Distance domination in partitioned graphs

by

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Abstract
For a graph $G$ with its vertex set partitioned into, say two sets $V(G) = V_1 \cup V_2$, bounds for $\gamma(G) + \gamma_G(V_1) + \gamma_G(V_2)$ have earlier been considered. This is generalized. We define a vertex set to distance $d$ dominate all vertices at distance at most $d$ from it. For partitioned graphs and any $d \geq 2$ we generalize theorems about ordinary distance one domination to distance $d$ domination. Further, we give bounds for distance 2 domination of a graph partitioned into three sets and state a conjecture.

Definitions
For $d \geq 1$ the vertex $x$ in a graph is said to distance $d$ dominate itself and all vertices at distance at most $d$ away from $x$. A set $D$ of vertices distance $d$ dominate $D$ and all vertices having distance at most $d$ to $D$. The distance $d$ domination number $\gamma_{\leq d}(G)$ of the graph $G$ is the cardinality of a smallest set $D$ which distance $d$ dominates all vertices in $G$. For $d = 1$ we get the usual domination number $\gamma_{\leq 1}(G) = \gamma(G) = |D|$.

Let $k \geq 2$ be an integer and $V_1, V_2, \ldots, V_k$ a partition of $V(G)$. For $i, 1 \leq i \leq k$, we shall by $\gamma_{\leq d}(G, V_i)$ denote the order of a smallest set of vertices in $G$ which distance $d$ dominates $V_i$. I.e. there exists $D_i \subseteq V(G)$ such that every vertex of $V_i$ either belongs to $D_i$ or in $G$ has distance at most $d$ to a vertex in $D_i$, and $\gamma_{\leq d}(G, V_i) = |D_i|$ for a smallest such $D_i$. Let $f_{\leq d}(k, G)$ denote the maximum taken over all partitions $V_1, \ldots, V_k$ of $V(G)$ of the sum $\gamma_{\leq d}(G) + \sum_{i=1}^{k} \gamma_{\leq d}(G, V_i)$. 

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For \( d = 1 \) we write \( \gamma(G, V_i) \) and \( f(k, G) \). When no misunderstanding is possible we may write \( \gamma \leq d(V_i) \), \( f \leq d(G) \) for short. Hartnell and Vestergaard gave upper bounds for \( f \leq d(k, G) = \gamma \leq d(G) + \sum_{i=1}^{k} \gamma \leq d(G, V_i) \), when \( d = 1 \). We shall generalize to \( d \geq 1 \).

For \( d = 1 \) and \( k = 2 \) we can slightly reformulate their result:

**Theorem 1.** \([2]\) Let \( G \) be a graph with at least 3 vertices in each component and let \( V_1, V_2 \) be any partition of \( V(G) \). Then

\[
\gamma(G) + \gamma(G, V_1) + \gamma(G, V_2) \leq \frac{5}{4}|V(G)|, \text{ i.e. } f(2, G) \leq \frac{5}{4}|V(G)|.
\]

Equality occurs if and only if each component of \( G \) satisfies

(i) the number of vertices is a multiple of four.

(ii) Every vertex has degree one or is adjacent to exactly one degree one vertex.

(iii) Every vertex of degree three or more is adjacent to exactly one degree two vertex having a degree one neighbour.

(iv) All degree one vertices are in one class \( V_1 \), all degree two vertices in the other class \( V_2 \) and vertices of degree \( \geq 3 \) can be in either class.

For \( d \geq 2 \) we have Theorem 2 below.

**Theorem 2.** Let \( d \geq 2 \) and let \( G \) be a graph with at least \( d + 2 \) vertices in each component. For any partition \( V_1, V_2 \) of \( V(G) \) we have

\[
\gamma \leq d(G) + \gamma \leq d(G, V_1) + \gamma \leq d(G, V_2) \leq \frac{6}{2d+3}|V(G)|
\]

and equality holds if and only if

(i) the order of each component if \( G \) is a multiple of \( 2d + 3 \) and

(ii) \( G \) can be constructed from a set of disjoint paths of lengths \( 2d + 2 \) by arbitrarily adding edges between their central vertices.

**Proof of inequality.**

It suffices to prove the inequality of Theorem 2 for trees. We shall use induction on \( n = |V(G)| \).

The inequality is true for \( n = d + 2 \), for consider, in fact, any tree \( T \) on \( n \geq d + 2 \) vertices and with diameter at most \( 2d \); then \( f \leq d(2, T) \leq 3 \), as we can place 3 dominators in the central vertex, when the diameter is an even number, and in an end vertex of the central edge when the diameter of \( T \) is an odd number. Obviously \( 3 \leq \frac{6}{2d+3}(d+2) \leq \frac{6}{2d+3}n \), so the inequality holds for small values of \( n \).

