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Robust Computation of Error Vector Magnitude for Wireless Standards

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Abstract

The modulation accuracy described by an error vector magnitude is a critical parameter in modern communication systems — defined originally as a performance metric for transmitters but now also used in receiver design and for more general signal analysis. The modulation accuracy is a measure of how far a test signal is from a reference signal at the symbol values when some parameters in a reconstruction model are optimized for best agreement. This paper provides an approach to computation of error vector magnitude as described in several standards from measured or simulated data. It is shown that the error vector magnitude optimization problem is generally non-convex. Robust estimation of the initial conditions for the optimizer is suggested, which is particularly important for a non-convex problem. A Bender decomposition approach is used to separate convex and non-convex parts of the problem to make the optimization procedure simpler and robust. A two step global optimization method is suggested where the global step is the grid method and the local method is the Newton method. A number of test cases are shown to illustrate the concepts.

Index Terms

Error vector magnitude, optimization, non-convex, communication systems, scientific computing

I. Introduction and State-of-the-Art

The concept of error vector magnitude (EVM) is used in modern communication systems to assess the modulation quality of a given signal — see e.g., [1], [2]. EVM is computed via a comparison of two signals — a reference and a test signal containing the information

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carrying complex symbol values. The test signal is normally formed by passing the reference signal through one or more functional blocks such as amplifiers, filters, mixers, converters or a complete system such as a transmitter. The test signal may thereby be affected by noise, distortion, interference etc. An error signal is then the vectorial difference between the reference signal and a compensated version of the test signal. The procedure takes into account the compensation possibilities of a data receiver. The receiver can, e.g., compensate for a constant gain and phase difference between the test and reference signals. A root mean square (RMS) EVM is then equal to square root of the ratio of average power of the error signal to the average power of the reference signal.

Determining the compensation model parameters is an optimization problem where the objective is to minimize the average power of the error signal. The EVM metric is used both as part of standardization and in design of both transmitters and receivers [1]. Although not intended as a receiver performance metric it provides a fast and easy to understand metric with the advantage that a data receiver is not needed as for bit or block error rate estimations. The link between error vector magnitude and bit error rate was studied for some communication systems in the papers by Hassun et al. [4], and Shafik et al. [5].

EVM is, e.g., used in WLAN (Wireless Local Area Network) [6], LTE (Long Term Evolution) [7], UMTS (Universal Mobile Telecommunications System) [8] and EDGE (Enhanced Data rates for GSM Evolution) [3]. In WLAN it is required to compensate for time offset, magnitude and phase offsets, and frequency offset. The E-UTRA (Evolved UMTS Terrestrial Radio Access) [7] also referred to as LTE requires compensation for the same parameters as in WLAN and also origin offset. The EDGE system has the same requirements to the compensation parameters as E-UTRA/LTE but also includes a parameter to compensate for power reduction over time. So in worst case it is necessary to first compensate for time offset, and then 6 real parameters must be optimized to reach the smallest possible power in the error signal. In principle each standard uses its own EVM calculation but as seen above, the core principle is the same. A measurement filter may also be needed; for example a raised cosine filter is used in EDGE [3]. The EVM measure as well as the extracted parameters can be used in design, validation and standard certification of transmitters.

A significant amount of research have been done in connection to EVM. Several work have focused on closed-form EVM calculations for a plural of settings. Pinto and Darwazeh [9] relates

EVM and magnitude/phase distortion in 8PSK modulated systems. Georgiadis [10] derived expressions for error vector magnitude in case the imperfection is gain/phase imbalance, and phase noise effects. Liu et al. [11] derived various equations for EVM given different kind of distortion mechanisms. Mahmoud and Arslan [12] gives equations of received EVM over an AWGN and Rayleigh fading channel. EVM is also used in the design phase. Liu et al. [13] optimizes for EVM to enable a peak-to-average-power reduction in OFDM systems to reduce power amplifier requirements. Forestier [14] used EVM in a joint EVM and power efficiency optimized design of a power amplifier.

