Correction to “Packetized Predictive Control of Stochastic Systems over Bit-Rate Limited Channels with Packet Loss”

Daniel E. Quevedo, Member, IEEE, Jan Østergaard, Senior Member, IEEE, Eduardo I. Silva, Member, IEEE, and Dragan Nešić, Fellow, IEEE

Abstract—We correct the results in Section V of the above mentioned manuscript.

In [1], we showed that a particular class of networked control system (NCS) with quantization, i.d.d. dropouts and disturbances can be described as a Markov jump linear system of the form

$$\theta_{k+1} = \tilde{A}(d_k)\theta_k + \tilde{B}(d_k)v_k,$$  \hspace{1cm} (1)

where

$$\theta_k \triangleq \left[ \begin{array}{c} \theta_{k-1} \\ d_k \end{array} \right] \in \mathbb{R}^{n+N}, \quad v_k \triangleq \left[ \begin{array}{c} w_k \\ n_k \end{array} \right] \in \mathbb{R}^{m+N}$$

and \( \{d_k\}_{k \in \mathbb{N}_0} \) is a Bernoulli dropout process, with

$$\text{Prob}(d_k = 1) = p \in (0, 1).$$

Throughout [1] we showed that properties of the NCS can be conveniently stated in terms of the expected system matrices

$$A(p) = \mathbb{E}\{\tilde{A}(d_k)\}, \quad B(p) = \mathbb{E}\{\tilde{B}(d_k)\} = [B_w \quad B_n(p)],$$

and the matrix \( \tilde{A} = \tilde{A}(1) - \tilde{A}(0) \). Unfortunately, Theorem 4 in Section V-A of [1] is incorrect. For white disturbances \( \{w_k\}_{k \in \mathbb{N}_0} \), the statement should be as given below. Non-white \( \{w_k\}_{k \in \mathbb{N}_0} \) can be accommodated by using standard state augmentation techniques; see, e.g., [2].

**Theorem 4:** Suppose that (1) is MSS and AWSS and that \( \{w_k\}_{k \in \mathbb{N}_0} \) is white with \( \sigma^2_w = \text{tr} R_w(0) \). Define

$$\mathcal{F}(z) \triangleq \left( zI - A(p) \right)^{-1},$$

$$\mathcal{C}(p) \triangleq \left( \sigma^2_w/m \right) B_w B_w^T + \left( \sigma^2_n/N \right) (1-p) \mathcal{E} \in \mathbb{R}^{(n+N) \times (n+N)},$$

where see [1, Sec.2] for definitions.

$$\mathcal{E} \triangleq \frac{B_w(p) B_w(p)^T}{(1-p)^2} = \left[ B_1 e_1^T (\Psi^T \Psi)^{-1} e_1 B_1^T \quad B_2 e_1^T (\Psi^T \Psi)^{-1} \right] - \left[ B_1 e_1^T (\Psi^T \Psi)^{-1} e_1 B_1^T \right] \cdot$$  \hspace{1cm} (2)

Then, the spectral density of \( \{\theta_k\}_{k \in \mathbb{N}_0} \) is given by

$$S_\theta(e^{j\omega}) = \mathcal{F}(e^{j\omega}) \left( p(1-p) \tilde{A} R_\theta(0) \tilde{A}^T + \mathcal{C}(p) \right) \mathcal{F}(e^{-j\omega}),$$  \hspace{1cm} (3)

where \( R_\theta(0) \) solves the following linear matrix equation:

$$R_\theta(0) = A(p)R_\theta(0)A(p)^T + p(1-p)\tilde{A} R_\theta(0) \tilde{A}^T + \mathcal{C}(p).$$  \hspace{1cm} (4)

Proof: See the appendix.