Assume the inequality to be true for trees with fewer than \( n \) vertices. If \( T \) has diameter \( \geq 2d + 3 \) there is an edge \( e \) in \( T \) such that \( T - e \) consists of two trees each having at least \( d + 2 \) vertices and the inequality holds. So we may assume \( T \) has diameter \( 2d + 1 \) or \( 2d + 2 \).
Case 1. Diam \((T) = 2d + 1\).

Let \(P = v_1v_2 \ldots v_{2d+2}\) be a diametrical path in \(T\). If \(T = P\), let \(D = \{v_{d+1}, v_{2d+2}\}\), let \(D_1, D_2\) both contain \(v_{d+1}\) and place \(v_{d+2}\) in \(D_i\) if \(v_{d+2} \in V_i, i = 1, 2\).

Then \(D\) dominates \(V(T)\), \(D_i\) dominates \(V_i\) for \(i = 1, 2\), and \(f_{\leq d}(T) \leq 5\). That satisfies the inequality as \(d \geq 2\) implies \(5 \leq \frac{6}{2d+3}(2d + 2)\).

Otherwise, \(n \geq 2d + 3\) and with \(D = D_1 = D_2 = \{v_{d+1}, v_{d+2}\}\) we obtain

\[
f_{\leq d}(2, T) \leq 6 \leq \frac{6}{2d+3}n.
\]

Case 2. Diam \((T) = 2d + 2\).

Let \(P = v_1v_2 \ldots v_{2d+3}\) be a diametrical path of \(T\). If \(\deg(v_i) \geq 3\) for any \(i \neq d + 2\) there is in \(T\) an edge \(e\) such that the two trees of \(T - e\) both have \(\geq d + 2\) vertices and the inequality holds.

So we may assume that on \(P\) no other vertex than \(v_{d+2}\) has degree more than two. Assume \(T - E(P)\) contains a path \(v_{d+2}x_1x_2 \ldots x_{d+1}\). If \(\deg(x_j) \geq 3\) for any \(j, 1 \leq j \leq d\), the two trees in \(T - v_{d+2}x_1\) both have \(\geq d + 2\) vertices and the inequality holds. So we may assume that \(\deg(x_j) = 2\) for \(1 \leq i \leq d\).

Thus \(T\) contains \(\alpha\) paths, \(\alpha \geq 2\), each of length \(d + 1\) and pendent from the central vertex \(v_{d+2}\) and possibly \(T\) also has other vertices, they all are within distance \(d\) from \(v_{d+2}\).

Case 2A. Assume \(T\) consists of \(\alpha\) paths of length \(d + 1\) pendent from \(v_{d+2}\). Then \(n = |V(T)| = 1 + \alpha(d + 1)\) and we see that \(f_{\leq d}(2, T) \leq 2\alpha + 2\) by placing \(\alpha\) vertices adjacent to \(v_{d+2}\) in \(D\), placing \(v_{d+2}\) in both \(D_1\) and \(D_2\) and placing the \(\alpha\) vertices at distance \(d + 1\) from \(v_{d+2}\) in \(D_i\) when they belong to \(V_i, i = 1, 2\). We certainly have \(2\alpha + 2 \leq \frac{6}{2d+3}(\alpha d + \alpha + 1)\) as \(\alpha \geq 2\).

Case 2B. Assume \(T\) consists of \(\alpha\) paths of length \(d + 1\) pendent from \(v_{d+2}\) and also of vertices \(y_1, y_2, \ldots, y_t, 1 \leq t\), such that for \(1 \leq i \leq t\), \(y_i\) has distance \(\leq d\) from \(v_{d+2}\).

Note that those of \(y_1, y_2, \ldots, y_t, 1 \leq t\) which are within distance \(d-1\) from \(v_{d+2}\) are dominated by the D-dominators already chosen in Case 2A. For the remaining vertices \(y_i\) at distance \(d\) from \(v_{d+2}\) there exists in \(T\) a path \(v_{d+2}y_1y_2 \ldots y_d\) and we have \(n \geq 1 + \alpha(d + 1) + d\). Taking the dominators from case 2A together with \(v_{d+2}\) added to \(D\) we obtain

\[
f_{\leq d}(2, T) \leq 2\alpha + 3 \leq \frac{6}{2d+3}(1 + \alpha(d + 1) + d) \leq \frac{6}{2d+3}n.
\]

This proves the inequality of Theorem 2. Finally, let \(f_{\leq d}(2, G) = \frac{6}{2d+3}|V(G)|\). Then deletion of edges from \(G\) to obtain a tree and smaller trees in the process of proving the inequality of Theorem 2 must at every stage preserve equality,
therefore the final components are paths $P_{2d+3}$ and if additional edges have ends at other vertices than centers of these paths, we get inequality. This proves Theorem 2. 

