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Vestergaard, Preben Dahl

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A short update on equipackable graphs

by

Preben Dahl Vestergaard

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A short update on equipackable graphs

P.D. Vestergaard
Department of Mathematical Sciences, Aalborg University
Fredrik Bajers Vej 7G, DK-9220 Aalborg Ø, Denmark
e-mail: pdv@math.aau.dk

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Abstract

A graph is called equipackable if every maximal packing in it is also maximum. This generalizes randomly packable graphs. We survey known results both on randomly packable graphs and on equipackable graphs. As a new result is given a characterization of $P_3$-equipackable graphs with all valencies at least two.

1 Definitions

Let $H$ be a subgraph of $G$. A collection $H_1, H_2, \ldots, H_k$ of edge disjoint subgraphs of $G$, each isomorphic to $H$, is called an $H$-packing of $G$, and it is maximal if $G - \bigcup_{i=1}^{k} E(H_i)$ contains no subgraph isomorphic to $H$. It is maximum if no $H$-packing in $G$ with more than $k$ copies of $H$ exists. Bosák
[3] wrote a book on packing, he used the equivalent term *decomposition*, and there are many papers on the topic. $G$ is called *$H$-packable* if there exists an $H$-packing of $G$ which uses all edges in $G$ and $G$ is called *randomly $H$-packable* if every maximal $H$-packing in $G$ uses all edges in $G$, i.e., if every $H$-packing can be extended to a decomposition of the edges of $G$ into copies of $H$. Note that not every graph $H$ produces a family of randomly $H$-packable graphs, if e.g. $H$ is the disjoint union of $K_2$ and $K_3$, no other graph than $H$ itself is randomly $H$-packable.

As a relaxation of random $H$-packability we define $G$ to be *$H$-equipackable* if every maximal $H$-packing is also a maximum $H$-packing. So the randomly $H$-packable graphs is contained as a subclass in the class of $H$-equipackable graphs. This paper focuses on $H = P_3$.

### 2 Notation

A graph $G$ has *order* $|V(G)|$ and *size* $|E(G)|$. The *path* and *circuit* on $k$ vertices is denoted by $P_k$ and $C_k$, respectively. By $C_m \bullet C_n$ we denote the graph of order $n + m - 1$ obtained from two circuits $C_m$ and $C_n$ by identifying one vertex from each. $S^{(r)}_{2k+1}$ denotes the graph obtained from $r$ paths $P_{2k+1}$ by identifying their center vertices. The *corona* $H \circ K_1$ on $H$ is is the graph of order $2|H|$ obtained by adding for each vertex $x$ of $H$ one new vertex $x'$ and a new edge $xx'$. By $H + G$ we denote the graph obtained from two disjoint graphs $H$ and $G$ by adding edges joining each vertex of $H$ to each vertex of $G$. A *matching* in the graph $G$ is a set of independent edges in $G$, it is *perfect* if it covers all vertices of $G$. By $M_t$, $t \geq 1$, we denote a matching having $t$ edges. The union of $k$ disjoint copies of a graph $G$ is denoted by $nG$, e.g. $M_t = tK_2$.

### 3 Results

An early result by Caro and Schönheim is
Lemma 1 ([4]) A connected graph $G$ is $P_3$-packable if and only if $G$ has even size.

Observation A connected graph $G$ of odd size contains an edge whose deletion leaves a connected graph, which necessarily is of even size and therefore by Lemma 1 is $P_3$-packable.

It is clear that a maximum $P_3$-packing in a connected $P_3$-equipackable graph contains either all or all but one edge of the graph.

It follows that a $P_3$-equipackable connected graph of even size is also randomly $P_3$-packable. Another useful observation is that if there is a maximal $P_3$-packing of a connected graph $G$ which omits at least two edges, then $G$ is not $P_3$-equipackable.

Ruiz characterized randomly $P_3$-packable graphs.

Theorem 1 ([11]) A connected graph $G$ is randomly $P_3$-packable if and only if $G \cong C_4$ or $G \cong K_{1,2k}, k \geq 1$.

Thus $P_3$-equipackable graphs of even size are quadrilaterals or stars. It remains to characterize $P_3$-equipackable graphs of odd size. Hartnell and Vestroagard did that for graphs of girth at least five.

