Aalborg Universitet

What is the problem in problem-based learning in higher education mathematics

Dahl, Bettina

Published in:
European Journal of Engineering Education

DOI (link to publication from Publisher):
10.1080/03043797.2017.1320354

Publication date:
2018

Document Version
Accepted author manuscript, peer reviewed version

Link to publication from Aalborg University

Citation for published version (APA):
Dahl, B. (2018). What is the problem in problem-based learning in higher education mathematics. *European Journal of Engineering Education*, *43*(1), 112-125. https://doi.org/10.1080/03043797.2017.1320354

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(Received 31 October 2016; Accepted 08 April 2017)

ABSTRACT

Problem and Project-Based Learning (PBL) emphasise collaborate work on problems relevant to society and emphases the relation between theory and practice. PBL fits engineering students as preparation for their future professions but what about mathematics? Mathematics is not just applied mathematics, but it is also a body of abstract knowledge where the application in society is not always obvious. Does mathematics, including pure mathematics, fit into a PBL curriculum? This paper argues that it does for two reasons: (1) PBL resembles the working methods of research mathematicians. (2) The concept of society includes the society of researchers to whom theoretical mathematics is relevant. The paper describes two cases of university PBL projects in mathematics; one in pure mathematics and the other in applied mathematics. The paper also discusses that future engineers need to understand the world of mathematics as well as how engineers fit into a process of fundamental-research-turned-into-applied-science.

KEYWORDS: PBL; mathematics education; higher education; engineering education; applied mathematics

Introduction

Aalborg University (AAU) in Denmark has since its establishment in 1974 run a problem and project-based learning (PBL) curriculum at all faculties. Half of the students' time is typically used in traditional courses and the other half in PBL groups of up to eight students. All curricula are based on shared PBL principles of student working problem oriented with projects that apply theory to solve or explain a problem from society. The students are responsible for their own learning including project planning, and they collaborate in teams and receive feedback from other students as well as supervisors who act like facilitators. The AAU PBL model also includes that the contents of the projects are not only mono-disciplinary but also when appropriate, they have an inter-disciplinary focus. The problems addressed in each study programme vary, but there is a tradition for contextualising disciplinary knowledge. The structure is usually that students find an initiating problem in society within the given thematic framework of a semester, then they analyse the problem in its context, formulate a more specific problem statement, and finally embark on a problem solution. In earlier semesters, students are to some degree being given the problems in project catalogues as their level of subject knowledge is often not high enough to be able to formulate relevant problems from nothing by themselves. The AAU PBL model is therefore a student-driven and motivated method working with exemplary problems that are typical for the professions in which the students would work after[AQ1] graduation (Boud and Feletti 1997; Kolmos, Fink, and Krogh 2004; De Graaff and Kolmos 2007; Du etal. 2009; Kolmos 2009; Barge 2010; Askehave etal. 2015).

AAU also has a study programme in mathematics at the Faculty of Engineering and Science. As De Graaff (2016, 397) argues, 'Working in a project is a natural preparation for a professional career in engineering. For other professions such a link to a project is less obvious.' Barrows (1996) describes how PBL was implemented in medicine and concluded that it fits very well and also Ulseth and Johnson (2015) show examples of how well PBL fits engineering. Mathematics is applied extensively outside of mathematics in, for instance, engineering, physics, chemistry, biology, and economics where mathematics is often a 'service subject' (Howson etal. 1988). However, mathematics itself is an abstract discipline dealing with theoretical concepts, structures, and logical connections – mathematics is not just a service subject, mathematics is also an abstract and pure science. The focus of this paper is to discuss if pure mathematics fits into an educational system that is organised around PBL. When discussing PBL in mathematics, the focus it is not on what Schoenfeld (1985) terms problem-solving by which he means the standard and non-standard tasks students typically solve as part of a course in mathematics. Savery (2006) and Norman (1988) also argue for not confusing PBL with the teaching of problem-solving. One might therefore argue that mathematics, in its pure form, would suffer in a PBL curriculum if mathematics is reduced to just being a tool when applied in other sciences and engineering, leading to lack of fundamental and deep understanding of mathematics and further development of mathematics. Applied mathematics relates to mathematics as seen in the wide use of mathematical models in society, but what about mathematics itself? Is it possible, in fact, to study formal pure mathematics at a PBL university? The main focus of this paper is therefore not on mathematics education for engineering students, but mathematics education for mathematics students at a PBL university – and how the latter is different from, but relating to, the former. It will also discuss why engineers of the future need to understand some of this pure world of mathematics. When discussing mathematics, the paper will also discuss its close relationship to physics which is the most fundamental science.

Thus, the overall question addressed is if PBL fits mathematics in a higher education curriculum. The paper therefore begins with a section that discusses what mathematics actually is and how it relates to PBL. This section ends with an argument that pure mathematics at a PBL university is close to the practices of professional mathematicians in two ways. These two ways are subsequently discussed in separate sections, including here a section about engineering mathematics. Then follows a section discussing what a problem in PBL in mathematics is, or could be. After this, two cases of project reports in mathematics from AAU are analysed to illustrate how PBL may look in pure and applied mathematics, respectively, and finally, the papers discusses and concludes with reference and suggestions to engineering education.

