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Robust Stability Analysis of Nonlinear Switched Systems with Filippov Solutions

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Abstract: This paper addresses the stability problem of a class of nonlinear switched systems with partitioned state-space and state-dependent switching. In lieu of the Carathéodory solutions, the general Filippov solutions are considered. This encapsulates solutions with infinite switching in finite time. Based on the theory of differential inclusions, a Lyapunov stability theorem is brought forward. These results are also extended to switched systems subject to polytopic uncertainty. Furthermore, the proposed stability theorems are reformulated using the sum of squares decomposition method which provides sufficient means to construct the corresponding Lyapunov functions via available semi-definite programming techniques.

1. INTRODUCTION

A large group of engineering applications give rise to systems which encompass both discrete and continuous dynamics. Mathematically, these systems are characterized by a collection of indexed differential or difference equations describing each subsystem and a switching rule between them. This rich family of systems is referred to switched or more generally hybrid systems. Examples of such systems in real world have been studied in open literature e.g. Wisniewski and Larsen [2008] and Larsen et al. [2007].

However despite numerous applications, their stability analysis has not been covered completely (Lin and Antsaklis [2009]). Several interesting phenomena arise when dealing with such systems; namely, even if all the subsystems are exponentially stable, one cannot guarantee the stability of the overall system (Branicky [1998]). Conversely, an appropriate switching law may contribute to stability even when all subsystems are unstable (Liberzon [2003]). Still, depending on the considered type of solutions (Carathéodory, Filippov, and etc.), discrepant stability phenomenon may follow. As an illustration, a switched system with stable Carathéodory solutions, may possess divergent Filippov solutions (see Leth and Wisniewski [2012]).

This research is mainly motivated by two contributions (Pranja and Papachristodoulou [2003]) and (Leth and Wisniewski [2012]). Leth and Wisniewski [2012] exploited the theory of differential inclusions (DI) and suggested Lyapunov-like stability theorem for piece-wise linear switched systems defined on polyhedral sets with Fillipov solutions. Pranja and Papachristodoulou [2003] proposed sum of squares based stability analysis tools for a class of hybrid systems; however, a unified stability theorem has not been suggested or fully established. Additionally, the solutions considered in the latter article are in the sense of Carathéodory which connotate the exclusion of solutions with infinite switching in finite time from the analysis. In the present paper, firstly, we extend the results reported in (Leth and Wisniewski [2012]) to the general nonlinear switched systems defined on regular sets by incorporating the theoretical notions of differential inclusions. Secondly, the robust stability problem of switched systems with polytopic uncertainty and Filippov solutions is addressed. Lastly, we propose sufficient conditions based on sum of squares (SOS) decomposition for the suggested stability theorems. This ensures computationally efficient means to investigate the stability of switched systems.

This paper is organized as follows. In the next section, the notations and some mathematical concepts adopted in this study are limned. The main results of this paper are brought forward in section 3. Section 4 demonstrates the accuracy of the proposed methodology via two examples. Finally, section 5 concludes the paper.

2. MATHEMATICAL PRELIMINARIES

The notations employed in this paper are relatively straightforward. $\mathbb{R}_{\geq 0}$ denotes the set $[0, \infty)$. $\| \cdot \|$ denotes the Euclidean vector norm on $\mathbb{R}^n$, $\langle \cdot \rangle$ the inner product, and $\mathbb{B}^n_\epsilon$ the closed ball of radius $\epsilon$ in $\mathbb{R}^n$ centered at origin. $\mathcal{P}$ accounts for the set of polynomial functions $p : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\mathcal{P}_{sos} \subset \mathcal{P}$ is the subset of polynomials with an SOS decomposition; i.e, $p \in \mathcal{P}_{sos}$ if and only if there are $p_i \in \mathcal{P}$, $i \in \{1, \ldots, k\}$ such that $p = p_1^2 + \cdots + p_k^2$. In this study, we consider a class of $n$-dimensional
In the sequel, we apply the following notions. For review in (Cortes [1998]).