Another area of work is in estimating EVM from measured data. McKinley et al. [15] derived EVM equations based on normalization of measured WLAN data, and Ku and Kenney [16] proposed a method to estimate EVM for a power amplifier by use of an intrinsic kernel function derived from two-tone intermodulation measurements. Freisleben [17] derived equations for a semi-analytical approach to estimate EVM for SAW filters with good experimental agreement. Yamanouch et al. [18] proposed a single tone measurement based technique to estimate EVM for power amplifiers for IEEE 802.11a OFDM systems. Acar et al. [19] proposed to use an enhanced EVM measurement technique for both receiver and transmitter testing. Mashhour and Borjak [20] used a steepest descend optimization technique with fixed initial conditions. Xia et al. [21] proposed methods for simple time offset correction and frequency offset estimation suitable for the EDGE communication standard.

Considering the existing methods, it appears that standard optimizers have been used in the literature without much concern for the problem at hand. To the best knowledge of the authors, it has not been recognized that the optimization problem is non-convex, which has severe impact on how the algorithms for solving the optimization problem should be designed. Also the behavior of the objective function is often such that many local minima exist with a very deep and narrow global minimum. This also means that using a data based approach for the initial conditions in the optimization method provides a basis for an efficient and robust optimization procedure.

The present paper provides an approach to EVM computation of measured or simulated data based on the EVM definition for various standards. The paper focuses on three main issues: 1) formulating and proving that the optimization problem is non-convex; 2) introduction of a decomposition method to handle this non-convex problem in an efficient and robust manner; and 3) obtaining reliable and robust start values of the parameters based on the data available.

The paper is organized as follows. First, section II describes the signal model and the problem of computing the error vector magnitude. In section III the time offset is determined by use of correlation techniques. Then section IV shows that determination of EVM is a 2-norm non-convex optimization problem with up to six real optimization variables, and we show how to use Benders decomposition to efficiently handle this non-convex problem. Following this, section V derives robust initial conditions for the optimization vector, which can be determined from the reference and test data. Validation, examples and comparisons are provided in section VI. Finally, section VII contains the conclusions.

II. Signal Model and EVM Definition

In a communication system, the transmitter sends a sequence of symbols, where each symbol can be represented at complex baseband [22] — for example as a vector. For practical reasons the transmitted signal is sampled as in-phase (I)/quadrature-phase (Q) signals while meeting the Shannon-Nyquist criterion for both the I and Q signal. Thereby it is possible to perfectly reconstruct the signal.

Depending on the standard eventually being used in the given situation it may be required to have a number of bursts (or time slots) where the model parameters are determined in each bursts, and where statistics is applied across a number of bursts. As an example, the EDGE system [3] requires at least 200 bursts to be included in the EVM determination where one burst consists of 147 (useful) symbols. An EVM is computed for each burst and a joint EVM is computed like $\mathrm{EVM}^2 = \frac{1}{200}\left(\mathrm{EVM}_1^2 + \cdots + \mathrm{EVM}_{200}^2\right)$ where the index refers to the burst number. Other systems do not use a frame/burst structure and only have one long 'burst'. We focus on the one burst/time slot formulation as the extension to multiple bursts is trivial.

The reference signal containing all N complex baseband symbols are described by the vector

$$\mathbf{r} \;=\; [r[1], \ldots, r[N]]^{\mathrm{T}} \;\in\; \mathbb{C}^{N\times 1} \tag{1}$$

where $r[n] = r(nT_{\mathrm{symb}})$ with T_{symb} being the symbol time. In (1), $\mathbf{r}$ is specified as a column vector as this is important for the algebra to follow.