What is mathematics?

In mathematics education literature, one often distinguishes between mathematics as a scholarly field of knowledge and school mathematics. The latter relates to the former, but it is not the same. School mathematics also depends on level as, for instance, third-grade primary level mathematics is very far from graduate level mathematics at university. A course for graduate students in mathematics is not just 'more difficult' than mathematics in the third grade, but in many ways it is a different topic. Tall (1991, 20) argues that moving from elementary to advanced mathematical thinking during the first year at university is a significant[AQ2] transition 'from *describing* to *defining*, from *convincing* to *proving* in a logical manner based on those definitions'. Hence, doing mathematics is qualitatively different in elementary and advanced mathematics. There are several frameworks to describe mathematics at the different education levels. In one sense, one may argue that students learn more topics such as numbers, geometry, functions within mathematics and these topics become increasingly abstract as students move through the education system. However, one might also argue that mathematics is more than a list of topics. According to Niss (2015, 36), 'mathematics should be perceived as an activity … coming to grips with what it means to be mathematically competent cannot be adequately captured by way of such lists'. It is equally important to describe the mathematical activities. OECD (2013) suggests seven aspects of what they term fundamental mathematical capabilities: Communication, Mathematising, Representation, Reasoning and argument, Devising strategies, Using symbolic, formal and technical language and operations, and Using mathematical tools. Almost similar descriptions of mathematical competencies across the whole education system are given by Niss and Jensen (2002) and NCTM (2000). In common is a desire to describe the whole of what it means to do mathematics in order for students to know mathematics. The mathematics described here appears as not only being a service subject but also a pure and abstract science involving, for instance, formal and abstract reasoning.

The concept of didactic transposition refers to the transformation a scholarly body of knowledge undergoes from when it is produced by researchers, then put into a curriculum and later taught at an educational institution (Chevallard 1992). It points to the fact that what is taught at schools origins in other institutions and practices and it becomes transformed when it is put into a curriculum and later appears in concrete lesson plans and lectures. Nunes, Carraher, and Schliemann (1993)[AQ3] made a further distinction between school mathematics and street mathematics, the latter denoting the often 'home-made' methods people apply when they solve mathematical type tasks on the job or in private settings. These methods are often different from the methods taught in school mathematics. Their study, for instance, found that Brazilian street children were very able to perform calculations at the market place, but they were not able to do what appeared to be the same calculations within a school mathematics setting.

PBL is one type of curriculum. This means that a specific type of transformation of the scholarly field of mathematics has been made to fit a PBL curriculum model. This paper argues that mathematics in a PBL university mathematics curriculum has similarities to the scholarly body of knowledge of mathematics. In one way, the above description of PBL seems different from mathematics as professional research mathematicians usually do not solve concrete problems in society; however, this paper will argue that learning pure mathematics at a PBL university is close to the practices of professional mathematicians in two ways:

1. The methods of working with mathematics are quite similar in PBL and in research.
2. The concept of society includes the society of researchers and professionals to whom pure mathematics is highly relevant.

The working methods of professional mathematicians

Research-based education

Universities in Denmark are by law (Uddannelses- og Forskningsministeriet 2015) required to provide so-called research-based education (*forskningsbaseret undervisning*). This has been an ideal for universities worldwide since the middle of the nineteenth century and the concept denotes several things (Laursen 1996). One definition is that teaching should consist of research results. However, this should not be the only criteria as university teaching would then not be much different from any other kind of teaching as most knowledge at some point was developed by researchers. A second definition is that teaching and research take place at the same institution. This is essential but it does not in itself ensure a high quality of teaching. A third definition points to that active researchers should teach and they should use their research when teaching. Laursen (1996) criticises this for being too focused on what the teacher does instead of what the students should do. One can argue that this argument is in agreement with, for instance, Tyler (1949, 63) who states that 'Learning takes place through the active behaviour of the student: it is what he does that he learns, not what the teacher does.' Laursen (1996) therefore prefers the fourth definition of research-based education, stating that students should learn to work research-like by collaborating with active researchers. This paper argues that studying in a PBL curriculum has many similarities to the way researchers work and students learn to work research-like when studying in a PBL curriculum. This is also stated by Savery (2006, 9) who argues that PBL 'is an instructional (and curricular) learner-centred approach that empowers learners to conduct research, integrate theory and practice, and apply knowledge and skills to develop a viable solution to a defined problem'. Furthermore, researchers in mathematics, and any other field, do usually not know the answer or solution to the enquiry set forth and the work process is adjusted along the way as the work progresses. Thus, mathematicians work differently from engineers since in the engineering problem-solving process, the engineer aims to find one solution. A researcher or scientist wants to know how and why it works. It is the difference between a convergent and a divergent process (Guilford 1967) where divergent thinking is the ability to draw on ideas from different disciplines to get a deeper understanding, while the engineering approach is more aligned with convergent thinking which is oriented towards getting the best answer to a clearly defined question. It emphasises speed and application of techniques already known.