For nonlinear switched systems $S = \{G, \mathcal{X}, I, \bar{\mathcal{F}}\}$ wherein $G$ is compact and defines the state-space, $\mathcal{X} = \{X_i\}_{i \in I}$ is the set containing (closed) partitions of $G$ with index set $I = \{1, 2, 3, \ldots, N\}$, and $\bar{\mathcal{F}} = \{F_i\}_{i \in I}$ a family of smooth functions ($F_i : U_i \rightarrow \mathbb{R}^n$ with $U_i$ an open neighborhood of $X_i$). Note that $G = \bigcup_{i \in I} X_i, X_i = \text{int}(X_i) \cup \partial X_i,$ $\text{int}(X_i) \neq \emptyset$ for all $X_i \in \mathcal{X}$, and $X_i \cap X_j \subset \partial X_i \cup \partial X_j.$ Also, denote $\bar{I} = \{(i, j) \in I \times I | X_i \cap X_j \neq \emptyset, i \neq j\}$ the set of index pairs which determines the partitions with non-empty intersections. Besides, each point of $G$ has an open neighborhood intersecting only finitely many elements of $\mathcal{X}$ (local finiteness). We also posit that for any $x \in \partial X_i$ and $v \in \mathbb{R}^n$ there is $a > 0$ and $j$ such that $x + at \in X_i$ for all $t \in [0, a]$ (characterization of a nice partitioning of the state space). Remark that partitioning by polyhedral sets assures that this latter property is satisfied. This is the case when considering switched systems defined on polyhedral sets, e.g. piecewise linear systems.

The global dynamics is described by the following differential inclusions

$$\dot{x}(t) \in \mathcal{F}(x(t)) \quad (1)$$

$$\dot{x}(t) \in \mathcal{F}^c(x(t)) \quad (2)$$

where the set-valued maps $\mathcal{F}$ and $\mathcal{F}^c$ are defined by

$$\mathcal{F} : G \rightarrow 2^G; x \mapsto \{v \in \mathbb{R}^n | v = F_i(x) \text{ if } x \in X_i\} \quad (3)$$

$$\mathcal{F}^c : G \rightarrow 2^G; x \mapsto \text{co}(\mathcal{F}(x)) \quad (4)$$

where $2^G$ denotes the power set of $G$, and the notation $\text{co}(-)$ signifies the convex hull. The choice of whether the dynamics is modeled by (1) or (2) depends on the nature of the motion to be considered (see Fig. 1). Pertaining to the solutions of discontinuous and switched dynamical systems, the interested reader is referred to the didactic review in (Cortes [1998]).

In the sequel, we apply the following notions. For $Q \subset \mathbb{R}^n$, $T_Q(x)$ denotes the Bouligand’s contingent cone of $Q$ at $x \in Q$. If $Q$ is convex then $T_Q(x)$ is closure of the cone spanned by $Q - \{x\}$. In addition, if $x$ is in the interior of $Q$, we have $T_Q(x) = \mathbb{R}^n$ (Aubin and Cellina [1984]). The upper contingent derivative of a function $U : \mathbb{R}^n \rightarrow \mathbb{R}$ at $x_0$ in the direction $v_0$ is defined as

$$D^+_q U(x_0)(v_0) = \lim_{l \rightarrow 0^+} \inf_{|x| \leq l} \left( \frac{U(x + lv_0) - U(x)}{l} \right)$$

Note in particular that, if $U(x)$ is Gâteaux differentiable then it holds that

$$D^+_q U(x_0)(v_0) = \frac{\partial U}{\partial x}(x, v)$$

where $\frac{\partial U}{\partial x}$ denotes the column vector with first-order partial derivatives of $U(x)$.