A test signal is similar in structure to the reference signal but may of course be exposed to noise, distortion and interference. Another difference is the sample rate $f_{\mathrm{samp}} = \Theta\, f_{\mathrm{symb}} = \Theta/T_{\mathrm{symb}}$ where Θ is a positive integer $\Theta \geq 1$. The raw test vector with all samples is then given

by

$$\mathbf{t}' \;=\; [t'[1],\ldots,t'[K]]^{\mathrm{T}} \;\in\; \mathbb{C}^{K\times 1} \tag{2}$$

where $t'[k] = t'(kT_{\mathrm{samp}})$ and $K \geq \Theta\,(N-1)+1$. Note that it is required that the test and reference signal is obtained in a synchronous setup. There may be some samples in offset between reference and test signals, which has to be determined (in multi-burst systems this offset is likely to be different for each burst). The only assumption is that $\mathbf{r}$ in a possibly distorted form is contained somewhere in $\mathbf{t}'$. The test symbols are extracted from the raw test samples as

$$t[n] \;=\; \Lambda(\mathbf{t}', \delta, n), \quad n = 1,\ldots,N \tag{3}$$

where Λ is a function that maps the raw samples to symbols and δ is an offset we need to determine. The exact definition of Λ is standard dependent. This offset is typically caused by group delay in filters or propagation effects. It could also be caused by some trigger delay between reference and test signals. Based on this extraction of symbols from the $t'[\cdot]$ signal, the test symbol vector $\mathbf{t}$ is formed as

$$\mathbf{t} \;=\; [t[1],\ldots,t[N]]^{\mathrm{T}} \;\in\; \mathbb{C}^{N\times 1}\,. \tag{4}$$

When computing the error vector magnitude following known standards, there are a number of effects a receiver may be able to compensate for. These are

1) A constant time delay, δ, between reference and test signal. This compensation must be done before the actual optimization is done.
2) Optimize the following parameters to achieve the smallest possible error vector magnitude:
   a) A constant gain and phase difference between reference and test signal.
   b) A constant loss factor.
   c) A constant frequency offset between reference and test signal.
   d) A constant origin offset.

Constant gain and phase as well as frequency offset are normally always compensated for in all standards. But for example the constant loss factor compensation is used in burst systems such as EDGE [3].

Translating the list above to manipulations on the signals, the following signal model is obtained for a residual complex error when the compensations are included

$$e_{\mathbf{x}}[n] = (x_1 + j\,x_2)\,t[n]\,\exp[-(x_3 + j\,\kappa\,x_4)\,(n-1)] - \;(x_5 + j\,x_6)\;-\;r[n], \quad n = 1,\ldots,N \tag{5}$$

where $\kappa = 2\pi T_{\text{symb}}$, $(x_1 + j\,x_2)$ is the complex gain factor, x_3 is the loss factor, x_4 is the frequency offset, and $(x_5 + j\,x_6)$ is the complex origin offset. Note that all variables $x_1 \cdots x_6$ represents a unique compensation parameter and, as a result, the model cannot be simplified. We use the notation $e_{\mathbf{x}}[n]$ to emphasize that the error signal is a function of the real parameters $\mathbf{x} = [x_1, \ldots, x_6]^{\mathrm{T}}$. Determination of the real parameters $x_1, \ldots, x_6$ that lead to the minimum RMS EVM is then an optimization problem described as

$$E_{\text{rms}} \;=\; \min_{\mathbf{x}} \sqrt{\frac{\frac{1}{N}\sum_{n=1}^{N} |e_{\mathbf{x}}[n]|^2}{\frac{1}{N}\sum_{n=1}^{N} |r[n]|^2}} \tag{6}$$

with solution(s) $\mathbf{x}^\star$. The definition (6) is used in, e.g., EDGE and UMTS [3], [8] and identifies/quantifies the best possible compensation *any* receiver can provide. Seen from an optimization point of view, the problem (6) is unconstrained. $x_1, x_2 \in \mathbb{R}$ meaning we cover the entire complex plane with these two variables. $x_3 \in \mathbb{R}$ where a positive x_3 means power reduction over time and a negative x_3 implies a power expansion over time. $x_4 \in \mathbb{R}$ is the frequency offset between reference and test — this may be both positive as well as negative. $x_5, x_6 \in \mathbb{R}$ meaning that we here cover the entire complex plane. Seen from a standardization point of view there are usually constraints on the frequency difference (x_4) that can be allowed. Also, most standards have a constraint on the magnitude of the origin offset. However, when we see this from an optimization point of view we have no constraints. To illustrative we could imagine a design phase where we make a poor decision on a the design or a component value leading to an extremely high x_4. Had we been limited by the standard we might not have seen the consequence on the EVM of this poor design decision. We have deliberately chosen x_1 and x_2 in real/imaginary form to avoid the constraint of a magnitude being positive and the phase being limited to, e.g., $-\pi \leq \kappa\, x_4 \leq \pi$. Avoiding the constraints also allows for a simpler optimization formulation. For consistency we have also chosen the remaining parameters as real. Note that it is also easier to handle the parameters $x_1 \cdots x_6$ independently in real form which is useful for the standards that only compensate using a subset of the presented compensation parameters.