The work processes of mathematics researchers in comparison to school work

In PBL the work is collaborative and in a study of professional mathematicians, Burton (2004) found that a majority worked co-operatively. Researchers, in any field, do not work like students often do in typical school mathematics setting. At school, students are usually presented to a topic in a textbook with carefully selected material presented in a logical order to help the students create an overview and understanding of various areas of the field. The teaching and assignments are often within what Olsen and Pedersen (2008) terms subject-oriented teaching, which is different from problem-based teaching. In subject-oriented teaching, students usually write an essay or a report where the purpose is to show overview and knowledge of a topic. There is not a clear focus as anything that might fit into the topic can be included into the work and there is also not a conclusion as such, other than perhaps a summary of main points of the theories. This paper does not want to argue that subject-oriented teaching is bad teaching. On the contrary, it is very essential to gain knowledge of central parts of a subject and its methods, but it is not research-based education in the sense of teaching students to work research-like. Subject-oriented work might become a part of a piece of research as research usually involves a literature review of existing research in a field, but it is not research in itself. Mathematics as a scholarly body of knowledge is developed through a process of research. The process can take years and experience alternating periods of success and failure. One well-known example is the search for a proof for Fermat's last theorem conjectured in 1637. Many mathematicians have attempted but failed to prove the theorem before Wiles did it in 1995 (Singh 1997). This is also reflected in the seven Millennium Prize Problems (CMI 2000) which are important classic problems that 'have resisted solution for many years'. This includes the Riemann hypothesis proposed in 1859 and the Poincaré conjecture stated in 1904.

What kind of mathematics the mathematics students need

Teaching through textbooks with logical presentations of the body of knowledge might therefore not be how *all* university teaching should be like; partly since students should meet a research-based education and partly because it might be hard to become knowledgeable about the body of scholarly knowledge of abstract and complex mathematics solely through well-structured textbooks. In a section entitled *Curriculum design in advanced mathematical learning*, Tall (1991) argues that students need help in getting insight into what goes on when they are in the difficult transition from pre-formal mathematics to more formal understanding of mathematical processes. Tall (1991) argues that students should learn the processes of mathematical thinking and not just its results.One may argue that university mathematics students therefore need to experience the process of research in mathematics themselves. In this sense, PBL may be a good tool as the teachingis bottom-up and explorative where the students are in unknown territory trying to find a way to reach some kind of conclusion to the issues they endeavour to investigate – and prove. Tall (1991, 16) in this contexts argues that if one views proofs as problem-solving, 'proof is actually[AQ4] the final stage of activity in which ideas are made precise. Yet so much teaching in university level mathematics *begins* with proofs'. Tall then quotes Poincaré (1913):

> To understand the demonstration of a theorem, is that to examine successively each of the syllogisms composing it and to ascertain its correctness to the rules of the game? … For some, yes; when they have done this, they will say: I understand. For the majority, no. Almost all are much more exacting they wish to know not merely whether all the syllogisms of a demonstrations are correct, but why they link together in this order rather than another. (Tall 1991, 16)

Hence, the teaching of advanced mathematics at university should not just involve teaching the products of mathematics, but also the processes and the deeper understanding of why, for instance, a theorem or piece of mathematics is true. In line with this, Skemp (1987, 207) compares mathematics to music:

> If we were to teach children music the way we teach mathematics, we would only succeed in putting most of them off for life. … For most of us mathematics, like music, needs to be expressed in physical actions and human interaction before its symbols can evoke the silent patterns of mathematical ideas (like musical notes), simultaneous relationships (like harmonies) and expressions or proofs (like melodies).

Teaching higher education mathematics is to some extent to teach students to become mathematicians. It is not a continuation of the subject-oriented education from high school and compulsory education and PBL can be exemplary learning, as it would resemble the work of a professional mathematician. This, however, does not mean that a PBL curriculum should not also have more traditional courses introducing the students to the field of knowledge and methods in general. It is important to have an extensive understanding of the main areas within ones field. Some of the knowledge introduced in courses might become part of the projects. This is also not contrary to how researchers work, as also researchers read papers or books giving a presentation of an area of knowledge; or take additional courses.

The society of researchers is the context of PBL projects in pure mathematics

What does 'society' means?

In an engineering curriculum, an initiating problem could be something like constructing a bridge across a given river. The analysis of the problem involves both engineering knowledge such as traffic analysis and calculations on materials needed to build the bridge, but it also involves environmental issues such as pollution from the traffic, effect of placing concrete blocks at the bottom of the river, and development of house prices in the areas close to the bridge. Such a process involves collaboration across disciplines and the ability to see the problem in its whole context in society. One may here argue that mathematics is different. It is pure and even though having proved a theorem has an effect on society as a whole since any adding to the body of scholarly knowledge affects society, determining the exact effect of any given theorem is very hard and speculative. Sometimes it takes decades before one knows how to apply a piece of mathematical knowledge in society.