For all $x \in \text{int}(X_i), i \in I, \mathcal{F}(x) = F_i(x)$ is a one point set and since each $F_i$ is continuous, $\mathcal{F}$ is upper semi-continuous at any $x \in \text{int}(X_i)$. Furthermore, for any $x \in bd X_i \cup bd X_{i+1} \cup \cdots \cup bd X_j, \mathcal{F}(x) = \{F_i(x), F_{i+1}(x), \ldots, F_j(x)\}$ is a multi-valued set. Because each $F_i$ is continuous, it follows that for all $x \in X_i$ and $\epsilon > 0$ there exists a $\sigma(\epsilon, x) \leq \epsilon$ such that $\forall \bar{x} \in x + B^\sigma_\epsilon \cap \mathcal{F}(x).$ To demonstrate that $\mathcal{F}$ is upper semi-continuous, it suffices to choose $\sigma = \min \sigma_i$. Additionally, because each of the maps $F_i, i \in I$, is continuous and $\mathcal{F}(x)$ is finite for all $x \in G$ (finiteness of $\mathcal{F}$ follows from partitioning with local finiteness property); then, from Lemma 16 in (p. 66, Filippov [1988]), it follows that $\mathcal{F}^c$ is also upper semi-continuous.

It is also worth noting that $\mathcal{F}$ cannot be lower semi-continuous at any point $x \in X_i \cap X_j, (i, j) \in \bar{I}$, on a boundary, since $\mathcal{F}$ is not a one point set. For $T > 0$, let $\bar{S}_T$ denote either $[0, T] \text{ or } [0, T]$. By a Carathéodory solution of DI (1) at $\zeta_0 \in G$, we understand an absolutely continuous function $\bar{S}_T \rightarrow G; t \mapsto \zeta(t)$ which solves the following Cauchy problem

$$\bar{\zeta}(t) \in \mathcal{F}(\zeta(t)) \quad \text{a.e.}, \quad \zeta(0) = \zeta_0 \quad (7)$$

A Filippov solution to DI (1) at $\zeta_0 \in G$ is a solution to (7) with $\mathcal{F}$ supplanted by $\mathcal{F}^c$ (Filippov [1988]).

We recall the following facts from the theory of differential inclusions. Let $W$ be some non-negative function defined on $\text{Graph}(\mathcal{F}) = \{x \times G | x \in \mathbb{R}^n \times G \}$. We shall say that a function $\Phi : G \rightarrow \mathbb{R}_0^+$ is a Lyapunov function for $\mathcal{F}$ with respect to $W$ if for all $x \in G$ and some $v \in \mathcal{F}(x)$ the following “Lyapunov property” holds

$$D^+_q \Phi(x)(v) + W(x, v) \leq 0 \quad (8)$$

The following proposition which is derived from Corollary 1 and 2 in (p. 292, Aubin and Cellina [1984]) illustrates that, in fact, the existence of an equilibrium can be inferred from the Lyapunov property.

**Proposition 2.** If there exist a continuous non-negative function $\Phi : G \rightarrow \mathbb{R}_0^+$ and a positive definite function $W : \text{Graph}(\mathcal{F}^c) \rightarrow \mathbb{R}_0^+$ satisfying (8) for all $x \in G$ and some $v \in \mathcal{F}^c(x)$, then there exists an equilibrium $x_* \in G$ for $\mathcal{F}^c$.

Indeed, if the conditions of Proposition 2 hold, then

![Fig. 1. The trajectories of a switched system. Notice that the motion follows either DI (1) or (2).](image)
\( \forall x \in G, \; F^c(x) \cap T_G(x) \neq \emptyset \) \hspace{1cm} (9)

Furthermore, if \( F^c(G) \) is bounded, from Theorem 1 in (p. 180, Aubin and Cellina [1984]), it follows that at any point \( x_0 \in G \), there exists a Filippov solution of (1) defined on \( S_\infty \) which remains in \( G \) (viability).

Now, we are ready to assert a stability condition for the set valued map \( F^c \) which is a direct result of applying Theorem 8.4 in (p. 176, Smirnov [2002]).

**Proposition 3.** Suppose \( 0 \in F^c(0) \). If there exist \( \epsilon > 0 \) and continuous positive definite functions \( V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) and \( W : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) such that for each \( x \in \mathbb{R}^n \)

\[
D_x V(x)(v) + W(x) \leq 0 \quad \text{for all} \quad v \in F^c(x) \quad (10)
\]

Then the equilibrium point 0 is asymptotically stable.