Before determining the compensation parameters some standards require a measurement filter to be applied. In UMTS it is required that the linear measurement filter $m[k]$ is applied to the test and reference signal *before* compensation [8]. We will call this the pre filtering case where we obtain the raw test samples as $t'[k] = m[k] * t''[k]$ and $*$ denotes convolution. Test signal at the symbols $\mathbf{t}$ is then obtained using (2) and (3). The procedure for the reference signal is the

same, however, δ is a known offset from where the useful part of the signal starts. In EDGE, the measurement filter is applied *after* compensation [3]. Since compensation is not a linear function $m[k] * (\exp[-(z(k-1))]t''[k]) \neq \exp[-(z(k-1))](m[k] * t''[k])$, the optimization objective is not the same as in the pre filtering case. We will call this the post filtering case. In the post filtering case we instead obtain the following definition

$$\bar{e}'_{\mathbf{x}}[k] = m[k] * \left[(x_1 + jx_2)t''[k] \exp[-(x_3 + j\kappa\, x_4)\frac{(k-1)}{\Theta}] - (x_5 + jx_6) - r'[k] \right] \tag{7}$$

where we scale by Θ such that $(x_3 + j\kappa\, x_4)$ is comparable with the definition (5). The signal $r'[k]$ is the raw reference signal and $r[n] = m[\Theta n] * r'[\Theta n]$ is the reference signal. The error signal at the symbols is then formed as

$$\bar{e}_{\mathbf{x}}[n] = \Lambda(\bar{\mathbf{e}}'(\mathbf{x}), \delta, n), \quad n = 1, \ldots, N \tag{8}$$

where

$$\bar{\mathbf{e}}'(\mathbf{x}) \;=\; [\bar{e}'_{\mathbf{x}}[1], \ldots, \bar{e}'_{\mathbf{x}}[K]]^{\mathrm{T}} \;\in\; \mathbb{C}^{K\times 1}\,. \tag{9}$$

Consult the relevant standard to see if the measurement filter should be used before or after compensation. See also the discussion in [20].

III. Time offset

The first parameter to determine is the time offset (or time delay), $\hat{\delta} \simeq \delta$, which is the delay leading to the best possible match between the test and reference signals. This must be done for all bursts independently. Determining $\hat{\delta} \simeq \delta$ is not trivial due to the model parameter x_3 describing a possible reduction in signal strength over time. Using (3) and assuming that the origin offset is modest for the error in (5) and we strive for $e_{\mathbf{x}}[n] = 0$, then taking the absolute value we have $|\exp[j\kappa\, x_4 \cdot (n-1)]| = 1$ and

$$|r[n]| \simeq \sqrt{x_1^2 + x_2^2}\, \exp[-x_3 \cdot (n-1)]\, |\Lambda(\mathbf{t}', \gamma, n)| \tag{10}$$

where we use γ as the running time offset. Depending on the quality of the used signal processing devices (e.g. power amplifier), the exponential coefficient may be low. For EDGE [3, Fig. B.2] the modulated signal should stay within a power envelope bound of $20+4-16.5 = 7.5\,\mathrm{dB}$ where $16.5\,\mathrm{dB}$ is the dynamic of the EDGE signal. The signal may be then be damped by as much as $\exp[-x_3\,(N-1)] \simeq 0.18$. At design time this can be even worse. Therefore the correction

made at symbol N may be substantial. Although it is obviously possible just to assume $x_3 = 0$, and thereby no influence from $\exp[-x_3 \cdot (n-1)]$, it would be better to estimate x_3, perform the correction in (10), and then determine the optimum time offset δ by determining the cross-correlation coefficient between the reference and compensated test signal. Note that $\sqrt{x_1^2 + x_2^2}$ in (10) does not matter for the correlation analysis as it affects all data points equally.