To further elucidate the situation, we note that (5) is linear and that its solution can be stated as the linear combination

$$R_\theta(0) = (\sigma^2_w/m)R_w^0(0) + (\sigma^2_n/N)R_n^0(0),$$  \hspace{1cm} (5)

where \( R_w^0(0) \) and \( R_n^0(0) \) satisfy

$$R_w^0(0) = A(p)R_w^0(0)A(p)^T + p(1-p)\tilde{A} R_w^0(0) \tilde{A}^T + B_w B_w^T,$n,$$

$$R_n^0(0) = A(p)R_n^0(0)A(p)^T + p(1-p)\tilde{A} R_n^0(0) \tilde{A}^T + (1-p)\mathcal{E}.$$

Therefore, the distortion \( D \) defined by (52) in [1] is given by

$$D \triangleq \text{tr}(\tilde{Q}(R_n^0(0))) + \lambda[0 \quad e_1^T] R_n(0) [0 \quad e_1^T]^T,$$

where \( \tilde{Q} \) is given in terms of the Kronecker product

$$\tilde{Q} \triangleq \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \mathcal{Q}.$$

Thus, \( D = \alpha \sigma^2_w + \beta \), with

$$\alpha = (1/N) \text{tr}(\tilde{Q}(R_w^0(0))) + (\lambda/N)[0 \quad e_1^T] R_w^0(0) [0 \quad e_1^T]^T,$$

$$\beta = (\sigma^2_n/m) \text{tr}(\tilde{Q}(R_n^0(0))) + (\lambda\sigma^2_n/m)[0 \quad e_1^T] R_n^0(0) [0 \quad e_1^T]^T.$$

The above expressions replace Lemma 11 of [1].

To derive a noise-shaping model, (6) can be substituted into (4) to provide

$$S_\theta(e^{j\omega}) = \mathcal{F}(e^{j\omega}) \left( \sigma^2_w/mK_wK_w^T + (\sigma^2_n/N)K_nK_n^T \right) \mathcal{F}(e^{-j\omega}),$$

where \( K_w \) and \( K_n \) are obtained from the factorizations

$$K_wK_w^T = B_w B_w^T + p(1-p)\tilde{A} R_w^0(0) \tilde{A}^T,$$\n
$$K_nK_n^T = (1-p)(\mathcal{E} + p\tilde{A} R_w^0(0) \tilde{A}^T).$$

If we define

$$\mathcal{H}(z) \triangleq \begin{bmatrix} I & 0 \end{bmatrix} \mathcal{F}(z),$$

then the above provides the noise-shaping model depicted in Fig. 2. The latter replaces Fig. 2 and Corollary 1 of [1].

**Remark 1:** We would like to emphasize that Theorem 4 can also be proven by adapting results in [3]–[5]. However, the noise shaping interpretation in Fig. 2 does not explicitly need an additional noise term to quantify second-order dropout effects, as opposed to what is done in [3]–[5].

The upper bound on the coding rate provided by Theorem 5 in [1] is also no longer correct, since it relied upon \( R_\theta(0) \). The new Theorem 5 is provided below:

**Theorem 5:** For any \( 1 \leq N \in \mathbb{N} \), the minimum bit-rate \( R \) of \( \tilde{u}_k \) satisfies

$$R(D) \leq \frac{1}{2} \log_2 \left( \det(I + (N/\sigma^2_n)R_\xi(0)) \right) + \frac{N}{2} \log_2 \left( \frac{\sigma^2}{6} \right) + 1,$$  \hspace{1cm} (7)

where

$$R_\xi(0) = \begin{bmatrix} \Gamma & 0 \end{bmatrix} R_\theta(0) [\Gamma \quad 0]^T.$$  \hspace{1cm} (8)

Proof: Follows immediately from (73) in [1] by omitting the last step where \( R_\xi(0) \) was written in terms of \( R_w(0) \) and (50) was used.
Fig. 2. Noise-Shaping Model of the NCS

Note that, in view of (6), the bound in (7) provides

\[
\lim_{\sigma_n^2 \to \infty} R(D) \leq \frac{1}{2} \log_2 \left( \det(I + [\Gamma \ 0] R_0(0) [\Gamma \ 0]^T) \right) + N \frac{\pi e}{6} + 1,
\]

expression, which is positively bounded away from zero and replaces (58) in [1].

