirt. The ABAQUS model contains approximately 200,000 degrees of freedom and the runtime is approximately 1 hour per stiffness coefficient on a 2.0 GHz P4 laptop computer. The BE/FE model is described in the next subsection.

**Results**

The non-dimensional values of $K_{VV}^0$ are given for three different cases:

*Different skirt lengths:* – The static stiffness $K_{VV}^0$ is given for various ratios between the foundation diameter $D$ and the length of the skirt $H$ in Table 2.1. The soil properties are $G_s = 1$ MPa and $\nu_s = 1/3$.

*Different Poisson’s ratios:* – The variation of $K_{VV}^0$ with respect to Poisson’s Ratio is shown in Table 2.1. $H/D=1$ and $G_s = 1$ MPa.

*Varying soil stiffness:* – $K_{VV}^0$ is given for different values of the shear modulus $G_s$ in the soil in Table 2.1. $H/D=1$ and $\nu_s = 1/3$.

The data are shown for fixed material properties of the foundation ($E_f = 210$ GPa, $\nu_f = 0.25$). The foundation radius is $R = 5$ m and the skirt thickness is $t = 50$ mm. In general there is a good agreement between the values of $K_{VV}^0$ computed by FE and BE/FE when it is taken into account that $K_{VV}^0$ has been calculated with two different methods of analysis and discretization. There is a tendency of increasing deviations with decreasing Poisson’s ratio and increasing skirt length. It should be noted that the static vertical stiffness for low values of $G_s$ (0.1 and 1.0 MPa) is equivalent to the stiffness of a suction caisson with rigid skirts, whereas high values of $G_s$ (approaching the shear modulus of the skirts) correspond to the behaviour of a rigid base surface foundation. The results agree with the work by Doherty and Deeks (2003) and Doherty et al. (2005) who employed the scaled boundary finite element method to analyse the static stiffness of suction caissons embedded in non-homogeneous elastic soil.

### 2.5.2 Reproduction of the vertical dynamic stiffness of a surface footing

Next, the FE/BE model of the suction caisson ($H/D = 1$) is tested against known analytical and numerical results for the vertical dynamic stiffness of a surface footing.

*Morten Liingaard*
Table 2.1: Vertical static stiffness

<table>
<thead>
<tr>
<th>$H/D$</th>
<th>$K_{0,V}^{V}$ FE</th>
<th>$K_{0,V}^{V}$ FE/BE</th>
<th>Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>7.25</td>
<td>7.39</td>
<td>−1.88 %</td>
</tr>
<tr>
<td>1</td>
<td>10.70</td>
<td>10.87</td>
<td>−1.60 %</td>
</tr>
<tr>
<td>2</td>
<td>14.61</td>
<td>14.99</td>
<td>−2.53 %</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\nu_s$</th>
<th>$K_{0,V}^{V}$ FE</th>
<th>$K_{0,V}^{V}$ FE/BE</th>
<th>Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>9.20</td>
<td>9.72</td>
<td>−5.38 %</td>
</tr>
<tr>
<td>0.2</td>
<td>9.74</td>
<td>10.13</td>
<td>−3.85 %</td>
</tr>
<tr>
<td>0.333</td>
<td>10.70</td>
<td>10.87</td>
<td>−1.60 %</td>
</tr>
<tr>
<td>0.4</td>
<td>11.32</td>
<td>11.39</td>
<td>−0.60 %</td>
</tr>
<tr>
<td>0.495</td>
<td>12.89</td>
<td>12.68</td>
<td>+1.64 %</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$G_s$</th>
<th>$K_{0,V}^{V}$ FE</th>
<th>$K_{0,V}^{V}$ FE/BE</th>
<th>Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^3$ Pa</td>
<td>10.73</td>
<td>10.91</td>
<td>−1.65 %</td>
</tr>
<tr>
<td>$10^6$ Pa</td>
<td>10.70</td>
<td>10.87</td>
<td>−1.60 %</td>
</tr>
<tr>
<td>$10^7$ Pa</td>
<td>10.65</td>
<td>10.48</td>
<td>+1.58 %</td>
</tr>
<tr>
<td>$10^8$ Pa</td>
<td>10.03</td>
<td>10.19</td>
<td>−1.62 %</td>
</tr>
<tr>
<td>$10^9$ Pa</td>
<td>7.85</td>
<td>8.01</td>
<td>−2.03 %</td>
</tr>
</tbody>
</table>

In the BE/FE model of the suction caisson, the skirt has been given material properties equal to the properties of the surrounding soil. The model of the suction caisson should then be able to reproduce the results obtained for a massless surface footing. The results obtained from the suction caisson model are compared with a BE/FE analysis of a surface footing and a known analytical solution. The analytical solution given by Veletsos and Tang (1987) is based on a perfect elastic half-space with Poisson’s ratio equal to $1/3$, and relaxed boundary conditions under the foundation are assumed, corresponding to the condition of ‘smooth’ contact.

Figure 2.3: BE/FE models of (a) surface foundation and (b) suction caisson ($H/D = 1$).
Boundary Element/Finite Element models

Due to symmetry only half the foundation is included. In the finite element region only half the model needs to be analysed when a plane of symmetry exists. The degrees of freedom in the plane of symmetry are simply eliminated in the system of equations in order to satisfy the conditions at the interface between the modelled and non-modelled part. The procedure for introducing a plane of symmetry in the BE region is more complex, and will not be given here. The procedure for BE analysis of problems with geometrical symmetry is discussed in details by Andersen and Jones (2001b).

The BE/FE model of the surface footing contains a massless circular foundation with the radius $R = 5 \, \text{m}$. The foundation is modelled by 40 quadrilateral finite elements employing quadratic interpolation. The thickness (height) of the foundation is one meter. The soil is discretized into a total of 152 boundary elements with quadratic interpolation. The model is illustrated in Figure 2.3a.

The BE/FE model of the suction caisson consists of four sections: a massless finite element section that forms the top of the foundation where the load is applied, a finite element section of the skirts, a boundary element domain inside the skirts and, finally, a boundary element domain outside the skirts that also forms the free surface. The skirt of the suction caisson is considered flexible, and the lid is assumed to be rigid. The lid is modelled as a solid finite element section with a thickness of one meter. Again, quadratic interpolation is employed. The models of the suction caisson and the subsoil contain approx. 100 finite elements and 350 boundary elements. The model is illustrated in Figure 2.3b. For both numerical models, the soil is linear elastic with $G_s = 1 \, \text{MPa}$, $\nu_s = 1/3$ and $\rho_s = 1000 \, \text{kg/m}^3$. The material of the surface foundation and the lid of the suction caisson is linear elastic with $G_s = 10^6 \, \text{MPa}$, $\nu_s = 1/3$ and $\rho_s = 0 \, \text{kg/m}^3$. The connection between soil and foundation corresponds to the condition of ‘rough’ contact since the foundation and the surrounding soil have common degrees of freedom.

The mesh of the free surface for the surface foundation has been truncated at a distance of 15 m (3 times radius $R$) from the centre of the footing, based on convergence studies. Regarding the suction caisson ($H/D = 1$), the mesh of the free surface is truncated at a distance of 30 m (6 times radius $R$) from the centre of the foundation. The truncation distance for the models of the suction caisson depends on the skirt embedment. Convergence studies for the worst case ($H/D = 2$) suggested a truncation distance of 30 m from the centre of the foundation. This length has been used for all the BE/FE analyses of the suction caisson, regardless of embedment depth of the skirt. Adaptive meshing could possibly improve the accuracy versus the number of degrees of freedom, but this facility is currently not available in the BE/FE software.

For a given excitation frequency a vertical load equal to 1 N is applied in the centre on top of the foundations and the complex displacements are computed. The complex vertical dynamic stiffness is then determined from the load and the displacement response. Note that load control has been used to generate the stiffness values. Displacement control would be more appropriate, but this feature is currently not available in the BE/FE software.

The models contain approximately 1000 (surface model) to 3000 (suction caisson model) degrees of freedom and the runtime is approximately 5 to 30 minutes for each excitation frequency on a 2.0 GHz P4 laptop computer.
Figure 2.4: Vertical dynamic stiffness for a surface foundation calculated by two different BE/FE models. The numerical results are compared with a known analytical solution.

Results

The BE/FE models have been utilized for 13 excitation frequencies in the range $a_0 \in [0;6]$. The results obtained from the numerical models are given in Figure 2.4 together with the known analytical solution reported by Veletsos and Tang (1987). The upper plot shows the normalized magnitude $|S_{VV}|/K_{VV}$, and the phase angle $\phi_{VV}$ is shown in the lower plot. First of all, the numerical models are able to reproduce the overall pattern of the frequency dependent stiffness of the analytical solution, when it is considered that the analytical solution by Veletsos and Tang (1987) is based on relaxed boundary conditions and the boundary element solutions corresponds to welded, or rough, contact. The same type of results have been reported by Alarcon et al. (1989). The results from the suction caisson model with ‘soil skirts’ match the results of the surface foundation model quite well, and it is concluded that the model of the suction caisson is able to reproduce the frequency dependent behaviour of a surface foundation without introducing errors due to the complexity of the model (two boundary element domains separated by a thin finite element structure).
2.6 Dynamic stiffness for vertical vibrations

In this section the dynamic stiffness is investigated for several different combinations of the mechanical properties of the soil–foundation system. The first case concerns the effects of Poisson’s ratio on the stiffness. In the second analysis the flexibility of the soil–foundation system is investigated for different ratios between the soil and the foundation stiffness. The third case is the variation of the stiffness due to a change in the skirt length. The first two analyses are carried out for the frequency range $a_0 \in [0;6]$, whereas the third analysis is extended to a larger frequency range $a_0 \in [0;12]$.

![Graph showing vertical dynamic stiffness variation of Poisson's ratio](image)

Figure 2.5: Vertical dynamic stiffness: variation of Poisson’s ratio. $H/D = 1$, $G_s = 1.0$ MPa and $\eta_s = 5\%$.
2.6 Dynamic stiffness for vertical vibrations

2.6.1 Vertical dynamic stiffness—variation of Poisson’s ratio

The dynamic stiffness for different Poisson’s ratios is presented in this section. The skirt length is fixed \((H/D = 1)\). The model properties are \(G_s = 1.0 \text{ MPa}, \rho_s = 1000 \text{ kg/m}^3, \eta_s = 5\%\), \(E_f = 210 \text{ GPa}, \nu_f = 0.25, \eta_f = 2\%\) and \(t = 50 \text{ mm}\). In order to model a massless foundation \(\rho_f = 0\) for the lid of the caisson and \(\rho_f = \rho_s\) for the skirt. In Figure 2.5, the results are shown for 5 different values of Poisson’s ratio for the frequency range \(a_0 \in [0;6]\). The dynamic stiffness is relatively insensitive to variations in \(\nu_s\) in the range from 0.1 to 0.4. When \(\nu_s\) approaches 0.5, the dynamic behaviour changes significantly. The main reason for the change in the dynamic behaviour for \(\nu_s\) close to 0.5 is the fact that \(c_P/c_S \in [\sqrt{2};2]\) for \(\nu_s \in [0;1/3]\), whereas \(c_P/c_S \rightarrow \infty\) for \(\nu_s \rightarrow 0.5\). Thus, for constant \(G_s\) the P-wave speed becomes infinite for \(\nu_s \rightarrow 0.5\). Note that it is possible to solve the BE system for \(\nu_s = 0.5\) by reordering the fundamental solution, however, here the range in Poisson’s ratio is thought to cover fully drained \((\nu_s = 0.1 - 0.2)\) to undrained \((\nu_s = 0.495)\) conditions.

2.6.2 Vertical dynamic stiffness—variation of soil stiffness

The influence of the ratio between the stiffness of the soil and the stiffness of the structure is evident from the analysis of the static stiffness, see Table 2.1. The influence on the dynamic behaviour is shown in Figure 2.6 for the frequency range \(a_0 \in [0;6]\). The fixed model properties are \(H/D = 1, \nu_s = 1/3, \rho_s = 1000 \text{ kg/m}^3, \eta_s = 5\%\), \(E_f = 210 \text{ GPa}, \nu_f = 0.25, \eta_f = 2\%\) and \(t = 50 \text{ mm}\). To model a massless foundation \(\rho_f = 0\) for the lid of the caisson and \(\rho_f = \rho_s\) for the skirt.

The shape of the curve for high values of \(G_s\) (1000 MPa) is approaching the shape of the frequency dependent behaviour of the surface foundation. When \(G_s\) decreases, the local oscillations become more distinct and the influence of the skirt flexibility vanishes, i.e. the caisson reacts as a rigid foundation. Rigid behaviour can be assumed for \(G_s \leq 1.0 \text{ MPa}\).

2.6.3 Vertical dynamic stiffness—high-frequency behaviour

The variation of the dynamic stiffness due to a change in the skirt length \(H\) is presented in the following. The BE/FE models for the analysis are similar to the model shown in Figure 2.3b. The model properties are \(G_s = 1 \text{ MPa}, \nu_s = 1/3, \rho_s = 1000 \text{ kg/m}^3, \eta_s = 5\%\), \(E_f = 210 \text{ GPa}, \nu_f = 0.25, \eta_f = 2\%\) and \(t = 50 \text{ mm}\). Note that \(\rho_f = 0\) for the lid of the caisson and \(\rho_f = \rho_s\) for the skirt. In order to get a picture of the high frequency behaviour of the suction caisson, the analyses have been performed for the frequency range \(a_0 \in [0;12]\). The components of the vertical dynamic stiffness for \(H/D = 1/4, 1\) and 2 are shown in Figure 2.7.

The vertical dynamic stiffness of the caisson with a relatively small embedment depth \((H/D = 1/4)\) varies smoothly with the frequency, whereas the magnitude for \(H/D = 1\) and 2 is characterized by distinct peaks, and it can be observed that the magnitude of dynamic stiffness overall increases with the skirt length.

The normalized magnitude of the impedance is characterized by repeated oscillations with local extremes. However, the average dynamic stiffness, measured over a wide range

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of frequencies, appears to be increasing monotonously with increasing frequency, similar to the situation for the surface footing in Figure 2.7.

In order to formulate the high-frequency behaviour of foundations by lumped-parameter models (see Appendix C a dashpot is used to describe the high-frequency impedance. The high-frequency behaviour is characterized by a phase angle approaching $\pi/2$ for $a_0 \to \infty$ and a linear relation that passes through origo in a frequency vs. magnitude diagram. The slope of the curve is equal to a limiting damping parameter $C_{VV}^\infty$ that describes the impedance for $a_0 \to \infty$, which in the case of the surface footing is given by

$$C_{VV}^\infty = \rho_s c_P A_b,$$

where $A_b$ is the area of the base of the foundation. It should be noted that $C_{VV}^\infty$ in Equation (2.16) is highly sensitive to $\nu_s$ due to the fact that $c_P$ enters the equation. For that reason $c_P$ may be inappropriate, and Gazetas and Dobry (1984) suggest the use

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of Lysmer’s analog ‘wave velocity’ $c_{La}=3.4c_S/\pi(1-\nu_s)$. Wolf (1994) suggests another approach where $c_P$ for $\nu_s \in [1/3;0.5]$ is constant, and equal to $c_P$ at $\nu_s = 1/3$.

At high frequencies the wavelengths are small compared with the dimensions of the source (or the vibrating surface). Thus, the soil immediately below the vibrating surface of a smooth surface footing is only exposed to P-waves. However, the skirts of the suction caisson generate additional S-waves due to a vertical high-frequency excitation. For that reason, the limiting damping parameter $C_{VV}^\infty$ of the suction caisson consists of two contributions: one from the vibration of the lid and one originating from the vibration of the skirt. $C_{VV}^\infty$ of the suction caisson is then given by

$$C_{VV}^\infty = \rho_s c_P A_{lid} + 2\rho_s c_S A_{skirt},$$

where $A_{lid}$ and $A_{skirt}$ are the vibrating surface areas of the lid and the skirt, respectively. Note that S-waves are generated both inside and outside the skirt, hence the factor ‘2’.

Figure 2.7: Vertical dynamic stiffness: high frequency behaviour. $G_s = 1.0$ MPa, $\nu_s = 1/3$ and $\eta_s = 5\%$. 

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in the latter contribution in Equation (C.40). The discussion of the proper choice of $c_P$ also applies here.

In order to verify that the high-frequency behaviour of the suction caisson, described by (C.40), the damping term $c_{VV}$ in Equation (2.14) is compared with $C_{VV}^\infty$. This is carried out by means of the dimensionless damping coefficient $\tilde{c}_{VV}$, given by

$$\tilde{c}_{VV} = \frac{R}{\nu_s} c_{VV}(a_0)$$

At high frequencies $\tilde{c}_{VV}$ should tend towards unity if the expressions in Equations (2.16) and (C.40) hold true (Dobry and Gazetas 1986). In Figure 2.8 the dimensionless damping coefficient $\tilde{c}_{VV}$ is plotted for the suction caisson data together with the results for a surface footing. It is evident that the dimensionless damping coefficient of the surface footing tends towards unity as the frequency increases. With respect to the suction caisson the problem is somewhat more complex. The high-frequency behaviour contains an infinite number of resonance peaks as $a_0 \rightarrow \infty$. However, this behaviour cannot be quantified by one single damping parameter, so the coefficient in Equation C.40 reflects the average behaviour of high-frequency vibrations. It should be emphasized that the purpose of determining the high-frequency dashpot parameters in Equations (2.16) and (C.40) is to control the lumped-parameter model approximation of the high-frequency vibrations. Note that $c_P$ at $\nu_s = 1/3$ has been used in Equations (2.16) and (C.40).

Figure 2.8: Vertical dimensionless damping coefficient $\tilde{c}_{VV}$. $G_s = 1.0$ MPa, $\nu_s = 1/3$ and $\eta_s = 5\%$.
2.7 Discussion

There are several observations associated with the oscillations of the impedance of the suction caissons:

♦ The peaks of the normalized magnitude are located at phase angles equal to $\pi/2$.
♦ The distance between the peaks is approximately $\Delta a_0 = 3.0 - 3.5$.
♦ The amplitude of the peaks increases significantly with skirt length.

However, the appearance of distinct peaks in the magnitude of the stiffness around certain frequencies cannot be explained by the variation of skirt length, Poisson’s ratio and the flexibility of the skirt. The fact that the oscillations are repeated for equal distances in frequency suggests that the phenomenon is due to wave interference in the soil inside the suction caisson. Since the amplitude of the peaks significantly increases with skirt length, it seems reasonable to examine the axial impedance of an infinite cylinder, in order to study the wave interference inside the caisson.

The dynamic stiffness per unit length of an infinite cylinder subjected to dynamic vertical excitation in the axial direction is shown in Figure 2.9. The dynamic stiffness is computed for $\eta_s=0.00$, 0.05 and 0.10, and the data are represented by the normalized magnitude and the phase angle. The slope of the dashed line in Figure 2.9 is equal to the limiting damping parameter $C^{\infty}_{zz}$ per unit length of the infinite cylinder. Note that the vertical motion of the infinitely long cylinder only generates S-waves, i.e. there is no contribution of P-waves. The solution for the impedance of the infinite cylinder subjected to dynamic vertical excitation in the axial direction is given in Appendix A.

The similarities of the impedance in Figure 2.7 and 2.9 are remarkable. However, the normalized magnitudes are not to scale, but the patterns of the magnitude and phase angle of the suction caissons ($H \geq 1$) are equivalent to those of the infinite cylinder for $\eta_s = 0.05$. The closed-form solution to the vertical dynamic stiffness $S_{VV}(\omega)$ of the infinite cylinder is given by

$$S_{VV}(\omega) = \frac{K^0_{VV}}{R J_0(k_S R) K_0(i k_S R)}, \quad K^0_{VV} = 2\pi R G_s,$$

where $J_0$ is the Bessel function of the first kind and order 0, $K_0$ is the modified Bessel function of the second kind and order 0, whereas $k_S = \omega/c_S$ is the wavenumber of S-waves. Recall that $G_s$ is the shear modulus of the soil. Note that Equation (2.19) is given by Equations (A.7) and (A.11) in Appendix A. As reported by Kitahara (1984), $J_0(k_S R)$ has a number of zeros for $\eta_s = 0$ and $k_S > 0$. At the corresponding cyclic frequencies, $S_{VV}(\omega)$ becomes singular and the stiffness becomes infinite. These anti-resonance frequencies are marked in Figure 2.9 by the vertical lines with the dash-dot signature. The distance between the lines tends towards $\pi$ for $\omega \to \infty$. Thus, the $n$th anti-resonance mode occurs at the non-dimensional frequency $a_0 \to \pi(n - 1/4)$ for $n \to \infty$. 

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2.8 Conclusion

In this chapter the vertical dynamic soil–structure interaction of the suction caisson foundations for offshore wind turbines has been evaluated by means of a dynamic three-dimensional coupled Boundary Element/Finite Element model. Benchmark tests have been performed to determine the capability of estimating the vertical static stiffness of the suction caisson by the BE/FE formulation, and there is good agreement between the estimation of $K_{VV}^0$ by FE and BE/FE. Furthermore, the BE/FE model of the suction caisson has been used to reproduce the vertical dynamic stiffness of a surface footing. The results from the suction caisson model with ‘soil skirts’ match the results of the surface foundation model, and the model of the suction caisson is able to reproduce the frequency dependent behaviour of a surface foundation without introducing errors due to the complexity of the model.

The dynamic stiffness has been investigated for several different combinations of the mechanical properties of the soil–foundation system, and the following observations can be made:

- The vertical dynamic stiffness changes with the skirt length. For a relatively small embedment depth ($H/D = 1/4$) the impedance varies smoothly with the frequency,
whereas the impedance for $H/D = 1$ and 2 is characterized by distinct peaks.

- The dynamic stiffness is relatively insensitive to variations in $\nu_s$ in the range from 0.1 to 0.4. When $\nu_s$ approaches 0.5, the dynamic behaviour changes significantly due to the fact that $c_P/c_S \to \infty$ for $\nu_s \to 0.5$.

- The impedance for high values of $G_s$ (1000 MPa) approaches the shape of the frequency dependent behaviour of the surface foundation. When $G_s$ decreases, the local oscillations become more distinct and the influence of the skirt flexibility vanishes, i.e. the caisson reacts as a rigid foundation. Rigid behaviour can be assumed for $G_s \leq 1.0$ MPa.

Furthermore, the high-frequency behaviour of the suction has been investigated. Here, the main conclusions are:

- Generally the magnitude of the impedance increases with the skirt length.

- The normalized magnitude of the impedance is characterized by repeated oscillations with local extremes for $a_0 \in [0;12]$. However, the average dynamic stiffness appears to be increasing monotonously with increasing frequency, similar to the situation for the surface footing.

- The phase angle for the suction caissons oscillate around $\pi/2$ for $a_0 > 4$, and it will eventually stabilize at higher frequencies.

- A limiting damping parameter $C_{zz}^\infty$ that describes the impedance for $a_0 \to \infty$ has been determined for applications involving lumped-parameter model approximation of the high-frequency vibrations.

The repeated oscillations in the impedance of the suction caisson are due to resonance and anti-resonance of the soil inside the suction caisson. This is concluded by comparing the vertical impedance characteristics of the suction caisson to those of an infinite cylinder subjected to dynamic vertical excitation in the axial direction.

This chapter has been focused on the analysis of the vertical component of the dynamic stiffness matrix $S$ and the preliminary benchmark testing to ensure that the numerical model is valid and able to capture the dynamic behaviour of the suction caisson. The analysis of the coupled horizontal and moment loading and the torsional loading conditions will be examined in Chapter 3.
Chapter 3
Dynamic stiffness of suction caissons—torsion, sliding and rocking

This chapter concerns the dynamic soil–structure interaction of steel suction caissons applied as foundations for offshore wind turbines. An emphasis is put on torsional vibrations and coupled sliding/rocking motion, and the influence of the foundation geometry and the properties of the surrounding soil is examined. The soil is simplified as a homogenous linear viscoelastic material and the dynamic stiffness of the suction caisson is expressed in terms of dimensionless frequency-dependent coefficients corresponding to the different degrees of freedom. The dynamic stiffness coefficients for the skirted foundation are evaluated by means of a three-dimensional coupled boundary element/finite element model. Comparisons with known analytical and numerical solutions indicate that the static and dynamic behaviour of the foundation are predicted accurately with the applied model. The analysis has been carried out for different combinations of the skirt length and the Poisson’s ratio of the subsoil. Finally, the high-frequency impedance has been determined for future use in lumped-parameter models of wind turbine foundations in aero-elastic codes.

3.1 Introduction

Modern offshore wind turbines are flexible structures with resonance frequencies as low as 0.15 Hz. Typically, this is close to the excitation frequencies related to waves and turbine blades passing the tower. Thus a small change in the structural stiffness may result in great changes in the response, for which reason a reliable computation of the structural stiffness is required. This necessitates an accurate prediction of the soil–structure interaction which is highly dependent on the properties of the soil as well as the geometry of the foundation. A novel foundation method for offshore wind turbines is the monopod suction caisson (Houlsby et al. 2005). For this particular kind of foundation, the vertical component of the dynamic stiffness has been discussed in Chapter 2. By contrast, the focus of the present analysis is the impedance related to torsional vibrations and coupled sliding/rocking motion. The previous work related to the analysis of torsional vibrations and coupled sliding/rocking is briefly presented in Section 3.2. Subsequently, a definition of the static and dynamic stiffnesses for the suction caisson is provided in
Section 3.3. The analysis of the torsional dynamic stiffness of the suction caisson is presented in Section 3.4 and the results obtained by analysing the coupled sliding and rocking motion are given in Section 3.5. The main conclusions of the chapter are given in Section 3.6. In this chapter the impedance is equal to the dynamic stiffness of the foundation, i.e. the impedance contains both a real and an imaginary part. The frequency dependent dynamic stiffness of the suction caisson is evaluated in the frequency domain by means of the three-dimensional coupled Boundary Element/Finite Element Method program BEASTS by Andersen and Jones (2001a). The basic concepts of the method and the preliminary benchmark tests to ensure that the applied numerical model is able to capture the dynamic behaviour of the suction caisson are described in Chapter 2.

3.2 Previous work

Luco and Westmann (1971) investigated the torsional vibrations of a circular massless footing resting on a homogeneous elastic half-space. They solved the system as a mixed boundary value problem with prescribed conditions under the foundation and zero traction at the remaining free surface. The integral equations of the mixed boundary value problems were evaluated and tabulated for a number of excitation frequencies. The effects of material damping on torsionally excited footings were reported by Veletsos and Damodaran Nair (1974), while Wong and Luco (1985) presented tables of horizontal, coupling, rocking, vertical and torsional impedance functions for rigid massless square foundations resting on layered viscoelastic soil. The impedance functions for rigid square foundations embedded in a uniform elastic half-space have been evaluated by means of a hybrid approach by Mita and Luco (1989). Emperador and J. (1989) applied the boundary element method for analysis of the dynamic response of axisymmetric embedded foundations. Approximate closed-form solutions for the torsional impedance of circular embedded foundations have been reported by Novak and Sachs (1973) and Avilés and Pérez-Rocha (1996). The coupled sliding/rocking vibrations of surface footings have been reported by e.g. Veletsos and Wei (1971). This work will be used as the reference solution for the subsequent analyses of the coupled sliding/rocking vibrations of the suction caissons. Bu and Lin (1999) have summarized the work with respect to analyses of coupled sliding/rocking vibrations of foundations and further references will not be repeated here.

3.3 Static and dynamic stiffness formulation

A massless rigid foundation has six degrees of freedom: one vertical, two horizontal (sliding), two rocking and one torsional. The six degrees of freedom and the corresponding forces and moments are shown in Figure 3.1, and in the general case all components of displacement may be coupled. However, in the particular case of axisymmetric foundations there is only a coupling between the horizontal sliding and rocking motion. Thus, the vertical and torsional motion are completely decoupled from each other and from the remaining degrees of freedom. Furthermore, for a circular footing with the radius \( R \) it is advantageous to represent the relationship between displacements/rotations and forces/moments in a non-dimensional form. For harmonic excitation with the cyclic
3.4 Dynamic stiffness for torsional vibrations

Figure 3.1: Degrees of freedom for a rigid surface footing: (a) displacements and rotations, and (b) forces and moments.

Here \( G_s \) is the shear modulus of the soil which is complex if material damping is introduced (see Chapter 2 for details). The coupling terms, \( S_{HM} \) and \( S_{MH} \), are assumed to be equal. This assumption is discussed in Subsection 3.5.2. The normalized dynamic stiffness depends on the cyclic frequency \( \omega \), and Poisson’s ratio of the soil, \( \nu_s \). A formulation that is independent of the mass density of the soil, \( \rho_s \), may be obtained by the introduction of the dimensionless frequency \( a_0 = \omega R/c_S \), where \( c_S = \sqrt{G_s/\rho_s} \) denotes the shear wave velocity of the soil. The normalized components of the dynamic stiffness matrix given in Equation (3.1) can then be written as

\[
S_{ij}(a_0) = K_{ij}^0 |k_{ij}(a_0) + i a_0 c_{ij}(a_0)|, \quad (i, j = H, M, T, V), \tag{3.2}
\]

where \( K_{ij}^0 \) are the corresponding components of the static stiffness matrix and \( i = \sqrt{-1} \) is the imaginary unit. The dimensionless dynamic stiffness and damping coefficients, \( k_{ij} \) and \( c_{ij} \), are both real. Both geometrical damping, i.e., the radiation of waves into the subsoil, and possibly also material dissipation contribute to \( c_{ij} \). The stiffness representation provided in terms of real and imaginary parts tends to be inconclusive in some situations. Instead it is convenient to examine the magnitude \( |S_{ij}| \) and phase angle \( \phi_{ij} \) of Equation (3.2). These are defined as

\[
|S_{ij}| = |K_{ij}^0| \sqrt{(k_{ij})^2 + (a_0 c_{ij})^2}, \quad \phi_{ij} = \arctan \left( \frac{a_0 c_{ij}}{k_{ij}} \right). \tag{3.3}
\]

This representation of the dynamic stiffness will be applied throughout the chapter.

3.4 Dynamic stiffness for torsional vibrations

In this section the torsional dynamic stiffness is investigated. The Poisson’s ratio has no impact on the torsional stiffness, since torsional vibrations of the suction caisson only
produce shear waves. Hence, the analysis only concerns the variation of the normalized torsional stiffness due to a change in the skirt length $H$. The geometry is sketched in Figure 3.2a. This Section consists of four parts. Firstly, the Boundary Element/Finite Element (BE/FE) model applied in the analysis is described. Secondly, the static torsional stiffness obtained by the BE/FE model is presented and compared with results from a static finite element analysis in ABAQUS (Abaqus 2003). Thirdly, the dynamic stiffness for torsional vibrations is examined, and the last subsection presents the asymptotic impedance behaviour in the high frequency range.

### 3.4.1 Boundary Element/Finite Element model

The BE/FE model of the suction caisson is divided into four sections: a finite element section that forms the top of the foundation (the lid), a finite element section of the skirt, a boundary element domain inside the skirt and, finally, a boundary element domain outside the skirt that also forms the free ground surface. Whereas the lid is massless, the skirt has a mass density corresponding to that of the soil. This produces a model which is directly comparable to a massless surface footing. The skirt of the suction caisson is considered flexible, and the lid is assumed to be rigid. The lid is modelled as a solid finite element section with a thickness of one meter. The elements utilized in the present study are 9-noded quadrilateral boundary elements and 26-noded isoparametric finite elements—both with quadratic spatial interpolation. The model of the suction caisson and the subsoil contains approx. 100 finite elements and 350 boundary elements. The mesh of the free surface is truncated at a distance of 30 m ($\sim 6R$) from the centre of the foundation (see subsection 2.5.2). The connection between the soil and the foundation corresponds to the condition of ‘rough’ contact since the foundation and the surrounding soil have common degrees of freedom. The model is illustrated in Figure 3.2b. Due to geometrical symmetry, only half the foundation is included in the model (see subsection 2.5.2). In the case of torsion, antisymmetric load and response are assumed. The BE/FE analysis has been carried out for 40 equally spaced excitation frequencies in the range $a_0 \in$
3.4 Dynamic stiffness for torsional vibrations

Table 3.1: Static torsional stiffness for different skirt lengths

<table>
<thead>
<tr>
<th>$H/D$</th>
<th>$K^0_{TT}$ FE</th>
<th>$K^0_{TT}$ BE/FE</th>
<th>Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>12.94</td>
<td>13.15</td>
<td>-1.63 %</td>
</tr>
<tr>
<td>1</td>
<td>32.36</td>
<td>32.43</td>
<td>-0.23 %</td>
</tr>
<tr>
<td>2</td>
<td>56.88</td>
<td>53.90</td>
<td>+5.52 %</td>
</tr>
</tbody>
</table>

For each frequency a pair of opposing horizontal forces are applied at the bottom of the lid in order to create a torque. The resulting torsional response is computed, and the complex dynamic stiffness is then determined as the ratio between the applied moment and the resulting amplitude of the rotation. Note that load control has been used to generate the stiffness values. Displacement control would be more appropriate, but this feature is currently not available in the BE/FE software.