The loss factor x_3 can be estimated in the following way. Suppose a given γ is chosen, and we then set up two equations with two unknowns from (10) at symbols n and $n + \Delta_n$. In this case, x_3 can be estimated as

$$\hat{x}_3(\gamma; n; \Delta_n) = \frac{-1}{\Delta_n} \ln\left\{ \left| \frac{\Lambda(\mathbf{t}', \gamma, n)}{\Lambda(\mathbf{t}', \gamma, n + \Delta_n)} \right| \left| \frac{r[n + \Delta_n]}{r[n]} \right| \right\} \tag{11}$$

where $n = 1, 2, \ldots, N-1$ and $\Delta_n = 1, 2, \ldots, N - n$. Averaging over n and Δ_n yields

$$\hat{x}_3(\gamma) = \frac{2}{N\,(N+1)} \sum_{n=1}^{N-1} \sum_{\Delta_n=1}^{N-n} \hat{x}_3(\gamma; n; \Delta_n)\,. \tag{12}$$

Thereby

$$|t_3'(\gamma; n)| \;=\; \exp[-\hat{x}_3(\gamma)\,(n-1)]\,|\Lambda(\mathbf{t}', \gamma, n)|\,. \tag{13}$$

This gives the cross-correlation coefficient as

$$\mathcal{C}_{|t_3'|,|r|}(\gamma) \;=\; -\frac{E_{|t_3'|}(\gamma)\,E_{|r|}}{\sigma_{|t'|}(\gamma)\,\sigma_{|r|}} + \frac{1}{N\,\sigma_{|t'|}(\gamma)\,\sigma_{|r|}} \sum_{n=1}^{N} |t_3'(\gamma; n)|\,|r[n]| \tag{14}$$

where

$$E_{|t_3'|}(\gamma) = \frac{1}{N} \sum_{n=1}^{N} |t_3'(\gamma; n)|, \qquad E_{|r|} = \frac{1}{N} \sum_{n=1}^{N} |r[n]| \tag{15}$$

$$\sigma_{|t_3'|}(\gamma) = \sqrt{\frac{1}{N} \sum_{n=1}^{N} \left\{ |t_3'(\gamma; n)| - E_{|t_3'|}(\gamma) \right\}^2}, \qquad \sigma_{|r|} = \sqrt{\frac{1}{N} \sum_{n=1}^{N} \left\{ |r[n]| - E_{|r|} \right\}^2}\,. \tag{16}$$

The first term in (14) is the crosscorrelation between absolute values of reference and test signals. The optimum value of γ can then be determined as the time delay leading to

$$\hat{\delta} = \underset{\gamma}{\operatorname{argmax}}\; \mathcal{C}_{|t_3'|,|r|}(\gamma),\; \gamma = 0, 1, \ldots, K - \Theta\,(N-1) - 1\,. \tag{17}$$

With the time offset, $\hat{\delta} \simeq \delta$, determined, the test symbol vector is created by use of (3) and (4). In terms of algorithmic complexity, (17) is a $\mathcal{O}((K - \Theta N)N^2)$ problem. We find that for large N, it is not necessary to compute $\hat{x}_3(\gamma)$ in (12) for each γ, instead the estimate $\hat{x}_3(0)$ is sufficiently

accurate. Since $\gamma = 0$ is usually a wrong time-offset we are averaging over intersymbol values to obtain the estimate $\hat{x}_3(0) = \hat{x}_3$. We subscribe that this can be a good estimate to the fact that we are estimating signal strength over time and intersymbols values are also affected by the signal strength over time. We however need larger N to accurately estimate x_3 since we are not evaluating the test signal exactly at the symbol points. In this case the algorithm complexity is reduced to $\mathcal{O}((K - \Theta N)N + N^2)$ which is much more tractable for large N.