Recall that if there exists an SOS decomposition for \( p(x) \), then it follows that \( p(x) \) is non-negative. Unfortunately, the converse does not hold in general; that is, there exist non-negative polynomials which do not have an SOS decomposition. An epitome of this class of non-negative polynomials is the Motzkin's polynomial (Motzkin [1965]) given by

\[
p(x) = 1 - 3x_1^2x_2^2 + x_1^2x_2^2 + x_1^4x_2^2 \quad (11)
\]

which is non-negative for all \( x \in \mathbb{R}^2 \). This imposes, more or less, some sort of conservatism when utilizing SOS based methods. The next proposition gives an interesting formulation to the SOS decomposition problem.

**Proposition 4.** (Chesi et al. [1996]). A polynomial \( p(x) \) of degree 2 belongs to \( \mathcal{P}_{sos} \) if and only if there exist a positive semi-definite matrix \( Q \) (known as the Gram matrix) and a vector of monomials \( Z(x) \) which contains all monomial of degree \( \leq d \) such that \( p(x) = Z^T(x)QZ(x) \).

Chesi et al. [1999] evinced that testing whether a polynomial is SOS can be formulated as a set of LMI feasibility tests. Subsequently, Parrilo [2003] demonstrated that the answer to the query that whether a given polynomial \( p(x) \) is SOS or not can be investigated via semi-definite programming methodologies.

**Proposition 5.** (Parrilo [2003]). Given a finite set \( \{p_i\}_{i=0}^m \in \mathcal{P} \), the existence of a set of scalars \( \{a_{ij}\}_{i,j=1}^m \in \mathbb{R} \) such that

\[
p_0 + \sum_{i=1}^m a_{ij}p_i \in \mathcal{P}_{sos} \quad (12)
\]

is an LMI feasibility problem.

The subsequent proposition formalizes the problem of constrained positivity of polynomials which is a direct result of applying Positivstellensatz method (Stengle [1994]).

**Proposition 6.** (Chesi [2010]). Let \( \{a_{ij}\}_{i=1}^m \) and \( \{b_{ij}\}_{i=1}^l \) belong to \( \mathcal{P} \), then

\[
p(x) \geq 0 \quad \forall x \in \mathbb{R}^n : a_i(x) = 0, \forall i = 1, 2, ..., k \quad \text{and} \quad b_j(x) \geq 0, \forall j = 1, 2, ..., l \quad (13)
\]

is satisfied, if the following holds

\[
\exists r_1, r_2, \ldots, r_k \in \mathcal{P} \quad \text{and} \quad \exists s_0, s_1, \ldots, s_l \in \mathcal{P}_{sos}
\]

\[
p = \sum_{i=1}^k r_ia_i + \sum_{i=1}^l s_ib_i + s_0 \quad (14)
\]

**Proposition 7.** The multivariable polynomial \( p(x) \) is strictly positive \( (p(x) > 0 \quad \forall x \in \mathbb{R}^n) \), if there exists a \( \lambda > 0 \) such that

\[
(p(x) - \lambda) \in \mathcal{P}_{sos} \quad (15)
\]

At this point, we are prepared to delineate the main contributions of this paper.

### 3. MAIN RESULTS

In this section, we will incorporate the mathematical notions given in the previous section to derive asymptotic stability conditions for the class of nonlinear switched systems with Filippov solutions defined on regular sets. Then, we present a theorem for robust asymptotic stability of switched systems with polytopic uncertainty. Finally, we bring forward sufficient conditions for stability using SOS techniques.

#### 3.1 Asymptotic Stability Conditions for Switched Systems

Consider the switched system \( S \) and let (2) describe the Filippov solutions of \( S \). It is assumed that 0 is an interior point of \( G \), and that it is located on some boundary of partitions. Note that 0 is not the equilibrium point, hence 0 is an equilibrium. Denote by \( \{V_i(x)\}_{i \in I} \) a family of positive definite and continuously differentiable (C1) functions \( (V_i : X_i \to \mathbb{R}_{\geq 0}) \).

We define a set valued map \( \Psi(x) \) as

\[
\Psi : G \to 2^\mathbb{R} : x \mapsto \{z \in \mathbb{R} \mid z = V_i(x) \quad \text{if} \quad x \in X_i\} \quad (16)
\]

Obviously, \( \Psi(x) \) can be considered as a switched system type of a candidate Lyapunov function.