### 3.4.2 Static stiffness

The static torsional stiffness $K^0_{TT}$ has been computed for three different ratios between the foundation diameter, $D = 2R$, and the skirt length, $H$. In all cases, the soil properties are $G_s = 1$ MPa and $\nu_s = 1/3$. The foundation material (steel) has the Young’s modulus $E_f = 210$ GPa and the Poisson’s ratio $\nu_f = 0.25$. The foundation radius is $R = 5$ m and the skirt thickness is $t = 50$ mm. The material properties of the soil and the foundation are identified by the subscripts $s$ and $f$, respectively. The results obtained with the BE/FE program BEASTS are listed in Table 3.1 for $H/D = 1/4$, 1 and 2. A comparison is made with the finite element solution provided by a three-dimensional ABAQUS model (see subsection 2.5.1). As indicated by Table 3.1, the two numerical models provide similar results, indicating that both the ABAQUS and BEASTS models are nearly converged. The deviation is properly due to the fact that better convergence has been obtained by the FE solution.

### 3.4.3 Dynamic stiffness

The normalized torsional dynamic stiffness, $|S_{TT}|/K^0_{TT}$, is analysed for the three normalized skirt lengths, $H/D = 1/4$, 1 and 2, and in the normalized frequency range $\alpha_0 \in [0;10]$. A comparison is made with two reference solutions. Firstly, the normalized torsional dynamic stiffness has been found for a surface footing. This result has been obtained by means of a three-dimensional BE/FE model with no skirt, i.e. with $H = 0$. Secondly, the dynamic stiffness per unit length of an infinite hollow cylinder subjected to dynamic excitation is evaluated by means of the two-dimensional coupled BE/FE program TEA by Jones, Thompson, and Petyt (1999). The hollow cylinder is modelled with 64 quadrilateral finite elements employing quadratic interpolation. The interior and exterior soil domains are modelled with 64 boundary elements each. The model is sketched in Figure 3.3, and plane strain is assumed. In all the analyses, the soil has the shear modulus $G_s = 1$ MPa, the Poisson’s ratio $\nu_s = 1/3$, the mass density $\rho_s = 1000$ kg/m$^3$ and the loss factor $\eta_s = 5\%$. Hysteretic material damping in the soil is assumed, i.e. the
loss factor is assumed to be constant for all frequencies. The foundation has the Young’s modulus $E_f = 210$ GPa, the Poisson’s ratio $\nu_f = 0.25$, the loss factor $\eta_f = 2\%$ and the skirt thickness $t = 50$ mm. In order to model a massless foundation, the mass density is $\rho_f = 0$ for the lid of the caisson and $\rho_f = \rho_s$ for the skirt. As indicated by Figure 3.4, the normalized magnitudes of the torsional impedance are similar for the surface footing, the caissons and the infinite cylinder in the frequency interval $a_0 \in [0;2]$. Note that the actual magnitude of the impedance for each skirt length is scaled by the static values given in Table 3.1. For $a_0 > 2$ the impedance of all the skirted foundations are greater than the impedance of the surface footing. The dynamic stiffness of the caisson with a relatively small embedment depth ($H/D = 1/4$) varies smoothly with the frequency. However, the normalized magnitudes for $H/D = 1$ and 2 are characterized by distinct peaks close to $a_0 = 4, 7$ and 10. The peaks become more pronounced when the skirt length is increased, and the behaviour corresponds well to that of the infinite cylinder. Between the peaks, the normalized torsional impedances for all skirt lengths are nearly identical in magnitude. This is even the case for the infinite cylinder. However, $K_{TT}^0$ (and therefore also $|S_{TT}|$) is increased significantly with an increase in the skirt length, cf. Table 3.1. Further, the local peaks in the normalized magnitude are associated with a significant change in the phase angle, $\phi_{TT}$. The fact that the oscillations are repeated for equal distances in frequency implies that the frequencies at the local peaks correspond to anti-resonance modes of the soil inside the suction caisson. This behaviour is similar to the observed behaviour for vertical vibrations as presented in Chapter 2.

Figure 3.3: Infinite hollow cylinder (a) and two-dimensional BE/FE model (b) of the cylinder where $\Omega_i$ and $\Omega_o$ are the inner and outer boundary element domains, respectively.
3.4 Dynamic stiffness for torsional vibrations

Figure 3.4: Torsional impedance: variation of skirt length. $G_s = 1.0$ MPa, $\nu_s = 1/3$ and $\eta_s = 5\%$.

3.4.4 High-frequency limit

The limiting damping parameter $C_{TT}^\infty$ of the suction caisson consists of two contributions: one from the vibration of the lid and one originating from the vibration of the skirt, see Chapter 2. $C_{TT}^\infty$ of the suction caisson is given by

$$C_{TT}^\infty = \rho_s c_s J_{lid} + (2\rho_s c_s A_{skirt}) R^2, \quad (3.4)$$

where $J_{lid}$ is the polar moment of inertia of the lid about the axis of rotation, and $A_{skirt}$ is the surface area of skirt. Note that S-waves are generated both inside and outside the skirt, hence the factor ‘2’ in the latter contribution in Equation (3.4). The radius $R$ is the distance from skirt to the axis of rotation.
3.5 Dynamic stiffness for coupled sliding–rocking vibrations

In this section the coupled sliding–rocking vibrations are investigated for several different combinations of the mechanical properties of the soil–foundation system. The first case concerns the effects of Poisson’s ratio on the stiffness. The second analysis investigates the variation of the stiffness due to a change in the skirt length. Finally, the limiting damping parameters for vibration in the high-frequency range are given.

3.5.1 Boundary Element/Finite Element model

The geometry and the discretization in the BE/FE models employed for the present analyses are as described in the previous section. However, the load is applied differently. For a given excitation frequency, two analyses are performed: one analysis with horizontal loading at the base of the lid of the caisson, and one analysis with a set of opposing vertical forces that are applied at each side of the foundation in order to create a rocking moment. The first analysis provides a relation between the horizontal force and the resulting displacements and rotations. The second analysis relates the applied moment to the resulting displacements and rotations. The system can be written as a subset of Equation 3.1, given as

\[
\begin{bmatrix}
H_1/G_s R^2 \\
M_2/G_s R^3
\end{bmatrix} =
\begin{bmatrix}
S_{HH} & -S_{HM} \\
-S_{MH} & S_{MM}
\end{bmatrix}
\begin{bmatrix}
U_1/R \\
\theta M_2
\end{bmatrix}.
\]

(3.5)

The two equations are then solved simultaneously, in order obtain the complex horizontal sliding impedance, \(S_{HH}\), the rocking moment impedance, \(S_{MM}\), and the coupling impedances, \(S_{HM}\) and \(S_{MH}\). As already mentioned and further discussed below, \(S_{HM} = S_{MH}\) within the precision of the model.

3.5.2 Static stiffness

The static stiffness coefficients of the coupled system have been determined by the BE/FE models for \(a_0 = 0.01\), and then compared with the results of static finite element analyses in ABAQUS. The non-dimensional values of \(K^0_{HH} \), \(K^0_{MM} \), \(K^0_{HM} \) and \(K^0_{MH}\) are given for two different cases:

Different skirt lengths: – The static stiffness components are given for various ratios between the foundation diameter \(D\) and the length of the skirt \(H\) in Table 3.2. The soil properties are \(G_s = 1\) MPa and \(\nu_s = 1/3\).

Different Poisson’s ratios: – The variation of static stiffness with respect to Poisson’s ratio is shown in Table 3.2. \(H/D = 1\) and \(G_s = 1\) MPa.

Note that the values in parentheses in Table 3.2 are obtained by the static finite element analyses in ABAQUS. The data are shown for fixed material properties of the foundation \((E_f = 210\) GPa, \(\nu_f = 0.25\)). The foundation radius is \(R = 5\) m and the skirt thickness is \(t = 50\) mm. In addition to the analyses listed above, it may be relevant to check the influence of the skirt flexibility. However, a preliminary study indicates that changes

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3.5 Dynamic stiffness for coupled sliding–rocking vibrations

Table 3.2: Coupled static stiffness.

<table>
<thead>
<tr>
<th>( H/D )</th>
<th>( K_{HM}^0 )</th>
<th>( K_{MM}^0 )</th>
<th>( K_{HM}^0 )</th>
<th>( K_{MH}^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>8.00 (7.47)</td>
<td>8.51 (8.41)</td>
<td>-3.13 (-2.68)</td>
<td>-2.78 (-2.68)</td>
</tr>
<tr>
<td>1</td>
<td>13.92 (12.98)</td>
<td>52.91 (49.73)</td>
<td>-18.28 (-16.11)</td>
<td>-17.20 (-16.12)</td>
</tr>
<tr>
<td>2</td>
<td>18.61 (18.47)</td>
<td>198.87 (193.41)</td>
<td>-44.80 (-43.02)</td>
<td>-43.54 (-43.12)</td>
</tr>
</tbody>
</table>

\( \nu_s = 0.1 \)

<table>
<thead>
<tr>
<th>( H/D )</th>
<th>( K_{HM}^0 )</th>
<th>( K_{MM}^0 )</th>
<th>( K_{HM}^0 )</th>
<th>( K_{MH}^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>12.49 (11.62)</td>
<td>49.75 (46.91)</td>
<td>-17.11 (-15.19)</td>
<td>-16.09 (-15.21)</td>
</tr>
<tr>
<td>1</td>
<td>13.01 (12.14)</td>
<td>50.83 (47.92)</td>
<td>-17.50 (-15.53)</td>
<td>-16.47 (-15.55)</td>
</tr>
<tr>
<td>2</td>
<td>13.92 (12.98)</td>
<td>52.91 (49.73)</td>
<td>-18.28 (-16.11)</td>
<td>-17.20 (-16.12)</td>
</tr>
</tbody>
</table>

\( \nu_s = 0.2 \)

<table>
<thead>
<tr>
<th>( H/D )</th>
<th>( K_{HM}^0 )</th>
<th>( K_{MM}^0 )</th>
<th>( K_{HM}^0 )</th>
<th>( K_{MH}^0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>14.54 (13.53)</td>
<td>54.42 (51.02)</td>
<td>-18.86 (-16.54)</td>
<td>-17.75 (-16.53)</td>
</tr>
<tr>
<td>1</td>
<td>15.74 (14.51)</td>
<td>57.79 (53.98)</td>
<td>-20.19 (-17.42)</td>
<td>-18.95 (-17.39)</td>
</tr>
</tbody>
</table>

The dynamic stiffness for different Poisson’s ratios is presented in this section. The skirt length is fixed \((H/D = 1)\), and the model properties are: \( G_s = 1.0 \) MPa, \( \rho_s = 1000 \) kg/m\(^3\), \( \eta_s = 5\% \), \( E_f = 210 \) GPa, \( \nu_f = 0.25 \), \( \eta_f = 2\% \) and \( t = 50 \) mm. In order to model a massless foundation \( \rho_f = 0 \) for the lid of the caisson and \( \rho_f = \rho_s \) for the skirt. In Figures 3.5–3.7, the results are shown for five different values of Poisson’s ratio and for the frequency range \( a_0 \in [0;6] \). Note that the range in Poisson’s ratio is thought to cover fully drained \((\nu_s = 0.1 – 0.2)\) to undrained \((\nu_s = 0.495)\) conditions. The analytical solution for a surface footing proposed by Veletsos and Wei (1971) is included as reference. Two numerical models of a massless surface footing are included for comparison with the analytical solution. The sliding and rocking impedance of the surface footing have been determined by a BE/FE model. In the case of the coupling between horizontal sliding and rocking, numerical experiments indicate that convergence of the impedance cannot be established with a reasonably low number of degrees of freedom in the BE/FE model. In particular it has been found that both the magnitude and the phase of the impedance is strongly dependent on the distance from the footing to the truncation edge of the free ground surface. Adaptive meshing could possibly improve the accuracy versus the
number of degrees of freedom, but this facility is currently not available in the BE/FE software. Therefore, instead of the coupled BE/FE model based on the Green’s function for the full-space, an alternative method proposed by Andersen and Clausen (2005) has been applied. Here the solution is established in the wavenumber domain, and the fundamental solution for a half-space is employed. Moreover, the impedance is computed directly by integration of the interaction forces between the footing and the subsoil. This is in contrast to the BE/FE approach, in which the impedance is found by inversion of the dynamic flexibility matrix. The latter approach may involve great inaccuracies with respect to the coupling term since $|S_{HM}|$ is much smaller than $|S_{HH}|$ and $|S_{MM}|$, in particular in the high-frequency range.

The sliding and rocking impedances are clearly dependent on Poisson’s ratio. The frequency at the first local extremum in the magnitude of the impedance in Figures 3.5 and 3.6 changes significantly with Poisson’s ratio. The first peak for $\nu_s = 0.1$ occurs at

Figure 3.5: Sliding impedance: variation of Poisson’s ratio. $G_s = 1.0 \text{ MPa}$ and $\eta_s = 5\%$. 

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Figure 3.6: Rocking impedance: variation of Poisson’s ratio. \( G_s = 1.0 \text{ MPa} \) and \( \eta_s = 5\% \).

\( a_0 = 3.2 \), whereas the first peak for \( \nu_s = 0.4 \) is placed close to \( a_0 = 4.5 \). However, the second local extremum is found at the frequency \( a_0 = 5.5 - 5.7 \) for all values of Poisson’s ratio. This behaviour is explained by the fact that sliding and rocking impedances are governed by both shear wave propagation and compression wave propagation. More specifically, the first peak in the response corresponds to antiresonance of P-waves inside the caisson, whereas the second peak corresponds to antiresonance of S-waves. The latter is independent of the Poisson’s ratio whereas an increase in \( \nu_s \) involves an increase in \( c_P \). Hence, the first peak in Figures 3.5–3.7 occurs at lower frequencies for lower Poisson’s ratios.

The coupling impedance in Figure 3.7 follows the pattern of the horizontal and moment impedances. Hence, an increase in the frequency provides an increase in the magnitude of the coupling impedance over the normalized frequency range \( a_0 \in [0;6] \). It is noted that the phase angle of the coupling impedance is close to \( \pi \) radians for \( a_0 = 0 \).
and slightly increasing with the frequency in the range \( a_0 \in [0;6] \). Accordingly the static stiffness components \( K_{HM}^0 \) and \( K_{MH}^0 \) are negative, see Table 3.2. It is generally observed that the coupling impedances of the suction caisson and the surface footing behave differently. Thus, in the case of the surface footing a decrease of both the magnitude and the phase of the coupling impedance with frequency is recorded in the interval \( a_0 \in [0;6] \).

A few remarks on the impedance of the surface footing: The sliding and rocking impedance determined by the BE/FE model agrees very well with the analytical solution reported by Veletsos and Wei (1971). Furthermore, the coupling terms obtained by the alternative method (Andersen and Clausen 2005) is consistent with the coupling reported by Veletsos and Wei (1971). Note that the analytical solution with respect to the coupling term is an approximation, due to fact that the boundary conditions in the interface between the soil and the footing are partly relaxed. Finally, it is emphasized that the problem of determining the coupling between horizontal sliding and rocking

Figure 3.7: Coupling impedance: variation of Poisson’s ratio. \( G_s = 1.0 \) MPa and \( \eta_s = 5\% \).
3.5 Dynamic stiffness for coupled sliding–rocking vibrations

Figure 3.8: Sliding impedance: variation of skirt length. \( G_s = 1.0 \text{ MPa}, \nu_s = 1/3 \) and \( \eta_s = 5\% \).

only is encountered for the surface footing. The coupling between horizontal sliding and rocking for the suction caisson is described satisfactorily by the BE/FE model.

3.5.4 Dynamic stiffness—variation of skirt length

The variation of the coupled dynamic stiffness components with respect to a change in the skirt length \( H \) is presented in the following. The model properties are \( G_s = 1 \text{ MPa}, \nu_s = 1/3, \rho_s = 1000 \text{ kg/m}^3, \eta_s = 5\%, E_f = 210 \text{ GPa}, \nu_f = 0.25, \eta_f = 2\% \) and \( t = 50 \text{ mm} \). Again, \( \rho_f = 0 \) for the lid of the caisson and \( \rho_f = \rho_s \) for the skirt in order to model a massless foundation. The magnitudes and the phase angles of the impedance for \( H/D = 1/4, 1 \) and 2 are shown in Figures 3.8–3.10 for the frequency range \( a_0 \in [0;12] \). The magnitudes are normalized with respect to the static stiffness coefficients listed in Table C.1, and the results achieved with two numerical models of a massless foundation.
The horizontal sliding impedance of an infinitely long hollow cylinder \((H/D = \infty)\) has been computed by application of the two dimensional BE/FE code TEA as described in Subsection 3.4.3 for the case of torsional vibrations. Evidently, a similar two-dimensional analysis cannot be performed for the rocking and coupling impedances. With reference to Figure 3.8, there is no indication of antiresonance of the waves inside the caisson with a relatively small embedment depth \((H/D = 1/4)\), i.e. there are no local peaks in the normalized magnitude of the impedance component for sliding. Thus the dynamic behaviour is similar to that of the surface footing, though the increase of the impedance with increasing frequency is more pronounced for the skirted foundation than the surface footing. However, the sliding impedances for \(H/D = 1\) and 2 are characterized by a number of local tips and dips. The peaks are not repeated with the normalized frequency interval \(\Delta a_0 = \pi\). This is the case for the vertical and torsional impedances, where
the location of the peaks are governed by the shear waves only. In contrast to this, the location of the peaks for the coupled sliding–rocking impedances are controlled by antiresonance of both shear waves and compression waves. Clearly, the locations of the peaks in the magnitude of the sliding impedance for \( H/D = 1 \) and 2 correspond to those for the infinitely long cylinder. Likewise, the variation of the phase angle \( \phi_{HH} \) is similar for \( H/D = 1, 2 \) and \( \infty \), cf. Figure 3.8. The magnitude of the horizontal impedance (Figure 3.8) seems to increase with skirt length. However, the change from \( H/D = 1/4 \) to \( H/D = 1 \) is significant, whereas only a small change is observed from \( H/D = 1 \) to \( H/D = 2 \). The magnitude of the impedance for \( H/D = 2 \) is actually below the impedance for \( H/D = 1 \) at high frequencies. This behaviour suggests that the horizontal vibrations are transmitted to the surrounding soil at relatively shallow depths. Hence, the effects of increasing the skirt length diminish with depth. This is not the case for the moment impedance in Figure 3.9, where the effects of increasing the skirt length

\[ G_s = 1.0 \text{ MPa}, \; \nu_s = 1/3 \text{ and } \eta_s = 5\%. \]
Chapter 3 – Dynamic stiffness of suction caissons—torsion, sliding and rocking

enlarge with depth. These tendencies are also evident in the static stiffness coefficients listed in Table 3.2. Finally, the coupling impedance in Figure 3.10 increases moderately with an increase of the skirt length, and again the phase angle is close to \( \pi \) radians for all frequencies. Otherwise, the overall response is similar to the horizontal and moment impedances.

3.5.5 High-frequency limit

In this subsection the high-frequency behaviour is formulated by limiting damping parameters (coefficients of a dashpot) with the intention of use in lumped-parameter models (see Appendix C. The total geometrical damping is equal to the sum of the waves radiating from the skirts and the lid of the caisson. The limiting damping parameter for the horizontal vibration (\( C_{HH}^\infty \)) consists of three contributions: shear waves radiating from the lid, shear waves radiating from the skirt parallel to the direction of loading, and compression waves radiating from the skirt perpendicular to the direction of loading. The high-frequency impedance for the rocking and coupling terms consist of similar contributions, see (Bu and Lin 1999; Gazetas and Dobry 1984; Gazetas and Tassoulas 1987; Fotopoulos et al. 1989; Wolf and Paronesso 1992) for further details. Assuming that both the lid and the skirts of the suction caisson are rigid, the limiting damping parameters \( C_{HH}^\infty, C_{MM}^\infty \) and \( C_{HM}^\infty \) of the suction caisson are given by

\[
C_{HH}^\infty = \rho_s c_S \pi R^2 + 2 \rho_s c_P \pi R H + 2 \rho c_P \pi R H, \tag{3.6a}
\]

\[
C_{MM}^\infty = \rho_s c_P \pi R^4 + 2 \rho_s c_P \frac{1}{3} \pi R H^3 + 2 \rho_s c_S \frac{1}{3} \pi R H^3 + 2 \rho_s c_S \pi R^3 H, \tag{3.6b}
\]

\[
C_{HM}^\infty = -2 \rho_s c_S \frac{1}{2} \pi R H^2 - 2 \rho_s c_P \frac{1}{2} \pi R H^2 = C_{MH}^\infty. \tag{3.6c}
\]

Note that waves radiate from both inside and outside the skirts, hence the factor ‘2’ in front of the appropriate contributions in Equations 3.6a–3.6c.

3.6 Conclusion

The impedance of suction caissons with respect to torsional vibrations and coupled sliding–rocking vibrations has been analysed numerically, employing a three-dimensional coupled Boundary Element/Finite Element model in the frequency domain.

3.6.1 Torsional vibrations

The torsional dynamic stiffness has been analysed with respect to the variation of the stiffness due to a change in the skirt length \( H \). The main conclusions are:

- The static torsional stiffness, \( K_{TT}^0 \), obtained with the BE/FE model has been compared with the results from a finite element analysis. There is good agreement between the estimations of \( K_{TT}^0 \) provided by the two methods with a maximum deviation of 5.52%.

- The magnitude of the static and dynamic torsional stiffness increases with skirt length.

Morten Liingaard
3.6 Conclusion

♦ The torsional impedance of the suction caisson with a relatively small embedment depth \((H/D = 1/4)\) varies smoothly with the frequency, whereas the torsional impedances for \(H/D = 1\) and 2 are characterized by distinct peaks in the normalized magnitude close to \(a_0 = 4, 7\) and 10.

♦ The oscillations are repeated for equal distances in frequency, corresponding to anti-resonance modes in the soil inside the suction caisson.

♦ The torsional impedance of the suction caisson has been compared with the impedance of an infinite cylinder subjected to a torsional moment. The change with frequency in the magnitude and the phase angle of the impedance are equivalent for the suction caisson and the infinite cylinder.

3.6.2 Coupled sliding–rocking vibrations

The impedance of the coupled sliding–rocking vibrations have been analysed with respect to the effects of Poisson’s ratio and the skirt length. The following conclusions can be made:

♦ The static stiffness has been calculated with a BE/FE model and a finite element model. The largest deviation of the results of the two models are 7.4%, 7.2% and 16.8% for the sliding, rocking and coupling terms, respectively.

♦ The two coupling terms between sliding and rocking are equal, i.e. \(K^0_{HM} = K^0_{MH}\), within the accuracy of the analysis. The maximum deviation between \(K^0_{HM}\) and \(K^0_{MH}\) is 11%.

♦ The sliding and rocking impedances are clearly dependent on the Poisson’s ratio of the soil, and the local extremum in the magnitude of the impedance changes significantly with Poisson’s ratio.

♦ The effects of increasing the skirt length diminish with depth with respect to the horizontal impedance. The effects of increasing the skirt length enlarge with depth with respect to the rocking impedance and the sliding–rocking coupling components.

♦ The coupled sliding–rocking impedances are characterized by a complex wave interference pattern in the soil inside the skirts. The local peaks in the magnitude of the impedance components are not repeated by \(\Delta a_0 = \pi\), which is the case for the vertical and torsional impedance components. The location of the peaks for the coupled sliding–rocking impedances are controlled by anti-resonance of both shear waves and compression waves.

♦ The analysis of the horizontal impedance for an infinite hollow cylinder clearly shows the anti-resonance frequencies of both shear waves and compression waves for the vibrating cylinder. The results agree very well with the horizontal impedance of the suction caissons.

Finally, it is noted that the high-frequency limits of the impedance components have been established for the skirted foundation. These will be applied in combination with the low-frequency impedances obtained with the BE/FE models in future formulations of lumped-parameter models of suction caissons.

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Chapter 4

Prototype bucket foundation for wind turbines—natural frequency estimation

The first full scale prototype bucket foundation for wind turbines has been installed in October 2002 at Aalborg University offshore test facility in Frederikshavn, Denmark. The suction caisson and the wind turbine have been equipped with an online monitoring system, consisting of 15 accelerometers and a real-time data-acquisition system. The chapter concerns the in service performance of the wind turbine, with focus on estimation of the natural frequencies of the structure/foundation. The natural frequencies are initially estimated by means of experimental Output-only Modal analysis. The experimental estimates are then compared with numerical simulations of the suction caisson foundation and the wind turbine. The numerical model consists of a finite element section for the wind turbine tower and nacelle. The soil-structure interaction of the soil-foundation section is modelled by lumped-parameter models capable of simulating dynamic frequency dependent behaviour of the structure-foundation system.

4.1 Introduction

The continuous development of wind turbine technology has resulted in great increases in both size and performance of the wind turbines during the last 25 years. The power output of wind turbines has improved by larger rotors and more powerful generators. In order to reduce the costs, the overall weight of the wind turbine components is minimized, meaning that the wind turbine structures are becoming more flexible and thus more sensitive to dynamic excitation. A modern offshore wind turbine (1.5 to 2 MW) is typically installed with a variable speed system so the rotational speed of the rotor varies from, for example, 10–20 RPM. This means that the excitation frequency of the rotor system varies. The first excitation frequency interval then becomes 0.17–0.33 Hz (for 10–20 RPM) and is referred to as the 1Ω frequency interval. The second excitation frequency interval corresponds to the rotor blade frequency that depends on the number of blades. For a three-bladed wind turbine the 3Ω frequency interval is equal to 0.5–1.0 Hz (for 10–20 RPM). Since the first resonance frequency ω₁ of the modern offshore wind turbines is placed between 1Ω and 3Ω, it is of utmost importance to be able to evaluate
the resonance frequencies of the wind turbine structure accurately as the wind turbines increase in size. At present, the wind turbine foundations are modeled simply by beam elements or static soil springs, which means that the foundation stiffness is frequency independent.

The purpose of this chapter is to investigate the natural frequencies of the Vestas 3.0 MW offshore wind turbine. The first part of this chapter concerns experimental estimation of the natural frequencies by means of experimental modal analysis of the structure. In the second part of this chapter, the natural frequencies are evaluated by a finite element model. The nacelle and wind turbine tower are modeled by two-dimensional beam members, and the soil-structure interaction is modeled by two types of foundation models. In the first approach, the soil–structure interaction is modeled by static springs for each degree of freedom at the foundation node. In the second approach, the frequency dependent behaviour of the structure-foundation system is taken into consideration by applying so-called lumped-parameter models.

It should be emphasized that the intention is to demonstrate an experimental and a numerical approach for estimating the response of the wind turbine. The discipline of finite element model updating is not considered. See e.g. Friswell and Mottershead (1995) and Datta (2002) regarding this topic.

4.1.1 The prototype in Frederikshavn

The suction caisson (also known as bucket foundation) is a relatively new type of foundation used to support offshore structures, see Houlsby et al. (2005). The concept has been developed over the past 5 years and has been utilized for the Vestas V90 3.0 MW offshore wind turbine at Aalborg University offshore test facility in Frederikshavn, Denmark. The concept is sketched in Figure 4.1.

In the initial phase of the installation process the skirt penetrates into the seabed due to the weight of the structure. In the second phase suction is applied to penetrate the skirt to the design depth. After installation the foundation acts a hybrid of a pile and a gravity based foundation. The stability of the foundation is ensured by a combination of earth pressures on the skirt and the vertical bearing capacity of the bucket. This foundation type is a welded steel structure and the fabrication/material costs are comparable to those of the monopile foundation concept. The installation phase does not require heavy pile hammers and the decommissioning is a relatively simple process where the foundation can be raised by applying pressure to the bucket structure.

The prototype of the suction caisson in Frederikshavn is designed with a diameter of 12 m and a skirt length of 6 m. The weight of the suction caisson is approx. 140 tons. The overall properties of the wind turbine is summarized in Table 4.1, see Vestas (2006) for further details. The foundation was placed late October 2002, and the actual installation period lasted approx. 12 hours. Det Norske Veritas (DNV) has certified the design of the prototype in Frederikshavn to B level. The turbine was installed on the foundation in December 2002. The design procedure for the prototype bucket foundation has been described in details by Ibsen et al. (2005).
Table 4.1: Properties of the Vestas V90 3.0 MW wind turbine

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hub height</td>
<td>80 m</td>
</tr>
<tr>
<td>Rotor diameter</td>
<td>90 m</td>
</tr>
<tr>
<td>Nominal revolutions</td>
<td>16.1 rpm</td>
</tr>
<tr>
<td>Operational interval</td>
<td>8.6-18.4 rpm</td>
</tr>
<tr>
<td>Weight nacelle</td>
<td>70 t</td>
</tr>
<tr>
<td>Weight rotor</td>
<td>41 t</td>
</tr>
<tr>
<td>Weight tower</td>
<td>160 t</td>
</tr>
</tbody>
</table>

Figure 4.1: The wind turbine on the bucket foundation (a). The levels indicate location of accelerometers. The overall geometry of the bucket foundation (b).
4.2 Experimental estimation of natural frequencies

The natural frequencies of the wind turbine have been estimated experimentally by means of experimental modal analysis of the structure. The monitoring system and analysis software are briefly introduced and the modal parameters are then presented. The experimental estimation technique is used to examine the natural frequencies for three various situations. These are:

♦ Idle conditions
♦ Wind turbine without wings
♦ Wind turbine without wings and nacelle

4.2.1 Modal identification technique

In this subsection the monitoring system, the analysis software, and the procedure for modal identification are introduced.

Monitoring system

The Vestas 3.0 MW prototype wind turbine is instrumented with 15 accelerometers and a real-time data-acquisition system. The sensors are Kinematics force balance accelerometers, model FBA ES-U (Kinematics 2002). The specifications are listed in Table 4.2. The accelerometers are placed at four different levels, three in the wind turbine tower and one in the compartments inside the bucket foundation, see Figure 4.1a. The positions, measuring directions and numbering are shown in Figure 4.2. The accelerometers are mounted on consoles that are attached to the steel structure by magnets. The online monitoring system consists of a DigiTexx PDAQ-8 portable data acquisition system with 16 channels and 16 bit resolution. The remote portable data acquisition system is placed inside the wind turbine and the DigiTexx RTMS-2001R Remote Client Software is used for real time data acquisition and monitoring at Aalborg University. The performance of the wind turbine is also monitored online by live web imaging.

<table>
<thead>
<tr>
<th>Table 4.2: Specifications of accelerometer</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Type:</strong></td>
</tr>
<tr>
<td><strong>Model:</strong></td>
</tr>
<tr>
<td><strong>Dynamic range:</strong></td>
</tr>
<tr>
<td><strong>Bandwidth:</strong></td>
</tr>
<tr>
<td><strong>Full-scale range:</strong></td>
</tr>
<tr>
<td><strong>Outputs:</strong></td>
</tr>
<tr>
<td><strong>Operating Temperature:</strong></td>
</tr>
</tbody>
</table>

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4.2 Experimental estimation of natural frequencies

[Diagram showing accelerometer locations at four levels: Level I: Seabed, Level II: 7 m above seabed, Level III: 40 m above seabed, Level IV: 83 m above seabed. Arrows indicate positive measuring direction.