Theorem 2 ([5]) A connected graph $G$ of girth 5 or more is $P_3$-equipackable if and only if $G$ satisfies one of the following:

(i) $G$ is a tree consisting of a single star (i.e., $K_{1,n}$)

(ii) $G$ is a tree which has two stems that are at distance 3, where the vertices on this shortest path are $w_1$ and $w_2$. Furthermore the stems are of odd parity and have no neighbours other than leaves and $w_1$ or $w_2$. In addition $w_1$ and $w_2$ are of degree two.

(iii) $G$ is a tree which has two stems that are at distance two where $w$ is the common neighbour of the stems. The two stems must be of different parity and neither stem has other neighbours than its leaves and $w$. Furthermore, the vertex $w$ must be of degree two.
(iv) \( G \) is a tree which has two stems that are adjacent where these stems are of the same parity and these stems have only each other and their leaves as neighbours.

(v) \( G \) is either \( C_7, C_5 \) or has \( 5+2m \) vertices where \( G \) consists of a circuit of length 5 along with \( 2m \) leaves attached to exactly one node on the 5-cycle.

The remaining problem is to characterize \( P_3 \)-equipackable graphs of girth 3 and 4. We shall consider graphs with \( \delta(G) \geq 2 \).

Theorem 3  Let \( G \) be a graph with \( \delta(G) \geq 2 \), which is connected and has a cutvertex. Then \( G \) is \( P_3 \)-equipackable if and only if either (1) \( G \) is of order eight and is obtained from two vertex disjoint quadrilaterals together with an edge joining a vertex in one quadrilateral to a vertex in the other or (2) \( G \) is a \( C_3 \bullet C_4 \).

Proof. The two graphs described can be checked to be \( P_3 \)-equipackable. Conversely, let \( x \) be a cutvertex in the \( P_3 \)-equipackable graph \( G \) and let \( A'_1, \ldots, A'_m \) be the components of \( G - x \), while \( A_1, \ldots, A_m \) are the graphs spanned in \( G \) by \( A'_i \cup \{x\} \). We first observe that \( |E(A_i)| \) can be odd for at most one \( i \). Assume namely \( A'_i \) is a component with \( |E(A_i)| \) odd. We shall by \( P_3 \)-removals isolate a subgraph of \( A'_i \) having odd size. If there is an even number of edges from \( x \) to \( A'_i \) we \( P_3 \)-remove them in pairs. This isolates \( A'_i \) with an odd number of edges. If there is an odd number of edges from \( x \) to \( A'_i \) we can \( P_3 \)-remove all but one of them, say \( xy \), and as \( \delta(G) \geq 2 \) we have that \( y \in V(A'_i) \) is incident with an edge \( yz, z \in A'_i \setminus \{y\} \). By removal of \( \{xy, yz\} \) we also in this case have isolated a subgraph of \( A'_i \) with an odd number of edges and \( |E(A_i)| \) can by the observation after Lemma 1 be odd for at most one index \( i \).

If all \( |E(A_i)| \) are even, then \( G \) has even size and, by the remark after Lemma 1, \( G \) is randomly \( P_3 \)-packable and hence by Theorem 1 is \( C_4 \) or a star, but \( C_4 \) has no cutvertex and the star violates \( \delta(G) \geq 2 \), so this case cannot occur.

Thus \( |E(A_1)| \), say, is odd and all other \( |E(A_i)|, i \geq 2 \), are even.
Subcase 1. Assume a bridge $xy$ of $G$ joins $x$ to $A'_1$. $P_3$-remove $yx, xz, z \in N(x) \cap V(A'_2)$. Then $A'_1$ is isolated and as a connected graph of even size, it is therefore randomly $P_3$-packable and by Theorem 1 isomorphic to $C_4$ as $\delta(G) \geq 2$. The graph spanned in $G$ by the union of $A_2, \ldots, A_m$ is connected, of even size and it is also randomly $P_3$-packable. By Theorem 1 it is isomorphic to $C_4$ and $G$ consists of two disjoint $C_4$’s joined by an edge as claimed in (1).

Subcase 2. Assume $x$ is joined to $A'_1$ by more than one edge. Again the graph spanned by the union of $A_2, \ldots, A_m$ is connected and of even size. It is randomly $P_3$-packable because we can by $P_3$-removals inside $A_1$ make sure that the unique non-removed edge is isolated inside $A'_1$. Thus all of $A_2, \ldots, A_m$ spans a $C_4$. We can deduce that $A_1$ is isomorphic to $C_3$, because $A_1$ minus an edge must be randomly $P_3$-packable and $C_4$ cannot be fitted in to ensure $P_3$-equipackability of $G$, so $A_1$ minus an edge must be a star and only $P_3$ with its two ends joined to $x$ will do. In this case $G$ is the disjoint union of a quadrilateral and a triangle with one vertex from each identified as claimed in (2). This proves Theorem 3.