But what does the concept 'society' means? Who is a member of the society? Is there only one society? Giddens (1993, 32) defines society as 'a *system of interrelationships* which connects individuals together[AQ5]'. One could therefore argue than the society consists of several groups of people who in different ways are connected and therefore form a society among themselves within the overall society. This paper argues for making a distinction between three groups within society. It is essential to be clear about what is meant by society before discussing the context and relevance of a given project.

First society: society in general

Each single person in, for instance, Denmark is part of the Danish society in general and has met school mathematics as part of compulsory education. They would all, either privately or professionally use mathematics to some extent. This, for instance, happens when adding sums during shopping, considerations of rates of interests regarding bank accounts, observations of graphs over various developments in news media etc. The mathematics being used is school mathematics, or perhaps street mathematics and it is mainly arithmetic.

Second society: society of professionals

This is the carpenter, plumber, engineer, IT technician, bank advisor, etc. who needs some understanding of mathematics in order to do their professional work. They would often use tools that are developed by mathematicians such as computer programs or formulas and many engineers would take part in developing the tools while applying mathematics. Others might not have much insight into the details of the mathematical models and algorithms behind the methods they apply daily, but they possess a good sense of school mathematics, particularly applied mathematics as well as mathematical reasoning and they would to some extent understand why the tools work and are able to use these tools in flexible manners.

Third society: society of researchers

This is the group of professional mathematicians as well as neighbouring professions such as computer science and physics who each develop pieces of mathematical or scholarly knowledge. This knowledge is discussed, shared, and further developed within the research community. Some of this knowledge is applied to develop tools for professionals or the society in general.

For instance, Google's PageRank-algorithm is based on the mathematical branch called linear discrete dynamical systems. Furthermore, it is, for instance, due to modern nanotechnology that it is possible to produce still faster processor chips for computers, but it is not the engineers doing research in nanotechnology who themselves manufacture the products they have made possible. Not all scholarly knowledge is applied outside the society of researchers directly. Most often the receivers of this knowledge are other researchers. Fundamental research in one area can facilitate new developments in a different field of research, and in this way a body of knowledge is here being gradually built. At some point, some of it is used to develop tools. Sometimes it takes decades before a practical use of scholarly knowledge is found.

The three societies and the concept of application

The three societies are not distinct since people in the two latter societies are also members of the first society and many times people in the two latter societies interact when developing tools. There is a process from developing abstract mathematical knowledge to later apply this knowledge when developing tools for other parts of the society. The three societies do not intend to be an exhaustive description of the society in total but to point to the fact that there are different types of users of mathematical knowledge. The same can be argued for other disciplines such as computer science and physics mentioned above.

A note on the concept of application is here needed. Mathematics is not just applied outside mathematics but also inside mathematics. One example is number theory which dates back to the times of Plato who in *Theaeteus* (1997, 164) tells us that Theodorus has proved that the square roots of 3, 5, etc. (up to 17), are irrational numbers or surds. For Millennia, number theory was without applications outside mathematics and the number-theorist Dickson (1874–1954) once said: 'Thank God that number theory is unsullied by any application' (Burr 1993). However, today number theory has countless applications in both mathematics itself but also outside mathematics in, for instance, computer science, cryptography, and steganography. Furthermore, Lützen (2013~~2004~~) argues that if one compares mathematics with physics in a historical perspective, one sees that these two subjects have interacted in a complex way. Mathematics was not just applied in physics but often unfinished mathematics has met unfinished physics and a complicated process has taken place where both subjects has benefitted, ending up in a satisfactory mathematical description of a physics phenomenon. An example here is Liouville's theorem for differentiation of arbitrary order. This theorem was developed in order to solve differential equations in physics and determine the fundamental microscopic force law. Hence a physics application here drove the development of mathematics. In physics AAU PBL project one furthermore finds quite narrow problem statements even at first year of study. For instance, an initiating problem could sound like: 'What are the optical and thermodynamic properties of gases?' and then part of the problem statement could be: 'What is the correlation between the velocity distribution and the temperature of the gas?' (Author's experience from being a co-supervisor for second-semester physics projects at AAU). Such questions are understandable and relevant to an audience of physics professionals but it is not likely to be understood by members of the general society, or even be relevant to them at this point – but this does not mean that such problems are not relevant at all – they are relevant in the third society.