Figure 4.2: Location of accelerometers at the four levels. The arrows indicate positive measuring directions. Accelerometer no. 1, 4 and 6 are mounted vertical with an upward measuring direction as positive.

Analysis software

The modal analysis of the wind turbine makes use of "Output-only modal identification" which is utilized when the modal properties are identified from measured responses only. The experimental modal analysis of the wind turbine prototype is performed by means of the software package ARTeMIS—Ambient Response Testing and Modal Identification Software (SVS 2006). The software is fully compatible with the hardware of the monitoring system described above. The software allows accurate modal identification under operational conditions and in situations where the structure is impossible or difficult to excite by externally applied forces. The typical outputs of the analyses are modal information about the natural frequencies, mode shapes and damping ratios. The modal analysis within this software is based on the assumptions that the underlying physical system of the structure is linear and time-invariant. The linearity imply that the physical system comply with the rules of linear superposition. The time-invariance implies that the underlying mechanical or structural system does not change in time. Within this

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frame the program is based on two different estimation techniques, one in time domain and one in frequency domain. The analyses described in this chapter are based on the frequency domain technique. The frequency domain estimation is a non-parametric model based on a Frequency Domain Decomposition (FDD) method. The FDD method is an extension of the well-known frequency domain approach, which is based on mode estimations directly from the Power Spectral Density (PSD) matrix, i.e. well separated modes can be identified at the peaks of the PSD matrix. The basic principle of the Frequency Domain Decomposition (FDD) technique is to perform an approximate decomposition of the system response into a set of independent single degree of freedom (SDOF) systems; each corresponding to an individual mode. In the FDD the Spectral Density matrix is decomposed by means of the Singular Value Decomposition (SVD) into a set of auto spectral density functions, each corresponding to a single degree of freedom system. The key feature is that the singular values are estimates of the Auto Spectral density of the SDOF systems, and the singular vectors are estimates of the mode shapes. The basic theory concerning identification by FDD is presented in appendix. For references, see (Brincker, Andersen, and Zhang 2000; Brincker, Zhang, and Andersen 2000).

**Modal identification procedure**

The natural frequencies of the wind turbine have been determined on a regular basis during the last three years of operation. The natural frequencies are estimated for idle conditions only, in order to avoid interference caused by rotating components of the wind turbine. The mode estimation for operational conditions is more complex, and requires information about all the possible "forced harmonic modes" from e.g. gears, generators, rotors and pitch systems. Furthermore, it should be noted that the structural system of an operational wind turbine is time-varying. Thus, errors are introduced in the modal identification, because the framework of the modal estimation relies on the assumptions that the underlying physical system of the structure is linear and time-invariant. In order to obtain reliable data for the modal analysis, the length of each time series corresponds to 1000 times the first natural period of the structure. The first natural frequency is approximately 0.3 Hz, which equals a first natural period of 3.3 seconds. Consequently, the length of data acquisition should be at least 3300 seconds. Finally, the FDD method has been applied for identifying the natural frequencies of the wind turbine.

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4.2 Natural frequencies for idle conditions

4.2.2 Natural frequencies for idle conditions

The singular values of the spectral density matrices determined by the Frequency Domain Decomposition method are given in Figure 4.3. When the wind turbine is stopped the structure is subjected to ambient excitation from the wind. The measured data used in the analysis was recorded February 15, 2005. The data set consists of a 1 hour measurement in 15 channels. The sampling frequency was 200 Hz and the data was decimated by an order of 20. The FDD technique was used for peak picking.

In Figure 4.3 the peaks for the first and second mode of the structure are shown. Note that there are closely spaced modes at the selected frequencies, which implies that there are two perpendicular modes at each natural frequency. The first resonance frequency is equal to 0.30 Hz and the second is 2.13 Hz. The peaks between the first and second mode of the wind turbine correspond to the resonance frequencies for the blades, i.e. the first modes of flap-wise and edgewise vibrations. The peak at 2.93 Hz appears to be a torsional mode of the structure.

4.2.3 Natural frequencies for wind turbine without wings

In the spring 2005 the nacelle of the wind turbine was replaced with a newer prototype version. In Figure 4.5 the wings have been removed prior to the replacement of the nacelle. During the period where the wings were removed, several data acquisition sequences have been performed. Figure 4.4 shows a representative plot of the singular values of the spectral density matrices for the wind turbine without wings. The measured data was recorded the March 21, 2005. The data set consists of a 1 hour measurement in
Figure 4.4: Singular values of the spectral density matrices determined by the Frequency Domain Decomposition method—without wings.

Figure 4.5: Replacement of nacelle in the spring 2005.
4.2 Experimental estimation of natural frequencies

Figure 4.6: Singular values of the spectral density matrices determined by the Frequency Domain Decomposition method—without wings and nacelle.

15 channels. The sampling frequency was 200 Hz and the data was decimated by an order of 20. In Figure 4.4 there are closely spaced modes at the selected frequencies, which again suggest two perpendicular modes at each natural frequency. The first resonance frequency is equal to 0.33 Hz and the second is 2.10–2.14 Hz.

Note that the local peaks (resonance frequencies for the blades) between the first and second mode have disappeared. Furthermore, the resonance frequency of the torsional mode has increased from 2.93 Hz to 3.43 Hz. The sharp peaks at 2 Hz and 4 Hz are forced harmonic vibrations, probably due to maintenance.

4.2.4 Natural frequencies for wind turbine without wings and nacelle

Figure 4.6 shows the singular values of the spectral density matrices for the wind turbine without wings and nacelle. The measured data was recorded the May 11, 2005. The data set consists of a 30 minutes measurement in 15 channels. The sampling frequency was 200 Hz and the data was decimated by an order of 20. The first and second natural frequency of the structure has changed significantly after the nacelle was removed. The first resonance frequency is equal to 0.72 Hz and the second is 2.88 Hz.

Subsequent experimental modal analyses show that the first and second natural frequency are equal to 0.29 Hz and 2.11 Hz, respectively. Thus, the replacement of the nacelle and wings resulted in marginal change of the natural frequencies of the structure.
4.3 Numerical estimation of natural frequencies

The natural frequencies of the wind turbine are estimated numerically by means of a Finite Element model of the wind turbine and the suction foundation. The soil–structure interaction of the structure is taken into account by means of two different approaches: static springs and frequency dependent lumped-parameter models.

Initially, the Finite Element (FE) model of the wind turbine is described. Secondly, the concepts of the two foundation models are briefly introduced, and thirdly, the natural frequencies of the wind turbine are estimated by the numerical model.

4.3.1 Finite Element model of the wind turbine

The finite element model of the wind turbine tower and the nacelle consists of two-dimensional beam members with three degrees of freedom for each node. The model properties of the finite element model are summarized in Table 4.3. The wind turbine tower is discretized by 31 linear elastic beam elements with varying length and section properties. The nacelle and rotor are modelled as point masses. The inertia of the blades is added as a mass moment of inertia in the rotor node. The finite element model is illustrated in Figure 4.7.

4.3.2 Foundation models

In the case of axisymmetric foundations there is only a coupling between the horizontal sliding and rocking motion. Thus, the vertical and torsional motion are completely decoupled from each other and from the remaining degrees of freedom. In this analysis,

<table>
<thead>
<tr>
<th>Property</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of elements</td>
<td>$N_{el}$</td>
</tr>
<tr>
<td>Number of nodes</td>
<td>$N_n$</td>
</tr>
<tr>
<td>Number of dofs</td>
<td>$N_{dof}$</td>
</tr>
<tr>
<td>Young’s modulus</td>
<td>$E_t$</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>$\nu_t$</td>
</tr>
<tr>
<td>Mass density</td>
<td>$\rho_t$</td>
</tr>
<tr>
<td>Loss factor</td>
<td>$\eta_t$</td>
</tr>
<tr>
<td>Section area</td>
<td>$A_t$</td>
</tr>
<tr>
<td>Section area moment of inertia</td>
<td>$I_t$</td>
</tr>
<tr>
<td>Point mass—nacelle</td>
<td>$m_{nacelle}$</td>
</tr>
<tr>
<td>Point mass—rotor</td>
<td>$m_{rotor}$</td>
</tr>
<tr>
<td>Mass moment of inertia—rotor</td>
<td>$J_{rotor}$</td>
</tr>
<tr>
<td>Point mass—foundation</td>
<td>$m_f$</td>
</tr>
</tbody>
</table>

The subscript $t$ denotes tower.

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the displacements/rotations and forces/moments are defined in one plane, i.e. the model can be formulated as a two-dimensional model, with no out-of-plane motions. Thus, the terms for coupled sliding and rocking perpendicular to the plane of motion can be omitted. Torsional motions are not considered.

The soil–structure interaction is modelled by two types of foundation models. In the first approach, the soil–structure interaction is modelled by static springs for each degree of freedom at the foundation node. In the second approach, the frequency dependent behaviour of the structure-foundation system is taken into consideration by applying lumped-parameter models. A fully fixed structure is used as reference. The foundation models are shown in Figure 4.8.

**Static springs**

The elastic static stiffness of the foundation can be expressed by dimensionless elastic stiffness coefficients corresponding to vertical \( K_{VV} \), sliding \( K_{HH} \) and rocking \( K_{MM} \).
4.3 Numerical estimation of natural frequencies

Figure 4.8: Foundation models for the finite element model. (a) Static springs, (b) lumped-parameter models, and (c) fixed (used for reference). No coupling terms are shown.

degrees of freedom. The coupling between sliding and rocking is given by \((K_{HM}^0)\). For the two-dimensional case, the elastic stiffness of the foundation system can be expressed as

\[
\begin{bmatrix}
\frac{H}{G_s R^2} \\
\frac{V}{G_s R^2} \\
\frac{M}{G_s R^3}
\end{bmatrix} =
\begin{bmatrix}
K_{HH}^0 & 0 & K_{HM}^0 \\
0 & K_{VV}^0 & 0 \\
K_{MH}^0 & 0 & K_{MM}^0
\end{bmatrix}
\begin{bmatrix}
\frac{U}{R} \\
\frac{W}{R} \\
\theta_M
\end{bmatrix},
\]

where \(R\) is the radius of the foundation and \(G_s\) is the shear modulus of the soil. \(H, V\) and \(M\) are sliding force, vertical force and rocking moment, respectively. \(U, W\) and \(\theta_M\) are the corresponding displacements/rotations. The shear modulus \(G_s\) is given by

\[
G_s = \frac{E_s}{2(1 + \nu_s)}
\]

where \(E_s\) is Young’s modulus and \(\nu_s\) is Poisson’s ratio. Note that the foundation is assumed to be rigid and the soil is linear elastic, i.e. the properties are given by \(G_s\) and \(\nu_s\). This means that the stiffness components in 4.1 are functions of Poisson’s ratio. The dimensionless elastic stiffness coefficients for the suction caisson are given in Section D.1 in Appendix D.

**Lumped-parameter models**

The investigations of frequency dependent behaviour of massless foundations often involves complicated three-dimensional elastodynamic analyses using rigorous methods, such as the finite element method or the boundary element method. The employed models typically consist of several thousand degrees of freedom, and the frequency dependent dynamic stiffness of the foundations are evaluated in the frequency domain. The requirement for real-time computations in the time domain in aero-elastic codes does not
conform with the use of e.g. a three-dimensional coupled Boundary Element/Finite Element Method, where the foundation stiffness is evaluated in the frequency domain.

In order to meet the requirements of real-time calculations and analysis in time domain, lumped-parameter models are particularly useful (Wolf 1994). A lumped-parameter model represents the frequency dependent soil-structure interaction of a massless foundation placed on or embedded into an unbounded soil domain. Prior to arranging the lumped-parameter models, the frequency dependent dynamic stiffness of the soil-foundation system must be obtained by a rigorous solution, see Chapters 2 and 3. The lumped-parameter models are then assembled by an arrangement of springs, dashpots and/or masses with initially unknown parameters. The unknown parameters are determined by curve fitting with respect to a known rigorous solution, i.e. the unknown parameters are determined by minimizing the total square error between the lumped-parameter model and the known rigorous solution. A key feature is that the models consist of real frequency-independent coefficients in a certain arrangement, which can be formulated into stiffness, damping and/or mass matrices. Thus, the lumped-parameter model can be incorporated into standard dynamic programs. Each degree of freedom at the foundation node of the structural model is coupled to a lumped-parameter model that may consist of additional internal degrees of freedom. Two simple lumped-parameter models are sketched in Figure 4.9. The lumped-parameter models are described in details in Appendix C. The calibration of the lumped-parameter models with respect to the suction caisson is shown in Section D.1 in Appendix D.

Properties of the foundation models

The model properties of the soil and the suction caisson used in the analyses of the static springs and lumped-parameter models (lpm) are given in Table 4.4. For details, see Appendix D. Note that the loss factor is assumed to be constant for all frequencies, i.e. hysteretic damping is assumed.

4.3.3 Numerical analysis of steady state response

To determine the steady state response, the wind turbine structure is subjected to a harmonic unit load with the circular frequency $\omega$. The unit load is applied as a horizontal
Table 4.4: Model properties for the foundation models

<table>
<thead>
<tr>
<th>Property</th>
<th>value</th>
<th>Static springs</th>
<th>lpm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Foundation radius</td>
<td>$R$</td>
<td>6 m</td>
<td>x</td>
</tr>
<tr>
<td>Skirt length</td>
<td>$H$</td>
<td>6 m</td>
<td>x</td>
</tr>
<tr>
<td>Skirt thickness</td>
<td>$t$</td>
<td>30 mm</td>
<td>x</td>
</tr>
<tr>
<td>Shear modulus (soil)$^1$</td>
<td>$G_s$</td>
<td>1, 10, 100 MPa</td>
<td>x</td>
</tr>
<tr>
<td>Poisson’s ratio (soil)</td>
<td>$\nu_s$</td>
<td>0.25</td>
<td>x</td>
</tr>
<tr>
<td>Mass density (soil)</td>
<td>$\rho_s$</td>
<td>1000 kg/m³</td>
<td>-</td>
</tr>
<tr>
<td>Loss factor (soil)</td>
<td>$\eta_s$</td>
<td>5 %</td>
<td>-</td>
</tr>
<tr>
<td>Young’s modulus (foundation)</td>
<td>$E_f$</td>
<td>210 GPa</td>
<td>x</td>
</tr>
<tr>
<td>Poisson’s ratio (foundation)</td>
<td>$\nu_f$</td>
<td>0.25</td>
<td>x</td>
</tr>
<tr>
<td>Mass density (foundation)$^2$</td>
<td>$\rho_f$</td>
<td>0/1000 kg/m³</td>
<td>-</td>
</tr>
<tr>
<td>Loss factor (foundation)</td>
<td>$\eta_f$</td>
<td>2 %</td>
<td>-</td>
</tr>
</tbody>
</table>

$^1$ The models are constructed for three values of $G_s$

$^2$ $\rho_f = 0$ for the lid of the caisson and $\rho_f = \rho_s$ for the skirt

point load at two levels, in order to excite both the first and second natural frequency of the wind turbine structure. To excite the first natural frequency, the load is applied at the nacelle node, and the second natural frequency is excited by applying the load at a mid-tower node. The steady state response is determined by solving the equation of motion for a harmonic response, given by

$$M\ddot{u} + C\dot{u} + Ku = fe^{i\omega t},$$ \hspace{1cm} (4.3)

where $M$, $C$ and $K$ are the mass, damping and stiffness matrix of the structure, respectively. $u$ is a column vector containing the nodal displacements and $f$ is a column vector of nodal forces. $t$ is time and $i$ is the imaginary unit, $i = \sqrt{-1}$. The equation of motion in Equation 4.3 is solved by direct analysis (Petyt 1998). The solution to Equation 4.3 is then

$$u = \left[K - \omega^2M + i\omega C\right]^{-1}fe^{i\omega t}$$ \hspace{1cm} (4.4)

The matrices $M$, $C$ and $K$ are assembled for the structural system. Subsequently, the boundary conditions are included, either by removing or adding components to the matrices. For the foundation model with static springs, the foundation stiffness for each degree of freedom is simply added to the associated degree of freedom for the structural system. Additional degrees of freedom are added for the lumped-parameter models, see Appendix D.3. Finally, the fixed degrees of freedom are removed from the system matrices for the reference case with a fully fixed foundation.

**Steady state response**

The steady state response of the wind turbine has been determined by means of the structural finite element combined with the foundation models shown in Figure 4.8. The
static springs and the lumped-parameter models have been determined for three different values of the shear modulus of the soil. That is $G_s$ equal to 1, 10 and 100 MPa. The steady state responses for the nacelle node and the mid-tower node are given for the magnitude of the node displacements as function of the frequency $f$ of the harmonic loading (note that $f = 2\pi/\omega$). Resonance of the structure when subjected to a unit load with a given frequency may be observed as local peaks in the magnitude of the node displacements. The steady state responses are shown in Figures 4.10 and 4.11.

The frequency intervals of the responses in Figures 4.10 and 4.11 correspond to the intervals, in which the first and second natural frequency of the structure should appear, according to the experimental findings.

The resonance of the structure is highly dependent on the stiffness of the soil. The natural frequencies are significantly reduced for soft soil conditions ($G_s = 1$ MPa). The first and second natural frequency estimated by the two foundation models tends toward the natural frequency of the fully fixed structure for stiff soil conditions ($G_s \geq 100$ MPa). The estimation of the natural frequencies for the two types of foundation models are shown in Table 4.5. The natural frequency estimations by applying the two foundation

<table>
<thead>
<tr>
<th>Soil stiffness $G_s$ [MPa]</th>
<th>first natural frequency [Hz]</th>
<th>second natural frequency [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Static LPM Fixed</td>
<td>Static LPM Fixed</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.205 0.204 - 1.47 1.41 -</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.307 0.307 - 1.97 1.95 -</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.329 0.331 - 2.16 2.16 -</td>
<td></td>
</tr>
<tr>
<td>$\infty$</td>
<td>- 0.331 - 2.19 -</td>
<td></td>
</tr>
</tbody>
</table>

models are very similar. The estimation of the first resonance frequency is identical for $G_s = 1$ and 10 MPa, and there is only minor deviations for $G_s = 100$ MPa. The estimation of the second resonance frequency shows greater, but insignificant variations between the two foundation models.

In contrast, the magnitude and shape of the response vary widely for the two foundation concepts. For a constant soil stiffness, the shape and magnitude of the resonance peaks in static spring response are determined by the amount of material damping in the wind turbine structure. In this case the structural damping has been estimated by a loss factor $\eta_t$ equal to 2 % (Table 4.3). If $\eta_t$ is decreased the resonance peak narrows down and the magnitude of the peak response increases. For high structural damping, the peak response of the static spring model becomes more broad-banded (bell-shaped) and the magnitude of the displacement decreases.
4.3 Numerical estimation of natural frequencies

Figure 4.10: Steady state response (nacelle node) of the wind turbine for different foundation models.

Figure 4.11: Steady state response (mid-tower node) of the wind turbine for different foundation models.
Now consider the response of the lumped-parameter models. Again, for a constant soil stiffness, the shape and magnitude of the resonance peaks are influenced by the amount of material damping in the wind turbine structure. Moreover, damping exists in the soil–structure interaction, contrary to the static spring model. The lumped-parameter models are based on the frequency dependent stiffness (impedance) of a massless foundation vibrating in a visco-elastic half-space. Thus, both geometrical damping, i.e. the radiation of waves into the subsoil, and material dissipation into the subsoil contribute to the overall damping of the structure. The material dissipation of the soil has been estimated by a loss factor $\eta_s$ equal to 5\% (Table 4.4).

It is evident that the implementation of both geometrical damping and material dissipation in the subsoil influence the peak response remarkably. The peak responses estimated by the lumped-parameter models are broad-banded and the magnitude at the peak is reduced significantly, especially for soft soil conditions ($G_s = 1$ MPa). As $G_s$ is increased, the peaks become more narrow-banded, and the effect of the soil–structure interaction is reduced. For $G_s = 100$ MPa, the response of the lumped-parameter model more or less coincides with that of the fixed model without any soil–structure interaction.

With respect to the lumped-parameter models, it is worth noticing that the peak response of the second natural frequency is heavily damped, compared to the peak response of the first natural frequency. This corresponds to the fact that the damping due to radiation of waves into the subsoil (geometrical damping) becomes more pronounced as the excitation frequency increases.

**Experimental vs. numerical**

The experimental modal analysis showed that the first and second natural frequency of the wind turbine are 0.30 Hz and 2.13 Hz, respectively. By inspection of Figures 4.10 and 4.11 these frequencies correspond to a soil shear modulus $G_s$ between 10 and 100 MPa. This observation agrees with the fact that $G_s$ is has been determined to 40–80 MPa at the site. The in-situ measurement of $G_s$ has been performed by cone penetration tests. The in-situ measurements are reported in Ibsen (2002).

**4.4 Conclusions**

The response of a Vestas 3.0 MW offshore wind turbine has been examined by means of an experimental and a numerical approach. The experimental estimation of the natural frequencies has been performed by experimental modal analysis of the structure. The numerical estimation of the response has been evaluated by a finite element model with two types of foundation models. One model, where the soil–structure interaction is modelled by static springs, and one model in which the frequency dependent behaviour of the structure-foundation system is taken into consideration by applying lumped-parameter models.

**4.4.1 Experimental approach**

An experimental modal analysis have been carried out with the intention of estimating the natural frequencies of the wind turbine. The main conclusions are:

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4.4 Conclusions

♦ The natural frequencies have been estimated for idle conditions only, in order to avoid interference caused by rotating components of the wind turbine. If mode estimation are to be performed for operational conditions, information about all the possible "forced harmonic modes" from e.g. gears, generators, rotors and pitch systems are required.

♦ The structural system of an operational wind turbine is time-varying. Thus, errors are introduced in the modal identification, because the framework of the modal estimation relies on the assumptions that the underlying physical system of the structure is linear and time-invariant.

♦ To obtain reliable data for the modal analysis, the length of each time series corresponds to 1000 times the first natural period of the structure.

♦ For idle conditions, the first and second natural frequency is equal to 0.30 Hz and 2.13 Hz, respectively. Resonance frequencies for the blades have been observed in the frequency interval between the first and second natural frequency of the structure.

♦ Replacement of the nacelle and blades in the spring 2005 resulted in marginal change of the natural frequencies of the structure.

4.4.2 Numerical approach

A finite element model of the wind turbine has been utilized to estimate the natural frequencies of the structure numerically. The soil-structure interaction has been simulated by two types of foundation models, static springs for each degree of freedom at the foundation node, and lumped-parameter models where the frequency dependent behaviour of the structure-foundation system is taken into account. The static springs and the lumped-parameter models have been determined for $G_s$ (shear modulus of the soil) equal to 1, 10 and 100 MPa. The following conclusions can be made:

♦ The resonance frequency of the structure is highly dependent on the stiffness of the soil. For soft soil conditions ($G_s = 1$ MPa) the first and second natural frequency are 0.20 Hz and 1.41 Hz, respectively. For stiff soil conditions ($G_s = 100$ MPa) the frequencies are 0.33 Hz and 2.16 Hz, respectively, close to the natural frequencies of a fully fixed structure (0.33 Hz and 2.19 Hz).

♦ The natural frequency estimations by applying the two foundation models are very similar. Insignificant variations on the estimation of the second resonance frequency have been observed.

♦ By using the lumped-parameter models, both geometrical damping and material dissipation into the subsoil contribute to the overall damping of the structure, in contrast to the static spring model where damping only exists in the wind turbine structure.

♦ The magnitude and shape of the response vary widely for the two foundation concepts. The peak responses estimated by the static spring model are narrow-banded, whereas the responses estimated by the lumped-parameter models are broad-banded.

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and the magnitude at the peak is reduced significantly, especially for soft soil conditions ($G_s = 1 \text{ MPa}$).

- The peak response of the second natural frequency is heavily damped, compared to the peak response of the first natural frequency, regarding the lumped-parameter models, suggesting that the damping due to radiation of waves into the subsoil becomes more pronounced as the excitation frequency increases.

### 4.4.3 Recommendations for future work:

- Parameter studies of the influence of soil damping, soil stiffness, structural mass and stiffness on the response of the wind turbine
- Studies of the effect of soil layering.
- Parameter studies and comparison of different foundation concepts
- Implement the lumped-parameter models of wind turbine foundations into aeroelastic codes, in order to test the composite structure–foundation system in a complex loading environment
The purpose of this thesis has been to evaluate the dynamic behaviour of a new innovative foundation concept; the suction caisson foundation. The frequency dependent stiffness of the suction caisson has been investigated by means of a three-dimensional coupled Boundary Element/Finite Element model. The dynamic stiffness (impedance) for each degree of freedom have been formulated into lumped-parameter models with frequency independent coefficients, suitable for implementation in standard dynamic finite element schemes. The lumped-parameter models for a suction caisson have been applied as the boundary conditions for a numerical model of an offshore wind turbine structure. The steady state response of the wind turbine has been evaluated by the numerical model and compared with experimental measurements of the resonance frequencies. The main conclusions of this work are presented in this chapter. Finally, directions for future work are given.

### 5.1 Conclusion—main findings

The conclusions are given with respect to the research aims stated in Section 1.4.

#### 5.1.1 Evaluation of the frequency dependent stiffness of suction caissons

The soil–structure interaction of steel suction caissons has been evaluated by means of static finite element analyses and dynamic analyses employing a three-dimensional coupled Boundary Element/Finite Element model. The static and dynamic stiffness have been investigated separately for each of the six degrees of freedom. The degrees of freedom are: one vertical, two horizontal (sliding), two rocking and one torsional. In the present case the foundation is axisymmetric, and there is only coupling between the horizontal sliding and rocking motion. Thus, the vertical and torsional motion are completely decoupled.

**Static results**

Initially, the static stiffness coefficients have been determined by means of a static finite element analysis in ABAQUS. The results agree with the work by Doherty and Deeks (2003) and Doherty et al. (2005) who employed the scaled boundary finite element method to analyse the static stiffness of suction caissons embedded in non-homogeneous elastic soil.
Modelling aspects

The reason for using the coupled formulation is that the inherent radiational damping phenomena of an unbounded soil domain can be modelled accurately by means of boundary elements, whereas the complex foundation geometry is best modeled by finite elements.

The static results have been used as convergence criteria for the element mesh size in the subsequent boundary element analyses of the dynamic stiffness. The reasoning for using the static stiffness as convergence criteria is that the shape of the impedance (location of peaks as function of frequency) converges faster than the actual magnitude of the impedance. Hence, it turns out that the magnitude of the impedance is the critical convergence parameter.

Due to symmetry only half the foundation is included in the coupled BE/FE model. The BE/FE model of the suction caisson consists of four sections: a massless finite element section that forms the top of the foundation where the load is applied, a finite element section of the skirts, a boundary element domain inside the skirts and, finally, a boundary element domain outside the skirts that also forms the free surface. The skirt of the suction caisson is considered flexible, and the lid is assumed to be rigid. Quadratic interpolation is employed. The BE/FE models of the suction caisson and the subsoil contain approx. 100 finite elements and 350 boundary elements. The BE/FE models are nearly converged when the static values are compared to those of the ABAQUS model.

The truncation distance for the models of the suction caisson depends on the skirt embedment. Convergence studies for the worst case \((H/D = 2)\) suggested a truncation distance of 30 m from the centre of the foundation. This length has been used for all the BE/FE analyses of the suction caisson, regardless of embedment depth of the skirt. Adaptive meshing could possibly improve the accuracy versus the number of degrees of freedom, but this facility is currently not available in the BE/FE software. Further, the applied mesh of the BE/FE models are capable of describing the impedance satisfactorily for a non-dimensional frequency \(a_0\) no higher than 12. Load control has been used to generate the stiffness values. Displacement control would be more appropriate, but this feature is currently not available in the BE/FE software.

The BE/FE models of the suction caisson contain approximately 3000 degrees of freedom and the runtime is approximately 30 minutes for each excitation frequency on a 2.0 GHz P4 laptop computer. Note that the analyses presented in this thesis are run with a loss factor of 5%. The runtime for the same model, but without material damping is considerable longer; the runtime for each excitation frequency for the results presented in Liingaard et al. (2005) is approximately 3 hours. Thus, by introducing small amounts of material damping, the runtime of the boundary element calculations is reduced remarkably.

Dynamic results

The dynamic stiffness of each degree of freedom has been investigated for several different combinations of the mechanical properties of the soil–foundation system. The dynamic stiffness (impedance) has been presented as a function of a non-dimensional frequency \(a_0\). The dynamic stiffness \(S_{ij}\) is complex, and therefore represented by the magnitude (complex modulus) and the phase angle of \(S_{ij}\).

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The impedance of suction caisson is characterized by repeated oscillations (distinct peaks) in the impedance for certain frequency intervals. This is due to resonance and anti-resonance of the soil inside the suction caisson. The coupled sliding–rocking impedances are characterized by a complex wave interference pattern in the soil inside the skirts. The local peaks in the magnitude of the impedance components are not repeated by $\Delta a_0 = \pi$, which is the case for the vertical and torsional impedance components. The location of the peaks for the coupled sliding–rocking impedances are controlled by antiresonance of both shear waves and compression waves. The peaks become more pronounced when the skirt length is increased, and the behaviour corresponds well to that of the infinite cylinder. This has been concluded by comparing the impedance characteristics of the suction caisson to those of an infinite cylinder subjected to dynamic excitation. The simulations of the vibrations of an infinite cylinder have been performed by means of an closed-form solution (vertical case) and a two-dimensional coupled BE/FE program for the torsional and sliding vibrations. The frequency intervals of the resonance peaks in the impedances of the infinite cylinder match peaks of the suction caisson very well. Other significant observations from Chapters 2 and 3, with respect to the impedance are given here:

- The dynamic stiffness changes with the skirt length. For a relatively small embedment depth ($H/D = 1/4$) the impedance varies smoothly with the frequency, whereas the impedance for $H/D = 1$ and 2 is characterized by distinct peaks. This observation applies for all degrees of freedom.

- Poisson’s ratio has no impact on the torsional stiffness, since torsional vibrations of the suction caisson only produce shear waves. The vertical dynamic stiffness is relatively insensitive to variations in $\nu_s$. Contrary, the sliding and rocking impedances are clearly dependent on the Poisson’s ratio of the soil, and the local extremum in the magnitude of the impedance changes significantly with Poisson’s ratio.

- The impedance for high values of $G_s$ (1000 MPa) approaches the shape of the frequency dependent behaviour of the surface foundation. When $G_s$ decreases, the local oscillations become more distinct and the influence of the skirt flexibility vanishes, i.e. the caisson reacts as a rigid foundation. Rigid behaviour can be assumed for $G_s \leq 1.0$ MPa. This observation applies for all degrees of freedom.

Finally, high-frequency limits of the impedance components have been established for the suction caissons. These high-frequency limits have been used in combination with the low-frequency impedances obtained with the BE/FE models in the formulations of lumped-parameter models of suction caissons.

5.1.2 Experimental estimation of resonance frequencies

A prototype suction caisson with a diameter of 12 meter and a skirt length of 6 meter has been installed as support for a Vestas V90 3.0 MW offshore wind turbine at Aalborg University offshore test facility in Frederikshavn, Denmark. The wind turbine has been instrumented with 15 accelerometers and a real-time data-acquisition system, and experimental modal analyses have been carried out with the intention of estimating the natural frequencies of the wind turbine.

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The natural frequencies have been estimated for idle conditions only, in order to avoid interference caused by rotating components of the wind turbine. Even though the structural system of an operational wind turbine is time-varying and errors may be introduced in the modal identification, estimations of the resonance frequencies and the $1\Omega$ and $3\Omega$ excitation frequency intervals are possible for operational conditions, see Ibsen and Liingaard (2005). However, the mode estimation for operational conditions is more complex. It is crucial to have information about all the possible "forced harmonic modes" from e.g. gears, generators, rotors and pitch systems.

For idle conditions, the first and second natural frequency of the Vestas V90 3.0 MW offshore wind turbine is equal to 0.30 Hz and 2.13 Hz, respectively. Resonance frequencies for the blades have been observed in the frequency interval between the first and second natural frequency of the structure. Replacement of the nacelle and blades in the spring 2005 resulted in marginal change of the natural frequencies of the structure.