Theorem 4 A connected graph $G$ with $\delta(G) \geq 2$ is $P_3$-equipackable if and only if $G$ is one of the graphs listed in Figure 1.

Proof. We can by inspection verify that the graphs in Fig. 1 all are $P_3$-equipackable. If $G$ has a cutvertex we know by Theorem 3 that $G \cong C_3 \cdot C_4$ or $G$ can be obtained by joining two quadrilaterals by an edge. So assume that $G$ is 2-connected and $P_3$-equipackable, we must prove that $G$ is one of the remaining graphs listed on Fig. 1. Let $C$ with length $\ell$ be a longest circuit in $G$. If $\ell = 3$ necessarily $G \cong C_3$, a graph in the family on top of Fig. 1. If $\ell = 4$ and $C$ has no diagonal the only possibility is $G \cong C_4$ since by 2-connectivity any $x$ in $V(G) \setminus V(C)$ must be joined to $C$ by two independent paths. Each path must be an edge, otherwise $G$ would contain a circuit longer than four. Also $x$ must have valency 2. Thus $G$ is of even size, that implies that $G$ is randomly $P_3$-packable and hence by Theorem 1 must be a $C_4$. If $\ell = 4$ and $C$ has a diagonal we can by a similar argument obtain that $G$ is a graph in the family on top of Fig. 1. If $\ell = 5$ and $G$ has order 5 we obtain the four graphs of Fig. 1. If $\ell = 5$ and $G$ has order $> 5$ each $x$ in $V(G) \setminus V(C)$ is by 2-connectivity joined by two independent paths to $C$. 

\[5\]
Again, not to produce a longer circuit, $x$ is joined by edges to two vertices at distance two on $C$, and $x$ has valency two. But now it is easy to see that we by $P_3$-deletions can isolate two edges, so this case cannot occur. If $\ell = 6$ we can find that $C$ must contain a triangular diagonal. If $G$ has order 6, either $G$ is this graph, included in Fig. 1, or we can by $P_3$-removals isolate two edges, a contradiction. In the remaining cases either $G$ is $C_7$, included in Fig. 1, or we can for each vertex $x$ on $C$ pairwise $P_3$-remove its adjacent edges in $E(G) \setminus E(C)$, so at most one edge besides the two circuit edges remain at $x$. If the end vertex $y$ of such an edge has a neighbour $z$ such that $yz \notin E(C)$, we $P_3$-remove $xy, yz$. We have thus isolated a component consisting of $C$ and possibly at some $C$-vertices one other edge in $E(G) \setminus E(C)$, a pendent edge or a diagonal. At most one vertex on $C$ has a pendent edge, otherwise we could by $P_3$-removals on $C$ isolate the two pendent edges, a contradiction. We can now see that we by further $P_3$-removals can isolate two edges, a contradiction. This proves the theorem.

![Figure 1: All $P_3$-equipackable graphs which are connected and have all valencies at least two.](image)

4 Randomly packable graphs

Equipackability is a relaxation of random packability, so let us mention a few results from packings, all have potential for generalizations to equipackability.
Theorem 1 by Ruiz [11] was later generalized from $P_3$ to $K_{1,r}$ by Barrientos, Bernasconi, Jeltsch, Tronisco and Ruiz:

**Theorem 5 ([1])** For $r \geq 2$ a connected graph is randomly $K_{1,r}$-packable if and only if it is $K_{r,r}$ or it is bipartite with all valencies in one partite set being multiples of $r$ and all valencies in the other set being less than $r$.