Remark 2: By using results in [6, Sec.5], the covariance matrix \( R_d(0) \) can be expressed explicitly in terms of Kronecker products and matrix inversions. Specifically, let

\[
G \triangleq A(p) \otimes A(p)^T + p(1-p) \bar{A} \otimes \bar{A}^T
\]

and let \( c \in \mathbb{R}^{(n+N)^2} \) be the vectorized version of the matrix \( C(p) \) given in (2). Then, the vectorized version of \( R_d(0) \) is simply given by \( r = (I - G)^{-1} c \). Using this approach, it is straightforward to numerically evaluate the rate and distortion in (7).

Fig. 3 illustrates the rate and distortion trade-off for different horizon lengths and a fixed packet loss probability \( p = 0.0085 \). It may be noticed that the distortion can be reduced by using a longer horizon length in addition to increasing the bit-rate. Fig. 4 shows that when the packet-loss probability increases, it is necessary to use a larger horizon length to guarantee stability and thereby reduce the distortion.

REFERENCES


APPENDIX

Proof of Theorem 4

Since \( \{ \nu_k \}_{k \in \mathbb{N}_0} \) is white and thus \( \mathbb{E}\{\theta_k \nu_k^T\} = 0 \), the system recursion (1) provides

\[
\mathbb{E}\{\theta_{k+1} \theta_{k+1}^T\} = \mathbb{E}\{\bar{A}(d_k) \theta_k \theta_k^T \bar{A}(d_k)^T\} + \mathbb{E}\{\bar{B}(d_k) \nu_k \nu_k^T \bar{B}(d_k)^T\}.
\]

Therefore, by conditioning on \( d_k \) and using the law of total expectation, we obtain:

\[
\mathbb{E}\{\theta_{k+1} \theta_{k+1}^T\} = \mathbb{E}\{\bar{A}(d_k) \theta_k \theta_k^T \bar{A}(d_k)^T\} \big| d_k = 1\}
\]

\[
+ (1-p) \mathbb{E}\{\bar{A}(d_k) \theta_k \theta_k^T \bar{A}(d_k)^T\} \big| d_k = 0\}
\]

\[
+ p \mathbb{E}\{\bar{B}(d_k) \nu_k \nu_k^T \bar{B}(d_k)^T\} \big| d_k = 1\}
\]

\[
+ (1-p) \mathbb{E}\{\bar{B}(d_k) \nu_k \nu_k^T \bar{B}(d_k)^T\} \big| d_k = 0\}
\]

\[
= \bar{A}(1) \mathbb{E}\{\theta_k \theta_k^T\} \bar{A}(1)^T + (1-p) \mathbb{E}\{\theta_k \theta_k^T\} \bar{A}(0)^T
\]

\[
+ p \bar{B}(1) \mathbb{E}\{\nu_k \nu_k^T\} \bar{B}(1)^T + (1-p) \mathbb{E}\{\nu_k \nu_k^T\} \bar{B}(0)^T,
\]

where we have used the fact that \( \{d_k\}_{k \in \mathbb{N}_0} \) is Bernoulli and \( \nu_k \) and \( \theta_k \) are independent of \( d_k \). Direct algebraic manipulations allow us to
rewrite the above as
\[ E\{\theta_k \theta_0^T\} = A(p)E\{\theta_k \theta_0^T\} A(p)^T \]
\[ + p(1-p)\tilde A E\{\theta_0 \theta_0^T\} A^T + C(p). \]  
(8)