5.1.3 Formulation and implementation of lumped-parameter models

The investigations of frequency dependent behaviour of foundations often involves complicated three-dimensional elastodynamic analyses using rigorous methods. The employed models typically consist of several thousand degrees of freedom, and the frequency dependent dynamic stiffness of the foundations are evaluated in the frequency domain. The requirement for real-time computations in commercial software packages for performance and loading analysis of wind turbines, do not conform with the use of e.g. a three-dimensional coupled Boundary Element/Finite Element Method, where the foundation stiffness is evaluated in the frequency domain. To meet the requirements of real-time calculations and analysis in time domain, lumped-parameter models are particularly useful. A lumped-parameter model represents the frequency dependent soil-structure interaction of a massless foundation placed on or embedded into an unbounded soil domain. A key feature is that the models consist of real frequency-independent coefficients in a certain arrangement, which can be formulated into stiffness, damping and/or mass matrices. Thus, the lumped-parameter model can be incorporated into standard dynamic finite element schemes.

In this thesis lumped-parameter model approximations have calibrated with respect to the impedance of a suction caisson. The impedance has been determined as stated in Chapters 2 and 3, and the lumped-parameter models have been constructed according to the procedure in section C.3. The frequency-independent coefficients for each degree of freedom are given in Appendix D.

The lumped-parameter models have been used to simulate the soil–structure interaction within a numerical finite element model of the Vestas V90 3.0 MW offshore wind turbine. To investigate the response for various soil conditions, the lumped-parameter models have been determined for $G_s$ equal to 1, 10 and 100 MPa. A foundation model with static springs only have been used for comparison. The output of the numerical simulations are steady state responses of the nacelle and mid-tower nodes as function of the frequency $f$ of a harmonic loading. Resonance of the structure when subjected to a unit load with a given frequency may be observed as local peaks in the magnitude of the node displacements. Following conclusions can be made:

♦ The resonance frequency of the structure is highly dependent on the stiffness of the...
soil. For soft soil conditions ($G_s = 1$ MPa) the first and second natural frequency are 0.20 Hz and 1.41 Hz, respectively. For stiff soil conditions ($G_s = 100$ MPa) the frequencies are 0.33 Hz and 2.16 Hz, respectively. The results for stiff soil conditions agrees with the natural frequencies of a fully fixed structure (0.33 Hz and 2.19 Hz).

By using the lumped-parameter models, both geometrical damping and material dissipation into the subsoil contribute to the overall damping of the structure, in contrast to a static spring foundation model where damping only exists in the wind turbine structure.

The peak responses estimated by the static spring model are narrow-banded, whereas the responses estimated by the lumped-parameter models are broad-banded and the magnitude at the peak is reduced significantly, especially for soft soil conditions ($G_s = 1$ MPa).

The peak response of the second natural frequency is heavily damped, compared to the peak response of the first natural frequency, regarding the lumped-parameter models, suggesting that the damping due to radiation of waves into the subsoil becomes more pronounced as the excitation frequency increases.

Overall, the simulations of the soil–structure interaction by means of lumped-parameter model approximations of the impedance have shown that the concept is useful for use in applications where the performance of the wind turbine are to be analysed. Furthermore, the use of lumped-parameter models introduces very few additional degrees of freedom in the numerical model. The two-dimensional numerical model, presented in Chapter 4, contains 111 degrees of freedom in total; only 15 of these equations are used to describe the frequency dependent behaviour of the soil–structure interaction.

The use of lumped-parameter models for simulating the actual frequency dependent behaviour of foundation is relatively inexpensive, especially for surface foundations resting on a homogeneous soil. In this particular case, rigorous solutions may be found in the literature, and simple lumped-parameter approximations already exist (see Section C.2 in Appendix).

## 5.2 Directions for future work

The directions are given with respect to experimental and numerical approaches.

### 5.2.1 Experimental approaches

The use of experimental modal analysis within this thesis is very limited. The intention have been to demonstrate an experimental approach for estimating the response of the wind turbine. However, comprehensive comparative analyses of experimental and numerical approaches are to be carried out for existing wind turbine structures.

It would be challenging to verify the impedance of suction caisson experimentally by means of model tests. Preliminary results have already be obtained by Houlsby et al. (2005) and Houlsby et al. (2006). The experimental setup should be able to generate vertical, horizontal, rocking and torsional excitation of the foundation. Moreover, the
loading system is to be able to vibrate the foundation with frequencies that excite the characteristic anti-resonance modes of the suction caisson.

### 5.2.2 Numerical approaches

The lumped-parameter model approximations are to be exercised in time-domain. The analyses in this thesis only involve analyses in the frequency domain.

Numerical results with respect to the impedance of suction caisson have been presented in this thesis. The impedance of surface footing is well described in the literature, at least with respect to footings on an elastic half-space. On the other hand, the dynamic behaviour of footings on layered soil are to be clarified. Furthermore, the dynamic behaviour of large diameter open piles are not documented in details.

Parameter studies of foundation impedances with respect to flexibility and mass of structure, damping of subsoil and layered soil profiles are to be carried out, in order to create a versatile toolbox for the commercial software packages for performance and loading analysis of wind turbines.

Finally, dynamic aspects have been covered to a certain level, however the behaviour during cyclic loading of offshore foundations needs to be examined in details. Both with respect to cyclic degradation/compaction and cyclic pore-pressure build-up.


Appendix
Consider an elastic full-space with material density $\rho$ and shear modulus $G$. As depicted on Figure A.1 the full-space is divided into two domains by an infinitely long circular cylinder, the centre axis of which coincides with the $x_3$-axis. The interior domain is coined $\Omega_1$, and the remaining part of the full-space, i.e. the exterior domain, is coined $\Omega_2$. The boundary of $\Omega_1$ is denoted $\Gamma_1$ and has the outward unit normal $\hat{n}_1(x)$, whereas the boundary of $\Omega_2$ is denoted $\Gamma_2$ and has the outward unit normal $\hat{n}_2(x)$. Evidently, $\Gamma_1$ coalesces with $\Gamma_2$, and $\hat{n}_1(x) = -\hat{n}_2(x)$ along the cylindrical interface, cf. Figure A.1.

The cylindrical interface between $\Omega_1$ and $\Omega_2$ is subject to a harmonically varying forced displacement with the cyclic frequency $\omega$ and applied in the $x_3$-direction, i.e. along the centre axis. This leads to pure antiplane shear wave propagation (SH-waves) in the elastic material, i.e. there is no displacement in the $x_1$- or $x_2$-direction. Depending on $\omega$, the geometry of the cylinder and the wave propagation velocity $c_S = \sqrt{G/\rho}$, the
where $K$ is the cylindrical interface, and $\phi$ is the excitation function. The two domains are reduced to since the surface is smooth along the entire interface, the boundary integral equations (2.7) are reduced to

\[ \frac{1}{2} V_1(x, \omega) + \int_{\Gamma_1} P^*(x, \omega; \xi)V_1(\xi, \omega)d\Gamma_\xi = \int_{\Gamma_1} V^*(x, \omega; \xi)P_1(\xi, \omega)d\Gamma_\xi, \quad (A.1a) \]

\[ \frac{1}{2} V_2(x, \omega) + \int_{\Gamma_2} P^*(x, \omega; \xi)V_2(\xi, \omega)d\Gamma_\xi = \int_{\Gamma_2} V^*(x, \omega; \xi)P_2(\xi, \omega)d\Gamma_\xi, \quad (A.1b) \]

where $V_1(x, \omega)$ and $V_2(x, \omega)$ are the displacements in the $x_3$-direction along the boundaries $\Gamma_1$ and $\Gamma_2$, respectively, whereas $P_1(x, \omega)$ and $P_2(x, \omega)$ are the corresponding surface tractions. Further, $P^*(x, \omega; \xi)$ is the surface traction related to the Green's function $V^*(x, \omega; \xi)$. In the case of antiplane shear waves, the fundamental solutions providing the response at the observation point $x$ to a harmonically varying point force at the source point $\xi$ are given as Domínguez (1993)

\[ V^*(x, \omega; \xi) = \frac{1}{2\pi G}K_0(ik_Sr), \quad P^*(x, \omega; \xi) = -\frac{k_S}{2\pi} \frac{\partial r}{\partial n}K_1(ik_Sr), \quad r = ||r||_2, \quad r = x - \xi, \quad (A.2) \]

where $K_m$ is the modified Bessel function of the second kind and order $m$, whereas $\partial r/\partial n$ defines the partial derivative of the distance $r$ between the source and observation point in the direction of the outward normal. With the definitions given on Figure A.1, and further introducing $\hat{n}(\xi) = \hat{n}_1(\xi) = -\hat{n}_2(\xi)$, it becomes evident that

\[ \frac{\partial r}{\partial n} \left\{ \begin{array}{ll} \hat{r}(x, \xi) \cdot \hat{n}(\xi) = \cos(\varphi) & \text{for } x \in \Gamma_1 \\ -\hat{r}(x, \xi) \cdot \hat{n}(\xi) = -\cos(\varphi) & \text{for } x \in \Gamma_2 \end{array} \right. \]

where $\hat{r}(x, \xi) = \frac{x - \xi}{||x - \xi||_2}$. \quad (A.3)

Here $\varphi$ is the angle between the distance vector $r$ and the normal vector $\hat{n}$. Finally, in Equation (A.1) $k_S$ is the wavenumber of S-waves. In the case of hysteretic material damping with the loss factor $\eta$,

\[ k_S = \frac{\omega}{c_S}, \quad c_S^2 = (1 + i\eta) \frac{G}{\rho}. \quad (A.4) \]

Now, the forced displacement is applied with constant amplitude $\hat{V}(\omega)$ and in phase along the cylindrical interface, $\Gamma \equiv \Gamma_1$. Accordingly, the traction on either side of the interface will be uniform and in phase. Continuity of the displacements across the interface then provides the result:

\[ V_1(x, \omega) = V_2(x, \omega) = \hat{V}(\omega), \quad P_1(x, \omega) = \hat{P}_1(\omega), \quad P_2(x, \omega) = \hat{P}_2(\omega), \quad x \in \Gamma. \quad (A.5) \]

Hence, Equation (A.1) may be rewritten as

\[ \hat{V}(\omega) \left( \frac{1}{2} + \int_{\Gamma} P^*(x, \omega; \xi)d\Gamma_\xi \right) = \hat{P}_1(\omega) \int_{\Gamma} V^*(x, \omega; \xi)d\Gamma_\xi, \quad (A.6a) \]

\[ \hat{V}(\omega) \left( \frac{1}{2} - \int_{\Gamma} P^*(x, \omega; \xi)d\Gamma_\xi \right) = \hat{P}_2(\omega) \int_{\Gamma} V^*(x, \omega; \xi)d\Gamma_\xi, \quad (A.6b) \]

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where use has been made of Equation (A.3). Addition of Equations (A.6a) and (A.6b) provides a measure of the dynamic stiffness per unit surface of the interface related to displacement along the cylinder axis. The stiffness per unit length of the infinite cylinder then becomes

\[ S_{VV}(\omega) = -L_{\Gamma} \frac{\hat{P}(\omega)}{V(\omega)} = -\frac{L_{\Gamma}}{2\alpha} \left( \hat{P}_1(\omega) + \hat{P}_2(\omega) \right), \quad (A.7) \]

where \( L_{\Gamma} \) is the length of the interface \( \Gamma \), measured in the \((x_1, x_2)\)-plane, and

\[ \alpha = \int_{\Gamma} V^*(x, \omega; \xi) d\Gamma_{\xi}. \quad (A.8) \]

Equations (A.7)–(A.8) hold for arbitrary geometries of the infinite cylinder. However, in what follows a restriction is made to an infinite circular cylinder with the radius \( R \), that is with \( L_{\Gamma} = 2\pi R \). In order to compute \( \alpha \), the cylindrical polar coordinates \((\varrho, \theta, z)\) are introduced such that

\[ x_1 = \varrho \cos \theta, \quad x_2 = \varrho \sin \theta, \quad x_3 = z. \quad (A.9) \]

In these coordinates, the boundary \( \Gamma \) is defined by \( \varrho = R \), \( 0 \leq \theta < 2\pi \), \( -\infty < z < \infty \). In particular, when an observation point \( x \) with the plane coordinates \((x_1, x_2) = (-1, 0)\) is considered (see Figure A.1), the distance \( r \) between the source and observation point becomes

\[ r = R \frac{\sin 2\varphi}{\sin \varphi} = 2R \cos \varphi. \quad (A.10) \]

Making use of the fact that \( \theta = 2\varphi \), Equation (A.8) may then be evaluated as

\[ \alpha = \frac{1}{2\pi G} \int_0^{2\pi} K_0(ik_S r) R d\theta = \frac{R}{\pi G} \int_0^\pi K_0(2ik_S R \cos \varphi) d\varphi = -\frac{R}{G} J_0(k_S R) K_0(ik_S R). \quad (A.11) \]

Here \( J_0 \) is the Bessel function of the first kind and order 0. It is noted that \( K_0(ik_S R) \to \infty \) for \( k_S \to 0 \). Hence, \( S_{VV}(\omega) \to 0 \) for \( \omega \to 0 \). Furthermore, \( J_0(k_S R) \) has a number of zeros for \( \eta = 0 \) and \( k_S > 0 \). At the corresponding circular frequencies, \( K_{VV}^*(\omega) \) becomes singular as reported by Kitahara (1984).
Appendix B

Experimental modal analysis

This appendix concerns the basic theory and principles for experimental modal analysis. The sections within the appendix are: Output-only modal analysis software (section B.1), general digital analysis (section B.2), basics of structural dynamics and modal analysis (section B.3) and system identification (section B.4).

B.1 Output-only Modal Analysis Software

The experimental modal analysis of the wind turbine prototype is performed by means of the software package ARTeMIS (Ambient Response Testing and Modal Identification Software). The ARTeMIS software is fully compatible with the hardware of the monitoring system. The software package consists of two tools, the ARTeMIS Testor and the ARTeMIS Extractor (SVS 2006).

B.1.1 Output-only Modal Identification

The experimental modal analysis of the wind turbine makes use of "Output-only modal identification" which is utilized when the modal properties are identified from measured responses only. "Output-only modal identification" is also known by the terms "ambient identification" or "ambient response analysis" within the field of civil engineering. The following description is based on SVS (2006).

Modal Identification

The basic principle in Modal identification is the determination of modal parameters from experimental data. The usual modal parameters are natural frequencies (the resonance frequencies), damping ratios (the degree to which the structure itself is able of damping out vibrations) and mode shapes (the way the structure moves at a certain resonance frequency). The common way is to use input-output modal identification where the modal parameters are found by fitting a model to a Frequency Response Function, a function relating excitation and response. The traditional techniques in input-output modal identification is described frequently in the literature, see for instance (Ewins 1995; Maia and Silva 1997).
Output-only modal Identification

When modal identification is based on the measured response (output) only, things become more complicated for several reasons, the excitation (input) is unknown and the measured response (output) is often noisy.

Output-only modal identification is used for analyzing large civil engineering structures, operating machinery or other structures that are not easily excited artificially. Large civil engineering structures are often excited by natural loads that cannot easily be controlled, for instance wave loads (offshore structures), wind loads (Buildings) or traffic loads (bridges). For operating machinery the problems are the same. They are also excited by natural sources like noise from bearings or vibrations from the environment around the structure. In these cases, it is an advantage to use output-only modal identification. Instead of exciting the structure artificially and dealing with the natural excitation as an unwanted noise source, the natural excitation is used as the excitation source. The idea of output-only modal identification is illustrated in Figure B.1.

The unknown loading conditions of the structure are assumed to be produced by a virtual system loaded by white noise. The white noise is assumed to drive both the real structural system and the virtual loading system as a total system and not only the structural system.

For that reason the user is identifying two types of modes, one type of modes that belongs to the real structural system and another type of "modes" that belong to the virtual loading system. The real structural modes are characterized by light damping,
Figure B.2: Geometry of wind turbine tower and foundation. Left, with opaque surfaces and right, without surfaces.

whereas the "modes" of the virtual loading system usually are heavily damped, see Figure B.1. Furthermore, the user might also identify computational modes that appear because the signals are contaminated with noise. This means, that it is of outmost importance that the real structural modes are separated from noise modes and excitation modes during the modal identification process.

B.1.2 ARTeMIS Testor

The ARTeMIS Testor is a test planning tool where the geometry of the structure and the sensor settings and locations are defined. There are three main tasks to be carried out: Geometry generation, hardware definition and test planning. The tasks are briefly described in the following.

Geometry generation

The geometry of the system consists of two subsets. The first subset of the geometrical model is the active master system defined by the coordinates of the actual sensors. In this case it is the $xyz$-coordinates of the 15 accelerometers (the positions are given in the main paper). The second subset is the slave system of nodes. The slave system represents the physical appearance of the structure. The displacements of the nodes of the slave system are coupled to the master system by means of slave equations. The slave equations are influence relations that states how much a slave node moves if the corresponding master node is displaced by 1 unit. Lines and opaque surfaces can be added into the geometry in order to make a realistic and uncomplicated representation of the structure in the subsequent analyses, see Figure B.2.
Hardware Definition

The hardware is defined by one or more virtual data acquisition units (a front-end) that each represents a measurement session. Each front-end unit holds as many transducer objects as there are measurement channels in the session. The front-end unit contains information about the number of data points, sampling frequency and the Nyquist frequency of the particular session. The Transducer object is a virtual measurement channel. This object contains the actual measurements of a single channel as well as the parameters necessary to describe them.

Test Planning

The Test Planning task is used to assign each of the transducer objects to the geometry. Each transducer object must be linked to one of the master nodes and the orientation of the transducer must be set as well, i.e. the degree of freedom (DOF). The location and orientation of the transducer objects are shown in Figure B.3

B.1.3 ARTeMIS Extractor

The ARTeMIS Extractor is the key application of the ARTeMIS software package. The software allows the user to perform accurate modal identification under operational conditions and in situations where the structure is impossible or difficult to excite by externally applied forces. The typical outputs of the analyses are modal information about the natural frequencies, mode shapes and damping ratios.

Analysis assumptions

The modal analysis within this software is based on the assumptions that the underlying physical system of the structure is linear and time-invariant. The linearity imply that the physical system comply with the rules of linear superposition. The time-invariance implies that the underlying mechanical or structural system does not change in time. Within this frame the program is based on two different estimation techniques, one in time domain and one in frequency domain, see Figure B.4.

Stochastic Subspace Identification

The time domain estimation is based on Stochastic Subspace Identification (SSI) technique. In the SSI techniques a parametric model is fitted directly to the raw time series data obtained from the accelerometers. The parametric models are characterized by the assumption of a mathematical model constructed from a set of parameters, where the mathematical model is a linear and time-invariant system of differential equations. The task of the SSI technique is to adjust the parameters in order to change the way the model fits to the data. In general the objective is to estimate a set of parameters that will minimize the deviation between the predicted system response (predicted transducer signal) of the model and measured system response (transducer signal). The parametric models and Stochastic Subspace Identification are described in Section B.4. For references, see (Andersen 1997; Brincker and Andersen 1999).
Frequency Domain Decomposition

The frequency domain estimation is a non-parametric model (also known as spectral models) based on a Frequency Domain Decomposition (FDD) method. The FDD method is an extension of the well-known frequency domain approach that is based on mode estimations directly from the Power Spectral Density (PSD) matrix, i.e. well separated modes can be identified at the peaks of the PSD matrix.

The basic principle of the Frequency Domain Decomposition (FDD) technique is to perform an approximate decomposition of the system response into a set of independent single degree of freedom (SDOF) systems; each corresponding to an individual mode. In the FDD the Spectral Density matrix is decomposed by means of the Singular Value Decomposition (SVD) into a set of auto spectral density functions, each corresponding to a single degree of freedom system. The steps of the FDD technique are illustrated in
Appendix B – Experimental modal analysis

Figure B.4: Modal analysis with ARTeMIS Extractor.

The key feature is that the singular values are estimates of the Auto Spectral density of the SDOF systems, and the singular vectors are estimates of the mode shapes. The basic theory concerning identification by FDD is given in Section B.4. For references, see (Brincker, Andersen, and Zhang 2000; Brincker, Zhang, and Andersen 2000).
Discrete data time series. $N$ is the number of channels

$G_{N1}(f)$ $G_{N2}(f)$ $G_{NN}(f)$

$G_{11}(f)$ $G_{12}(f)$ $G_{1N}(f)$

$G_{21}(f)$ $G_{22}(f)$ $G_{2N}(f)$

$N \times N$ Cross Spectral Density functions estimated from $N$ data time series

N Singular Value functions estimated from $G(f)$

Mode shape for peak value of the SVD plot

Figure B.5: Main steps of the Frequency Domain Decomposition (FDD) technique.

December 4, 2006
Example—Peak picking by FDD

The following example shows the main output of peak picking by FDD. The FDD peak picking is based on measured data recorded February 15, 2005. The data set consists of a 1 hour measurement in 15 channels. The sampling frequency was 200 Hz and the data was decimated by an order of 20. The main screen image for the FDD modal identification technique is shown in Figure B.6. Note that four modes are identified. The second screen image shows the first mode shape of the wind turbine.
Figure B.6: Top: main screen image for the FDD modal identification technique. Bottom: screen image of the first mode shape of the wind turbine.
B.2 General digital data analysis

This section explains the basic digital operations that are required prior to the main estimation of the modal parameters by the two identification procedures. Before the measured data is useful as input data to the modal identification estimation routines several pre-processing procedures are required. The typical pre-processing steps are described in the following.

B.2.1 Data Sampling and aliasing

The analog input signals from the transducers are continuous and processed by means of an analog filter and an analog-to-digital conversion in order to manage the information on a digital computer. The process prior to the digital signal processing is shown in Figure B.7.

![Diagram of filtering and A/D conversion of analog input signal prior to digital signal processing.](image)

Figure B.7: Filtering and A/D conversion of analog input signal prior to digital signal processing.

B.2.2 Structure of measured data

The measured data from the accelerometers are considered as sample records of a random process, i.e. the data are physical realizations of a random process. It is assumed that the random process is stationary, which means that the loading and structural system is assumed to be time invariant.

The measured data are digital representations of a continuous signal from the transducers. Two data time series $x$, $y$ are illustrated in Figure B.8. The time series are sampled with a fixed sample frequency $f_s$. The equally spaced time interval between the data points is denoted the sampling interval $\Delta t$. $\Delta t$ is equal to $1/f_s$. It is assumed that

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the recorded time series can be separated into data segments $x_n$ and $y_n$ of the length $T$ containing $N$ numbers of data points. By this segmentation of the time series the data are assumed to be periodic with a return period of $T$.

![Digital representation of a continuous signal](image)

Figure B.8: Digital representation of a continuous signal. The time series $x$ and $y$ are digitized signals with equally spaced time intervals $\Delta t$.

**B.2.3 Nyquist frequency**

To present the frequency content of the data the Fourier Transform $X(f)$ of $x(t)$ is imposed. Each frequency component (or cycle) of the original data requires at least two samples, which means that the highest frequency that can be defined by a sampling rate of $f_s = 1/\Delta t$ is $f_s/2$. This particular band-limiting frequency is denoted the Nyquist frequency (or folding frequency):

$$f_{nyq} = \frac{f_s}{2} = \frac{1}{2\Delta t}$$  \hspace{1cm} (B.1)

**B.2.4 Aliasing**

Frequencies or vibration cycles above $f_{nyq}$ in the original data will appear below $f_{nyq}$ in the frequency domain and could be misinterpreted as low frequency content, see Figure B.9. This phenomenon is known as aliasing. To avoid aliasing the frequency content of the original data above $f_{nyq}$ should be removed prior to the subsequent signal processing procedures. The high frequency information can be removed by "anti-aliasing filters" by applying a low pass filter that cuts off frequency content higher than $f_{nyq}$. Real filters does not have an infinitely sharp cut-off shape, so the anti-aliasing filter cut-off
frequency is set to approx. 80 % of \( f_{nyq} \) to assure that any data at frequencies above \( f_{nyq} \) are strongly suppressed (Bendat and Piersol 1986).
Figure B.9: Aliased power spectrum due to folding.
B.2.5 Signal processing (digital data analysis)

The traditional non-parametric models for system identification are primarily based on spectral analysis that makes use of Fourier Transform techniques. The spectral analysis is employed for analysing stochastically excited systems and in this case the excitation and the system response can be characterized by spectral densities in frequency domain. The basic signal processing steps in the non-parametric methods are described in the following.

Spectral Analysis—example

When operating with spectral analysis techniques the shape of the time domain waveform of the vibrating structure is not dealt with; the key information is the frequency, phase and amplitude of the component sinusoids. The Discrete Fourier Transform (DFT) technique is used to extract this information. This general concept is shown by an example, based on a description from Smith (1997).

The measuring device is a transducer (here an accelerometer) where the data is sampled by a rate of 200 Hz and thereby a Nyquist frequency of 100 Hz. An analog low-pass filter (anti-aliasing filter) is used to remove all frequencies above 100 Hz, and the cut-off frequency is set to 80 % of the Nyquist frequency. A sample of 1024 data point of a measured signal is shown in Figure B.10(a). This corresponds to a data segment of a time series, as shown in Figure B.8. The DFT technique makes use of the Fast Fourier transform (FFT) algorithm. When the FFT is applied for transforming a sample of 1024 data points, this result in a 513 point frequency spectrum in the frequency domain, i.e. the frequency range from 0 to 100 Hz is divided into 513 frequency points. By using the FFT algorithm it is assumed that the signal to be transformed is periodic within the transformation window (here corresponding to the 1024 samples). Many types of signals, such as random signals are non-periodic in the transformation window, which may lead to distortion of the frequency spectrum. This distortion is referred to as "spectral leakage" and results in inaccurate spectral information of the measured signal. To suppress the spectral leakage the measured signal is tapered before applying the FFT, so the discontinuities at the edges of the transformation window are reduced. This time history tapering is done by multiplying the measured signal in Figure B.10(a) by a suitable time window as shown in Figure B.10(b). This specific window is denoted a Hamming Window but other time windows are available, see e.g. Bendat and Piersol (1986). The resulting signal is shown in Figure B.10(c), where the samples near the ends have been reduced in amplitude. The windowed signal in Figure B.10(c) is transformed by means of DFT into a 513 point frequency spectrum (here a Power Spectral Density spectrum) as shown in Figure B.10(d). This plot is filled with noise because there is not enough information in the original 1024 points to obtain a well defined spectrum. The noise is not reduced by refining the FFT to 2048 points (=1025 point frequency spectrum), because using a longer FFT provides better frequency resolution, but the same noise level.

In order to reduce the noise more data is needed. This is carried out by separating the data into multiple 1024 data point segments. Each segment is multiplied by the Hamming Window, processed by the 1024 FFT algorithm and converted into a frequency spectrum. The resulting spectrum is constructed by averaging all the frequency spectra.
as shown in Figure B.10(e). Here the spectrum is an average of 200 spectra. The noise level has been reduced and the relevant features of the signal can be investigated. It should be noted that the number of segments should be sufficiently large, e.g. 100 or more.
Figure B.10: Example of spectral analysis of a signal. (a) shows 1024 samples taken from a transducer with a sample frequency of 200 Hz. The signal is multiplied by a Hamming window (b), resulting in the windowed signal in (c). The Power Spectral Density (PSD) of the windowed signal is calculated by means of the Discrete Fourier Transform (DFT) and followed by multiplication in frequency domain, as displayed in (d). When averaging e.g. 200 of these spectra the random noise is reduced, resulting in the averaged spectrum shown in (e).

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Spectral analysis—General concepts

The spectral analysis is used for identifying the frequency composition of random signals in the frequency domain. In the following some basic descriptive properties used for describing random signals are presented. These are:

♦ Autocorrelation functions
♦ Cross-correlation functions
♦ Spectral density functions

Autocorrelation Function

The definition of an autocorrelation function is: The expected value of the product of a random variable or signal realization with a time-shifted version of itself. The autocorrelation function contains information about how quickly random signals or processes change with respect to time, and whether the process has a periodic component and what the expected frequency might be.

A pair of random variables from the same process \( x(t) \) is considered, that is \( x_1 = x(t_1) \) and \( x_2 = x(t_2) \). Then the autocorrelation \( \Pi_{xx}(t_1, t_2) \) of \( x_1 \) and \( x_2 \) can be written as:

\[
\Pi_{xx}(t_1, t_2) = E[x_1 x_2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 p(x_1, x_2) dx_1 dx_2 \tag{B.2}
\]

where \( p(x_1, x_2) \) is the joint probability density function of \( x_1 = x(t_1) \) and \( x_2 = x(t_2) \). The above equation is valid for stationary and non-stationary random processes. For a stationary process the expression can be generalized, and it can be proven that the expected values of the random process will be constant and independent of time. Therefore, the autocorrelation function will depend only on the time difference and not the absolute time. The time difference is introduced as \( \tau = t_1 - t_2 \) and the autocorrelation \( \Pi_{xx}(\tau) \) can be expressed as:

\[
\Pi_{xx}(t, t + \tau) = \Pi_{xx}(\tau) = E[x(t)x(t + \tau)] \tag{B.3}
\]

Usually the whole random process is not available or described properly. In these cases, the autocorrelation can be estimated for a given interval, 0 to \( T \) seconds, of the sample function of the random process. The estimation of the autocorrelation \( \hat{\Pi}_{xx}(\tau) \) is given as:

\[
\hat{\Pi}_{xx}(t, t + \tau) = \frac{1}{T - \tau} \int_{0}^{T - \tau} x(t)x(t + \tau) dt \tag{B.4}
\]

This is given for the continuous case. It is usually not possible to describe the complete continuous-time function for the random signals, so the equation is modified in order to treat the information in a discrete-time formula. The discrete-time formulation for estimating the autocorrelation is as follows:

\[
\hat{\Pi}_{xx}[m] = \frac{1}{N - m} \sum_{n=0}^{N-m-1} x[n]x[n + m] \tag{B.5}
\]
where $N$ is the number of data points in the sample, $m$ is the data point corresponding to $\tau$ and $n$ the data point corresponding to $t$.

**Cross-correlation Function**

When dealing with multiple random processes, it is important to describe the relationship between the processes. This may for example occur if more than one random signal is applied to a system. The cross correlation function is defined as the expected value of the product of a random variable from one random process with a time-shifted, random variable from a different random process.

For a stationary process the expression for the cross-correlation can be written in terms of $\tau = t_1 - t_2$ since the expected values of the random process will be constant and independent of time, as described for the autocorrelation. Consider two random processes $x(t)$ and $y(t)$. Then the cross correlation function is defined as:

$$\Pi_{xy}(t, t + \tau) = \Pi_{xy}(\tau) = E[x(t)y(t + \tau)]$$  \hspace{1cm} (B.6)

The cross-correlation can be estimated for a given interval, 0 to $T$ seconds, of the sample functions of the random processes. The estimation of the cross-correlation $\hat{\Pi}_{xy}(\tau)$ is given as:

$$\hat{\Pi}_{xy}(\tau) = \frac{1}{T - \tau} \int_{0}^{T-\tau} x(t)y(t + \tau)dt$$  \hspace{1cm} (B.7)

The discrete-time formulation for estimating the cross-correlation is as follows:

$$\hat{\Pi}_{xy}[m] = \frac{1}{N - m} \sum_{n=0}^{N-m-1} x[n]y[n + m]$$  \hspace{1cm} (B.8)

where $N$ is the number of data points in the sample, $m$ is the data point corresponding to $\tau$ and $n$ the data point corresponding to $t$.

