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Rate-distortion in closed-loop LTI systems

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Abstract—We consider a networked LTI system subject to an average data-rate constraint in the feedback path. We provide upper bounds to the minimal source coding rate required to achieve mean square stability and a desired level of performance. In the quadratic Gaussian case, an almost complete rate-distortion characterization is presented.

I. INTRODUCTION

This paper focuses on the interplay between average data-rate constraints (in bits per sample) and stationary performance for a networked control system comprising a noisy LTI plant and an average data-rate constraint in the feedback path. In such a setup, the results of [8] guarantee that it is possible to find causal encoders and decoders such that the resulting closed loop system is mean square stable, if and only if the average data-rate is greater than the sum of the logarithm of the absolute value of the unstable plant poles. This result has been extended in several directions (see, e.g., [7], [9]). However, when performance bounds subject to average data-rate constraints are sought, there are relatively fewer results available. Indeed, to our knowledge, there are no computable characterizations of the optimal encoding policies in networked control scenarios [1], [3], [5], [9], [13].

In this note, we present upper and lower bounds on the average data-rate across the channel is defined as

\[ R \triangleq \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} R(i), \]  

where \( R(i) \) refers to the expected length (in nats) of \( y_c(i) \).

We do not restrict the complexity of the encoder or the decoder a priori, and only assume them to be causal, and to have access to independent side information \( S_E \) and \( S_D \). Our aim is characterizing

\[ R(D) \triangleq \inf_{\sigma_e^2 \leq D} R, \]  

where \( \sigma_e^2 \triangleq \text{trace } \{ P_e \} \), \( P_e \) is the stationary variance matrix of \( e \), \( D > 0 \) is a desired level of performance, and the optimization is carried out with respect to all causal encoders \( E \) and decoders \( D \) that render the resulting NCS (asymptotically) mean square stable (MSS), i.e., that render \( (x, u, d) \) jointly second-order and asymptotically wide-sense stationary processes.

III. AN INFORMATION-THEORETIC LOWER BOUND ON AVERAGE DATA-RATES

Theorem 3.1: Consider the NCS of Figure 1. Under suitable assumptions,

\[ R \geq I_\infty(y \to u) \geq I_\infty(y_G \to u_G), \]  

where \( I_\infty(\alpha \to \beta) \) denotes the mutual information rate [6] between \( \alpha \) and \( \beta \), and \( (y_G, u_G) \) are jointly Gaussian processes with the same second order statistics as \( (y, u) \).

Thus, in order to bound \( R(D) \) from below, it suffices to minimize the directed mutual information rate that would appear across the source coding scheme, when all signals in the loop are jointly Gaussian.

Lemma 3.1: Suppose that \( (y^k, u^k) \) in Fig. 1 are second order and jointly Gaussian random sequences. Then \( u^k \) can be constructed from \( y^k \) as

\[ u(i) = L_i(y^i, u^{i-1}) + s(i), \quad i = 1, \ldots, k \]  

where, for each \( i = 1, \ldots, k \), \( s(i) \) is a zero-mean Gaussian random variable such that \( s(i) \perp (u^{i-1}, y^{i-1}, s^{i-1}) \), and

![Networked control system](image-url)
where $L_i : \mathbb{R}^{i\times(i-1)} \to \mathbb{R}$ is a linear operator such that $L_i(y_i, u_i)$ is the minimum mean-square error estimator of $u(i)$ given $(y_i, u_i)$.

We conclude from the above that, for a given performance level $D$, the minimum of $I_\infty(y_G \to u_G)$ over all causal encoders and decoders is achievable by an encoder/decoder pair which behaves as a linear system plus additive white Gaussian noise $s^k$ such that $s(i) \perp \perp (y_i, u_i^{-1})$, $\forall i$.

IV. LOWER AND UPPER BOUNDS ON $\mathcal{R}_D$

We next define the class of linear source coding schemes, which are capable of yielding a relationship between $y$ and $u$ of the form given by (4).

**Definition 4.1:** A source coding scheme is said to be linear if and only if, when used around a noiseless digital channel, is such that its input and output $u$ are related via

$$u = Fw, \quad w = q + v, \quad v = K \text{diag} \left\{ z_i^{-1}, 1 \right\} \begin{bmatrix} w \\ y \end{bmatrix},$$

where $v$ and $w$ are auxiliary signals, $q$ is a second-order zero-mean i.i.d. sequence, both $F$ and $K$ are proper LTI systems, and $q$ is independent of $(x_0, d)$.

When a linear source coding scheme is used in the NCS of Figure 1, the LTI feedback system of Figure 2 arises.

**Lemma 4.1:** Consider the NCS of Figure 1 and assume that the encoder $\mathcal{E}$ and the decoder $\mathcal{D}$ form a linear source coding scheme. Under suitable assumptions, $I_\infty(y \to u) = I_\infty(v \to w)$ and

$$\frac{1}{4\pi} \int_{-\pi}^{\pi} \log \frac{S_w(e^{j\omega})}{\sigma_q^2} \, d\omega \leq I_\infty(v \to w),$$

where $S_w$ is the stationary power spectral density of $w$ and $\sigma_q^2$ is the variance of the auxiliary noise $q$.

Linear source coding schemes have sufficient degrees of freedom to allow one to whiten $w$ without compromising optimality. Thus, our results lead to:

**Theorem 4.1:** Consider the NCS of Figure 1 under suitable assumptions. Define, with reference to the feedback scheme of Figure 2, the infimal signal-to-noise ratio function

$$\gamma(D) \triangleq \inf_{\sigma_q^2 \leq D} \frac{\sigma_v^2}{\sigma_q^2},$$

where $\sigma_v^2$, $\alpha \in \{v, q, e\}$, is the stationary variance of $\alpha$ in Figure 2, and the optimization is carried out with respect to all $\sigma_q^2 \in \mathbb{R}^+$ and all proper LTI filters $F$ and $K$ which render the feedback system of Figure 2 internally stable and well-posed. Then:

$$\frac{1}{2} \log \left(1 + \gamma(D)\right) \leq \mathcal{R}(D).$$

Moreover, there exists a linear source coding scheme such that

$$\mathcal{R}(D) < \frac{1}{2} \log \left(1 + \gamma(D)\right) + \frac{1}{2} \log \left(\frac{2\pi e}{12}\right) + \log 2.$$  

Theorem 4.1 characterizes the minimal average data-rate that guarantees a given stationary performance level, in terms of $\gamma(D)$, i.e., in terms of the minimal SNR that guarantees the desired performance level in a related LTI architecture. Interestingly, the upper bound in (9) is valid even if one removes the assumption of $(x_0, d)$ being Gaussian.

To find $\gamma(D)$, one can resort to the results in [4]. A case where an explicit solution is available is when $D \to \infty$, i.e., when only stabilization is sought. In that case, it follows from Theorem 4.1 and [12] that

$$\gamma(\infty) = \left(\prod_{i=1}^{n_p} |p_i|^2\right) - 1,$$

where $p_1, \ldots, p_{n_p}$ are the unstable poles of $P$. If one uses (10) in (8) and (9), then one recovers, within a modest gap, the absolute minimal average data-rate compatible with stability derived in [8].

REFERENCES