**Proposition 8.** If \( V_i(x) = V_i(x) \) for all \( x \in X_i \times X_j \) and all \( (i, j) \in I \), then \( \Psi(i) \) is real-valued (\( \Psi : G \to \mathbb{R} \)) and locally Lipschitzian.

Notice that, Proposition 8 does not impose any restrain on the structure of \( \{V_i(x)\}_{i \in I} \), e.g. homogenous or quadratic forms as was done in Leth and Wisniewski [2012]. This considerably mitigates the conservatism in finding the family of Lyapunov functions \( \{V_i(x)\}_{i \in I} \).

**Proposition 9.** Suppose

I) \( \frac{\partial \Psi_i(x)}{\partial x} F_i(x) < 0 \) for all \( x \in X_i \setminus \{0\} \) and all \( i \in I \),

II) \( \frac{\partial \Psi_i(x)}{\partial x} F_j(x) < 0 \) for all \( x \in X_i \cap X_j \setminus \{0\} \) and all \( (i, j) \in I \).

Then there exists a continuous positive definite function \( W : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) such that

III) \( \frac{\partial \Psi_i(x)}{\partial x} F_i(x) < -W(x) \) for all \( x \in X_i \setminus \{0\} \) and all \( i \in I \),

IV) \( \frac{\partial \Psi_i(x)}{\partial x} F_j(x) < -W(x) \) for all \( x \in X_i \cap X_j \setminus \{0\} \) and all \( (i, j) \in I \).

**Proof.** Suppose for each \( X_i \), with \( i \in I \), there exist an open neighborhood \( T_i \) of \( X_i \) such that condition (I) holds due to the compactness of \( X_i \). Then, the collection of such open neighborhoods \( \{T_i\}_{i \in I} \) is an open cover of \( G \) such that \( G \subseteq \bigcup_{i \in I} T_i \). Therefore, there exists a partition of unity subordinate to the cover \( \{T_i\}_{i \in I} \); i.e., a family of continuous functions \( \{\psi_i : G \to [0, 1]\}_{i \in I} \) with \( \text{supp}(\psi_i) \subseteq T_i \) such that for any point \( x \in G \), there is a neighborhood
of $x$ where all but finite number of functions $\{\psi_i\}_{i \in I}$ are equal to 0, and such that $\sum_{i \in I} \psi_i(x) = 1$. Thus, let

$W_1(x) = -\sum_{i \in I} \psi_i(x)\frac{\partial V_i(x)}{\partial x} F_i(x)$ which satisfies (III).

In a similar manner, for all $x \in \mathbb{R}^n$ with $(i,j) \in \tilde{I}$ and $l \in \{1,2,\ldots,M\}$ where $M$ is the number of members in $\tilde{I}$, there exist open neighborhoods $\mathcal{Y}_l$ whose collection $\{\mathcal{Y}_l\}_{l \in L}$ is an open cover to the closed set $G' \subset G$. Because $G'$ is a closed subset of $G$, $G'$ is also compact. So, there exists a partition of unity subordinate to the cover $\{\mathcal{Y}_l\}_{l \in L}$ characterized by $\{\phi_l : \mathcal{Y}_l \rightarrow [0,1]\}_{l \in L}$. At this point, it suffices to let $W_2(x) = -\sum_{l \in L} \phi_l(x) \Gamma_l(x)$ where $\Gamma_l(x) = (\frac{\partial V_l(x)}{\partial x}, F_l(x))$ if $x \in \mathcal{Y}_l$, $l \in L$. Obviously, $W_2(x)$ satisfies (IV). Finally, we can select the map $W(x) = \max\{W_1(x), W_2(x)\}$, and this completes the proof.

The next proposition provides a Lyapunov-like stability theorem for the class of switched systems under study.