Engineering mathematics

When discussing application of mathematics, one also needs to discuss the concept of engineering mathematics which is a term used to describe the mathematics applied by engineers. Engineering mathematics is therefore applied mathematics, for instance, mathematical modelling on complex problems. Christensen (2008) describes examples of engineering projects applying mathematics at AAU. The work in engineering mathematics belongs to the second society but uses methods and theories developed by researchers in the third society. Alpers etal. (2013) describe and discuss the SEFI (European Society for Engineering Education) framework for mathematics curricula in engineering education in which they discuss the mathematics competencies also discussed above in this paper. A main message of their work is that the description of the mathematics competencies is not only relevant to mathematics education, but it is also a useful tool for mathematics education for engineers. They argue that even though content remains important, and students need some familiarity with the mathematics concepts and procedures prior to application, the knowledge ought to be embedded in view of the mathematical competencies in order to avoid that mathematics courses for engineering students are mainly restricted to contents. They also argue that 'mathematics education must be integrated in the surrounding engineering study course to really achieve the ability to use mathematics in engineering contexts' (Alpers etal. 2003[AQ6], 7). The question of how mathematics curricula should be organised in engineering education is beyond the scope of this paper; however, what is essential for this paper is the emphasis on the application side of mathematics while maintaining that some level of knowledge of mathematics concepts and procedures are necessary too. It is also interesting that the SEFI framework uses a framework from mathematics in order to place limitations on how much mathematics content engineering students should learn. As argued by Antonsen (2009), the eight competencies from Niss and Jensen (2002) are different as four of them may be termed inner mathematical competencies (thinking, reasoning, representing, symbols, and formalism) as their focus from a mathematical point of view are relating to pure mathematics. Thus, in other contexts, the discussion of mathematics competencies are used to protect mathematics from being just a service subject where mathematics is mainly being used as a tool in other disciplines.

When doing PBL in mathematics, this paper argues that PBL problems could be found in all three societies but that the problems are different and their context and audience is therefore also different. Most often the problems and applications of pure mathematics in higher education are found in the third society. In PBL, students need to be aware in which society they are working and if possible have ideas of its relation to the other societies.

What is a 'problem' in PBL in mathematics

The term 'problem' might indicate that it is something negative that has to be improved or fixed, but it is not necessarily the case. In fact, in the literature about PBL, sometimes 'P' indicates a 'problem' and other times a 'project' and there are many PBL models 'ranging from PBL lectures, where the teacher builds his presentation around a case from practice to self-organized group work outside formal education' (De Graaff 2016, 396) and also a multitude of ways of structuring a PBL curriculum (Dahl etal. 2016). At AAU, the 'P' in PBL indicates both problem and project but both approaches stem from similar educational theories. In project management literature (Jensen and Dinitzen 2011), a project is usually defined as a task that has a specific goal, the time frame and resources are limited, the organisation about the project is temporary, the solution requires multiple disciplines, and the task requires more than one person to complete. A problem could also be a case from a professional practice, which is often used in medical education programmes in PBL (here denoted case-based learning). As argued by De Graaff (2016), a problem, on the other hand, could be something that initiates the learning process, for instance, a description of a natural phenomenon. The student group should then find or provide an explanation of that phenomenon, which satisfies the scientific criteria of the specific subject. Problems are here knowledge problems belonging to the realm of fundamental science. Adolphsen (2007) writes about theoretical problems in PBL and argues that these are the object in fundamental science and can be perceived as anomalies in our understanding, explanations, and theories about the world. The answer to such problems provides us with new insight. Also scientists work with problems. Popper in his book *All Life Is Problem Solving* (1999, 3) argues that

> the natural as well as the social sciences always start from *problems*, from the fact that something inspires *amazement* in us, as the Greek philosophers used to say. To solve these problems, the sciences use … the method of *trial and error*.

This means that science does not begin with observations but with problems which the scientists attempt to solve through a process of conjectures and refutations in which the original problem change character. Problems are here knowledge problems but the initiating problems behind such problems can both be practical and empirical.

Data: two cases of PBL projects in higher education mathematics

This section will describe and discuss two examples of project reports of student groups at the second semester of pure mathematics at AAU from spring 2014. The author was co-supervisor for both groups, but the groups also had a main supervisor from the mathematics department. Both reports were by the examiners judged to be good. At AAU, students begin studying their area of speciality already from first semester and there is not a core curriculum or disciplinary breath requirement, which is the case in, for instance, the U.S.A.

The purpose of the two cases is not to provide a full analysis of the content of the written reports, or argue that they are representative for all student reports, but to provide illustrative examples of what PBL in higher education mathematics may look like. The students in these two cases all worked within the branch of mathematics called discrete mathematics with the common theme, *Combinatorics: graph theory and optimisation*. The problems in the project reports were either real-life problems such as route optimisation for trucks collecting eggs from farmers or theoretical problems of general route optimisation in networks.

Methodology and analytical framework

The method of analysis in this paper was quantitative content analysis counting pages of different chapters, number of times the students used proofs, examples and figures in order to determine the quality of the reports. The analysis furthermore used a deductive schemata application as the three worlds of Tall (2013) were used to analyse the reports (Titscher etal. 2000).