IV. The Optimization Problem

A. Starting point

The optimization problem is to minimize the error vector magnitude as described by (6) for each burst. Using vector notation for the error values yields

$$\sum_{n=1}^{N} |e_{\mathbf{x}}[n]|^2 = \|\mathbf{e}(\mathbf{x})\|_2^2 = f(\mathbf{x}) \tag{18}$$

where

$$\mathbf{e}(\mathbf{x}) \;=\; [e_{\mathbf{x}}[1], \ldots, e_{\mathbf{x}}[N]]^{\mathrm{T}} \;\in\; \mathbb{C}^{N\times 1}\,. \tag{19}$$

Using

$$E(\mathbf{x}) = \sqrt{\frac{\frac{1}{N}\sum_{n=1}^{N} |e_{\mathbf{x}}[n]|^2}{\frac{1}{N}\sum_{n=1}^{N} |r[n]|^2}} = \sqrt{\frac{f(\mathbf{x})}{\sum_{n=1}^{N} |r[n]|^2}} \tag{20}$$

we obviously have that minimization of the RMS EVM (6) is equivalent to the problem

$$\underset{\mathbf{x}}{\text{minimize}} \quad f(\mathbf{x}) \tag{21}$$

with solution $\mathbf{x}^\star$ (not necessarily unique) such that $f(\mathbf{x}) \geq f(\mathbf{x}^\star)$, $\forall \mathbf{x} \in \mathbb{R}^6$.

B. Analysis of the optimization problem

The complex exponential function in the objective, see (5) and (18), means that the optimization is extremely difficult. Even a small difference between the test and reference frequencies lead to much different signals after several tens or hundreds of symbols. This is further complicated by the observation that the objective function f is not convex. A closer analysis reveals, not surprisingly, that multiplication with the complex exponential function in (5) is the cause of this.

This is an important observation for the optimization problem. Next, we formalize the previous observations. We see that

$$f(\mathbf{x}) = \sum_{n=1}^{N} |e_{\mathbf{x}}[n]|^2 = \sum_{n=1}^{N} \Re^2\{e_{\mathbf{x}}[n]\} + \sum_{n=1}^{N} \Im^2\{e_{\mathbf{x}}[n]\}\,. \tag{22}$$

Following the pre filtering case, see (5), we can, e.g., rewrite the real part as

$$\Re^2\{e_{\mathbf{x}}[n]\} = \Big(\Re\{(x_1 + jx_2)t[n]\exp[-(x_3 + j\kappa\, x_4)(n-1)]\} - x_5 - \Re\{r[n]\}\Big)^2 \tag{23}$$

and imaginary part as

$$\Im^2\{e_{\mathbf{x}}[n]\} = \Big(\Im\{(x_1 + jx_2)t[n]\exp[-(x_3 + j\kappa\, x_4)(n-1)]\} - x_6 - \Im\{r[n]\}\Big)^2. \tag{24}$$

Define

$$f(\mathbf{x}) = f(\mathbf{x}_{\rm v}; \mathbf{x}_{\rm r}), \quad \mathbf{x}_{\rm v} = [x_3,\, x_4]^{\rm T} \,\in\, \mathbb{R}^2, \quad \mathbf{x}_{\rm r} = [x_1,\, x_2,\, x_5,\, x_6]^{\rm T} \,\in\, \mathbb{R}^4 \tag{25}$$

in which case we can write the objective function as

$$f(\mathbf{x}) = f(\mathbf{x}_{\rm v}; \mathbf{x}_{\rm r}) = \|\boldsymbol{\mathcal{A}}(\mathbf{x}_{\rm v})\mathbf{x}_{\rm r} - \mathbf{r}_{\rm r}\|_2^2 \tag{26}$$