Proposition 10. Let $\{V_i(x)\}_{i \in I}$ be a family of $C^1$ Lyapunov functions. The switched system $\mathcal{S}$ is asymptotically stable at the origin if the following conditions hold

$$V_i(x) > 0 \quad \forall x \in X_i \setminus \{0\} \quad (17)$$

for all $i \in I$,

$$\left(\frac{\partial V_i(x)}{\partial x}, F_i(x)\right) < 0 \quad \forall x \in X_i \setminus \{0\} \quad (18)$$

$$V_i(x) = V_j(x) \quad \forall x \in X_i \cap X_j \setminus \{0\} \quad (19)$$

for all $(i,j) \in \tilde{I}$.

**Proof.** The proof follows the same lines of Proposition 10 in (Leth and Wcisnewski [2012]). It is necessary to show that Proposition 10 holds. From (16),(17),(20), and Proposition 8, we conclude that there exists a continuous, locally Lipschitzian, single-valued, and positive definite function $\psi_i(x)$ satisfying (III). Subsequently, from (18),(19) and Proposition 9 it follows that there exists a positive definite function $W(x)$ satisfying III and IV.

Given a set of $C^1$ functions $\{V_i(x)\}_{i \in I}$ and from the definition of partitioning, it follows that for any $x \in G$ and $v \in \mathbb{R}^n$, there is $a > 0$ such that $x + at \in X_j$ for any $t \in [0,a]$. On the other hand, $D_+V_i(x)(v) = \lim_{h \to 0_+} \frac{V_i(x+hv) - V_i(x)}{h} = \left(\frac{\partial V_i(x)}{\partial x}, v\right)$. Then, from Proposition 9 and (18) it follows that

$$D_+\psi_i(x)(u) = D_+V_i(x)(u)$$

$$= D_+V_i(x)\left(\sum_{k \in I} \alpha_k F_k(x)\right)$$

$$= \left(\frac{\partial V_i(x)}{\partial x}, \sum_{k \in I} \alpha_k F_k(x)\right)$$

$$= \sum_{k \in I} \alpha_k \left(\frac{\partial V_k(x)}{\partial x}, F_k(x)\right)$$

$$\leq -\sum_{k \in I} \alpha_k W(x)$$

$$\leq -W(x)$$

in which, we applied (19), condition IV and Proposition 9. Thus, 0 is an asymptotically stable equilibrium.

3.2 Robust Asymptotic Stability of Switched Systems with Polytopic Uncertainty

At this stage, we extend our results to a class of switched systems with polytopic uncertainty $\mathcal{S} = \{G, X, I, \tilde{\mathcal{F}}\}$ with $\tilde{\mathcal{F}} = \{F_i(x, \theta^i)\}_{i \in I}$ and

$$F_i(x, \theta^i) = \sum_{l=1}^{L_i} \theta_{d,l} f_{il}(x) \quad (21)$$

where, $f_{il} : U_l \rightarrow \mathbb{R}^n, l = 1,2,\ldots,L_i$ ($U_l$ is an open neighborhood of $X_i$) is a family of smooth functions, and $\theta^i, i \in I$ are uncertain constant parameter vectors ($\theta^i = [\theta_{1,i}, \theta_{2,i}, \ldots, \theta_{L_i,i}]^T \in \mathbb{R}^{L_i}$) satisfying

$$\theta^i \in \Theta^i \triangleq \left\{ \theta^i \in \mathbb{R}^{L_i} \mid \theta_{d,l} \geq 0, l = 1,2,\ldots,L_i,\right. \right.$$
and 11 presents stability conditions for general nonlinear switched systems; nonetheless, the above assumptions on the structure of Lyapunov functions and subsystems should be made so that providing a computational efficient method for stability analysis becomes doable. In order to achieve this goal, we need computational efficient methods to check the positivity of a given polynomial over a specific set. The positivity test can be performed using two main approaches; i.e., the slack variable approach and the SOS approach. The former method has been characterized for positivity analysis of polynomials over hyper-rectangles (Sato [2009]); however, this scheme lacks a computational efficient algorithm. On the other hand, currently there are well-developed computational tools for SOS decomposition e.g. SOSTOOLS (Prajna et al. [2004]).