Tall's (2013) theory of the three worlds of mathematics is applied to describe what type of mathematical concepts was seen in the reports. The first world is that of conceptual embodiment, which is an object-based understanding of mathematics reflecting on the senses to observe, describe and deduce properties such as that subtraction means to 'take away' something in a physical sense hence the sensory meaning of a concept. The second world is an action-based proceptual-symbolic world that compresses action-schemas into thinkable concepts operating dually as processes and concepts (procepts). This is when symbols change their meaning from being something to calculate using an algorithm to exist by itself. For instance, 2^3 which is $2 \cdot 2 \cdot 2$ as repeated multiplication but which is then extended to x^3. The third is a property-based formal-axiomatic world of concept definitions and set-theoretical proofs. There are also other types of proofs. Some are based on the first world embodiment seen in, for instance, Euclidian proofs or the symbolic second world where proofs world be algebraic using the rules of arithmetic. Many proofs would have elements of all worlds but in higher education projects in pure mathematics, one would expect the majority of the report to be within the third world.

Overall structure of the two case reports

Both reports first discuss the problem and formulated a problem statement, and then they have theoretical chapters describing the mathematical content. The introduction includes the initiating problem, the problem analysis, and problem statement. One might discuss how long this part should be. In general at AAU, the length of this part varies a lot. In some engineering projects, the problem analysis needs around half the report since, as illustrated in the example above about the bridge, it is necessary to consider a number of factors in order to establish what the problem is exactly. In mathematics (and physics), usually this part would be a minor part of the report. The exact length of a problem analysis would vary from problem to problem, but in most cases, it would have to consist of a motivation explaining the purpose of working within the given problem area, it should establish the overall theme or problem field as well as how the group gets from the overall theme to the specific problem, it should discuss possible ways to solve the problem and describe, discuss, and justify the specific methods the group chooses. This includes describing the context and society of the problem, and for whom the problem is relevant.

Case A – applied mathematics: route optimisation of egg collection

Six students (Mørk etal. 2014) worked with a problem of route optimisation of a transportation company which collects eggs from farmers in Denmark. The eggs are afterwards distributed to various supermarkets. In this case, the society is the second type mentioned above since it would be useful information for the company's transportation planners. The report was 94 pages, including references[AQ7] etc. (Table 1).

Table 1. Overview of a student report – applied mathematics and the second society.

Introduction 4 pages	The group starts by discussing the importance of transportation for the general society particularly for companies and in this context they introduce the Travelling Salesman's Problem (TSP). They then describe a specific company which collects eggs from farmers across the country and show a map of all egg producers in Denmark. They then introduce graph theory as a method to optimise routes, and they discuss the concept of an algorithm as well as mention three specific ones they intend to use. The problem statement was: 'How can graph theory, including the algorithms Christofides, Nearest Neighbour, Nearest Insertion, and TSP be used to optimize the route of picking to egg produces in Mid Jutland [Danish region]?' (Author translation) The group then discusses various demarcations in their project, including choice of region, that they are not considering rules of when drivers should have a break, rush hours, etc.

Graph theory 22 pages	Then follows a chapter about graph theory included oriented and unoriented graphs, weights, neighbours, isomorph graphs, pairing, roads, trees, and Euler. In total, the chapter contains 18 formal definitions and 6 theorems with proofs as well as a number of examples.
Hamilton and TSP 6 pages	Then follows a chapter with the theorems of Ores and Dirac as well as Optimal Hamilton cycle. In total the chapter contains 1 formal definition and 3 theorems with proofs as well as some examples.
Algorithms 8 pages	The next chapter is about algorithms and the complexity of algorithms, including the Big O, Big Ω, and Big θ notations. In total the chapter contains 4 formal definitions and 1 theorem with proof as well as some examples.
Types of algorithms 14 pages	The next chapter proves the algorithms of Christofides, Dijkstra, Kruskal, Minimum perfect pairing, Euler path, and Hamilton cycle. It also includes the Nearest Neighbour and Nearest Insertion algorithms. In total the chapter contains 2 algorithms whose pseudocodes are shown and then proved, 2 theorems with proofs and 2 theorems that are not proved as well as a number of examples.
The case 14 pages	The group then applies the three above-mentioned algorithms on their case. In total they show 25 figures as well as a number of tables and matrices.
Discussion 2 pages	The group discusses its results as the three algorithms give different results. Christofides finds both the shortest and the fastest route; however, this algorithm is of greater complexity than the two other. They discuss the usefulness of their results for the company as well as the effect of the demarcations made in the introduction. Here they further discuss that they have not considered the capacity of the trucks and how different sizes of trucks might affect the results.
Conclusion 2 pages	Here the group summarises their main conclusions and relates it to the problem statement.
Perspectives 4 pages	The group has a longer discussion about how to make mathematical models and how accurate they are as it is always needed to make simplifications.

The problem analysis above is not very long since it is obvious that a transportation company would benefit from using more optimal routes and this might also reduce pollution. Here the problem analysis focuses on what type of mathematical tools already exist in order to do optimisation. This means a discussion of which algorithms are potentially useful and which type of mathematical scholarly knowledge would be needed in order to give an answer to the problem. Such discussion aids to the formulation of a more specific problem that a group of students can manage to do within a given timeframe.