**Spectral density Function**

The spectral density functions can be defined in several ways. These are:

- Definition via correlation functions
- Definition via finite Fourier transforms
- Definition via filtering-squaring-averaging operations

Only the first two definitions will be mentioned here, for further details, see Bendat and Piersol (1986). The most common way to define the spectral density function is by means of the correlation function described in the previous section. The spectral density function is defined by taking a single Fourier Transform of the correlation function. The auto-spectral density function $S_{xx}(f)$ is thus defined as:

$$S_{xx}(f) = \int_{-\infty}^{\infty} \Pi_{xx}(\tau)e^{-i2\pi f\tau}d\tau$$  \hspace{1cm} (B.9)

Where $i = \sqrt{-1}$ is the imaginary unit and $f$ is the frequency. This approach gives a two-sided spectral density function $S_{xx}(f)$, which is defined for $f \in [-\infty, \infty]$. It
should be noted that the auto-spectral density function $S_{xx}(f)$ also is denoted the "power
spectral density function". As well as the auto-spectral density function is defined for a
autocorrelation function there exists a cross-spectral density function $S_{xy}(f)$, defined as:

$$S_{xy}(f) = \int_{-\infty}^{\infty} \Pi_{xy}(\tau)e^{-i2\pi f\tau}d\tau \quad (B.10)$$

The second way of defining the spectral density function is based on finite Fourier
Transforms of the original data series. Two random processes $x(t)$ and $y(t)$ are considered.
For a finite time interval $0 \leq t \leq T$ the spectral density function can be defined as:

$$S_{xx}(f, T) = \frac{1}{T}X^*(f, T)X(f, T) \quad (B.11a)$$
$$S_{xy}(f, T) = \frac{1}{T}X^*(f, T)Y(f, T) \quad (B.11b)$$

where

$$X(f, T) = \int_{0}^{T} x(t)e^{-i2\pi ft}dt \quad (B.12a)$$
$$Y(f, T) = \int_{0}^{T} y(t)e^{-i2\pi ft}dt \quad (B.12b)$$

$X(f, T)$ and $Y(f, T)$ represent finite Fourier transforms of $x(t)$ and $y(t)$, respectively,
and $X^*(f, T)$ is the complex conjugate of $X(f, T)$. The estimate of $S_{xx}(f)$ or $S_{xy}(f)$
when $T$ tends toward infinity is given by:

$$S_{xx}(f) = \lim_{T \to \infty} E[S_{xx}(f, T)] \quad (B.13a)$$
$$S_{xy}(f) = \lim_{T \to \infty} E[S_{xy}(f, T)] \quad (B.13b)$$

It can be shown that (B.13) is equal to (B.9) and (B.10) (Bendat and Piersol 1986).

It is not convenient to describe the frequency composition in the frequency range
from $-\infty$ to $\infty$. Hence, the spectral density function $S(f)$ is converted into a one-
sided spectral density function $G(f)$ where $f \in [0, \infty]$. The one-sided auto-spectral and
cross-spectral density functions $G_{xx}(f)$ and $G_{xy}(f)$ are defined as:

$$G_{xx}(f) = 2S_{xx}(f) \quad 0 \leq f \leq \infty \quad \text{otherwise zero} \quad (B.14a)$$
$$G_{xy}(f) = 2S_{xy}(f) \quad 0 \leq f \leq \infty \quad \text{otherwise zero} \quad (B.14b)$$
B.3 Basics of structural dynamics and modal analysis

In this section the basic equations for structural dynamics and modal analysis are presented. The section is based on Bendat and Piersol (1986) and Andersen (1997).

B.3.1 Dynamic model of second-order structural system

Dynamic structural systems subjected to external loading are often modelled as a lumped mass-spring-dashpot parameter model given by:

\[ M \ddot{z}(t) + C \dot{z}(t) + Kz(t) = f(t) \]  

(B.15)

\( M, C \) and \( K \) are the mass, damping and stiffness matrices with the dimensions \( p \times p \). \( z(t) \) and \( f(t) \) are \( p \times 1 \) displacement and force vectors at the mass points, respectively. Equation (B.15) is a second-order differential equation that represents the force equilibrium of the structural system. The inertial forces \( M \ddot{z} \) are balanced by a set of linear-elastic restoring forces \( Kz \), viscous damping forces \( C \dot{z} \) and the external forces \( f(t) \).

The general solution of the linear constant-parameter can be described by a weighting function \( h(\tau) \), also known as the unit impulse response function, which is defined as the output of the system at any time to a unit impulse input applied a time before (Bendat and Piersol 1986). \( h(\tau) \) has the dimension \( p \times p \). If it is assumed that the initial conditions are zero, i.e. \( z(0) = 0 \) and \( \dot{z}(0) = 0 \), then the solution can be written in terms of the following convolution integral:

\[
z(t) = \int_{0}^{t} h(\tau)f(t - \tau)d\tau, \quad \begin{cases} z(0) = 0 \\ \dot{z}(0) = 0 \end{cases} \]  

(B.16)

The expression in (B.16) states that the output \( z(t) \) is given as a weighted linear sum over the entire history of the input \( f(t) \).

The unit impulse response function \( h(\tau) \) describes the system in time domain. The system can also be described in the frequency domain by means of a frequency response function \( H(f) \). If the system parameters are constant and the system is linear then \( H(f) \) is defined as the Fourier Transform of \( h(\tau) \):

\[
H(f) = \int_{0}^{\infty} h(\tau)e^{-j2\pi ft}d\tau, \quad \text{or} \quad H(\omega) = \int_{0}^{\infty} h(\tau)e^{-j\omega \tau}d\tau \]  

(B.17)

where \( f \) is frequency, \( \omega \) is angular frequency and \( i \) is the imaginary unit.

B.3.2 Modal Analysis

Within the field of system identification is assumed that the estimated models can serve as a basis for a subsequent modal analysis of the structure. In the following it is shown how the modal information can be extracted from the second-order structural system in (B.15). For a particular mode, the \( j \)th mode, can be represented by various modal parameters. These are (Andersen 1997):

- Modal frequency:
The different parameters will be introduced and explained as they appear in the description.

**State space representation of the dynamic system**

The vibrations of the system in (B.15) are assumed to be viscously damped, and for that reason it is necessary to evaluate the eigenvalue problem of the system as complex in order to determine the modal parameters. The solution of the complex eigenvalue problem requires the construction of a $2p \times 2p$ system matrix and a $2p$ response vector. The response vector is denoted as the state vector of the system in (B.15). The state vector is defined in terms of displacements and velocities of the system:

$$x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}$$

(B.18)

By means of the state vector in (B.18) the second-order system in (B.15) can be reduced to a first-order differential equation system:

$$A \dot{z}(t) + Bz(t) = u(t)$$

$$A = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}, \quad B = \begin{bmatrix} K & 0 \\ 0 & -M \end{bmatrix}, \quad u(t) = \begin{bmatrix} f(t) \\ 0 \end{bmatrix}$$

(B.19)

The differential equation in (B.19) is denoted as the "state space representation" of the dynamic system.
**Free vibrations of the dynamic system**

The free vibration of the system in (B.19) is given by:

\[ \mathbf{A} \ddot{z}(t) + \mathbf{B} z(t) = 0 \quad (B.20) \]

The solution for (B.20) is assumed to be on the following form:

\[ x(t) = \Psi e^{\lambda t} \quad (B.21) \]

Where \( \Psi \) is a complex vector with dimensions \( 2p \times 1 \) and \( \lambda \) is a complex constant. When (B.21) is inserted into (B.20) it shows that (B.21) is a solution if and only if \( \Psi \) is a solution to the first-order eigenvalue problem given as:

\[ (\lambda \mathbf{A} + \mathbf{B}) \Psi = 0 \quad (B.22) \]

This leads to the characteristic polynomial of the eigenvalue problem:

\[ \det (\lambda \mathbf{A} + \mathbf{B}) = 0 \quad (B.23) \]

The order of the polynomial is \( 2p \) and has \( 2p \) roots \( \lambda_j, j = 1, \ldots, 2p \). For each of the roots \( \lambda_j \) a non-trivial solution \( \Psi_j \) to (B.23) exists. The solution vector \( \Psi_j \) is denoted an eigenvector. The system is assumed to be underdamped (typical for a broad range of civil engineering structures) and this means that the eigenvalues \( \lambda_j \) can be represented by complex conjugated pairs, given by:

\[ \lambda_j, \lambda_{j+1} = -2\pi f_j \zeta_j \pm i2\pi f_j \sqrt{1 - \zeta_j^2} = -\omega_j \zeta_j \pm i\omega_j \sqrt{1 - \zeta_j^2} \quad (B.24) \]

\[ \zeta_j < 1, \quad j = 1, 3, \ldots, 2p - 1 \]

Where \( f_j \) is the natural eigenfrequency, \( \omega_j \) is the angular eigenfrequency and \( \zeta_j \) the damping ratio of the \( j \)th underdamped mode. Note that both \( \lambda_j \) and \( \lambda_{j+1} \) is given in (B.24). From (B.18) and (B.19) it follows that the eigenvector has the form:

\[ \Psi_j = \begin{bmatrix} \Phi_j \\ \lambda_j \Phi_j \end{bmatrix}, \quad j = 1, 2, \ldots, 2p \quad (B.25) \]

The standard eigenvalue problem of the second-order system can be formulated if \( \mathbf{A} \) and \( \mathbf{B} \) and (B.25) is inserted into (B.22). This gives the following equation:

\[ (\lambda_j^2 \mathbf{M} + \lambda_j \mathbf{C} + \mathbf{K}) \Phi_j = 0, \quad j = 1, 2, \ldots, 2p \quad (B.26) \]

The vector \( \Phi_j \) is the non-trivial solution for the standard eigenvalue problem of the second-order system, and is denoted as the "mode shape". The eigenvectors \( \Psi_j \) for all the modes from \( j = 1 \) to \( 2p \) can be assembled in one matrix \( \Psi \) which defines the complex modal matrix for the system. \( \Psi \) is given as:

\[ \Psi = \begin{bmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_{2p} \\ \lambda_1 \Phi_1 & \lambda_2 \Phi_2 & \cdots & \lambda_{2p} \Phi_{2p} \end{bmatrix} \quad (B.27) \]
B.3 Basics of structural dynamics and modal analysis

The matrix in (B.28) has important orthogonal properties with respect to the matrix $A$. The special properties are as follows:

$$
\Psi^T A \Psi = M_d, \quad \Psi^T B \Psi = \Lambda M_d,
$$

where $\Lambda$ and $M_d$ are diagonal matrices containing $2p$ eigenvalues $\lambda_j$ and damped modal masses $m_j$, respectively.

The solution to the system in (B.15) is given by the convolution integral in (B.16). This function can conveniently be expressed in terms of the modal decomposed system as:

$$
h(\tau) = \sum_{j=1}^{2p} \frac{\Phi_j \Phi_j^T}{m_j} e^{\lambda_j \tau} = \sum_{j=1}^{2p} R_j e^{\lambda_j \tau}
$$

$m_j$ is the $j$th diagonal element of $M_d$ and $R_j$ is the residue matrix that corresponds to the $j$th eigenvalue.

The mode shapes are called "shapes" because they are unique in shape but not in value. That is, the mode shape vector for $\Phi_j$ each mode $j$ does not have unique values. The mode shape vector can be arbitrary scaled to any set of values, but relationship of one shape component to another is unique. In the system masses are known it is possible to scale the mode shapes so that the modal masses are unity. However, when the modal data is obtained from experimental spectral analyses (from experimental frequency transfer function measurements), no mass matrix is available for scaling. Without the mass matrix the experimental mode shapes can still be scaled to unit modal masses by using the relationship between residues and mode shapes:

$$
R_j = \alpha_j \Phi_j \Phi_j^T, \quad \alpha_j = \frac{1}{m_j \omega_j}
$$

where $\alpha_j$ is a scaling constant for the $j$th mode. The relation between $\alpha_j$ and $m_j$ is also shown in (B.30).

B.3.3 Spectral analysis of dynamic excited system

Stochastically excited system can be analysed in the frequency domain, if certain characteristics are satisfied. It is assumed that the system is linear and the applied excitation $f(t)$ is a stationary zero mean Gaussian distributed stochastic process. In this case the response $z(t)$ of the system is also a Gaussian distributed stochastic process. Since $f(t)$ is assumed zero mean, it can be fully described by its correlation function $\Pi_{ff}(\tau)$. The system is assumed linear so the response $z(t)$ of the system is also described by its correlation function $\Pi_{zz}(\tau)$.

By using (B.9) the auto spectral density functions $S_{ff}(\omega)$ and $S_{zz}(\omega)$ for $f(t)$ and $z(t)$ can be established. Note that $\omega$ is an arbitrary angular frequency. By introducing the frequency response function $H(\omega)$ from (B.17) it is possible to describe $S_{zz}(\omega)$ by means of $S_{ff}(\omega)$ in the following way:

$$
S_{zz}(\omega) = H(\omega) S_{ff}(\omega) H^H(\omega)
$$
where the superscript $^H$ is the Hermitian conjugate (equal to complex conjugate and transpose). In (B.17) it is shown that the frequency response function $H(\omega)$ is the Fourier transform of the unit impulse response function $h(\tau)$. Using this relation means that (B.29) can be transformed into frequency domain and the frequency response function $H(\omega)$ can be given as a partial fraction expansion:

$$H(\omega) = \sum_{j=1}^{2p} \frac{R_j}{i\omega - \lambda_j}$$  \hspace{1cm} (B.32)

where $\lambda_j$ and $R_j$ are the poles and residues of the partial fraction expansion, respectively. Suppose that the input $f(t)$ is Gaussian white noise, then the auto spectral density function $S_{ff}(\omega)$ is constant intensity matrix denoted by $F$. The spectral density function $S_{zz}(\omega)$ of the response $z(t)$ of a Gaussian white noise excited second-order system is then given by:

$$S_{zz}(\omega) = H(\omega)F H^H(\omega) = \sum_{j=1}^{2p} \sum_{k=1}^{2p} \frac{R_j F R_j^H}{(i\omega - \lambda_j)(i\omega - \lambda_k)}$$  \hspace{1cm} (B.33)
B.4 System Identification

The system identification by ARTeMIS (SVS 2006) operates with two different identification techniques, one in time domain and one in frequency domain. The two models are:

- Frequency Domain Decomposition (FDD)
- Stochastic Subspace Iteration (SSI)

The models are described briefly in the following.

B.4.1 ID by Frequency Domain Decomposition (FDD)

The frequency domain estimation is a non-parametric model (also known as spectral models) based on a Frequency Domain Decomposition (FDD) method. The FDD method is an extension of the well-known frequency domain approach that is based on model estimations directly from the Power Spectral Density (PSD) matrix, i.e. well separated modes can be identified at the peaks of the PSD matrix.

The basics of the identification algorithm are as follows. The estimate of power spectral density matrix $\hat{G}_{yy}(f)$ is determined by means of signal processing of the measured accelerations. $\hat{G}_{yy}(f)$ is a $N \times N$ matrix where $N$ is the number of channels, known at discrete frequencies $f = f_i$. The estimate of power spectral density matrix $\hat{G}_{yy}(f_i)$ is then decomposed by means of a Singular Value Decomposition (SVD) into a matrix of the form:

$$\hat{G}_{yy}(f_i) = U_i \Sigma_i U_i^H$$  \hspace{1cm} (B.34)

where $U_i = [u_{i1}, u_{i2}, \ldots, u_{iN}]$ is unitary matrix containing $N$ singular vectors $u_{ij}$. $\Sigma_i$ is a diagonal matrix containing $N$ scalar singular values $\sigma_{ij}^2$:

$$\Sigma_i = \begin{bmatrix}
\sigma_{i1}^2 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & \sigma_{iN}^2
\end{bmatrix}$$  \hspace{1cm} (B.35)

According to the theory of Frequency Domain Decomposition the first singular vector $u_{i1}$ is an estimate of the mode shape $\hat{\Phi}$ (Brincker et al. 2000; Brincker et al. 2000):

$$\hat{\Phi} = u_{i1}$$  \hspace{1cm} (B.36)

The corresponding singular value $\sigma_{iN}^2$ is then part of a power spectral density function of an equivalent single degree of freedom (SDOF) system. The relation in (B.36) may not seem obvious, but it becomes clear when (B.34) is compared to (B.33). Remember that $\hat{G}_{yy}(f_i)$ is a one-sided spectrum equal to $2\hat{S}_{yy}(f_i)$. In (B.33) the spectrum is given in terms of residues, and these residues are again given by mode shapes. This means that the modes shapes are related to the singular vectors in (B.34).

The power spectral density function of the SDOF system is identified around a peak (mode $k$ in Figure B.11) by comparing the mode shape estimate $\hat{\Phi}$ with singular vectors.
for the frequencies around the mode. The comparison is done by means of a Modal Assurance Criterion (MAC):

$$\text{MAC}(\Phi_r, \Phi_s) = \frac{|\Phi_r^H \Phi_s|^2}{|\Phi_r^H \Phi_r||\Phi_s^H \Phi_s|}$$ (B.37)

The MAC value is the square of correlation between two modal vectors $\Phi_r$ and $\Phi_s$. If the MAC value is unity the two vectors are identical within a modal scale factor. Further information about modal indicators is given in Zhang et al. (2001).

Figure B.11: Plot of singular values of the power spectral density matrix as a function of frequency. The singular values around the $k$th mode of the system (structure) belong to the SDOF power spectral density function.

If the singular vectors for the frequencies around the peak have a high MAC value with respect to the mode shape estimate $\hat{\Phi}$, the corresponding singular values belong to the SDOF density function. This is illustrated in Figure B.11 where the red part of the power spectral density function is a SDOF density function.

When the SDOF power spectral density function has been estimated for a mode, the corresponding singular vectors are averaged together to obtain an improved estimate of the mode shape. The natural frequency and the damping ratio of the mode is estimated by transforming the SDOF Spectral Bell to time domain by inverse FFT. This results in a SDOF Correlation Function, and by simple regression analysis the estimates of both the natural frequency as well as the damping ratio can be obtained.

### B.4.2 ID by Stochastic Subspace Iteration (SSI)

In the SSI techniques a parametric model is fitted directly to the raw time series data obtained from the accelerometers. The parametric models are characterized by the assumption of a mathematical model constructed from a set of parameters, where the mathematical model is a linear and time-invariant system of differential equations.

*Morten Liingaard*
A dynamic structural model can be described by a set of linear second-order differential equations with constant coefficients as stated in the previous section. The model is reproduced here:

$$M \ddot{z}(t) + C \dot{z}(t) + Kz(t) = f(t) \quad (B.38)$$

$M$, $C$ and $K$ are the mass, damping and stiffness matrices, and $z(t)$ and $f(t)$ displacement and force vectors, respectively. $(B.38)$ can be rewritten as a first-order system (rearrangement of $(B.19)$), given by:

$$\dot{x}(t) = A_c x(t) + B_c u(t) \quad x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}$$

$$\begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ -M^{-1}B_2 \end{bmatrix}, \quad f(t) = B_2 u(t) \quad (B.39)$$

Where $A_c$ is the state matrix and $B_c$ the system control influence coefficient matrix. Note that the excitation force $f(t)$ is factorized into a matrix $B_2$ describing the inputs in space and a vector $u(t)$ describing the inputs in time.

In practice, not all the degrees of freedom are monitored. The measurements (accelerations, velocity or displacement) are evaluated at a subsystem of nodes (or locations). The observation equation for the measurements is given by:

$$y(t) = C_a \ddot{z}(t) + C_v \dot{z}(t) + C_d z(t) \quad (B.40)$$

where $y(t)$ corresponds to the output in the monitored subsystem. $C_a$, $C_v$ and $C_d$ are the output matrices for acceleration, velocity and displacement, respectively. The output vector $y(t)$ can be transformed into:

$$y(t) = \Omega x(t) + Du(t)$$

$$\Omega = \begin{bmatrix} C_d - C_a M^{-1}K & C_v - C_a M^{-1}C \end{bmatrix}, \quad D = C_a M^{-1}B_2 \quad (B.41)$$

where $\Omega$ the output matrix and $D$ is a direct transmission matrix. $(B.39)$ and $(B.41)$ constitute a continuous-time state-space model of a dynamic system. Since experimental data are discrete in nature the continuous system is reformulated into a discrete system. The measurements are then available at discrete time instants $k\Delta t$. The discrete state space model is then given by:

$$x_{k+1} = Ax_k + Bu_k$$

$$y_k = \Omega x_k + Du_k \quad (B.42)$$

where $x(k\Delta t)$ is the discrete time state vector, $A = \exp(A_c \Delta t)$ is the discrete state matrix and $B = [A - I]A_c^{-1}B_c$ is the discrete input matrix. The equation in $(B.42)$ forms the discrete-time state space model of a dynamic system. The model in $(B.42)$ does not contain any uncertainties, such as process and measurement noise. There is always noise in practice, so the model in $(B.42)$ is extended by including stochastic components. The noise is included by two components, $w_k$ and $v_k$, where $w_k$ is process noise due to disturbances and modeling inaccuracies and $v_k$ is measurement noise due to sensor
Appendix B – Experimental modal analysis

inaccuracy. When the stochastic components are included the following discrete-time state space model is obtained:

\[ x_{k+1} = Ax_k + Bu_k + w_k \]
\[ y_k = \Omega x_k + Du_k + v_k \]  
(B.43)

In (B.43) it is assumed that the input \( u_k \) is known. This is not the case when the input is an unmeasurable stochastic process. In the case of such ambient vibrations it is impossible to distinguish the input term \( u_k \) from the noise terms \( w_k \) and \( v_k \). Modeling the input term \( u_k \) by the noise terms \( w_k \) and \( v_k \) results in a purely stochastic system:

\[ x_{k+1} = Ax_k + w_k \]
\[ y_k = \Omega x_k + v_k \]  
(B.44)

The equation in (B.44) constitutes the basis for the time-domain system identification technique, based on output only. The Stochastic Subspace Identification (SSI) technique is a class of techniques that are formulated and solved using the state space formulation in (B.44).

**Principle of SSI solution**

In order to solve (B.44), the system is reformulated. This includes three steps. First step is to determine \( x_k \). \( x_k \) is denoted as Kalman sequences that in SSI are found by means of a so-called orthogonal projection technique, see e.g. Van Overschee and De Moor (1996). Second step is to solve the regression problem for the matrices \( A \) and \( \Omega \) and for the residual noise components \( w_k \) and \( v_k \). The third step is to estimate a so-called Kalman gain matrix \( K_k \) so that the system can be written as a full covariance equivalent model:

\[ \dot{x}_{k+1} = Ax_k + K_k e_k \]
\[ y_k = \Omega x_k + e_k \]  
(B.45)

where \( K_k \) is the Kalman gain matrix, \( e_k \) is called the innovation and is a zero-mean Gaussian white noise process and \( x_k \) is the predicted state vector. It can be shown that by performing a modal decomposition of the \( A \) matrix as \( A = V [\mu_j] V^{-1} \) and introducing a new state vector \( z_k \) the equation in (B.45) can also be written as:

\[ z_{k+1} = [\mu_j] z_k + \Psi e_k \]
\[ y_k = \Phi z_k + e_k \]  
(B.46)

where \([\mu_j]\) is a diagonal matrix containing the discrete eigenvalues. The natural frequencies \( f_j \) and damping ratios \( \zeta_j \) are extracted using the following definition:

\[ \mu_j = \exp \left( -2\pi f_j \left( \zeta_j \pm \sqrt{1 - \zeta_j^2} \right) \Delta t \right) \]  
(B.47)

where \( \Delta t \) is the sampling interval. The mode shape that are associated with the \( j \)th mode is given by the \( j \)th column of the matrix \( \Phi \). The last matrix \( \Psi \) that completes the modal decomposition contains a set of row vectors. The \( j \)th row vector corresponds to the \( j \)th mode. This vector distributes the white noise excitation \( e_k \) in modal domain to all the degrees of freedom.

**Morten Liingaard**
A lumped-parameter model represents the frequency dependent soil-structure interaction of a massless foundation placed on or embedded into an unbounded soil domain. The lumped-parameter model development have been reported by (Wolf 1991b; Wolf 1991a; Wolf and Paronesso 1991; Wolf and Paronesso 1992; Wolf 1994; Wolf 1997; Wu and Lee 2002; Wu and Lee 2004).

In this appendix the the steps of establishing a lumped-parameter model are presented. Following sections are included in this appendix: Static and dynamic formulation (Section C.1), Simple lumped-parameter models (Section C.2) and Advanced lumped-parameter models (Section C.3).

C.1 Static and Dynamic Stiffness Formulation

The elastic behaviour of foundations is relevant in several situations. The elastic response of footings is used to evaluate of deformations during working loads (in serviceability conditions) and may be used as the "elastic zone" for advanced elasto-plastic macro-models of foundations, see e.g. Martin and Houlsby (2001) and Houlsby and Cassidy (2002). The dynamic response of the wind turbine structure (e.g. eigen frequencies/modes) are affected by the properties of the foundation. The purpose of this research is to provide accurate means of evaluation of the dynamic properties of the foundation, so that it can be properly included in a composite structure-foundation system. The typical approach is that each analysis of the composite system should employ a complete analysis (using for instance finite-element method) of both the structure and foundation. Such an approach is, however, inefficient and time consuming, as for practical purposes the foundation system can be treated as a substructure with predetermined properties. The interactions between the foundation and structure are then expressed purely in terms of force and moment resultants, and their conjugate displacements and rotations, see Figure C.1.

C.1.1 Static stiffness

The elastic static stiffness of the foundation can be expressed by dimensionless elastic stiffness coefficients corresponding to vertical ($K^0_{VV}$), horizontal ($K^0_{HH}$), moment ($K^0_{MM}$) and torsional ($K^0_{TT}$) degrees of freedom. Cross-coupling between horizontal and moment
loads exists so an additional cross-coupling term \( K_{MH}^0 \) is necessary. Under general (combined) static loading (see Figure C.1) the elastic stiffness of the foundation system can be expressed as

\[
\begin{bmatrix}
 V/G_s R^2 \\
 H_1/G_s R^2 \\
 H_2/G_s R^2 \\
 T/G_s R^3 \\
 M_1/G_s R^3 \\
 M_2/G_s R^3
\end{bmatrix}
= \begin{bmatrix}
 K_{VV}^0 & 0 & 0 & 0 & 0 & 0 \\
 0 & K_{HH}^0 & 0 & 0 & 0 & -K_{MH}^0 \\
 0 & 0 & K_{HH}^0 & 0 & K_{MM}^0 & 0 \\
 0 & 0 & 0 & K_{TT}^0 & 0 & 0 \\
 0 & 0 & K_{MH}^0 & 0 & K_{MM}^0 & 0 \\
 0 & -K_{MH}^0 & 0 & 0 & 0 & K_{MM}^0
\end{bmatrix}\begin{bmatrix}
 W/R \\
 U_1/R \\
 U_2/R \\
 \theta_T \\
 \theta_{M1} \\
 \theta_{M2}
\end{bmatrix},
\] (C.1)

where \( R \) is the radius of the foundation and \( G_s \) is the shear modulus of the soil. The shear modulus \( G_s \) is given by

\[
G_s = \frac{E_s}{2(1 + \nu_s)}
\] (C.2)

where \( E_s \) is Young’s modulus and \( \nu_s \) is Poisson’s ratio. Note that the foundation is assumed to be rigid and the soil is linear elastic, i.e. the properties are given by \( G_s \) and \( \nu_s \). This means that the stiffness components \( K_{ij}^0 \) \((i, j = H, M, T, V)\) in C.1 are functions of Poisson’s ratio.

### C.1.2 Dynamic stiffness

It is assumed that the foundation is excited with a harmonic vibrating force with the circular frequency \( \omega \). The dynamic system for a vertical vibrating surface footing with no mass is shown in Figure C.2(a). For each degree of freedom the dynamic stiffness of the system can be represented by a frequency dependent spring and dashpot, as shown in Figure C.2(b).

A generalized massless axisymmetric rigid foundation has six degrees of freedom: one vertical, two horizontal, two rocking and one torsional. The six degrees of freedom and the corresponding forces and moments are shown in Figure C.1. The dynamic stiffness matrix \( S \) is related to the vector of forces and moments \( R \) and the vector of displacements and rotations \( U \) as follows:

\[
R = SU
\] (C.3)

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The component form of Equation (C.3) can be written as:

\[
\begin{bmatrix}
\frac{V}{G_s R^2} \\
\frac{H_1}{G_s R^2} \\
\frac{H_2}{G_s R^2} \\
\frac{T}{G_s R^3} \\
\frac{M_1}{G_s R^3} \\
\frac{M_2}{G_s R^3}
\end{bmatrix}
= \begin{bmatrix}
S_{VV} & 0 & 0 & 0 & 0 & 0 \\
0 & S_{HH} & 0 & 0 & 0 & -S_{MH} \\
0 & 0 & S_{HH} & 0 & S_{MH} & 0 \\
0 & 0 & 0 & S_{TT} & 0 & 0 \\
0 & 0 & S_{MH} & 0 & S_{MM} & 0 \\
0 & -S_{MH} & 0 & 0 & 0 & S_{MM}
\end{bmatrix}
\begin{bmatrix}
W/R \\
U_1/R \\
U_2/R \\
\theta_T \\
\theta_M \theta_1 \\
\theta_M \theta_2
\end{bmatrix}
\]

where \( R \) is the radius of the foundation and \( G_s \) is the shear modulus of the soil. The components in \( \mathbf{S} \) are functions of the cyclic frequency \( \omega \), and \( \mathbf{S} \) reflects the dynamic stiffness of the soil for a given shape of the foundation. The components of \( \mathbf{S} \) can be written as:

\[
S_{ij}(\omega) = K_{ij}(\omega) + i \omega C_{ij}(\omega), \quad (i, j = H, M, T, V),
\]

where \( K_{ij} \) and \( C_{ij} \) are the dynamic stiffness and damping coefficients with respect to \( \omega \), respectively, and \( i \) is the imaginary unit, \( i = \sqrt{-1} \). It is convenient to use dimensionless frequency \( a_0 = \omega R/c_S \) that is normalized by the ratio of the foundation radius \( R \) and the shear wave velocity of the soil \( c_S \). The dynamic stiffness components can then be written as:

\[
S_{ij}(a_0) = K_{ij}^0[k_{ij}(a_0) + i a_0 c_{ij}(a_0)], \quad (i, j = H, M, T, V),
\]

where \( K_{ij}^0 \) is the static value of \( ij \)th stiffness component, \( k_{ij} \) and \( c_{ij} \) are the dynamic stiffness and damping coefficients with respect to \( a_0 \), respectively. The non-dimensional dynamic stiffness and damping coefficients, \( k_{ij} \) and \( c_{ij} \), are both real. Both geometrical damping, i.e. the radiation of waves into the subsoil, and possibly also material dissipation contribute to \( c_{ij} \).

In some situations it is useful to examine the magnitude and phase angle of Equation (C.6) in addition to the real and imaginary parts of the dynamic stiffness. The
magnitude (complex modulus) of $S_{ij}$ is given by

$$|S_{ij}| = |K^0_{ij}| \sqrt{|(k_{ij})^2 + (a_0c_{ij})^2|}, \quad \text{(C.7)}$$

and the phase angle $\phi_{ij}$ of $S_{ij}$ is given as

$$\phi_{ij} = \arctan \left( \frac{a_0c_{ij}}{k_{ij}} \right). \quad \text{(C.8)}$$

Note that the above-mentioned stiffness formulations are based on the fact that the foundation is rigid. This means that components in $S$ are functions of Poisson’s ratio $\nu_s$ and the circular frequency $\omega$ (if dynamic) for a given shape of the foundation.

### C.2 Simple lumped-parameter models

The frequency dependency of the foundation stiffness is taken into account, by applying lumped-parameter models. Two types of models are categorized as simple models: The standard lumped-parameter model and the fundamental lumped-parameter model. The presentation of the models is based on Wolf (1994).

#### C.2.1 Standard lumped-parameter model

The standard lumped-parameter model contains three coefficients, $K$, $C$ and $M$, for each degree of freedom, see Figure C.3. The spring stiffness $K$ is equal to the static stiffness coefficient for the elastic half-space, thus $K$ is given by the expressions in section C.1.1.

The dashpot and mass coefficients, $C$ and $M$, do not have physical meaning but are solely curve fitting parameters, used to reproduce the dynamic stiffness of the foundation. The parameters $C$ and $M$ are given by two non-dimensional coefficients $\gamma$ and $\mu$ by

$$C = \frac{R}{cS} \gamma K \quad \text{(C.9a)}$$
$$M = \frac{R^2}{c^2S} \mu K. \quad \text{(C.9b)}$$

The values of $K$, $\gamma$ and $\mu$ for a circular disk with mass on a elastic half-space are given in Table C.1 (Reproduced from Wolf (1994)).