Let the set
\[ \{ x \in G \mid \xi_{ik}(x) \geq 0 \text{ for } k = 1, 2, \ldots, n_{X_i} \} \tag{28} \]
represent the partition \( X_i \), where \( \xi_{ik} \in \mathcal{P} \) and \( n_{X_i} \) denotes the number of polynomial inequalities required to completely characterize \( X_i \). The boundary between the partitions is defined as
\[ X_i \cap X_j = \{ x \in G \mid \gamma_{ij}(x) = 0 \} \tag{29} \]
where \( \gamma_{ij} \in \mathcal{P} \), with \( (i, j) \in I \).

Condition (18) can be guaranteed if there exist polynomial functions \( p_{ij}(x) \in \mathcal{P} \) such that
\[ V_i(x) + \mu_{ij}(x) = V_j(x) \text{ for all } (i, j) \in I \tag{30} \]
Using the generalized S-procedure (Polić and Terlaky [2007]), the above discussions, and Proposition 7, it follows that (17) and (18) are satisfied if the following SOS problem is feasible
\[ V_i(x) - \sum_{i=1}^{n_{X_i}} q_{ik}(x)\xi_{ik}(x) - \mu_i \in \mathcal{P}_{sos} \tag{31} \]
\[ -\langle \frac{\partial V_i(x)}{\partial x}, F_i(x) \rangle - \sum_{i=1}^{n_{X_i}} w_{ik}(x)\xi_{ik}(x) - \mu_i \in \mathcal{P}_{sos} \tag{32} \]
for some SOS polynomials \( \xi_{ik}(x), w_{ik}(x) \) and positive scalars \( \mu_i, \gamma_{ij} \).

Condition (19) can also be reformulated using Proposition 6 as (see (14)) wherein \( b_i = 0, r_i(x) = r(x) \in \mathcal{P}, p = -\langle \frac{\partial V_i(x)}{\partial x}, F_i(x) \rangle \), and noting that \( s_0 \in \mathcal{P}_{sos} \)
\[ -\langle \frac{\partial V_i(x)}{\partial x}, F_i(x) \rangle - r(x)\gamma_{ij}(x) - \nu_{ij} \in \mathcal{P}_{sos} \tag{33} \]
Thus far, we have found SOS representations for all conditions in Proposition 10. The following proposition summarizes the above discussions.

**Proposition 12.** Consider the nonlinear switched system \( \mathcal{S} \). If there exist a family of polynomials \( \{ V_i(x) \}_{i \in I} \) with \( V_i(0) = 0 \) if \( 0 \in X_i \), \( r_{ij} \in \mathcal{P}, p_{ij} \in \mathcal{P} \) with \( (i, j) \in I \), \( q_{ik} \in \mathcal{P}_{sos}, w_{ik} \in \mathcal{P}_{sos} \) with \( k = 1, 2, \ldots, n_{X_i} \), and a set of positive scalars \( \lambda_i, \mu_i \) with \( i \in I \), and \( \nu_{ij} \) with \( (i, j) \in \hat{I} \), such that
\[ V_i(x) - \sum_{i=1}^{n_{X_i}} q_{ik}(x)\xi_{ik}(x) - \lambda_i \in \mathcal{P}_{sos} \tag{34} \]
\[ -\langle \frac{\partial V_i(x)}{\partial x}, F_i(x) \rangle - \sum_{i=1}^{n_{X_i}} w_{ik}(x)\xi_{ik}(x) - \mu_i \in \mathcal{P}_{sos} \tag{35} \]
for all \( i \in I \), and
\[ -\langle \frac{\partial V_i(x)}{\partial x}, F_i(x) \rangle - r_{ij}(x)\gamma_{ij}(x) - \nu_{ij} \in \mathcal{P}_{sos} \tag{36} \]
\[ V_i(x) + p_{ii}(x)\gamma_{ij}(x) = V_j(x) \tag{37} \]
for all \( (i, j) \in \hat{I} \). Then, the equilibrium 0 is asymptotically stable.