Case B – pure mathematics: flow in a network

If the problem does not deal with a situation where one needs to solve an unfortunate situation, the problem could be in unknown territory where the consequences and applications are either unknown or where one is within the body of scholarly knowledge and further develop a piece of this knowledge in areas that so far are un-discovered. The latter is to some extent not possible for students since developing new mathematical knowledge is hard, but they can do something close where they mirror a development previously done by research mathematicians. The society would here be (other) researchers in mathematics and the problem analysis would contain a description of why researchers find this problem mathematically interesting and relevant as well as an overview of what has been done so far and what mathematical knowledge would be useful in order to answer this question as well as discussing potential uses in the first and second society. This would again aid the group in formulating a more specific problem statement that is possible for the group to answer within the timeframe. Below is an example of a group of seven students (Støttrup etal. 2014). The report was 76 pages, including[AQ8] references etc. (Table 2).

Table 2. Overview of a student report – pure mathematics and the third society.

Introduction 2 pages	The group introduces graph theory and describes which types of real problem can be solved using graph theory. Here they mention transportation problems. After discussing this, the group states that transportation problems need not always be as concrete as, for instance, transporting people from one place to another, it can also be information in a computer network being transported to multiple receivers. The group then describes that they wish to focus on multi-casting, which is how two pieces of information *a* and *b* is being sent from *s* to *t*l and *t*l. They then list and describe the type of theorems and algorithms needed to answer this problem. The problem statement was: 'In a given network, how is the maximum flow determined? How can the max-flow min-cut theorem be used to solve the network coding problem with multi casting?' (Author translation).
Graph theory 12 pages	Then follows a chapter providing definitions, theorems, and proofs of central concepts in graph theory relevant for the problem (graphs, degree, oriented graphs, weighted graphs, matrix representation, trees). In total, the chapter contains 13 formal definitions, 6 theorems and 2 corollaries with proofs as well as some examples.
Algorithms 8 pages	The next chapter is about algorithms and consists of the notions of Big O, Big Ω, and Big θ and different algorithms such as *Depth-first* search and *Breadth-first search*. In total, the chapter contains 3 formal definitions, 3 theorems, 1 corollary, and 2 lemmas with proofs as well as pseudocodes and some examples.
Networks 30 pages	Then follows a chapter that defines concepts such as network and flow as well as the max-flow min-cut theorem and max-flow min-cut algorithms such as Ford–Fulkerson and Edmonds–Karp. Pseudocodes are shown. Then Matlab codes are applied to the algorithms. In total, the chapter contains 9 formal definitions, 6 theorems, and 1 corollary with proofs, a proof of an algorithm from pseudocode and some examples.
Fields 4 pages	Then follows a chapter introducing fields as the group argues that they need to be able to calculate in fields when later working with network coding and multi-casting. In total the chapter contains 3 formal definitions and 2 theorems with proofs as well as some examples.

Network coding and multi-casting 12 pages	This chapter is about network coding and multi-casting as well as the Jaggi–Sanders algorithm. This algorithm is applied. In total the chapter contains 5 formal definitions, 2 theorems, and 1 lemma with proofs as well as some examples.
Conclusion 2 pages	The students conclude that they are able to determine the max-flow in a given network and that the Edmonds-Karp algorithm finds the maximum flow in polynomial time. They also describe that the Jaggi–Sanders algorithm finds a solution even though it does not use max-flow min-cut theorem directly.
Perspectives 2 pages	They have a general discussion about the process of making mathematical models and how it fits the work they have done. They also discuss that their project mainly works with vertical mathematisation (Van den Heuvel-Panhuizen 2003) as they solve a theoretical problem.

It is evident that the students are able to find a relevant theoretical problem that is solvable within graph theory. They are also very able to describe how their problem is different from more concrete problems. The problem analysis is shorter than seen in Case A which dealt with a problem in the second society, hence a more practical and real problem. However, this might not always be the case since a theoretical problem might be very complex.

Results: types of knowledge in the case reports

The introduction chapter (Case A) shows multi-disciplinary knowledge as the students were able to draw on information from outside of mathematics to, for instance, describe the routines of truck drivers and that this also involves legal aspects such as how long time a driver can drive before he by law needs a rest. The knowledge is also what one might term meta-mathematical as the students in, for instance, the chapter *Perspectives* are able to look at their own process from above and discuss what a mathematical modelling process is and that such a process would always involve, for instance, simplifications. They are also able to reflect on (Case B) that their work is vertical mathematisation (and therefore in the third society) and not horizontal mathematisation (Van den Heuvel-Panhuizen 2003) where mathematics is applied to the outside world.