Note that the inertia of the disk $m$ (mass moment of inertia for rocking vibrations) enters the expressions for $\gamma$ with respect to rocking and torsional vibrations in the expressions given by Wolf (1994). However, it is possible to construct the parameters for a massless foundation.

Based on the three coefficients, $K$, $C$ and $M$, the dynamic stiffness for each degree of freedom can be formulated as

$$S(\omega) = K - \omega^2 M + i\omega C. \quad \text{(C.10)}$$

The dynamic stiffness in Equation (C.10) can be rewritten in terms of the non-dimensional frequency $a_0$ as

$$S(a_0) = K [k(a_0) + ia_0c(a_0)] \quad \text{(C.11)}$$

\textit{Morten Liingaard}
Table C.1: Non-dimensional coefficients for the standard lumped-parameter model. The coefficients correspond to a disk with mass on an elastic half-space.

<table>
<thead>
<tr>
<th></th>
<th>Static stiffness $K$</th>
<th>Dashpot coeff. $\gamma$</th>
<th>Mass coeff. $\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal</td>
<td>$\frac{8G_s R}{2-\nu_s}$</td>
<td>0.58</td>
<td>0.095</td>
</tr>
<tr>
<td>Vertical</td>
<td>$\frac{4G_s R}{1-\nu_s}$</td>
<td>0.85</td>
<td>0.27</td>
</tr>
<tr>
<td>Rocking</td>
<td>$\frac{8G_s R^3}{3(1-\nu_s)} \left(1 + \frac{3(1-\nu_s)\mu}{8R^4\rho_s}\right)$</td>
<td>0.3</td>
<td>0.24</td>
</tr>
<tr>
<td>Torsional</td>
<td>$\frac{16G_s R^3}{3} \left(1 + \frac{3m}{R^2\rho_s}\right) \sqrt{\frac{m}{R^2\rho_s}}$</td>
<td>0.433</td>
<td>0.045</td>
</tr>
</tbody>
</table>

By comparing Equations (C.9) and (C.10) with Equation (C.11) it becomes evident that the spring and damping coefficients $k(a_0)$ and $c(a_0)$ can be written as

$$k(a_0) = 1 - \mu a_0^2$$  \hspace{1cm} (C.12a)
$$c(a_0) = \gamma.$$ \hspace{1cm} (C.12b)

It turns out that the damping term $c(a_0)$ of the standard lumped-parameter model is constant. This behaviour is not well suited to represent the damping of a footing, in particular with respect to the torsional and rocking vibrations. Further, the normalized real part of Equation (C.11) given by $k(a_0)$ in Equation (C.12) describes a parabolic shape of the dynamic stiffness. The parabolic shape may represent the actual dynamic stiffness of a given foundation at low frequencies, but is inadequate for modelling the dynamic stiffness at intermediate and high frequencies. The standard lumped-parameter approximation of the vertical dynamic stiffness of a massless circular rigid footing is illustrated in Figure C.4. The approximation is compared with a rigorous solution provided by Veletsos and Tang (1987).

The main advantage of the standard lumped-parameter is that no additional degrees of freedom are introduced. However, the frequency dependent representation of the dynamic stiffness is very simple. Thus, the model is restricted to be used in the low-frequency range.

![Figure C.3: Standard lumped-parameter model for translation motion.](image-url)
C.2.2 **Fundamental lumped-parameter models**

The fundamental lumped-parameter model consists of one static stiffness parameter and four free parameters, found by curve fitting. As opposed to the standard lumped-parameter model, this type of model contains one additional internal degree of freedom. The fundamental lumped-parameter model can be assembled in several ways by combining spring, dashpots and masses. Two examples are shown in Figure C.5. The spring stiffness \( K \) is equal to the static stiffness coefficient for the elastic half-space, given by the expressions in section C.1.1. The remaining four *free parameters* are obtained by curve fitting. The spring-dashpot model in Figure C.5a is represented by the parameters \( M_0, C_0, K_1 \) and \( C_1 \), whereas the monkey-tail model in Figure C.5b is represented by the parameters \( M_0, C_0, M_1 \) and \( C_1 \). Consider the monkey-tail model. The four free parameters \( M_0, C_0, M_1 \) and \( C_1 \) can be formulated by means of the non-dimensional coefficients \( \mu_0, \gamma_0, \mu_1 \) and \( \gamma_1 \) as

\[
M_0 = \frac{R^2}{c_S^2} \mu_0 K \\
C_0 = \frac{R}{c_S} \gamma_0 K \\
M_1 = \frac{R^2}{c_S^2} \mu_1 K \\
C_1 = \frac{R}{c_S} \gamma_1 K
\]

The values of \( K, \mu_0, \gamma_0, \mu_1 \) and \( \gamma_1 \) for a circular disk on an elastic half-space are given in Table C.2 (Reproduced from Wolf (1994)). Most of the coefficients, except for torsional vibrations, depend on \( \nu_s \). Note that some of the non-dimensional coefficients may be

*Figure C.4: Vertical dynamic stiffness of a massless circular footing on an elastic half-space. The rigorous solution is compared with the approximation of a standard lumped-parameter model.*
Table C.2: Non-dimensional coefficients for the fundamental lumped-parameter model. The coefficients correspond to a disk on an elastic half-space.

<table>
<thead>
<tr>
<th></th>
<th>Static stiffness</th>
<th>Dashpots</th>
<th>Masses</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$K$</td>
<td>$\gamma_0$</td>
<td>$\gamma_1$</td>
</tr>
<tr>
<td>Horizontal</td>
<td>$\frac{8G_s R}{2-v_s}$</td>
<td>0.78-0.4$v_s$</td>
<td>—</td>
</tr>
<tr>
<td>Vertical</td>
<td>$\frac{4G_s R}{1-v_s}$</td>
<td>0.8</td>
<td>0.34-4.3$v_s^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$v_s &gt; \frac{1}{3}$</td>
</tr>
<tr>
<td>Rocking</td>
<td>$\frac{8G_s R^3}{3(1-v_s)}$</td>
<td>—</td>
<td>0.42-0.3$v_s^2$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$v_s &gt; \frac{1}{3}$</td>
</tr>
<tr>
<td>Torsional</td>
<td>$\frac{16G_s R^3}{3}$</td>
<td>0.017</td>
<td>0.291</td>
</tr>
</tbody>
</table>

missing for some of the vibration modes.

The dynamic stiffness $S(\omega)$ of the fundamental lumped-parameter model (for harmonic loading) can be established by formulating the equilibrium equation for each of the two degrees of freedom, $u_0(\omega)$ and $u_1(\omega)$ in Figure C.5b. The two equilibrium equations are

\begin{align*}
-\omega^2 M_1 u_1(\omega) + i \omega C_1 \left[u_1(\omega) - u_0(\omega)\right] &= 0, \tag{C.14a} \\
-\omega^2 M_0 u_0(\omega) + i \omega (C_0 + C_1) u_0(\omega) - i \omega C_1 u_1(\omega) + K u_0(\omega) &= P_0(\omega), \tag{C.14b}
\end{align*}

where $u_0(\omega)$ is the displacement amplitude related to the applied load amplitude $P_0(\omega)$. By eliminating $u_1(\omega)$ in Equations (C.14a) and (C.14b) the relation between $P_0(\omega)$ and
Figure C.6: Vertical dynamic stiffness of a massless circular footing on an elastic half-space. The rigorous solution is compared with the approximation of a fundamental lumped-parameter model.

\[ P_0(\omega) = K \left[ 1 - \frac{\omega^2 M_0}{K} - \frac{\omega^2 M_0}{K} C_1 \right] + i \omega \left( \frac{M_1}{C_1} \gamma_1 + \frac{C_0}{K} \right) u_0(\omega). \]  

(C.15)

The dynamic stiffness in Equation (C.15) can be rewritten in terms of the non-dimensional frequency \( a_0 \) as stated in Equation (C.11). By substituting Equation (C.13) into Equation (C.15), the spring and damping coefficients \( k(a_0) \) and \( c(a_0) \) of the fundamental lumped-parameter model (monkey-tail version) can determined as

\[ k(a_0) = 1 - \frac{\mu_1 a_0^2}{1 + \frac{\mu_1}{\gamma_1} a_0^2} - \mu_0 a_0^2 \]  

(C.16a)

\[ c(a_0) = \frac{\mu_1}{\gamma_1} \frac{a_0^2}{1 + \frac{\mu_1}{\gamma_1} a_0^2} + \gamma_0. \]  

(C.16b)

As opposed to the standard lumped-parameter model, the fundamental lumped-parameter model is double-asymptotic, meaning that the approximation of \( S(a_0) \) is exact for the static limit, \( a_0 \to 0 \), and for the high-frequency limit, for \( a_0 \to \infty \). The fundamental lumped-parameter approximation of the vertical dynamic stiffness of a massless circular rigid footing is illustrated in Figure C.6. The approximation is compared with a rigorous solution provided by Veletsos and Tang (1987). By including an additional degree of freedom this approximation has approved when comparing with the standard lumped-parameter model in the previous section. Note that the procedure for establishing the formulation for spring-dashpot model is similar to that of the monkey-tail model. The only difference is the characteristics of the non-dimensional coefficients.
Figure C.7: Vertical dynamic stiffness of a massless circular footing on an elastic half-space. The rigorous solution is compared with the approximation of both the standard and the fundamental lumped-parameter model.

The approximation of the fundamental lumped-parameter model is compared with that of the standard lumped-parameter model in Figure C.7. The approximations are shown for the real and imaginary part of the dynamic stiffness, as well as for the magnitude and phase angle.
C.3 Advanced lumped-parameter models

The investigations of frequency dependent behaviour of massless foundations often involves complicated three-dimensional elastodynamic analyses using rigorous methods, such as the finite element method or the boundary element method. The employed models typically consist of several thousand degrees of freedom, and the frequency dependent dynamic stiffness of the foundations are evaluated in the frequency domain. The requirement for real-time computations in the time domain in aero-elastic codes do not conform with the use of e.g. a three-dimensional coupled Boundary Element/Finite Element Method, where the foundation stiffness is evaluated in the frequency domain.

In order to meet the requirements of real-time calculations and analysis in time domain, lumped-parameter models are particularly useful. A lumped-parameter model represents a unbounded soil domain, and the soil-structure interaction of a massless foundation can be modelled by relatively few springs, dashpots and masses, all with real frequency-independent coefficients. Each degree of freedom at the foundation node of the structural model is coupled to a lumped-parameter model that may consist of additional internal degrees of freedom. A systematic procedure to obtain consistent lumped-parameter models with real coefficients has been suggested by Wolf (1991b). The procedure is as follows:

♦ Determine the frequency dependent impedance or dynamic stiffness \( S(a_0) \) by means of the finite element method or the boundary element method.

♦ Decompose the dynamic stiffness \( S(a_0) \) into a singular part \( S_s(a_0) \) and a regular part \( S_r(a_0) \). The singular part \( S_s(a_0) \) represents the asymptotic value of the dynamic stiffness for \( a_0 \to \infty \). The difference between \( S(a_0) \) and \( S_s(a_0) \) is the regular part \( S_r(a_0) \).

♦ Approximate the regular part \( S_r(a_0) \) by the ratio of two polynomials \( P \) and \( Q \). The degree of the polynomial in the denominator is \( M \) and one less \( (M - 1) \) in the the numerator. The approximation of the regular part \( S_r(a_0) \) now contains \( 2M - 1 \) unknown real coefficients, which are determined by a curve-fitting technique based on the least-squares method.

♦ Establish the lumped-parameter model from the \( 2M - 1 \) real coefficients. The lumped-parameter model may contain several constant/linear, first-order and second-order discrete-element models. Finally, the lumped-parameter model is formulated into stiffness, damping and mass matrices, which can be incorporated into standard dynamic programs.

The four steps in the procedure are explained in the following sections. It should be noted that the lumped-parameter models do not provide any information of the stresses or strains in the embedded foundations or in the surrounding subsoil. The models are macro-models of the entire soil-structure interface.

C.3.1 Dynamic stiffness obtained from rigorous methods

The classical methods for analysing vibrations of foundations are based on analytical solutions for massless circular foundations resting on an elastic half-space. The classical
solutions by Reissner, Quinlan and Sung were obtained by integration of Lamb’s solution for a vibrating point load on a half-space (Richart et al. 1970; Das 1993). The mixed boundary value problems with prescribed conditions under the foundation and zero traction at the remaining free surface were investigated by Veletsos and Wei (1971) and Luco and Westmann (1971). The integral equations of the mixed boundary value problems were evaluated and tabulated for a number of excitation frequencies. A closed-form solution has been presented by Krenk and Schmidt (1981).

Whereas analytical and semi-analytical solutions may be formulated for surface footings with a simple geometries, numerical analysis is required in the case of flexible embedded foundations with complex geometry. The Finite Element Method (FEM) is very useful for the analysis of structure with local inhomogeneities and complex geometries. However, only a finite region can be discretized. Hence, at the artificial boundaries of the unbounded domain, e.g. soil, transmitting boundary conditions must be applied as suggested by Higdon (1990), Higdon (1992) and Krenk (2002). Numerous concepts, including the Scaled Boundary Finite Element Method are presented by Wolf and Song (1996), and Andersen (2002) gave a brief overview of different solutions techniques. In the case of analyses by coupled boundary element/finite element models, wave radiation into the subsoil is ensured by a coupling with the boundary element method. If the full-space fundamental solution is utilized, both the soil–foundation interface and the free soil surface must be discretized. A smaller numerical model, i.e. a model with fewer degrees of freedom, may be obtained with the use of other types of solutions, e.g. half-space solutions. However, this comes at the cost that the fundamental solution can be very complicated, and often a closed-form solution cannot be found. The work within the boundary element formulation of dynamic soil–structure interaction has been reported by, for example, Domínguez (1993), Beskos (1987) and Beskos (1997).

C.3.2 Decomposition of the dynamic stiffness

The complex frequency dependent dynamic stiffness coefficient for each degree of freedom is denoted by $S(a_0)$. In the following the indices are omitted for simplicity. $S(a_0)$ are then decomposed into a singular part $S_s(a_0)$, and a regular part $S_r(a_0)$, given by

$$S(a_0) = S_s(a_0) + S_r(a_0), \quad \text{(C.17)}$$

where

$$S_s(a_0) = K^0 [k^\infty + i a_0 c^\infty]. \quad \text{(C.18)}$$

For the limit $a_0 \to \infty$, the second term on the right-hand side dominates, leading to a high-frequency limit

$$S_s(a_0) \approx K^0 [i a_0 c^\infty]. \quad \text{(C.19)}$$

The high-frequency behaviour of a surface footing is characterized by a phase angle approaching $\pi/2$ for $a_0 \to \infty$ and a linear relation that passes through origo in a frequency vs. magnitude diagram. The slope of the curve is equal to a limiting damping parameter $c^\infty$ that describes the impedance for $a_0 \to \infty$. For example, the vertical limiting damping

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parameter in terms of $\omega$ of a circular surface footing is given by

$$C^\infty_{VV} = \rho_s c_P A_b,$$  \hspace{1cm} (C.20)

where $A_b$ is the area of the base of the foundation. The vertical limiting damping parameter in terms of $a_0$ can be found by multiplying by the right-hand side in Equation (C.20) by $\frac{c_s}{K^0 R}$. (Recall that $a_0 = \omega R/c_s$). The vertical limiting damping parameter $c^\infty_{VV}$ is then given as

$$c^\infty_{VV} = [\rho_s c_P A_b] \frac{c_s}{K^0 R}. \hspace{1cm} (C.21)$$

It should be noted that $C^\infty_{VV}$ or $c^\infty_{VV}$ are highly sensitive to $\nu_s$ due to the fact that $c_P$ enters the equation. For that reason $c_P$ may be inappropriate, and Gazetas and Dobry (1984) suggest the use of Lysmer’s analog ‘wave velocity’ $c_{La} = 3.4c_s/\pi(1 - \nu_s)$. Wolf (1994) suggests another approach where $c_P$ for $\nu_s \in [1/3; 0.5]$ is constant, and equal to $c_P$ at $\nu_s = 1/3$.

Note that the singular part $S_s(a_0)$ is relatively simple to determine. $S_s(a_0)$ is a function of the mass density of the soil, the wave velocity of the soil, and the base area or moment of inertia of the foundation. The base area enters the equation for translational degrees of freedom, whereas the moment of inertia and the polar moment of inertia enters for the rocking and the torsional degree of freedom, respectively.

The remaining part $S_r(a_0)$ is found by subtracting $S_s(a_0)$ from $S(a_0)$. The regular part is used as input for the curve-fitting procedure described in the next section.

### C.3.3 Polynomial-fraction approximation

The regular part $S_r(a_0)$ of the dynamic stiffness is approximated by the ratio of two polynomials $P$ and $Q$. Furthermore, it is assumed that the polynomial-fraction approximation can be established in terms of $i a_0$. The approximation of $S_r(a_0)$ in terms of $P$ and $Q$ is then

$$S_r(a_0) \approx S_r(i a_0) = \frac{P(i a_0)}{Q(i a_0)} = K^0 \frac{K^a - K^b}{K^0} + p_1(i a_0) + p_2(i a_0)^2 + \cdots + p_{M-1}(i a_0)^{M-1},$$

where $p_i$ and $q_i$ are the $2M - 1$ unknown real coefficients to be determined by curve-fitting. Note that the degree of the polynomial in the denominator is $M$, and $M - 1$ in the the numerator.

The total approximation of $S(a_0)$ is found by adding Equations (C.18) and (C.22) as stated in Equation (C.17). The approximation has two important characteristics: The approximation of $S(a_0)$ is exact for the static limit, where $S(a_0) \rightarrow K^0$ for $a_0 \rightarrow 0$, and for the high-frequency limit, where $S(a_0) \rightarrow S_s(a_0)$ for $a_0 \rightarrow \infty$, since $S_r(a_0) \rightarrow 0$ for $a_0 \rightarrow \infty$. This means that the approximation is double-asymptotic.

The $2M - 1$ unknown real coefficients in Equation (C.22) are computed by a MATLAB routine. The inputs are: the complex values of $S_r(a_0)$, the corresponding frequencies, and the degrees of the polynomials in the denominator and the numerator of Equation (C.22). The routine returns the real coefficients $p_i$, $q_i$ of $S_r(a_0)$. 

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C.3.4 Discrete models for partial-fraction expansions

The polynomial-fraction approximation in Equation (C.22) can be formulated by a partial-fraction expansion, given by

\[
S_r(i\alpha_0) = \sum_{l=1}^{M} \frac{A_l}{i\alpha_0 - s_l},
\]

where \( s_l \) are the poles of \( S_r(i\alpha_0) \), and \( A_l \) are the residues at the poles. In order to obtain a stable system, the real part of all the poles must be negative, otherwise the approximation may become unstable. This criterion can be handled by using an iterative algorithm to find a stable approximation to the system.

The polynomial coefficients \( p_i, q_i \) of Equation (C.22) can be converted into a partial-fraction expansion by routines in e.g. MATLAB (use the function \texttt{residue} that converts between partial-fraction expansion and polynomial coefficients). Some of the poles \( s_l \) may be complex, resulting in complex conjugate pairs of \( s_l \). Consequently, the corresponding residues \( A_l \) also appear as complex conjugate pairs. When two complex conjugate pairs are added, a second-order term with real coefficients appears. For \( J \) conjugate pairs, Equation (C.23) can be rewritten as

\[
S_r(i\alpha_0) = \sum_{l=1}^{J} \frac{\beta_{1l}i\alpha_0 + \beta_{0l}}{(i\alpha_0)^2 + \alpha_{1l}i\alpha_0 + \alpha_{0l}} + \sum_{l=1}^{M-2J} \frac{A_l}{i\alpha_0 - s_l}.
\]

The coefficients \( \alpha_{0l}, \alpha_{1l}, \beta_{0l}, \beta_{1l} \) are given by

\[
\begin{align*}
\alpha_{0l} &= s_{1l}^2 + s_{2l}^2, \\
\alpha_{1l} &= -2s_{1l}, \\
\beta_{0l} &= -2(A_{1l}s_{1l} + A_{2l}s_{2l}), \\
\beta_{1l} &= 2A_{1l},
\end{align*}
\]

where the real and imaginary parts of the complex conjugate poles are denoted by \( s_{1l} \) and \( s_{2l} \), respectively. Similar, real and imaginary parts of the complex conjugate residues are denoted by \( A_{1l} \) and \( A_{2l} \), respectively.

By adding the singular term in Equation (C.18) to the expression in Equation (C.23), the total approximation of the dynamic stiffness can be written as

\[
\frac{S(i\alpha_0)}{K^0} = k^\infty + i\alpha_0c^\infty + \sum_{l=1}^{J} \frac{\beta_{1l}i\alpha_0 + \beta_{0l}}{(i\alpha_0)^2 + \alpha_{1l}i\alpha_0 + \alpha_{0l}} + \sum_{l=1}^{M-2J} \frac{A_l}{i\alpha_0 - s_l}.
\]

The total approximation of the dynamic stiffness in Equation (C.26) consists three characteristic types of terms: a constant/linear term, first-order terms and second-order terms. These terms are given as

\[
\begin{align*}
\text{Constant/linear term} & \quad k^\infty + i\alpha_0c^\infty \\
\text{First-order term} & \quad \frac{A}{i\alpha_0 - s} \\
\text{Second-order term} & \quad \frac{\beta_{1l}i\alpha_0 + \beta_{0l}}{(i\alpha_0)^2 + \alpha_{1l}i\alpha_0 + \alpha_{0l}}.
\end{align*}
\]
The number of first- or second-order terms in the approximation depends on the choice of polynomial degree \( M \). Each term can be represented by a discrete-element model, similar to those in Figures C.3 and C.5. The discrete-element models for the three types of terms in equation (C.27) are introduced in the next subsections.

**Constant/linear term**

The constant/linear term given by Equation (C.27a) consists of two known parameters, \( k^\infty \) and \( c^\infty \), that represents the singular part of the dynamic stiffness. The discrete-element model for the constant/linear term is shown in Figure C.8. The equilibrium formulation of node 0 (for harmonic loading) is as follows

\[
[kK] u_0(\omega) + i\omega \left[ \frac{R}{c_S}K \right] u_0(\omega) = P_0(\omega) \tag{C.28}
\]

Recalling that \( a_0 = \omega R/c_S \) the equilibrium formulation in Equation (C.28) results in a force-displacement relation given by

\[
\frac{P_0(a_0)}{K} = (\kappa + ia_0\gamma) u_0(a_0). \tag{C.29}
\]

By comparing Equation (C.27a) and Equation (C.29) it is evident that the non-dimensional coefficients, \( \kappa \) and \( \gamma \), are equal to \( k^\infty \) and \( c^\infty \), respectively.

**First-order term**

The first-order term given by Equation (C.27b). The model has two known parameters, \( A \) and \( s \). The layout of the discrete-element model is shown in Figure C.9(a). The model is constructed by a spring \( (-\kappa K) \) in parallel with another spring \( (\kappa K) \) and dashpot \( (\gamma \frac{R}{c_S}K) \) in series. The serial connection between the spring \( (\kappa K) \) and the dashpot \( (\gamma \frac{R}{c_S}K) \) results in an internal node 1 (internal degree of freedom). The equilibrium formulations for node 0 and 1 (for harmonic loading) are as follows

\[
\text{node 0 : } [kK] (u_0(\omega) - u_1(\omega)) + [-kK] u_0(\omega) = P_0(\omega) \tag{C.30a}
\]
\[
\text{node 1 : } [kK] (u_1(\omega) - u_0(\omega)) + i\omega \left[ \frac{R}{c_S}K \right] u_1(\omega) = 0. \tag{C.30b}
\]
By eliminating $u_1(\omega)$ in Equations (C.30a) and (C.30b) the force-displacement relation of the first-order model is given by

$$\frac{P_0(a_0)}{K} = \frac{-\kappa^2}{\alpha a_0 + \frac{s}{\gamma}} u_0(a_0). \quad (C.31)$$

By comparing Equations (C.27b) and (C.31) $\kappa$ and $\gamma$ are identified as

$$\kappa = \frac{A}{s}, \quad (C.32a)$$
$$\gamma = -\frac{A}{s^2}. \quad (C.32b)$$

It should be noted that the first-order term also could be represented by a monkey-tail model. This turns out to be advantageous in situations where $\kappa$ and $\gamma$ in Equation (C.32) are negative, which may be the case when $A$ is positive ($s$ is negative). To avoid negative coefficients of springs and dashpots, the monkey-tail model is applied, and the resulting coefficients are positive. By inspecting the equilibrium formulations for node 0 and 1, see Figure C.9(b), the coefficients can be identified as

$$\gamma = \frac{A}{s^2}, \quad (C.33a)$$
$$\mu = -\frac{A}{s^3}. \quad (C.33b)$$

Second-order term

The second-order term given by Equation (C.27c). The model has four known parameters, $a_0$, $a_1$, $b_0$ and $b_1$. An example of a second-order discrete-element model is shown in Figure C.10(a). This particular model has two internal nodes. The equilibrium
formulations for nodes 0, 1 and 2 (for harmonic loading) are as follows

node 0: \[ \kappa_1 K (u_0(\omega) - u_1(\omega)) + [-\kappa_1 K] u_0(\omega) = P_0(\omega) \]  
(C.34a)

node 1: \[ \kappa_1 K (u_1(\omega) - u_0(\omega)) + i \omega \left[ \gamma_1 \frac{R}{c_s K} \right] (u_1(\omega) - u_2(\omega)) = 0 \]  
(C.34b)

node 2: \[ \kappa_2 K u_2(\omega) + i \omega \left[ \gamma_2 \frac{R}{c_s K} \right] u_2(\omega) + i \omega \left[ \gamma_1 \frac{R}{c_s K} \right] (u_2(\omega) - u_1(\omega)) = 0. \]  
(C.34c)

After some rearrangement and elimination the internal degrees of freedom, the force-displacement relation of the second-order model is given by

\[ \frac{P_0(a_0)}{K} = \frac{-\kappa_1^2 \gamma_1 + \kappa_2^2 \gamma_2}{(i a_0)^2 + \left( \kappa_1 \frac{\gamma_1 + \gamma_2}{\gamma_1 + \gamma_2} + \frac{\kappa_2}{\gamma_1} \right) i a_0 + \frac{\kappa_1 \kappa_2}{\gamma_1} a_0} u_0(a_0). \]  
(C.35)

The four coefficients in Equation (C.35) can be identified as

\[ \kappa_1 = -\frac{\beta_0}{\alpha_0} \]  
(C.36a)

\[ \gamma_1 = -\frac{\alpha_0 \beta_1 - \alpha_1 \beta_0}{\alpha_0^2} \]  
(C.36b)

\[ \kappa_2 = \frac{\beta_0}{\alpha_0^2} \left( -\alpha_0 \beta_1 + \alpha_1 \beta_0 \right)^2 - \frac{\alpha_0^2 \beta_1^2 - \alpha_1 \beta_0 \beta_1 + \beta_0^2}{\alpha_0^2 \alpha_1 \beta_0 \beta_1 + \beta_0^2} \]  
(C.36c)

\[ \gamma_2 = \frac{\beta_0^2}{\alpha_0^2} \left( -\alpha_0 \beta_1 + \alpha_1 \beta_0 \right) \]  
(C.36d)

by comparison of Equations (C.27c) and (C.35).

By introducing a second-order model with springs, dampers and a mass, it is possible to construct a second-order model with only one internal degree of freedom. The model

\[ \begin{align*}
\begin{array}{c}
-\kappa_1 K \\
\kappa_1 K \\
\kappa_2 K \\
\end{array} \\
\begin{array}{c}
0 \\
\gamma_1 \frac{R}{c_s K} \\
\gamma_2 \frac{R}{c_s K} \\
\end{array}
\end{align*} \]  
(a)

\[ \begin{align*}
\begin{array}{c}
P_0 \\
\kappa_1 K \\
-\gamma_2 \frac{R}{c_s K} \\
\kappa_2 K \\
\end{array} \\
\begin{array}{c}
0 \\
1 \\
\mu \frac{R^2}{c_s^2 K} \\
\gamma_2 \frac{R}{c_s K} \\
\end{array}
\end{align*} \]  
(b)

Figure C.10: The discrete-element model for the second-order term. (a) Spring-dashpot model with two internal degrees of freedom, and (b) Spring-dashpot-mass model with one internal degree of freedom.

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is sketched in Figure C.10(b). The force-displacement relation of the alternative second-order model is given by

\[
P_0(a_0) = \frac{2 \left( \frac{\kappa_1 \gamma}{\mu} + \frac{\gamma^2}{\mu^2} \right) i a_0 - \frac{\kappa^2}{\mu^2} + \frac{(\kappa_1 + \kappa_2)\gamma^2}{\mu^2}}{(ia_0)^2 + 2 \frac{\kappa}{\mu} i a_0 + \frac{\kappa_1 + \kappa_2}{\mu}} u_0(a_0).
\]  

(C.37)

By equating the coefficients in Equation (C.37) to the terms of the second-order model in Equation (C.27c), the four parameters \( \kappa_1, \kappa_2, \gamma \) and \( \mu \) can be determined. In order to calculate \( \mu \), a quadratic equation has to be solved. The quadratic equation for \( \mu \) is

\[
a \mu^2 + b \mu + c = 0 \quad \text{where}
\]

\[
a = \alpha_4^1 - 4 \alpha_0 \alpha_1^2 \quad \text{(C.38b)}
\]

\[
b = -8 \alpha_1 \beta_1 + 16 \beta_0 \quad \text{(C.38c)}
\]

\[
c = 16 \frac{\beta_1^2}{\alpha_1^2}. \quad \text{(C.38d)}
\]

Equation (C.38a) results in two solutions for \( \mu \). To ensure real values of \( \mu \), \( b^2 - 4ac \geq 0 \) or \( \alpha_0 \beta_1^2 - \alpha_1 \beta_0 \beta_1 + \beta_0^2 \geq 0 \). When \( \mu \) has been determined, the three remaining coefficients can be calculated by

\[
\kappa_1 = \frac{\mu \alpha_1^2}{4} - \frac{\beta_1}{\alpha_1} \quad \text{(C.39a)}
\]

\[
\kappa_2 = \mu \alpha_0 - \kappa_1 \quad \text{(C.39b)}
\]

\[
\gamma = \frac{\mu \alpha_1}{2}. \quad \text{(C.39c)}
\]
C.3.5 Example — vertical dynamic stiffness of a suction caisson

Consider a suction caisson with a diameter, $D = 2R$, and the skirt length, $H$. The vertical dynamic stiffness of the suction caisson has been determined by a three-dimensional coupled boundary element/finite element model, see Section 2.6 for details. The real and imaginary part of the dynamic stiffness for a suction caisson with $H/D = 1$ is shown in Figure C.11. The total dynamic stiffness $S_{VV}(a_0)$ is obtained by numerical analysis, and the singular part $S_{s,VV}(a_0)$ is represented by a limiting damping parameter $C_{VV}^\infty$ that describes the impedance for $a_0 \to \infty$, which in the case of the suction caisson is given by (Section 2.6)

$$C_{VV}^\infty = \rho_s c_P A_{lid} + 2\rho_s c_S A_{skirt}, \quad (C.40)$$

where $A_{lid}$ and $A_{skirt}$ are the vibrating surface areas of the lid and the skirt, respectively. $c_P$, $c_S$ and $\rho_s$ are the primary (dilatation) wave velocity, shear wave velocity and mass density of the soil, respectively. Finally, the regular part $S_r(a_0)$ is found by subtracting $S_s(a_0)$ from $S(a_0)$, according to Equation (C.17). The real and imaginary part of $S_{VV}(a_0)$, $S_{s,VV}(a_0)$ and $S_r(a_0)$ are illustrated in Figure C.11.

