Rate-distortion in closed-loop LTI systems

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Abstract—We consider a networked LTI system subject to an average data-rate constraint in the feedback path. We provide upper bounds to the minimal source coding rate required to achieve mean square stability and a desired level of performance. In the quadratic Gaussian case, an almost complete rate-distortion characterization is presented.

I. INTRODUCTION

This paper focuses on the interplay between average data-rate constraints (in bits per sample) and stationary performance for a networked control system comprising a noisy LTI plant and an average data-rate constraint in the feedback path. In such a setup, the results of [8] guarantee that it is possible to find causal encoders and decoders such that the resulting closed loop system is mean square stable, if and only if the average data-rate is greater than the sum of the logarithm of the absolute value of the unstable plant poles. This result has been extended in several directions (see, e.g., [7], [9]). However, when performance bounds subject to average data-rate constraints are sought, there are relatively fewer results available. Indeed, to our knowledge, there are no computable characterizations of the optimal encoding policies in networked control scenarios [1], [3], [5], [9], [13].

In this note, we present upper and lower bounds on the minimal average data-rate that allows one to attain a given performance level (as measured by the stationary variance of the plant output). From a source-coding perspective, we are aiming at characterizing the rate-distortion function in closed-loop systems. This extends beyond causal rate-distortion theory [2] due to being subject to a stability constraint. Our results exploit a framework for networked control system design subject to average data-rates developed in [10], [11].

II. PROBLEM SETUP

Consider the NCS of Figure 1, where $P$ is an LTI plant with state $x \in \mathbb{R}^{n_x}$ and initial state $x_0$, $u \in \mathbb{R}$ is the control input, $y \in \mathbb{R}$ is a sensor output, $e \in \mathbb{R}^{n_e}$ is a signal related to closed loop performance, and $d \in \mathbb{R}^{n_d}$ is a disturbance. We assume that $(x_0, d)$ are jointly second-order and Gaussian (with finite entropies). The feedback path in Figure 1 comprises a delay-free noiseless digital channel, a causal encoder whose output $y_c$ is a sequence of binary words, and a causal decoder. The process $y_c$ is a signal related to closed loop performance, and $y \in \mathbb{R}$ is a sensor output, $e \in \mathbb{R}^{n_e}$ is a signal related to closed loop performance, and $d \in \mathbb{R}^{n_d}$ is a disturbance. We assume that $(x_0, d)$ are jointly second-order and Gaussian (with finite entropies). The feedback path in Figure 1 comprises a delay-free noiseless digital channel, a causal encoder whose output $y_c$ is a sequence of binary words, and a causal decoder. The

average data-rate across the channel is defined as

$$ R \triangleq \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} R(i), $$

where $R(i)$ refers to the expected length (in nats) of $y_c(i)$.

We do not restrict the complexity of the encoder or the decoder a priori, and only assume them to be causal, and to have access to independent side information $S_E$ and $S_D$. Our aim is characterizing

$$ R(D) \triangleq \inf_{\sigma^2 \leq D} R, $$

where $\sigma^2 \triangleq \text{trace} \{P_e\}$, $P_e$ is the stationary variance matrix of $e$, $D > 0$ is a desired level of performance, and the optimization is carried out with respect to all causal encoders $E$ and decoders $D$ that render the resulting NCS (asymptotically) mean square stable (MSS), i.e., that render $(x, u, d)$ jointly second-order and asymptotically wide-sense stationary processes.

III. AN INFORMATION-THEORETIC LOWER BOUND ON AVERAGE DATA-RATES

Theorem 3.1: Consider the NCS of Figure 1. Under suitable assumptions,

$$ R \geq I_\infty(y \to u) \geq I_\infty(y_G \to u_G), $$

where $I_\infty(\alpha \to \beta)$ denotes the mutual information rate [6] between $\alpha$ and $\beta$, and $(y_G, u_G)$ are jointly Gaussian processes with the same second order statistics as $(y, u)$. $\blacksquare$

Thus, in order to bound $R(D)$ from below, it suffices to minimize the directed mutual information rate that would appear across the source coding scheme, when all signals in the loop are jointly Gaussian.