In a similar way, one can derive that
\[ E\{\theta_{k+\ell} \theta_k^T\} = E\{\tilde A(d_{k+\ell}) \theta_{k+\ell} + \tilde B(d_{k+\ell})\nu_{k+\ell}\theta_k^T\} \]
\[ = E\{\tilde A(d_{k+\ell}) \theta_k \theta_k^T\} + E\{\tilde B(d_{k+\ell})\nu_{k+\ell}\theta_k^T\} \]
\[ = A(p)E\{\theta_{k+\ell} \theta_k^T\} + B(p)E\{\nu_{k+\ell} \theta_k^T\} \]
\[ = A(p)E\{\theta_{k+\ell} \theta_k^T\}, \quad \forall \ell \in \mathbb{N}_0, \]  
(9)

since \( \{\nu_k\}_{k \in \mathbb{N}_0} \) is white and \( \theta_k \) and \( \theta_{k+\ell} \) are independent of \( d_{k+\ell} \) for non-negative values of \( \ell \). Equation (9) gives the explicit expression
\[ E\{\theta_{k+\ell} \theta_k^T\} = A(p)^\ell E\{\theta_0 \theta_0^T\}, \quad \forall \ell \in \mathbb{N}_0. \]  
(10)

Since the system is AWSS, we have \( \lim_{n \rightarrow -\infty} E\{\theta_{n+1} \theta_0^T\} = R_0(0) \), the stationary covariance matrix of \( \{\theta_k\}_{k \in \mathbb{N}_0} \). By (8) and results in [7], [8], the latter is given by the solution to (5).

On the other hand, in steady state, (10) gives that the covariance function
\[ R_0(\ell) = A(p)^\ell R_0(0), \quad \forall \ell \in \mathbb{N}_0. \]  
(11)

Consequently, the positive real part of the spectrum of \( \{\theta_k\}_{k \in \mathbb{N}_0} \) is given by
\[ S^+_\theta(z) = \frac{1}{2} R_0(0) + \sum_{\ell=1}^{\infty} R_0(\ell) z^{-\ell} \]
\[ = (1/2)I + A(p)(zI - A(p))^{-1} R_0(0), \]
where we have used the fact that, by assumption, (1) is MSS and AWSS, thus \( A(p) \) is Schur (see Lemma 4 in [1]) and the geometric series
\[ \sum_{n=0}^{\infty} (A(p)z^{-1})^n = (I - A(p)z^{-1})^{-1}. \]

Since \( \{\theta_k\}_{k \in \mathbb{N}_0} \) is AWSS, its spectrum satisfies [9]
\[ S_\theta(z) = S^+_\theta(z) + (S^+_\theta(z^{-1}))^T \]
\[ = R_0(0) + A(p)(zI - A(p))^{-1} R_0(0) \]
\[ + R_0(0)(z^{-1}I - A(p))^{-T} A(p)^T, \]

Therefore, we have
\[ (zI - A(p)) S_\theta(z) (z^{-1}I - A(p))^T \]
\[ = (zI - A(p)) R_0(0) (z^{-1}I - A(p))^T \]
\[ + (zI - A(p)) (zI - A(p))^{-1} R_0(0) (z^{-1}I - A(p))^T \]
\[ + (zI - A(p)) R_0(0) (z^{-1}I - A(p))^{-T} A(p)^T (z^{-1}I - A(p))^T \]
\[ = (zI - A(p)) R_0(0) (z^{-1}I - A(p))^T \]
\[ + A(p) R_0(0) (z^{-1}I - A(p))^T + (zI - A(p)) R_0(0) A(p)^T, \]

since \( (zI - A(p)) A(p)(zI - A(p))^{-1} = A(p) \). Thus,
\[ F^{-1}(z) S_\theta(z) F^{-T}(z^{-1}) \]
\[ = (zR_0(0) - A(p) R_0(0)) (z^{-1}I - A(p))^T + z^{-1} A(p) R_0(0) \]
\[ - A(p) R_0(0) A(p)^T + zR_0(0) A(p)^T - A(p) R_0(0) A(p)^T \]
\[ = R_0(0) - z^{-1} A(p) R_0(0) - z R_0(0) A(p)^T \]
\[ + A(p) R_0(0) A(p)^T + z^{-1} A(p) R_0(0) - A(p) R_0(0) A(p)^T \]
\[ + zR_0(0) A(p)^T - A(p) R_0(0) A(p)^T \]
\[ = R_0(0) - A(p) R_0(0) A(p)^T, \]
and (5) establishes (4). \( \square \)