Next, the polynomial-fraction approximation of the regular part $S_r(a_0)$ is applied (Equation C.22). The polynomial degree of the denominator and the numerator is set to

\[\text{Figure C.11: Total, regular and singular terms of the vertical dynamic stiffness of a suction caisson.} \ G_s = 1.0 \text{ MPa, } \nu_s = 1/3.\]
### Table C.3: Coefficients for the partial-fraction expansion

<table>
<thead>
<tr>
<th>Poles $s_l$</th>
<th>Residues $A_l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l = 1$</td>
<td>$-1.8459 + 6.0094i$</td>
</tr>
<tr>
<td>$l = 2$</td>
<td>$-1.8459 - 6.0094i$</td>
</tr>
<tr>
<td>$l = 3$</td>
<td>$-0.2544 + 5.7003i$</td>
</tr>
<tr>
<td>$l = 4$</td>
<td>$-0.2544 - 5.7003i$</td>
</tr>
<tr>
<td>$l = 5$</td>
<td>$-0.5547 + 2.4330i$</td>
</tr>
<tr>
<td>$l = 6$</td>
<td>$-0.5547 - 2.4330i$</td>
</tr>
</tbody>
</table>

6 and 5, respectively. Thus, $2 \times 6 - 1 = 11$ coefficients are to be determined by curve-fitting. The polynomial-fraction approximation is applied for $a_0 \in [0;6]$. The polynomial degree is simply too low to fit the data well for $a_0 \in [0;20]$.

The 11 polynomial coefficients have been determined by curve-fitting (based on least squares method) and converted into a partial-fraction expansion. The results of the curve-fitting are given in Table C.3. It turns out that the partial-fraction expansion of $S_r(ia_0)$ for this particular foundation is given by three second-order terms, in addition to the singular (const/linear) term. This is due to the fact that the poles and residues of the partial-fraction expansion all are complex. Note that Table C.3 contains three complex conjugate pairs of poles and residues.

The poles and residues in Table C.3 are then converted into the appropriate coefficients, according to the expressions in Equation (C.35). The coefficients of the three second-order discrete-elements are shown in Table C.4.

### Table C.4: Coefficients of the three second-order discrete-elements

<table>
<thead>
<tr>
<th>$l$</th>
<th>$\kappa_{1l}$</th>
<th>$\gamma_{1l}$</th>
<th>$\kappa_{2l}$</th>
<th>$\gamma_{2l}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l = 1, 2$</td>
<td>$-5.3860$</td>
<td>$-0.7766$</td>
<td>$+3.4420$</td>
<td>$+0.5901$</td>
</tr>
<tr>
<td>$l = 3, 4$</td>
<td>$+2.8613$</td>
<td>$+0.0667$</td>
<td>$-0.0503$</td>
<td>$-0.0663$</td>
</tr>
<tr>
<td>$l = 5, 6$</td>
<td>$+1.5247$</td>
<td>$-0.4224$</td>
<td>$-0.4082$</td>
<td>$+0.2366$</td>
</tr>
</tbody>
</table>

The total approximation of the dynamic stiffness can then be formulated by means of Equation (C.26). The coefficients for the three second-order elements are given in Table C.4, and the last coefficient to be determined is $c^\infty$ of the singular part (if $k^\infty$ vanishes, see Equation (C.19)). $c^\infty$ is found by multiplying Equation (C.40) with $K_{i0}$. In the case $c^\infty$ is equal to 2.8935. All the components have now been determined. The complete lumped-parameter model is shown in Figure C.12, and the approximation of the total dynamic stiffness $S_{V,V}(a_0)$ is shown in Figure C.13. Note that the approximation fits very well for $a_0 \in [0;6]$, and tends towards the high-frequency limit for $a_0 > 6$. The high-frequency limit corresponds to $S_{s,V,V}(a_0)$ in Figure C.11.
Figure C.12: Complete lumped-parameter model. The parameters $K$ and $\frac{R}{c^2}K$ are omitted on the $\kappa$ and $\gamma$ terms, respectively.

Figure C.13: Lumped-parameter model approximation of the vertical dynamic stiffness of a suction caisson. The approximation is based on data for $a_0 \in [0;6]$. $G_s = 1.0$ MPa, $\nu_s = 1/3$.

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Finally, the frequency-independent stiffness and damping matrices of the lumped-parameter model are assembled. There is no mass matrix, since the lumped-parameter model only consists of springs and dampers. The matrices are constructed from the equilibrium equations for each node \((0, a, b, c, d, e, f)\) of the model. The equilibrium formulations (for harmonic loading) are as follows

node 0:

\[
\begin{align*}
[K_{11}] (u_0 - u_a) + [-K_{11}] u_0 + [K_{13}] (u_0 - u_c) + [-K_{13}] u_0 + \\

[K_{15}] (u_0 - u_e) + [-K_{15}] u_0 + i\omega \left[ \frac{R}{cS} K \right] u_0 = P_0 \quad (C.41a)
\end{align*}
\]

node a:

\[
\begin{align*}
[K_{11}] (u_a - u_0) + i\omega \left[ \frac{\gamma_{11} R}{cS} K \right] (u_a - u_b) = 0 \quad (C.41b)
\end{align*}
\]

node b:

\[
\begin{align*}
[K_{21}] u_b + i\omega \left[ \frac{\gamma_{21} R}{cS} K \right] u_b + i\omega \left[ \frac{\gamma_{11} R}{cS} K \right] (u_b - u_a) = 0 \quad (C.41c)
\end{align*}
\]

node c:

\[
\begin{align*}
[K_{13}] (u_c - u_0) + i\omega \left[ \frac{\gamma_{13} R}{cS} K \right] (u_c - u_d) = 0 \quad (C.41d)
\end{align*}
\]

node d:

\[
\begin{align*}
[K_{23}] u_d + i\omega \left[ \frac{\gamma_{23} R}{cS} K \right] u_d + i\omega \left[ \frac{\gamma_{13} R}{cS} K \right] (u_d - u_c) = 0 \quad (C.41e)
\end{align*}
\]

node e:

\[
\begin{align*}
[K_{15}] (u_e - u_0) + i\omega \left[ \frac{\gamma_{15} R}{cS} K \right] (u_e - u_f) = 0 \quad (C.41f)
\end{align*}
\]

node f:

\[
\begin{align*}
[K_{25}] u_f + i\omega \left[ \frac{\gamma_{25} R}{cS} K \right] u_f + i\omega \left[ \frac{\gamma_{15} R}{cS} K \right] (u_f - u_e) = 0 \quad (C.41g)
\end{align*}
\]
By rearranging the equations with respect to the degrees of freedom, the force-displacement relation for harmonic loading is given by

\[(K + i\omega C) U = P,\]  \hspace{1cm} (C.42)

where \(K, C, U\) and \(P\) are given as

\[
K = \begin{bmatrix}
0 & -\kappa_{11} & 0 & -\kappa_{13} & 0 & -\kappa_{15} & 0 \\
-\kappa_{11} & \kappa_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \kappa_{21} & 0 & 0 & 0 & 0 \\
-\kappa_{13} & 0 & 0 & \kappa_{13} & 0 & 0 & 0 \\
0 & 0 & 0 & \kappa_{23} & 0 & 0 & 0 \\
-\kappa_{15} & 0 & 0 & 0 & \kappa_{15} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \kappa_{25} & 0
\end{bmatrix}
\]

\[
C = \frac{R}{c_s K} \begin{bmatrix}
\infty & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \gamma_{11} & -\gamma_{11} & 0 & 0 & 0 & 0 \\
0 & -\gamma_{11} & \gamma_{11} + \gamma_{21} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma_{13} & -\gamma_{13} & 0 & 0 \\
0 & 0 & 0 & -\gamma_{13} & \gamma_{13} + \gamma_{23} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \gamma_{15} & -\gamma_{15} \\
0 & 0 & 0 & 0 & 0 & -\gamma_{15} & \gamma_{15} + \gamma_{25}
\end{bmatrix}
\]

\[
U = \begin{bmatrix}
u_0 \\
u_a \\
u_b \\
u_c \\
u_d \\
u_e \\
u_f
\end{bmatrix}, \hspace{1cm} P = \begin{bmatrix}
P_0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\hspace{1cm} (C.43a, b, c)
Appendix D
Application of lumped-parameter models

The general procedure for determining the coefficients for a lumped-parameter model has been given in Appendix C. This Appendix concerns the lumped-parameter models for a suction caisson with a ratio between skirt length and foundation diameter equal to 1/2, embedded into a viscoelastic soil. The models are presented for three different values of the shear modulus of the subsoil (section D.1). Subsequently, the assembly of the dynamic stiffness matrix for the foundation is considered (section D.2), and the solution for obtaining the steady state response, when using lumped-parameter models is given (section D.2).

D.1 Lumped-parameter models for the suction caisson

The lumped-parameter models have been constructed according to the procedure in section C.3. After a brief summary of the modelling procedure for determining the exact solution, the lumped-parameter models for each degree of freedom are given. Note: the lumped-parameter models for torsional vibrations are not used in the numerical simulations in Chapter 4. They are included to provide a full three-dimensional approximation of the suction caisson.

D.1.1 Determination of the exact solution for the dynamic stiffness

The frequency dependent dynamic stiffness coefficients are determined by means of a dynamic three-dimensional coupled Boundary Element/Finite Element (BE/FE) program BEASTS by Andersen and Jones (2001a). The evaluation of the impedance of suction caisson foundations for offshore wind turbines have been reported in details in Chapters 2 and 3.

The BE/FE model of the suction caisson consists of four sections: a massless finite element section that forms the top of the foundation where the load is applied, a finite element section of the skirts, a boundary element domain inside the skirts and, finally, a boundary element domain outside the skirts that also forms the free surface. Again, quadratic interpolation is employed. The models of the suction caisson and the subsoil contain approx. 100 finite elements and 350 boundary elements. The mesh of the free surface is truncated at a distance of 30 m (6 times radius $R$) from the centre of the
foundation. The model is illustrated in Figure D.1. The properties of the soil and the suction caisson used in the BE/FE analyses are given in Table D.1. Note that \( \rho_f \) of the lid of the caisson foundation is zero and that \( \rho_f = \rho_s \) for the skirt, in order to model a massless foundation.

The dynamic behaviour of the caisson is influenced by ratio between the stiffness of the soil and the stiffness of the structure, see Chapter 2. For low values of \( G_s \) the influence of the skirt flexibility vanishes, i.e. the caisson reacts as a rigid foundation. Rigid behaviour can be assumed for \( G_s \leq 1.0 \text{ MPa} \) (\( E_f \) is constant). On the other hand, the dynamic behaviour of the suction caisson tends towards the frequency dependent behaviour of the surface foundation for high values of \( G_s \) (1000 MPa). To show the effects of \( G_s \) on the dynamic behaviour of the caisson, the sliding (horizontal) impedance for three values of \( G_s \) is shown in Figure D.2. Note that the impedance changes as the shear modulus of the soil \( G_s \) increases. The impedance for \( G_s = 1 \text{ MPa} \) and \( G_s = 10 \text{ MPa} \)  

![Figure D.1: Geometry (a) and BE/FE model (b) of the suction caisson.](image)

Table D.1: Model properties for the BE/FE analyses

<table>
<thead>
<tr>
<th>Property</th>
<th>value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Foundation radius</td>
<td>( R ) 6 \text{ m}</td>
</tr>
<tr>
<td>Skirt length</td>
<td>( H ) 6 \text{ m}</td>
</tr>
<tr>
<td>Skirt thickness</td>
<td>( t ) 30 \text{ mm}</td>
</tr>
<tr>
<td>Shear modulus (soil)</td>
<td>( G_s ) 1,10,100 \text{ MPa}</td>
</tr>
<tr>
<td>Poisson’s ratio (soil)</td>
<td>( \nu_s ) 0.25</td>
</tr>
<tr>
<td>Mass density (soil)</td>
<td>( \rho_s ) 1000 \text{ kg/m}^3</td>
</tr>
<tr>
<td>Loss factor (soil)</td>
<td>( \eta_s ) 5 %</td>
</tr>
<tr>
<td>Young’s modulus (foundation)</td>
<td>( E_f ) 210 \text{ GPa}</td>
</tr>
<tr>
<td>Poisson’s ratio (foundation)</td>
<td>( \nu_f ) 0.25</td>
</tr>
<tr>
<td>Mass density (foundation)</td>
<td>( \rho_f ) 0/1000 \text{ kg/m}^3</td>
</tr>
<tr>
<td>Loss factor (foundation)</td>
<td>( \eta_f ) 2 %</td>
</tr>
</tbody>
</table>

\( \dagger \) The models are constructed for three values of \( G_s \)

\( \ddagger \) \( \rho_f = 0 \) for the lid of the caisson and \( \rho_f = \rho_s \) for the skirt

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corresponds to that of a rigid suction caisson where the influence of the skirt flexibility vanishes. In contrast, the impedance for $G_s = 100$ MPa corresponds more or less to the behaviour of a surface footing.

Figure D.2: Sliding impedance: variation of soil stiffness. $\nu_s = 0.25$ and $\eta_s = 5\%$. 

*December 4, 2006*
Table D.2: Vertical: Type and numbers of internal degrees of freedom for the lumped-parameter models

<table>
<thead>
<tr>
<th>$G_s$</th>
<th>Type</th>
<th>No. of internal dofs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>3 second-order (pcm$^{1}$)</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>2 second-order (pcm$^{1}$) + 1 first-order (pcm$^{1}$)</td>
<td>3</td>
</tr>
<tr>
<td>100</td>
<td>2 second-order (pcm$^{1}$) + 1 first-order (pcm$^{1}$)</td>
<td>3</td>
</tr>
</tbody>
</table>

† Spring-dashpot-mass model, see Figure C.10(b)
‡ Spring-dashpot-mass model, see Figure C.9(b)

D.1.2 Lumped-parameter models for vertical vibrations

The type of approximation for the vertical lumped-parameter models is summarized in Table D.2 and the approximation is compared with the rigorous solution in Figure D.3. The pole-residue coefficients, the stiffness, damping and mass matrices of the models are given in the following.

Pole-residue coefficients

Table D.3: Vertical: Poles and residues

<table>
<thead>
<tr>
<th>Poles $s$</th>
<th>Residues $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_s = 1$ MPa</td>
<td>$-3.6431 + 5.0238i$</td>
</tr>
<tr>
<td></td>
<td>$-3.6431 - 5.0238i$</td>
</tr>
<tr>
<td></td>
<td>$-1.2197 + 2.8101i$</td>
</tr>
<tr>
<td></td>
<td>$-1.2197 - 2.8101i$</td>
</tr>
<tr>
<td></td>
<td>$-0.5940 + 0.9980i$</td>
</tr>
<tr>
<td></td>
<td>$-0.5940 - 0.9980i$</td>
</tr>
<tr>
<td>$G_s = 10$ MPa</td>
<td>$-2.5113$</td>
</tr>
<tr>
<td></td>
<td>$-0.8520 + 4.5455i$</td>
</tr>
<tr>
<td></td>
<td>$-0.8520 - 4.5455i$</td>
</tr>
<tr>
<td></td>
<td>$-0.7600 + 2.2086i$</td>
</tr>
<tr>
<td></td>
<td>$-0.7600 - 2.2086i$</td>
</tr>
<tr>
<td>$G_s = 100$ MPa</td>
<td>$-23.8012$</td>
</tr>
<tr>
<td></td>
<td>$-1.1905 + 2.2720i$</td>
</tr>
<tr>
<td></td>
<td>$-1.1905 - 2.2720i$</td>
</tr>
<tr>
<td></td>
<td>$-0.9607 + 4.7741i$</td>
</tr>
<tr>
<td></td>
<td>$-0.9607 - 4.7741i$</td>
</tr>
</tbody>
</table>

Matrices for the models

The resulting matrices of the models are given by Equations D.1 and D.2. The model structure stated in Equation D.1 corresponds to the lumped-parameter model with three
complex conjugate poles \((G_s = 1 \text{ MPa})\), whereas the model structure stated in Equation D.2 corresponds to the lumped-parameter models with one real and two complex conjugate poles \((G_s = 10 \text{ MPa} \text{ and } 100 \text{ MPa})\). The corresponding coefficients are listed in Table D.4.

\[
K_{VV} = K_{VV}^0 \begin{bmatrix} \frac{2^2}{\mu_1} + \frac{2^2}{\mu_2} + \frac{2^3}{\mu_3} & -\kappa_1 & -\kappa_3 & -\kappa_5 \\ -\kappa_1 & \kappa_1 + \kappa_2 & 0 & 0 \\ -\kappa_3 & 0 & \kappa_3 + \kappa_4 & 0 \\ -\kappa_5 & 0 & 0 & \kappa_5 + \kappa_6 \end{bmatrix} \quad \text{(D.1a)}
\]

\[
C_{VV} = \frac{R}{c_s} K_{VV}^0 \begin{bmatrix} c^\infty & -\gamma_1 & -\gamma_2 & -\gamma_3 \\ -\gamma_1 & 2\gamma_1 & 0 & 0 \\ -\gamma_2 & 0 & 2\gamma_2 & 0 \\ -\gamma_3 & 0 & 0 & 2\gamma_3 \end{bmatrix} \quad \text{(D.1b)}
\]

\[
M_{VV} = \frac{R^2}{c_s^2} K_{VV}^0 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_3 \end{bmatrix} \quad \text{(D.1c)}
\]

Note that the limiting damping parameter for \(G_s = 100 \text{ MPa}\) has been fitted manually. Since the impedance for high values of \(G_s\) approaches the frequency dependent behaviour of the surface footings, the solution in Equation C.40 in Chapter 2 is not valid. \(c^\infty\) for \(G_s = 100 \text{ MPa}\) in Table D.4 is in between the value for the suction caisson and a surface footing.
### Table D.4: Vertical: Model coefficients

<table>
<thead>
<tr>
<th>$G_s$</th>
<th>$\kappa$ coeff.</th>
<th>Value</th>
<th>$\gamma$ coeff.</th>
<th>Value</th>
<th>$\mu$ coeff.</th>
<th>Value</th>
<th>misc</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 MPa</td>
<td>$\kappa_1$</td>
<td>11.8449</td>
<td>$\gamma_1$</td>
<td>2.8176</td>
<td>$\mu_1$</td>
<td>0.7734</td>
<td>$c^\infty$</td>
<td>2.2581</td>
</tr>
<tr>
<td></td>
<td>$\kappa_2$</td>
<td>17.9400</td>
<td>$\gamma_2$</td>
<td>0.0037</td>
<td>$\mu_2$</td>
<td>0.0062</td>
<td>$K_{VV}$</td>
<td>7.9747</td>
</tr>
<tr>
<td></td>
<td>$\kappa_3$</td>
<td>0.6215</td>
<td>$\gamma_3$</td>
<td>0.2296</td>
<td>$\mu_3$</td>
<td>0.1882</td>
<td>$K_{VV}$</td>
<td>7.4508</td>
</tr>
<tr>
<td></td>
<td>$\kappa_4$</td>
<td>-0.3958</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\kappa_5$</td>
<td>2.3510</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\kappa_6$</td>
<td>-0.5848</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 MPa</td>
<td>$\kappa_1$</td>
<td>2.5145</td>
<td>$\gamma_1$</td>
<td>1.3043</td>
<td>$\mu_1$</td>
<td>1.5309</td>
<td>$c^\infty$</td>
<td>2.3107</td>
</tr>
<tr>
<td></td>
<td>$\kappa_2$</td>
<td>30.2269</td>
<td>$\gamma_2$</td>
<td>0.8653</td>
<td>$\mu_2$</td>
<td>1.1385</td>
<td>$K_{VV}$</td>
<td>7.9333</td>
</tr>
<tr>
<td></td>
<td>$\kappa_3$</td>
<td>2.2228</td>
<td>$\gamma_3$</td>
<td>0.0916</td>
<td>$\mu_3$</td>
<td>0.0365</td>
<td>$K_{VV}$</td>
<td>6.4658</td>
</tr>
<tr>
<td></td>
<td>$\kappa_4$</td>
<td>3.9882</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100 MPa</td>
<td>$\kappa_1$</td>
<td>-0.4212</td>
<td>$\gamma_1$</td>
<td>0.0107</td>
<td>$\mu_1$</td>
<td>0.0111</td>
<td>$c^\infty$</td>
<td>0.4208†</td>
</tr>
<tr>
<td></td>
<td>$\kappa_2$</td>
<td>0.6852</td>
<td>$\gamma_2$</td>
<td>0.0145</td>
<td>$\mu_2$</td>
<td>0.0122</td>
<td>$K_{VV}$</td>
<td>6.4658</td>
</tr>
<tr>
<td></td>
<td>$\kappa_3$</td>
<td>0.4132</td>
<td>$\gamma_3$</td>
<td>0.1583</td>
<td>$\mu_3$</td>
<td>0.0067</td>
<td>$K_{VV}$</td>
<td>6.4658</td>
</tr>
<tr>
<td></td>
<td>$\kappa_4$</td>
<td>-0.3329</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

† Manual fit.
Figure D.3: Vertical impedance: Boundary element solution and the corresponding lumped-parameter approximation. $\nu_s = 0.25$ and $\eta_s = 5\%$. 

December 4, 2006
Table D.5: Sliding: Type and numbers of internal degrees of freedom for the lumped-parameter models

<table>
<thead>
<tr>
<th>$G_s$</th>
<th>Type</th>
<th>No. of internal dofs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>3 second-order (kcm)</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>3 second-order (kcm)†</td>
<td>3</td>
</tr>
<tr>
<td>100</td>
<td>2 second-order (kcm)† + 1 first-order (kcm)‡</td>
<td>3</td>
</tr>
</tbody>
</table>

† Spring-dashpot-mass model, see Figure C.10(b)
‡ Spring-dashpot-mass model, see Figure C.9(b)

D.1.3 Lumped-parameter models for sliding vibrations

The type of approximation for the horizontal lumped-parameter models is summarized in Table D.5 and the approximation is compared with the rigorous solution in Figure D.4. The pole-residue coefficients, the stiffness, damping and mass matrices of the models are given in the following.

Pole-residue coefficients

Table D.6: Sliding: Poles and residues

<table>
<thead>
<tr>
<th>Poles $s$</th>
<th>Residues $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_s = 1$ MPa</td>
<td></td>
</tr>
<tr>
<td>$-3.1835 + 4.8983i$</td>
<td>$-14.2305 - 32.9079i$</td>
</tr>
<tr>
<td>$-3.1835 - 4.8983i$</td>
<td>$-14.2305 + 32.9079i$</td>
</tr>
<tr>
<td>$-0.5497 + 5.7479i$</td>
<td>$-1.6722 + 4.9951i$</td>
</tr>
<tr>
<td>$-0.5497 - 5.7479i$</td>
<td>$-1.6722 - 4.9951i$</td>
</tr>
<tr>
<td>$-1.0329 + 4.4915i$</td>
<td>$+7.9207 + 11.3350i$</td>
</tr>
<tr>
<td>$-1.0329 - 4.4915i$</td>
<td>$+7.9207 - 11.3350i$</td>
</tr>
<tr>
<td>$G_s = 10$ MPa</td>
<td></td>
</tr>
<tr>
<td>$-2.9289 + 7.0308i$</td>
<td>$-6.6629 - 23.0006i$</td>
</tr>
<tr>
<td>$-2.9289 - 7.0308i$</td>
<td>$-6.6629 + 23.0006i$</td>
</tr>
<tr>
<td>$-0.5447 + 5.7685i$</td>
<td>$-0.8154 + 3.0212i$</td>
</tr>
<tr>
<td>$-0.5447 - 5.7685i$</td>
<td>$-0.8154 - 3.0212i$</td>
</tr>
<tr>
<td>$-0.8437 + 3.7649i$</td>
<td>$-2.4717 + 4.9915i$</td>
</tr>
<tr>
<td>$-0.8437 - 3.7649i$</td>
<td>$-2.4717 - 4.9915i$</td>
</tr>
<tr>
<td>$G_s = 100$ MPa</td>
<td></td>
</tr>
<tr>
<td>$-14.9506$</td>
<td>$+45.7048$</td>
</tr>
<tr>
<td>$-0.6453 + 5.5078i$</td>
<td>$-0.0784 + 0.6483i$</td>
</tr>
<tr>
<td>$-0.6453 - 5.5078i$</td>
<td>$-0.0784 - 0.6483i$</td>
</tr>
<tr>
<td>$-1.2456 + 3.0948i$</td>
<td>$-0.9869 + 2.9066i$</td>
</tr>
<tr>
<td>$-1.2456 - 3.0948i$</td>
<td>$-0.9869 - 2.9066i$</td>
</tr>
</tbody>
</table>

Matrices for the models

The resulting matrices of the models are given by Equations D.3 and D.4. The model structure stated in Equation D.3 corresponds to the lumped-parameter model with three

Morten Liingaard
complex conjugate poles ($G_s = 1$ and 10 MPa), whereas the model structure stated in Equation D.4 corresponds to the lumped-parameter model with one real and two complex conjugate poles ($G_s = 100$ MPa). The corresponding coefficients are listed in Table D.7.

\[
\begin{align*}
K_{HH} &= K^0_{HH} \begin{bmatrix}
\frac{2^2}{\mu_1} + \frac{2^2}{\mu_2} + \frac{2^2}{\mu_3} & -\kappa_1 & -\kappa_3 & -\kappa_5 \\
-\kappa_1 & \kappa_1 + \kappa_2 & 0 & 0 \\
-\kappa_3 & 0 & \kappa_3 + \kappa_4 & 0 \\
-\kappa_5 & 0 & 0 & \kappa_5 + \kappa_6
\end{bmatrix} \\
C_{HH} &= \frac{R}{c_s} K^0_{HH} \begin{bmatrix}
c^\infty & -\gamma_1 & -\gamma_2 & -\gamma_3 \\
-\gamma_1 & 2\gamma_1 & 0 & 0 \\
-\gamma_2 & 0 & 2\gamma_2 & 0 \\
-\gamma_3 & 0 & 0 & 2\gamma_3
\end{bmatrix} \\
M_{HH} &= \frac{R^2}{c_s^2} K^0_{HH} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \mu_1 & 0 & 0 \\
0 & 0 & \mu_2 & 0 \\
0 & 0 & 0 & \mu_3
\end{bmatrix}
\end{align*}
\] (D.3a, D.3b, D.3c)

\[
\begin{align*}
K_{HH} &= K^0_{HH} \begin{bmatrix}
\frac{2^2}{\mu_1} + \frac{2^2}{\mu_2} + \frac{2^2}{\mu_3} & -\kappa_1 & -\kappa_3 & 0 \\
-\kappa_1 & \kappa_1 + \kappa_2 & 0 & 0 \\
-\kappa_3 & 0 & \kappa_3 + \kappa_4 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \\
C_{HH} &= \frac{R}{c_s} K^0_{HH} \begin{bmatrix}
c^\infty & -\gamma_1 & -\gamma_2 & -\gamma_3 \\
-\gamma_1 & 2\gamma_1 & 0 & 0 \\
-\gamma_2 & 0 & 2\gamma_2 & 0 \\
-\gamma_3 & 0 & 0 & \gamma_3
\end{bmatrix} \\
M_{HH} &= \frac{R^2}{c_s^2} K^0_{HH} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \mu_1 & 0 & 0 \\
0 & 0 & \mu_2 & 0 \\
0 & 0 & 0 & \mu_3
\end{bmatrix}
\end{align*}
\] (D.4a, D.4b, D.4c)

Note that the limiting damping parameter for $G_s = 100$ MPa has been fitted manually. Since the impedance for high values of $G_s$ approaches the frequency dependent behaviour of the surface footings, the solution in Equation 3.6a in Chapter 3 is not valid. $c^\infty$ for $G_s = 100$ MPa in Table D.7 is in between the value for the suction caisson and a surface footing.
Table D.7: Sliding: Model coefficients

<table>
<thead>
<tr>
<th></th>
<th>$\chi$ coeff.</th>
<th>Value</th>
<th>$\gamma$ coeff.</th>
<th>Value</th>
<th>$\mu$ coeff.</th>
<th>Value</th>
<th>misc</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_s = 1$ MPa</td>
<td>$\kappa_1$</td>
<td>18.5077</td>
<td>$\gamma_1$</td>
<td>4.4095</td>
<td>$\mu_1$</td>
<td>1.3851</td>
<td>$c^\infty$</td>
<td>2.1480</td>
</tr>
<tr>
<td></td>
<td>$\kappa_2$</td>
<td>28.7646</td>
<td>$\gamma_2$</td>
<td>0.0862</td>
<td>$\mu_2$</td>
<td>0.1569</td>
<td>$K_{HH}$</td>
<td>9.4540</td>
</tr>
<tr>
<td></td>
<td>$\kappa_3$</td>
<td>3.0895</td>
<td>$\gamma_3$</td>
<td>0.5374</td>
<td>$\mu_3$</td>
<td>0.5203</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\kappa_4$</td>
<td>2.1411</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\kappa_5$</td>
<td>-7.1135</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\kappa_6$</td>
<td>18.1655</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_s = 10$ MPa</td>
<td>$\kappa_1$</td>
<td>8.9522</td>
<td>$\gamma_1$</td>
<td>2.2798</td>
<td>$\mu_1$</td>
<td>0.7784</td>
<td>$c^\infty$</td>
<td>2.2035</td>
</tr>
<tr>
<td></td>
<td>$\kappa_2$</td>
<td>36.2012</td>
<td>$\gamma_2$</td>
<td>0.0344</td>
<td>$\mu_2$</td>
<td>0.0631</td>
<td>$K_{HH}$</td>
<td>9.2162</td>
</tr>
<tr>
<td></td>
<td>$\kappa_3$</td>
<td>1.5156</td>
<td>$\gamma_3$</td>
<td>0.1821</td>
<td>$\mu_3$</td>
<td>0.2158</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\kappa_4$</td>
<td>0.6945</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\kappa_5$</td>
<td>3.0832</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\kappa_6$</td>
<td>0.1297</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$G_s = 100$ MPa</td>
<td>$\kappa_1$</td>
<td>0.1224</td>
<td>$\gamma_1$</td>
<td>0.0013</td>
<td>$\mu_1$</td>
<td>0.0021</td>
<td>$c^\infty$</td>
<td>0.9275$^*$</td>
</tr>
<tr>
<td></td>
<td>$\kappa_2$</td>
<td>-0.0590</td>
<td>$\gamma_2$</td>
<td>0.0423</td>
<td>$\mu_2$</td>
<td>0.0339</td>
<td>$K_{HH}$</td>
<td>7.8288</td>
</tr>
<tr>
<td></td>
<td>$\kappa_3$</td>
<td>0.8450</td>
<td>$\gamma_3$</td>
<td>0.2045</td>
<td>$\mu_3$</td>
<td>0.0137</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\kappa_4$</td>
<td>-0.4672</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$^*$ Manual fit.
Figure D.4: Sliding impedance: Boundary element solution and the corresponding lumped-parameter approximation. $\nu_s = 0.25$ and $\eta_s = 5\%$. 

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D.1.4 Lumped-parameter models for rocking vibrations

The type of approximation for the rocking lumped-parameter models is summarized in Table D.8 and the approximation is compared with the rigorous solution in Figure D.5. The pole-residue coefficients, the stiffness, damping and mass matrices of the models are given in the following.