Comment. The bound of Theorem 2 is best possible, but only slightly better than the crude evaluation $f_{\leq d}(2, G) \leq 3 \cdot \gamma_{\leq d}(G) \leq 3 \frac{1}{d+1}|V(G)|$. (cf. [4])

For partition into 3 classes, a best possible inequality is given by Hartnell and Vestergaard [2].

Theorem 3. [Hartnell, Vestergaard 2003] Let $n \geq 3$ be an integer. Let $T$ be a tree on $n$ vertices such that $T \notin \{P_4, P_7\}$ and let $\{V_1, V_2, V_3\}$ be a partition of $V(T)$. Then

$$\gamma(T) + \gamma_T(V_1) + \gamma_T(V_2) + \gamma_T(V_3) \leq \frac{7n}{5}.$$ 

For distance 2 domination of a tree $T$ with its vertex set partitioned into 3 sets we shall prove.

Theorem 4. Let $n \geq 4$ be an integer. Let $T$ be a tree on $n$ vertices and let $\{V_1, V_2, V_3\}$ be a partition of $V(T)$. Then

$$\gamma_{\leq 2}(T) + \gamma_{\leq 2}(V_1) + \gamma_{\leq 2}(V_2) + \gamma_{\leq 2}(V_3) \leq n.$$ 

Proof. It is enough to prove the theorem for trees. By induction on $n$ it is enough to prove the theorem for trees $T$ with diameter $\leq 6$, since otherwise, $T$ has an edge $e$ such that both trees in $T - e$ have $\geq 4$ vertices. If $T$ has diameter 2 or 4 it suffices to place its central vertex in each of $D_1, D_2, D_3$. Similarly, if $T$ has diameter 3 we can place an end vertex of the central edge in each of the four dominating sets. In these cases we have $f_{\leq 2}(3, T) \leq 4 \leq n$.

If $T$ has diameter 5, let $v_1 \ldots v_6$ be a diametrical path. Place 4 dominators in $v_4$ and for each vertex $x$ at distance 3 from $v_4$, $x \in V_i$, place a $D_i$-dominator in $x$ and a $D$-dominator in $b$, the second last vertex on the unique path $xabv_4$ from $x$ to $v_4$. In all cases we obtain $f_{\leq 2}(3, T) \leq n$.

Assume $T$ has diameter 6. Let $P = v_1v_2\ldots v_7$ be a diametrical path in $T$. If $\text{deg}(v_i) \geq 3$ for $i \neq 4$ there is an edge $e$ in $T$ such that the two trees in $T - e$ have at least 4 vertices and by induction the result follows. So we may assume that $\text{deg}(v_2) = \text{deg}(v_3) = \text{deg}(v_5) = \text{deg}(v_6) = 2$. We easily see that a path $P_7$ on seven vertices has $f_{\leq 2}(3, P_7) = 7$, i.e. $P_7$ satisfies Theorem 4, so assume $\text{deg}(v_4) \geq 3$.

Let $l$ denote the length of a longest path emanating from $v_4$ in $T - E(P)$, $l \leq 3$. For $l = 1$ we place 4 dominators in each of $v_3, v_5$. For $l = 2, 3$ we place 4 dominators in $v_3$ and each vertex $x$ at distance 3 from $v_4$ is chosen to
class-dominate itself, while we on $xav_4$, the unique path from $x$ to $v_4$ choose $b$ for $D$-domination. That gives $f_{\leq 2}(3, T) \leq n$. This proves Theorem 4. ■

The inequality of Theorem 4 is best possible as shown by the following examples.

$$f_{\leq 2}(3, P_7) = 7, \ f_{\leq 2}(3, P_8) = 8.$$ 

A path on 9 vertices with a pendent edge from its central vertex has $f_{\leq 2}(3, T) = 10 = n$.

However, it can be proven that $f_{\leq 2}(3, T_{11}) \leq 10$ for any tree on $n$ vertices and $f_{\leq 2}(3, T_{12}) \leq 11$ for any tree on 12 vertices. For any tree $T_{13}$ on 13 vertices we have $f_{\leq 2}(3, T_{13}) \leq 12$. So possibly there is a stronger inequality for trees with sufficiently many vertices. Some references to domination in partitioned graphs are given below.

References