The choice of the specific content to be included in the project report was guided by the problem statement. It is evident that the majority of the student reports consisted of mainly formal-axiomatic mathematics belonging to the third world of Tall (2013). This was also the case with the project in applied mathematics (Case A). The proofs were not derived from set-theoretical definitions using epsilon–delta notation but was nevertheless formal proofs using (or attempting to use) correct mathematical syntax. Epsilon–delta notation is not introduced until the third semester. Some of the knowledge presented was also in the embodied symbolic world seen in the many examples and figures. Considering the seven mathematical capabilities (OECD 2013), one also sees that to some extent they are all present in the report. In terms of communication, the groups use the report as a means of communicating in first instance to the examiners, but also to each other, thus preparing themselves for mathematical communication in their later professional life. They also do mathematising as they present a problem in a mathematical way and here create a model. Both reports use mathematical representation, reasoning and argument, and symbolic language during, for instance, proofs and other justification. They both devise strategies based on their problem statement and also use mathematical tools such as Matlab. Case A from applied mathematics can furthermore be argued to show that applied and pure mathematics does not need to be seen as opposites as the students were obviously able to both produce Tall's (2013) third world of mathematics reasoning as well as apply mathematics. The part with pure mathematics took up the majority of the report as the students were in a mathematics programme. One might argue that engineering students might also embark on solving similar optimisation problems using perhaps more convergent methods not learning or describing the proofs of the algorithm but still able to apply the algorithms.

Discussion and conclusion

Mathematics is a body of abstract and complex knowledge, which in its purest form is not relating to reality. Plato in *Republic VI* (1997, 1129–1132) even argued for mathematics solely belonging to the ideal realm. However, even though mathematics is a body of abstract knowledge, PBL is still a fitting curriculum for developing higher level mathematics as students would experience the processes of (re)inventing mathematical knowledge, which sometimes is applicable to the general society, sometimes to the society of professionals, or to the society of researchers. The didactic transposition of mathematics as a scholarly field of knowledge to a PBL curriculum therefore includes that students learn to work research-like in PBL. Examples of the two latter societies, hence pure and applied mathematics are given above, where both problems were from a second-semester PBL project in mathematics. Both are within the overall theme of combinatorics, graph theory, and optimisation within the field of discrete mathematics. Case B from pure mathematics points to the fact that a relevant problem in PBL does not need to be a problem relevant to all of society. This is rarely the case in fundamental research within the natural sciences and mathematics. This means that the problem in PBL in fact only really makes sense and appears relevant within the context of a specific scientific society. To others, it is not considered relevant.

In general, when one considers the dichotomy of mathematics as a tool versus mathematics as an abstract discipline, history (Lützen 2013~~2004~~) shows that it is not that simple. Applied mathematics in physics has to some extend steered the development of the abstract discipline of mathematics just as much as the abstract discipline in some cases has remained abstract for millennia until it found a concrete application as, for instance, seen in computer science applying number theory. But one also needs to discuss the concept of application. Mathematics is applied not only in what above is termed the first and the second society, but also in the third society where mathematicians apply theorems and methods developed by other mathematicians in the development of more abstract mathematics as well as in the development of tools for the first and second society. So the picture is more complex than just abstract real mathematics versus applied mathematics in the real world. In PBL, students of mathematics have the opportunity to apply mathematics in all three societies.

The paper began by stating that working in projects might not suit every profession, but that it fits engineers quite well. However, this might not be sufficient for engineers in the twenty-first century. Acosta etal. (2010, 9) argue that the twenty-first century means a new world for engineers:

> As always, technical expertise, operational skills, and creative problem solving will be absolutely necessary for a successful career. But, increasingly those traditional abilities will be necessary but not sufficient. Understanding the situations that surround the development and implementation of new technologies, from organizational processes to cultural guidelines and constraints, also will be crucial.

If Acosta etal. (2010) are correct, new engineers also need to understand how the development of new technologies takes place. A part of such a development takes place within a more pure science field of mathematics, and engineers of the future therefore need to understand this world of mathematics as well as where they fit into the process of fundamental-research-turned-into-applied-science.

Acknowledgements

The author would very much like to thank Olav Geil, Esben Skovsen, Erik de Graaff, Ljerka Jukić Matić, and my students for providing valuable feedback on the ideas and various versions of the paper.

Disclosure statement

No potential conflict of interest was reported by the author[AQ9].

Notes on contributor

Bettina Dahl Søndergaard is Associate Professor in mathematics and engineering education at Aalborg University (Denmark) and a professor of mathematics education at the University of Bergen (Norway). She has a M.Sc. (cand.scient.) in Mathematics from Aalborg University, a M.Sc. in Educational Research Methodology from University of Oxford (U.K.) and her Ph.D. is in Mathematics Education from Roskilde University (DK). Previously, she was Associate Professor of Aarhus University (DK) working with educational development at higher education in science and mathematics. Before that she was Assistant Professor at Virginia Tech (U.S.A.) working with mathematics teacher training for secondary level and before this, she was Advisor at the Norwegian Centre for Mathematics Education at the Norwegian University of Science and Technology.

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