where

$$\boldsymbol{\mathcal{A}}(\mathbf{x}_{\rm v}) = \begin{bmatrix} \boldsymbol{\tau}_{\Re}(\mathbf{x}_{\rm v}) & -\boldsymbol{\tau}_{\Im}(\mathbf{x}_{\rm v}) & -\boldsymbol{\upsilon} & \mathbf{0}_N \\ \boldsymbol{\tau}_{\Im}(\mathbf{x}_{\rm v}) & \boldsymbol{\tau}_{\Re}(\mathbf{x}_{\rm v}) & \mathbf{0}_N & -\boldsymbol{\upsilon} \end{bmatrix} \in \mathbb{R}^{2N\times 4}, \quad \mathbf{r}_{\rm r} \quad = \begin{bmatrix} \mathbf{r}_{\Re} \\ \mathbf{r}_{\Im} \end{bmatrix} \in \mathbb{R}^{2N} \tag{27}$$

and

$$\boldsymbol{\tau}_{\Re}(\mathbf{x}_{\rm v}) = [\Re\{\tau(\mathbf{x}_{\rm v}; 1)\}, \ldots, \Re\{\tau(\mathbf{x}_{\rm v}; N)\}]^{\rm T}, \boldsymbol{\tau}_{\Im}(\mathbf{x}_{\rm v}) = [\Im\{\tau(\mathbf{x}_{\rm v}; 1)\}, \ldots, \Im\{\tau(\mathbf{x}_{\rm v}; N)\}]^{\rm T} \tag{28}$$

$$\boldsymbol{\upsilon} = [\upsilon[1], \ldots, \upsilon[N]]^{\rm T}, \mathbf{0}_N = [0, \ldots, 0]^{\rm T} \in \{0\}^N \tag{29}$$

$$\mathbf{r}_{\Re} = [\Re\{r[1]\}, \ldots, \Re\{r[N]\}]^{\rm T}, \mathbf{r}_{\Im} = [\Im\{r[1]\}, \ldots, \Im\{r[N]\}]^{\rm T}\,. \tag{30}$$

For the pre filtering case we then have

$$\tau(\mathbf{x}_{\rm v}; n) = t[n]\, \exp[-(x_3 + j\,\kappa\, x_4)(n-1)], \qquad \upsilon[n] = 1\,. \tag{31}$$

In the post filtering case, see (7), we define

$$\boldsymbol{\tau}'(\mathbf{x}_{\rm v}) = [\tau'(\mathbf{x}_{\rm v}, 1), \ldots \tau'(\mathbf{x}_{\rm v}, K)]^{\rm T}, \qquad \tau'(\mathbf{x}_{\rm v}, k) = m[k] * \left(t''[k]\, \exp\left[-\frac{(x_3 + j\,\kappa\, x_4)}{\Theta}\,(k-1)\right]\right) \tag{32}$$

and

$$\boldsymbol{\upsilon}' = [\upsilon'[1], \ldots, \upsilon'[K]]^{\rm T}, \qquad \upsilon'[k] = m[k] * 1\,. \tag{33}$$

In the post filtering case, the variables τ and υ are then given as

$$\tau(\mathbf{x}_\mathrm{v}; n) = \Lambda(\boldsymbol{\tau}'(\mathbf{x}_\mathrm{v}), \delta, n), \qquad \upsilon[n] = \Lambda(\boldsymbol{\upsilon}', \delta, n)\,. \tag{34}$$

Note that we can only form the post filtering case in this form if $\Lambda(\mathbf{t}', \delta, n)$ is a linear function for $\mathbf{t}' \in \mathbb{C}^K$.