Lemma 3.1: Suppose that $(y^k, u^k)$ in Fig. 1 are second order and jointly Gaussian random sequences. Then $u^k$ can be constructed from $y^k$ as

$$ u(i) = L_i(y^i, u_i^{i-1}) + s(i), \quad i = 1, \ldots, k $$

where, for each $i = 1, \ldots, k$, $s(i)$ is a zero-mean Gaussian random variable such that $s(i) \perp (u^{i-1}, y^{i-1}, s^{i-1})$, and
where \( L_i : \mathbb{R}^{ix(i-1)} \to \mathbb{R} \) is a linear operator such that \( L_i(y^t, u^{t-1}) \) is the minimum mean-square error estimator of \( u(i) \) given \( y^t, u^{t-1} \).

We conclude from the above that, for a given performance level \( D \), the minimum of \( I_\infty(y_\mathcal{G} \to u_\mathcal{G}) \) over all causal encoders and decoders is achievable by an encoder/decoder pair which behaves as a linear system plus additive white Gaussian noise \( s^b \) such that \( s(i) \perp (y^t, u^{t-1}) \), \( \forall i \).

IV. LOWER AND UPPER BOUNDS ON \( \mathcal{R}_D \)

We next define the class of linear source coding schemes, which are capable of yielding a relationship between \( y \) and \( u \) of the form given by (4).

**Definition 4.1:** A source coding scheme is said to be linear if and only if, when used around a noiseless digital channel, is such that its input and output \( u \) are related via

\[
u = Fw, \quad w = q + v, \quad v = K \text{ diag } (z_{-1}, 1) \begin{bmatrix} w \\ y \end{bmatrix},
\]

(5)

where \( v \) and \( w \) are auxiliary signals, \( q \) is a second-order zero-mean i.i.d. sequence, both \( F \) and \( K \) are proper LTI systems, and \( q \) is independent of \( (x_0, d) \).

When a linear source coding scheme is used in the NCS of Figure 1, the LTI feedback system of Figure 2 arises.

**Lemma 4.1:** Consider the NCS of Figure 1 and assume that the encoder \( \mathcal{E} \) and the decoder \( \mathcal{D} \) form a linear source coding scheme. Under suitable assumptions, \( I_\infty(y \to u) = I_\infty(v \to w) \) and

\[
\frac{1}{4\pi} \int_{-\pi}^{\pi} \log \frac{S_w(e^{j\omega})}{\sigma_q^2} \, d\omega \leq I_\infty(v \to w),
\]

where \( S_w \) is the stationary power spectral density of \( w \) and \( \sigma_q^2 \) is the variance of the auxiliary noise \( q \).

Linear source coding schemes have sufficient degrees of freedom to allow one to whiten \( w \) without compromising optimality. Thus, our results lead to:

**Theorem 4.1:** Consider the NCS of Figure 1 under suitable assumptions. Define, with reference to the feedback scheme of Figure 2, the infimal signal-to-noise ratio function

\[
\gamma(D) \triangleq \inf_{\sigma_q^2 \leq D} \frac{\sigma_q^2}{\sigma_e^2},
\]

(7)

where \( \sigma_o^2, \sigma_q^2, \sigma_e^2 \in \mathbb{R}^+ \) and all proper LTI filters \( F \) and \( K \) which render the feedback system of Figure 2 internally stable and well-posed. Then:

\[
\frac{1}{2} \log \left( 1 + \gamma(D) \right) \leq \mathcal{R}(D).
\]

Moreover, there exists a linear source coding scheme such that

\[
\mathcal{R}(D) < \frac{1}{2} \log \left( 1 + \gamma(D) \right) + \frac{1}{2} \log \left( \frac{2\pi e}{12} \right) + \log 2.
\]

Theorem 4.1 characterizes the minimal average data-rate that guarantees a given stationary level of performance, in terms of \( \gamma(D) \), i.e., in terms of the minimal SNR that guarantees the desired performance level in a related LTI architecture. Interestingly, the upper bound in (9) is valid even if one removes the assumption of \( (x_0, d) \) being Gaussian.

To find \( \gamma(D) \), one can resort to the results in [4]. A case where an explicit solution is available is when \( D \to \infty \), i.e., when only stabilization is sought. In that case, it follows from Theorem 4.1 and [12] that

\[
\gamma(\infty) = \left( \prod_{i=1}^{n_e} |p_i|^2 \right) - 1, \quad (10)
\]

where \( p_1, \ldots, p_{n_e} \) are the unstable poles of \( P \). If one uses (10) in (8) and (9), then one recovers, within a modest gap, the absolute minimal average data-rate compatible with stability derived in [8].

REFERENCES