It is worth noting that condition (36) can be further relaxed by just considering those boundaries possessing attractive Filippov solutions (instead of checking (36) for all \( (i, j) \in \hat{I} \)). One can infer the existence of an attractive Filippov solution by checking
\[ -\langle \frac{\partial r_{ij}(x), F_i(x) \rangle - \langle \frac{\partial r_{ij}(x)}{\partial x}, F_j(x) \rangle < 0 \ \forall x \in X_i \cap X_j \] or, in terms of an SOS decomposition problem, if the following holds
\[ -\langle \frac{\partial r_{ij}(x)}{\partial x}, F_i(x) \rangle - \langle \frac{\partial r_{ij}(x)}{\partial x}, F_j(x) \rangle - l(x)\gamma_{ij}(x) - \kappa_{ij} \in \mathcal{P}_{sos} \tag{38} \]
for some \( l(x) \in \mathcal{P} \) and some positive scalar \( \kappa_{ij} \).

It should be noted that Proposition 12 only provides sufficient conditions. Indeed, given a nonlinear switched system \( \mathcal{S} \), one can search for the corresponding candidate Lyapunov functions via semi-definite programming schemes; if the problem is feasible, then the switched system \( \mathcal{S} \) is asymptotically stable.

Based on similar arguments for Proposition 12, we can characterize an SOS representation for conditions in Proposition 11.

**Proposition 13.** Consider the uncertain switched system \( \mathcal{S} \). If there exist a family of polynomials \( \{ V_i(x) \}_{i \in I} \) with \( V_i(0) = 0 \) if \( 0 \in X_i \), \( r_{ij} \in \mathcal{P}, p_{ij} \in \mathcal{P} \) with \( (i, j) \in \bar{I} \), \( q_{ik} \in \mathcal{P}_{sos}, w_{ik} \in \mathcal{P}_{sos} \) with \( k = 1, 2, \ldots, n_{X_i} \), and a set of positive scalars \( \lambda_i, \mu_i \) with \( i \in I \), and \( \nu_{ij} \) with \( (i, j) \in \bar{I} \), \( I = 1, 2, \ldots, L_i \), such that
\[ V_i(x) - \sum_{i=1}^{n_{X_i}} q_{ik}(x)\xi_{ik}(x) - \lambda_i \in \mathcal{P}_{sos} \tag{39} \]
\[ -\langle \frac{\partial V_i(x)}{\partial x}, F_i(x) \rangle - \sum_{i=1}^{n_{X_i}} w_{ik}(x)\xi_{ik}(x) - \mu_i \in \mathcal{P}_{sos} \tag{40} \]
holds for all for all \( i \in I \) and \( l = 1, 2, \ldots, L_i \),
\[ -\langle \frac{\partial V_i(x)}{\partial x}, F_i(x) \rangle - r_{ij}(x)\gamma_{ij}(x) - \nu_{ij} \in \mathcal{P}_{sos} \tag{41} \]
and (37) holds for all for all \( (i, j) \in \bar{I} \) and \( l = 1, 2, \ldots, L_j \).

Then, the origin is robustly asymptotically stable.

4. SIMULATION EXAMPLES

4.1 Example 1

Consider the following switched system described by
\[ \dot{x} \in \mathcal{G}(x) \tag{42} \]
\[ \dot{x} \in \mathcal{G}^*(x) \tag{43} \]
with \( \mathcal{G} : \mathbb{R}^2 \to 2^{\mathbb{R}^2}; x \to \{ v \in \mathbb{R}^2 \mid v = g_i(x) \text{ if } x \in X_i \} \) \((\mathcal{G}^* = \text{co}(\mathcal{G}))\), wherein \( i \in I = \{ 1, 2 \} \), the partitions defined as
where $\mathcal{H}(x, \theta)$ is the family of homogeneous polynomials of degree 3.

The systems satisfying the conditions of Proposition 13 are stable.

5. CONCLUSION

In this paper, a Lyapunov-like stability theorem for nonlinear switched systems with partitioned state-space and state-dependent switching is brought forward. This result has been exploited to formulate conditions on robust asymptotic stability of switched systems with polytopic uncertainty. Since the analysis is based on the theory of differential inclusions, the proposed stability analysis scheme includes Filippov solutions. Furthermore, in order to provide a computationally efficient method to implement the suggested stability theorem, the results are reformulated using SOS decomposition techniques which can be realized based on semi-definite programming tools.

REFERENCES


Fig. 3. Trajectories of the switched system in Example 2 with four starting points in different quadrants.