### 4.1 Randomly path-packable graphs

Beineke, Goddard, Hamburger [2] generalized Theorem 1 from $P_3$ to $P_k$, $3 \leq k \leq 6$, and Molina with coauthors [9, 10] extended to $k \leq 10$:

| [11] | Randomly $P_2$-packable | Trivially every graph |
| [11] | Randomly $P_3$-packable | $C_4$ and stars of even size |
| [2]  | Randomly $P_4$-packable | $P_4, K_4, K_{2,3}, C_6, C_3 \bullet C_3$ |
| [9, 10] | Randomly $P_5$-packable | $P_5, K_{2,4}, C_4 \bullet C_4, C_8, S_5^{(k)}, k \geq 2$ |
| [2]  | Randomly $P_6$-packable | $P_6, C_{10}, C_5 \bullet C_5, K_4 \circ K_1$ and the graph obtained by joining two new vertices by an edge to the same valency 2 vertex of a $K_{2,4}$ |
| [9, 10] | Randomly $P_7, P_8, P_9,$ and $P_{10}$-packable | Families of graphs whose descriptions become increasingly complex with growing $k$ |

### 4.2 Randomly matching-packable graphs

Ruiz characterized randomly $M_2$-packable graphs:

**Theorem 6 ([11])** A graph is randomly $M_2$-packable if and only if it one of the following: $C_4, K_4, 2K_3, K_3 \cup K_{1,3}, 2K_{1,n}$ or $2nK_2, n \geq 1$. 7
This was generalized to randomly $M_t$-packable graphs by Beineke, Goddard, Hamburger, but only for graphs with sufficiently many edges:

**Theorem 7 ([2] Th. 2.5)** For a given integer $t \geq 2$, a graph with at least $2t^3 - t^2$ edges is randomly $M_t$-packable if and only if it is isomorphic to $tH$, where $H$ is either $nK_2$ or $K_{1,n}$ for some $n \geq 1$.

Those graphs in which each matching can be extended to a perfect matching are called *randomly matchable* and if each matching extends to a maximum matching, which is not necessarily perfect, they are called *equimatchable*. Sumner characterized randomly matchable graphs:

**Theorem 8 ([12])** A connected graph $G$ is randomly matchable if and only if $G \cong K_{2n}$ or $G \cong K_{n,n}$, where $n$ is a positive integer.

Lesk, Plummer and Pulleyblank [6] gave a characterization of equimatchable graphs in terms of the Gallai-Edmonds structure theorem (described in [7]). Define a *total matching* to be a subset $X$ of $E(G) \cup V(G)$ such that no two elements of $X$ are adjacent or incident. Topp and Vestergaard characterized totally equimatchable graphs:

**Theorem 9 ([13])** A connected graph $G$ is totally equimatchable if and only if $G$ is $K_n, K_{n,n}$ or $K_1 + \bigcup_{i=1}^n K_{2m_i}$, where $n$ and $m_1, m_2, \ldots, m_n$ are any positive integers.

### 4.3 Randomly $K_n$-packable graphs

Beineke, Hamburger and Goddard proved

**Theorem 10 ([2] Th. 3.1)** A graph $G$ is randomly $K_n$-packable if and only if every edge is in precisely one copy of $K_n$ in $G$.

McSorley and Porter ([8]) have considered a vertex variant. They let
(*) \( \{G_{\alpha n}\}_{\alpha=1}^{\infty} = G_n, G_{2n}, \ldots, G_{\alpha n}, \ldots \) be a sequence of graphs such that \( G_1 \sim K_n, G_{\alpha n} \) has order \( \alpha n \) and \( G_{\alpha n} - K_n \sim G_{(\alpha-1)n} \) for any complete subgraph \( K_n \) in \( G_{\alpha n} \), \( \alpha \geq 2 \).

For \( 0 \leq \lambda \leq n \) they call (*) a \( (K_n, \lambda) \)-removable sequence if \( G_{\alpha n} \) is \( (n-1) + (\alpha - 1)\lambda \)-regular, and they prove that for a fixed \( n \geq 2 \) there is a unique \( (K_n, \lambda) \)-removable sequence for \( \lambda = 0, n - 1 \) or \( n \).

**Theorem 11 ([8])** For a fixed \( n \geq 2 \),

\( \{\alpha K_n\}_{\alpha=1}^{\infty} \) is the unique \( (K_n, 0) \)-removable sequence.

\( \{K_{\alpha, \alpha, \alpha, \ldots, \alpha}\}_{\alpha=1}^{\infty} \) is the unique \( (K_n, n - 1) \)-removable sequence.

\( \{K_{\alpha n}\}_{\alpha=1}^{\infty} \) is the unique \( (K_n, n) \)-removable sequence.

### 5 Open problems

Molina and coauthors in [10] posed a still unsolved problem: Does the characterizations become easier if only 2-connected graphs are considered? Their examples of 2-connected randomly \( P_k \)-packable graphs contain only two copies of \( P_k \) and with three exceptions have a vertex of odd degree, and they asked whether that holds generally.

### References