Pole-residue coefficients

Table D.9: Rocking: Poles and residues

<table>
<thead>
<tr>
<th>$G_s$</th>
<th>Poles $s$</th>
<th>Residues $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_s = 1 \text{ MPa}$</td>
<td>$-2.2574$</td>
<td>$3.0119$</td>
</tr>
<tr>
<td></td>
<td>$-0.4660 + 4.2593i$</td>
<td>$+0.2815 + 0.9699i$</td>
</tr>
<tr>
<td></td>
<td>$-0.4660 - 4.2593i$</td>
<td>$+0.2815 - 0.9699i$</td>
</tr>
<tr>
<td></td>
<td>$-0.2503 + 6.2918i$</td>
<td>$+0.0514 - 0.2789i$</td>
</tr>
<tr>
<td></td>
<td>$-0.2503 - 6.2918i$</td>
<td>$+0.0514 + 0.2789i$</td>
</tr>
<tr>
<td>$G_s = 10 \text{ MPa}$</td>
<td>$-8.2898 + 5.8728i$</td>
<td>$-11.3577 - 30.0471i$</td>
</tr>
<tr>
<td></td>
<td>$-8.2898 - 5.8728i$</td>
<td>$-11.3577 + 30.0471i$</td>
</tr>
<tr>
<td></td>
<td>$-0.9062$</td>
<td>$+0.1849$</td>
</tr>
<tr>
<td></td>
<td>$-0.7761 + 4.2620i$</td>
<td>$+0.8639 + 1.9198i$</td>
</tr>
<tr>
<td></td>
<td>$-0.7761 - 4.2620i$</td>
<td>$+0.8639 - 1.9198i$</td>
</tr>
<tr>
<td>$G_s = 100 \text{ MPa}$</td>
<td>$-21.2318$</td>
<td>$+30.5256$</td>
</tr>
<tr>
<td></td>
<td>$-2.3226 + 0.4374i$</td>
<td>$-1.2139 - 4.1473i$</td>
</tr>
<tr>
<td></td>
<td>$-2.3226 - 0.4374i$</td>
<td>$-1.2139 + 4.1473i$</td>
</tr>
<tr>
<td></td>
<td>$-0.6393 + 4.3133i$</td>
<td>$+0.4135 + 0.2652i$</td>
</tr>
<tr>
<td></td>
<td>$-0.6393 - 4.3133i$</td>
<td>$+0.4135 - 0.2652i$</td>
</tr>
</tbody>
</table>
Matrices for the models

The resulting matrices of the models are given by Equation D.5. The model structure stated in Equation D.5 corresponds to the lumped-parameter models with one real and two complex conjugate poles. The corresponding coefficients are listed in Table D.10.

\[
\begin{align*}
K_{MM} &= K_{MM}^0 \begin{bmatrix}
\gamma_1^2 + \frac{\gamma_2^2}{\mu_1} + \frac{\gamma_3^2}{\mu_3} & -\kappa_1 & -\kappa_3 & 0 \\
-\kappa_1 & \kappa_1 + \kappa_2 & 0 & 0 \\
-\kappa_3 & 0 & \kappa_3 + \kappa_4 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} & (D.5a) \\
C_{MM} &= \frac{R}{c_S} K_{MM}^0 \begin{bmatrix}
\gamma_1^\infty & -\gamma_1 & -\gamma_2 & -\gamma_3 \\
-\gamma_1 & 2\gamma_1 & 0 & 0 \\
-\gamma_2 & 0 & 2\gamma_2 & 0 \\
-\gamma_3 & 0 & 0 & \gamma_3 \\
\end{bmatrix} & (D.5b) \\
M_{MM} &= \frac{R^2}{c_S^2} K_{MM}^0 \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \mu_1 & 0 & 0 \\
0 & 0 & \mu_2 & 0 \\
0 & 0 & 0 & \mu_3 \\
\end{bmatrix} & (D.5c)
\end{align*}
\]

Table D.10: Rocking: Model coefficients

<table>
<thead>
<tr>
<th>$G_s$</th>
<th>$\kappa$ coeff.</th>
<th>Value</th>
<th>$\gamma$ coeff.</th>
<th>Value</th>
<th>$\mu$ coeff.</th>
<th>Value</th>
<th>misc</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 MPa</td>
<td>$\kappa_1$</td>
<td>-0.1161</td>
<td>$\gamma_1$</td>
<td>0.3572</td>
<td>$\mu_1$</td>
<td>1.4275</td>
<td>$c_\infty$</td>
<td>0.8055</td>
</tr>
<tr>
<td></td>
<td>$\kappa_2$</td>
<td>56.7137</td>
<td>$\gamma_2$</td>
<td>0.0202</td>
<td>$\mu_2$</td>
<td>0.0433</td>
<td>$K_{MM}^0$</td>
<td>16.5930</td>
</tr>
<tr>
<td></td>
<td>$\kappa_3$</td>
<td>-0.5946</td>
<td>$\gamma_3$</td>
<td>0.5910</td>
<td>$\mu_3$</td>
<td>0.2618</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\kappa_4$</td>
<td>1.3887</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 MPa</td>
<td>$\kappa_1$</td>
<td>11.9561</td>
<td>$\gamma_1$</td>
<td>1.2770</td>
<td>$\mu_1$</td>
<td>0.1540</td>
<td>$c_\infty$</td>
<td>0.8415</td>
</tr>
<tr>
<td></td>
<td>$\kappa_2$</td>
<td>3.9427</td>
<td>$\gamma_2$</td>
<td>0.0561</td>
<td>$\mu_2$</td>
<td>0.0722</td>
<td>$K_{MM}^0$</td>
<td>15.8830</td>
</tr>
<tr>
<td></td>
<td>$\kappa_3$</td>
<td>-1.0696</td>
<td>$\gamma_3$</td>
<td>0.2252</td>
<td>$\mu_3$</td>
<td>0.2485</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\kappa_4$</td>
<td>2.4251</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100 MPa</td>
<td>$\kappa_1$</td>
<td>-0.5945</td>
<td>$\gamma_1$</td>
<td>0.0820</td>
<td>$\mu_1$</td>
<td>0.1283</td>
<td>$c_\infty$</td>
<td>0.3959$^\dagger$</td>
</tr>
<tr>
<td></td>
<td>$\kappa_2$</td>
<td>3.033</td>
<td>$\gamma_2$</td>
<td>8.6772</td>
<td>$\mu_2$</td>
<td>3.8865</td>
<td>$K_{MM}^0$</td>
<td>11.8941</td>
</tr>
<tr>
<td></td>
<td>$\kappa_3$</td>
<td>19.9167</td>
<td>$\gamma_3$</td>
<td>0.0677</td>
<td>$\mu_3$</td>
<td>0.0032</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\kappa_4$</td>
<td>0.1989</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$^\dagger$ Manual fit.

Note that the limiting damping parameter for $G_s = 100$ MPa has been fitted manually. Since the impedance for high values of $G_s$ approaches the frequency dependent behaviour of the surface footings, the solution in Equation 3.6b in Chapter 3 is not valid. $c_\infty$ for $G_s = 100$ MPa in Table D.10 is in between the value for the suction caisson and a surface footing.
Figure D.5: Rocking impedance: Boundary element solution and the corresponding lumped-parameter approximation. $\nu_s = 0.25$ and $\eta_s = 5\%$. 

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Table D.11: Coupling: Type and numbers of internal degrees of freedom for the lumped-parameter models

<table>
<thead>
<tr>
<th>$G_s$</th>
<th>Type</th>
<th>No. of internal dofs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>2 second-order (kcm) + 1 first-order (kcm)</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>2 second-order (kcm) + 1 first-order (kcm)</td>
<td>3</td>
</tr>
<tr>
<td>100</td>
<td>2 second-order (kcm) + 1 first-order (kcm)</td>
<td>3</td>
</tr>
</tbody>
</table>

† Spring-dashpot-mass model, see Figure C.10(b)
‡ Spring-dashpot-mass model, see Figure C.9(b)

D.1.5 Lumped-parameter models for the coupling term

The type of approximation for the coupling lumped-parameter models is summarized in Table D.11 and the approximation is compared with the rigorous solution in Figure D.6. The pole-residue coefficients, the stiffness, damping and mass matrices of the models are given in the following.

Pole-residue coefficients

Table D.12: Coupling: Poles and residues

<table>
<thead>
<tr>
<th>Poles $s$</th>
<th>Residues $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_s = 1$ MPa</td>
<td>-3.2542 + 9.7824</td>
</tr>
<tr>
<td></td>
<td>-0.6757 + 4.2024i + 1.4116 + 3.6383i</td>
</tr>
<tr>
<td></td>
<td>-0.6757 - 4.2024i + 1.4116 - 3.6383i</td>
</tr>
<tr>
<td></td>
<td>-0.3401 + 5.9793i + 1.0812 + 1.3791i</td>
</tr>
<tr>
<td></td>
<td>-0.3401 - 5.9793i + 1.0812 - 1.3791i</td>
</tr>
<tr>
<td>$G_s = 10$ MPa</td>
<td>-2.9049 + 5.5089</td>
</tr>
<tr>
<td></td>
<td>-0.5912 + 4.1399i + 1.9160 + 2.2139i</td>
</tr>
<tr>
<td></td>
<td>-0.5912 - 4.1399i + 1.9160 - 2.2139i</td>
</tr>
<tr>
<td></td>
<td>-0.4251 + 6.1778i + 2.2902 - 0.0207i</td>
</tr>
<tr>
<td></td>
<td>-0.4251 - 6.1778i + 2.2902 + 0.0207i</td>
</tr>
<tr>
<td>$G_s = 100$ MPa</td>
<td>-3.5564 + 8.5065i + 21.7153 + 3.5724i</td>
</tr>
<tr>
<td></td>
<td>-3.5564 - 8.5065i + 21.7153 - 3.5724i</td>
</tr>
<tr>
<td></td>
<td>-1.2170 + 0.2659</td>
</tr>
<tr>
<td></td>
<td>-0.9167 + 3.5203i + 2.7409 + 1.3198i</td>
</tr>
<tr>
<td></td>
<td>-0.9167 - 3.5203i + 2.7409 - 1.3198i</td>
</tr>
</tbody>
</table>

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Matrices for the models

The resulting matrices of the models are given by Equation D.6. The model structure stated in Equation D.6 corresponds to the lumped-parameter models with one real and two complex conjugate poles. The corresponding coefficients are listed in Table D.13.

\[
\begin{align*}
K_{HM} &= K_0^{\text{HM}} \begin{bmatrix}
\frac{\gamma_1}{\mu_1} + \frac{\gamma_2}{\mu_2} + \frac{\gamma_3}{\mu_3} & -\kappa_1 & -\kappa_4 & 0 \\
-\kappa_1 & \kappa_1 + \kappa_2 & 0 & 0 \\
-\kappa_3 & 0 & \kappa_3 + \kappa_4 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \tag{D.6a} \\
C_{HM} &= \frac{R}{c_s} K_0^{\text{HM}} \begin{bmatrix}
e^\infty & -\gamma_1 & -\gamma_2 & -\gamma_3 \\
-\gamma_1 & 2\gamma_1 & 0 & 0 \\
-\gamma_2 & 0 & 2\gamma_2 & 0 \\
-\gamma_3 & 0 & 0 & \gamma_3
\end{bmatrix} \tag{D.6b} \\
M_{HM} &= \frac{R^2}{c_s^2} K_0^{\text{HM}} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \mu_1 & 0 & 0 \\
0 & 0 & \mu_2 & 0 \\
0 & 0 & 0 & \mu_3
\end{bmatrix} \tag{D.6c}
\end{align*}
\]

Table D.13: Coupling: Model coefficients

<table>
<thead>
<tr>
<th>(G_s)</th>
<th>(\kappa) coeff.</th>
<th>Value</th>
<th>(\gamma) coeff.</th>
<th>Value</th>
<th>(\mu) coeff.</th>
<th>Value</th>
<th>misc</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 MPa</td>
<td>(\kappa_1)</td>
<td>-3.1170</td>
<td>(\gamma_1)</td>
<td>0.1836</td>
<td>(\mu_1)</td>
<td>0.5399</td>
<td>e^\infty</td>
<td>1.3253</td>
</tr>
<tr>
<td></td>
<td>(\kappa_2)</td>
<td>22.4813</td>
<td>(\gamma_2)</td>
<td>0.0931</td>
<td>(\mu_2)</td>
<td>0.1377</td>
<td>(K_0^{\text{HM}})</td>
<td>-6.4765</td>
</tr>
<tr>
<td></td>
<td>(\kappa_3)</td>
<td>-2.0263</td>
<td>(\gamma_3)</td>
<td>0.9238</td>
<td>(\mu_3)</td>
<td>0.2839</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\kappa_4)</td>
<td>4.5215</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 MPa</td>
<td>(\kappa_1)</td>
<td>-5.0128</td>
<td>(\gamma_1)</td>
<td>0.8799</td>
<td>(\mu_1)</td>
<td>2.0697</td>
<td>e^\infty</td>
<td>1.4061</td>
</tr>
<tr>
<td></td>
<td>(\kappa_2)</td>
<td>84.3772</td>
<td>(\gamma_2)</td>
<td>0.2917</td>
<td>(\mu_2)</td>
<td>0.4935</td>
<td>(K_0^{\text{HM}})</td>
<td>-6.1043</td>
</tr>
<tr>
<td></td>
<td>(\kappa_3)</td>
<td>-3.0686</td>
<td>(\gamma_3)</td>
<td>0.6528</td>
<td>(\mu_3)</td>
<td>0.2247</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\kappa_4)</td>
<td>11.6986</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100 MPa</td>
<td>(\kappa_1)</td>
<td>-0.4212</td>
<td>(\gamma_1)</td>
<td>0.0107</td>
<td>(\mu_1)</td>
<td>0.0111</td>
<td>e^\infty</td>
<td>0.4208↑</td>
</tr>
<tr>
<td></td>
<td>(\kappa_2)</td>
<td>0.6852</td>
<td>(\gamma_2)</td>
<td>0.0145</td>
<td>(\mu_2)</td>
<td>0.0122</td>
<td>(K_0^{\text{HM}})</td>
<td>-4.0359</td>
</tr>
<tr>
<td></td>
<td>(\kappa_3)</td>
<td>0.4132</td>
<td>(\gamma_3)</td>
<td>0.1583</td>
<td>(\mu_3)</td>
<td>0.0067</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(\kappa_4)</td>
<td>-0.3329</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

† Manual fit.

Note that the limiting damping parameter for \(G_s = 100\) MPa has been fitted manually. Since the impedance for high values of \(G_s\) approaches the frequency dependent behaviour of the surface footings, the solution in Equation 3.6c in Chapter 3 is not valid. \(e^\infty\) for \(G_s = 100\) MPa in Table D.13 is in between the value for the suction caisson and a surface footing.

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Figure D.6: Coupling impedance: Boundary element solution and the corresponding lumped-parameter approximation. $\nu_s = 0.25$ and $\eta_s = 5\%$. 

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Table D.14: Torsion: Type and numbers of internal degrees of freedom for the lumped-parameter models

<table>
<thead>
<tr>
<th>$G_s$</th>
<th>Type</th>
<th>No. of internal dofs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0</td>
<td>2 second-order (kcm$^1$) + 1 first-order (kcm$^1$)</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>2 second-order (kcm$^1$) + 1 first-order (kcm$^1$)</td>
<td>3</td>
</tr>
<tr>
<td>100</td>
<td>2 second-order (kcm$^1$) + 1 first-order (kcm$^1$)</td>
<td>3</td>
</tr>
</tbody>
</table>

† Spring-dashpot-mass model, see Figure C.10(b)
‡ Spring-dashpot-mass model, see Figure C.9(b)

D.1.6 Lumped-parameter models for the torsional term

The type of approximation for the torsional lumped-parameter models is summarized in Table D.14 and the approximation is compared with the rigorous solution in Figure D.7. The pole-residue coefficients, the stiffness, damping and mass matrices of the models are given in the following.

Pole-residue coefficients

Table D.15: Torsion: Poles and residues

<table>
<thead>
<tr>
<th>$G_s$</th>
<th>Poles $s$</th>
<th>Residues $A$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_s = 1$ MPa</td>
<td>$-2.0852 + 4.7267i$</td>
<td>$-0.9261 - 3.4940i$</td>
</tr>
<tr>
<td></td>
<td>$-2.0852 - 4.7267i$</td>
<td>$-0.9261 + 3.4940i$</td>
</tr>
<tr>
<td></td>
<td>$-1.3704$</td>
<td>$+0.8947$</td>
</tr>
<tr>
<td></td>
<td>$-0.5230 + 4.4196i$</td>
<td>$-0.0782 + 1.6683i$</td>
</tr>
<tr>
<td></td>
<td>$-0.5230 - 4.4196i$</td>
<td>$-0.0782 - 1.6683i$</td>
</tr>
<tr>
<td>$G_s = 10$ MPa</td>
<td>$-2.8905 + 5.2170i$</td>
<td>$-1.6770 - 4.9680i$</td>
</tr>
<tr>
<td></td>
<td>$-2.8905 - 5.2170i$</td>
<td>$-1.6770 + 4.9680i$</td>
</tr>
<tr>
<td></td>
<td>$-1.2508$</td>
<td>$+0.6362$</td>
</tr>
<tr>
<td></td>
<td>$-0.5122 + 4.3775i$</td>
<td>$-0.1489 + 1.5255i$</td>
</tr>
<tr>
<td></td>
<td>$-0.5122 - 4.3775i$</td>
<td>$-0.1489 - 1.5255i$</td>
</tr>
<tr>
<td>$G_s = 100$ MPa</td>
<td>$-4.6430$</td>
<td>$+8.7857$</td>
</tr>
<tr>
<td></td>
<td>$-0.6051 + 4.2483i$</td>
<td>$+0.4685 + 1.4270i$</td>
</tr>
<tr>
<td></td>
<td>$-0.6051 - 4.2483i$</td>
<td>$+0.4685 - 1.4270i$</td>
</tr>
<tr>
<td></td>
<td>$-0.4184 + 7.1604i$</td>
<td>$+0.9069 + 1.0041i$</td>
</tr>
<tr>
<td></td>
<td>$-0.4184 - 7.1604i$</td>
<td>$+0.9069 - 1.0041i$</td>
</tr>
</tbody>
</table>
Matrices for the models

The resulting matrices of the models are given by Equation D.7. The model structure stated in Equation D.7 corresponds to the lumped-parameter models with one real and two complex conjugate poles. The corresponding coefficients are listed in Table D.16.

\[
K_{TT} = K_0^{TT} \begin{bmatrix}
\gamma_1^2 + \frac{\gamma_2^2}{\mu_1} + \frac{\gamma_3^2}{\mu_3} & -\kappa_1 & -\kappa_3 & 0 \\
-\kappa_1 & 0 & 0 & 0 \\
-\kappa_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\] (D.7a)

\[
C_{TT} = \frac{R}{c_S} K_0^{TT} \begin{bmatrix}
\gamma_1 & -\gamma_2 & -\gamma_3 \\
-\gamma_1 & 2\gamma_1 & 0 \\
-\gamma_2 & 0 & 2\gamma_2 \\
-\gamma_3 & 0 & 0 & \gamma_3
\end{bmatrix}
\] (D.7b)

\[
M_{TT} = \frac{R^2}{c_S} K_0^{TT} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \mu_1 & 0 & 0 \\
0 & 0 & \mu_2 & 0 \\
0 & 0 & 0 & \mu_3
\end{bmatrix}
\] (D.7c)

Table D.16: Torsion: Model coefficients

<table>
<thead>
<tr>
<th>$G_s$</th>
<th>$\kappa$ coeff.</th>
<th>Value</th>
<th>$\gamma$ coeff.</th>
<th>Value</th>
<th>$\mu$ coeff.</th>
<th>Value</th>
<th>misc</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 MPa</td>
<td>$\kappa_1$</td>
<td>1.9481</td>
<td>$\gamma_1$</td>
<td>0.7212</td>
<td>$\mu_1$</td>
<td>0.3459</td>
<td>$e^\infty$</td>
<td>0.7257</td>
</tr>
<tr>
<td></td>
<td>$\kappa_2$</td>
<td>7.2834</td>
<td>$\gamma_2$</td>
<td>0.0008</td>
<td>$\mu_2$</td>
<td>0.0015</td>
<td>$K_0^{TT}$</td>
<td>19.4817</td>
</tr>
<tr>
<td></td>
<td>$\kappa_3$</td>
<td>0.1500</td>
<td>$\gamma_3$</td>
<td>0.4764</td>
<td>$\mu_3$</td>
<td>0.3477</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\kappa_4$</td>
<td>-0.1199</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10 MPa</td>
<td>$\kappa_1$</td>
<td>2.5375</td>
<td>$\gamma_1$</td>
<td>0.6772</td>
<td>$\mu_1$</td>
<td>0.2345</td>
<td>$e^\infty$</td>
<td>0.7382</td>
</tr>
<tr>
<td></td>
<td>$\kappa_2$</td>
<td>5.7962</td>
<td>$\gamma_2$</td>
<td>0.0032</td>
<td>$\mu_2$</td>
<td>0.0063</td>
<td>$K_0^{TT}$</td>
<td>19.1516</td>
</tr>
<tr>
<td></td>
<td>$\kappa_3$</td>
<td>0.2923</td>
<td>$\gamma_3$</td>
<td>0.4066</td>
<td>$\mu_3$</td>
<td>0.3251</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\kappa_4$</td>
<td>-0.1697</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>100 MPa</td>
<td>$\kappa_1$</td>
<td>-2.1190</td>
<td>$\gamma_1$</td>
<td>0.1165</td>
<td>$\mu_1$</td>
<td>0.2784</td>
<td>$e^\infty$</td>
<td>0.5363</td>
</tr>
<tr>
<td></td>
<td>$\kappa_2$</td>
<td>16.4413</td>
<td>$\gamma_2$</td>
<td>0.0292</td>
<td>$\mu_2$</td>
<td>0.0482</td>
<td>$K_0^{TT}$</td>
<td>16.5191</td>
</tr>
<tr>
<td></td>
<td>$\kappa_3$</td>
<td>-0.7566</td>
<td>$\gamma_3$</td>
<td>0.4075</td>
<td>$\mu_3$</td>
<td>0.0878</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\kappa_4$</td>
<td>1.6437</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$^\dagger$ Manual fit.

Note that the limiting damping parameter for $G_s = 100$ MPa has been fitted manually. Since the impedance for high values of $G_s$ approaches the frequency dependent behaviour of the surface footings, the solution in Equation 3.4 in Chapter 3 is not valid. $c^\infty$ for $G_s = 100$ MPa in Table D.16 is in between the value for the suction caisson and a surface footing.

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Figure D.7: Torsional impedance: Boundary element solution and the corresponding lumped-parameter approximation. $\nu_s = 0.25$ and $\eta_s = 5\%$. 
D.2 Assembly of the global dynamic stiffness matrix

The dynamic stiffness for each degree of freedom is given by three matrices $K_{\text{dof}}$, $C_{\text{dof}}$ and $M_{\text{dof}}$. The subscript 'dof' denotes the degree of freedom, which is either $VV$, $HH$, $MM$, $TT$ or $HM$. The matrices describing the dynamic stiffness for each of the degrees of freedom are denoted as local matrices in the following. Each local matrix contains frequency independent coefficients, which are determined by the procedure applied in the previous sections. The size of $K_{\text{dof}}$, $C_{\text{dof}}$ and $M_{\text{dof}}$ are given by the numbers and types of discrete elements used to approximate the dynamic stiffness. The size of the local matrices are denoted by $n_{\text{dof}}$.

D.2.1 Structure of the local dynamic stiffness matrices

Each local matrix can be divided into four sections. The first section contain the stiffness, damping or mass coefficient of the external node of the lumped-parameter model, i.e, the coefficient that enters the finite element formulation of the structural system. The second section contains the coefficients of the internal nodes of the lumped-parameter model, and finally, the third and fourth section contain coefficients that link the external and internal nodes. The structure of $K_{\text{dof}}$, $C_{\text{dof}}$ and $M_{\text{dof}}$ are given as

\[
K_{\text{dof}} = \begin{bmatrix}
k_{11,\text{dof}} & k_{12,\text{dof}} \\
k_{21,\text{dof}} & k_{22,\text{dof}}
\end{bmatrix}, \quad
C_{\text{dof}} = \begin{bmatrix}
c_{11,\text{dof}} & c_{12,\text{dof}} \\
c_{21,\text{dof}} & c_{22,\text{dof}}
\end{bmatrix}, \quad
M_{\text{dof}} = \begin{bmatrix}
m_{11,\text{dof}} & m_{12,\text{dof}} \\
m_{21,\text{dof}} & m_{22,\text{dof}}
\end{bmatrix}.
\]

The sub-matrices, denoted by the subscript $11$, contain only one component ($1 \times 1$ matrices). The size of the sub-matrices denoted by the subscript $22$ is $(n_{\text{dof}} - 1) \times (n_{\text{dof}} - 1)$, and the size of the sub-matrices denoted by the subscript $12$ and $21$ are $1 \times (n_{\text{dof}} - 1)$ and $(n_{\text{dof}} - 1) \times 1$, respectively.

D.2.2 Structure of the global dynamic stiffness matrices

The dynamic stiffness relation for a generalized massless axisymmetric rigid foundation with six degrees of freedom (one vertical, two horizontal, two rocking and one torsional) is given in section C.1.2. The stiffness formulation is given by an impedance matrix, $S_{ij}(a_0)$, relating the displacements and forces acting on the foundation. $S_{ij}(a_0)$ is a frequency dependent matrix with complex components, which does not fit into the framework of ordinary finite element codes. However, the lumped-parameter model represents a unbounded soil domain, and the soil-structure interaction of a massless foundation can be modelled by relatively few springs, dashpots and masses, all with real frequency-independent coefficients. Each degree of freedom at the foundation node of the structural model is coupled to a lumped-parameter model that may consist of additional internal degrees of freedom.

In this subsection the structure of the global dynamic stiffness matrices, based on the lumped-parameter models, will be explained. The global dynamic stiffness matrices are given for two- and three-dimensional problems.
Global dynamic stiffness matrices for 2D

A two-dimensional beam member is capable of axial deformation and ending in one principal plane. Each node in the finite element formulation is described by three degrees of freedom. For details, see Petyt (1998). The global matrices, $K^{2D}$, $C^{2D}$ and $M^{2D}$, representing the dynamic stiffness of a two-dimensional foundation are as follows:

$$
K^{2D} = \begin{bmatrix}
k_{11}^{HH} & 0 & k_{11}^{HM} \\
0 & k_{11}^{VV} & 0 \\
k_{11}^{HM} & 0 & k_{11}^{MM}
\end{bmatrix}
\begin{bmatrix}
k_{12}^{HH} & 0 & 0 & k_{12}^{HM} \\
0 & k_{12}^{VV} & 0 & 0 \\
0 & 0 & k_{12}^{MM} & 0 \\
0 & 0 & 0 & k_{12}^{HM}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & k_{12}^{MM} & 0 \\
0 & 0 & 0 & k_{12}^{HM}
\end{bmatrix}
$$

$$
C^{2D} = \begin{bmatrix}
c_{11}^{HH} & 0 & c_{11}^{HM} \\
0 & c_{11}^{VV} & 0 \\
c_{11}^{HM} & 0 & c_{11}^{MM}
\end{bmatrix}
\begin{bmatrix}
c_{12}^{HH} & 0 & 0 & c_{12}^{HM} \\
0 & c_{12}^{VV} & 0 & 0 \\
0 & 0 & c_{12}^{MM} & 0 \\
0 & 0 & 0 & c_{12}^{HM}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & c_{12}^{MM} & 0 \\
0 & 0 & 0 & c_{12}^{HM}
\end{bmatrix}
$$

$$
M^{2D} = \begin{bmatrix}
m_{11}^{HH} & 0 & m_{11}^{HM} \\
0 & m_{11}^{VV} & 0 \\
m_{11}^{HM} & 0 & m_{11}^{MM}
\end{bmatrix}
\begin{bmatrix}
m_{12}^{HH} & 0 & 0 & m_{12}^{HM} \\
0 & m_{12}^{VV} & 0 & 0 \\
0 & 0 & m_{12}^{MM} & 0 \\
0 & 0 & 0 & m_{12}^{HM}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & m_{12}^{MM} & 0 \\
0 & 0 & 0 & m_{12}^{HM}
\end{bmatrix}
$$

The upper left part of the matrices are to be added to the foundation node of the structural finite element model. The remaining components of the matrices correspond to the additional internal degrees of freedom, arising from the lumped-parameter models. The number of additional degrees of freedom for the two-dimensional model is $(n_{VV} - 1) + (n_{HH} - 1) + (n_{MM} - 1) + 2(n_{HM} - 1)$, i.e. the sum of the additional internal degrees of freedom.

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The assembly between the global matrices of the foundation and the system matrices of the structural system is sketched in Figure D.8.

**Global dynamic stiffness matrices for 3D**

A three-dimensional beam member is capable of axial deformation, bending in two principal planes and torsion about the beam axis. Each node in the finite element formulation is described by six degrees of freedom. For details, see Petyt (1998). The global matrices, $K^{3D}$, $C^{3D}$ and $M^{3D}$, representing the dynamic stiffness of a three-dimensional foundation are as follows:

$$
K^{3D} = \begin{bmatrix}
{k_{11}^{11}} & {k_{12}^{12}} \\
{k_{21}^{11}} & {k_{22}^{12}}
\end{bmatrix},
C^{3D} = \begin{bmatrix}
{c_{11}^{11}} & {c_{12}^{12}} \\
{c_{21}^{11}} & {c_{22}^{12}}
\end{bmatrix},
M^{3D} = \begin{bmatrix}
{m_{11}^{11}} & {m_{12}^{12}} \\
{m_{21}^{11}} & {m_{22}^{12}}
\end{bmatrix} \tag{D.10}
$$
where \( \bar{k}^{11} \), \( \bar{k}^{12} \), \( \bar{k}^{21} \) and \( \bar{k}^{22} \) are given as

\[
\bar{k}^{11} = \begin{bmatrix}
0 & 0 & 0 & 0 & -k_{11}^{11} \\
0 & k_{11}^{11} & 0 & 0 & 0 \\
0 & 0 & k_{11}^{11} & 0 & -k_{11}^{11} \\
0 & 0 & 0 & k_{11}^{11} & 0 \\
0 & -k_{11}^{11} & 0 & 0 & 0 \\
\end{bmatrix}
\] (D.11a)

\[
\bar{k}^{12} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & -k_{12}^{12} \\
0 & k_{12}^{12} & 0 & 0 & 0 & 0 \\
0 & 0 & k_{12}^{12} & 0 & 0 & -k_{12}^{12} \\
0 & 0 & 0 & k_{12}^{12} & 0 & 0 \\
0 & 0 & 0 & 0 & k_{12}^{12} & 0 \\
0 & 0 & 0 & 0 & 0 & k_{12}^{12} \\
\end{bmatrix}
\] (D.11b)

\[
\bar{k}^{21} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & k_{21}^{21} & 0 & 0 & 0 & 0 \\
0 & 0 & k_{21}^{21} & 0 & 0 & 0 \\
0 & 0 & 0 & k_{21}^{21} & 0 & 0 \\
0 & 0 & 0 & 0 & k_{21}^{21} & 0 \\
0 & 0 & 0 & 0 & 0 & k_{21}^{21} \\
\end{bmatrix}
\] (D.11c)

\[
\bar{k}^{22} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & k_{22}^{22} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & k_{22}^{22} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & k_{22}^{22} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & k_{22}^{22} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & k_{22}^{22} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & k_{22}^{22} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & k_{22}^{22} \\
\end{bmatrix}
\] (D.11d)

The sub-matrices in \( C^{3D} \) and \( M^{3D} \) are similar to those for \( K^{3D} \), given by the equations in D.11. The sub-matrices are obtained by replacing \( k \) by \( c \) and \( m \), respectively. The number of additional degrees of freedom for the three-dimensional model is \( (n_{VV} - 1) + 2(n_{HH} - 1) + (n_{TT} - 1) + 2(n_{MM} - 1) + 4(n_{HM} - 1) \). Note that the rows in \( \bar{k}^{11} \) (and hence \( \bar{c}^{11} \) and \( \bar{m}^{11} \)) can be interchanged, depending on the arrangement of the degrees of freedom in the structural finite element formulation. Appropriate rearrangement of the remaining sub-matrices (\( \bar{k}^{12} \), \( \bar{k}^{21} \) and \( \bar{k}^{22} \)) should then be performed as well.

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D.3 Direct analysis of the steady state response for lumped-parameter models

The steady state response is determined by solving the equation of motion for a harmonic response, given by

\[ M\ddot{u} + C\dot{u} + Ku = fe^{i\omega t}, \]  

where \( M, C \) and \( K \) are the mass, damping and stiffness matrices of the vibrating structure, respectively. \( M, C \) and \( K \) are assembled from the global matrices of the foundation and the system matrices of the structural system, as sketched in Figure D.9. \( u \) is a column vector containing the nodal displacements and \( f \) is a column vector of nodal forces. \( t \) is time and \( i \) is the imaginary unit, \( i = \sqrt{-1} \). The equation of motion in Equation D.12 is solved by direct analysis (Petyt 1998). The solution to Equation D.12 is then

\[ u = [K - \omega^2M + i\omega C]^{-1} fe^{i\omega t} \]  

Components of the structural model

Components of the foundation model

Figure D.9: Structure of the matrices and vectors for the direct analysis.