Proof of convexity of $f(\mathbf{x}_\mathrm{v}; \mathbf{x}_\mathrm{r})$ in (25) w.r.t. $\mathbf{x}_\mathrm{r}$: Following the definition of convexity [23], and furthermore using that $\boldsymbol{\mathcal{A}}(\mathbf{x}_\mathrm{v})(\alpha\mathbf{x}_\mathrm{r} + \beta\tilde{\mathbf{x}}_\mathrm{r}) = \alpha\boldsymbol{\mathcal{A}}(\mathbf{x}_\mathrm{v})\mathbf{x}_\mathrm{r} + \beta\boldsymbol{\mathcal{A}}(\mathbf{x}_\mathrm{v})\tilde{\mathbf{x}}_\mathrm{r}$ we have

$$\begin{aligned} f(\mathbf{x}_\mathrm{v}; \theta\mathbf{x}_\mathrm{r}+(1-\theta)\tilde{\mathbf{x}}_\mathrm{r}) &= \|\theta\,(\boldsymbol{\mathcal{A}}(\mathbf{x}_\mathrm{v})\mathbf{x}_\mathrm{r}-\mathbf{r}_\mathrm{r}) + (1-\theta)(\boldsymbol{\mathcal{A}}(\mathbf{x}_\mathrm{v})\tilde{\mathbf{x}}_\mathrm{r}-\mathbf{r}_\mathrm{r})\|_2^2 \\ &\leq \|\theta\,(\boldsymbol{\mathcal{A}}(\mathbf{x}_\mathrm{v})\mathbf{x}_\mathrm{r}-\mathbf{r}_\mathrm{r})\|_2^2 + \|(1-\theta)(\boldsymbol{\mathcal{A}}(\mathbf{x}_\mathrm{v})\tilde{\mathbf{x}}_\mathrm{r}-\mathbf{r}_\mathrm{r})\|_2^2 \\ &\leq \theta f(\mathbf{x}_\mathrm{v}; \mathbf{x}_\mathrm{r}) + (1-\theta) f(\mathbf{x}_\mathrm{v}; \tilde{\mathbf{x}}_\mathrm{r}) \end{aligned} \tag{35}$$

for all $\mathbf{x}_\mathrm{v} \in \mathbb{R}^{2\times 1}$, $\mathbf{x}_\mathrm{r}, \tilde{\mathbf{x}}_\mathrm{r} \in \mathbb{R}^{4\times 1}$, $\theta \in [0; 1]$ so $f(\mathbf{x}_\mathrm{v}; \mathbf{x}_\mathrm{r})$ is a convex (quadratic) function in $\mathbf{x}_\mathrm{r}$ (for $\mathbf{x}_\mathrm{v}$ fixed). □

For $\mathbf{x}_\mathrm{v}$ we have $\boldsymbol{\mathcal{A}}(\theta\mathbf{x}_\mathrm{v} + (1-\theta)\tilde{\mathbf{x}}_\mathrm{v})\mathbf{x}_\mathrm{r} \neq \theta\boldsymbol{\mathcal{A}}(\mathbf{x}_\mathrm{v})\mathbf{x}_\mathrm{r} + (1-\theta)\boldsymbol{\mathcal{A}}(\tilde{\mathbf{x}}_\mathrm{v})\mathbf{x}_\mathrm{r}$ in general, which indicates that $f(\mathbf{x}_\mathrm{v}; \mathbf{x}_\mathrm{r})$ is not convex in $\mathbf{x}_\mathrm{v}$.

Proof of non-convexity of $f(\mathbf{x}_\mathrm{v}; \mathbf{x}_\mathrm{r})$ in (25) w.r.t. $\mathbf{x}_\mathrm{v}$: To prove non-convexity w.r.t. $\mathbf{x}_\mathrm{v}$, it is sufficient to provide just one simple example where the convexity inequality is violated [23]. Let $N = 2$, $r[0] = r[1] = t[0] = t[1] = 1$, $\mathbf{x}_\mathrm{r} = [1,\, 0,\, 0,\, 0]^\mathrm{T}$ and $\theta = \frac{1}{2}$. We pick $\mathbf{x}_\mathrm{v} = [0,\, 0]^\mathrm{T}$, $\tilde{\mathbf{x}}_\mathrm{v} = [1,\, \frac{1}{2}f_\mathrm{symb}]^\mathrm{T}$. This gives $\theta f(\mathbf{x}_\mathrm{v}; \mathbf{x}_\mathrm{r}) + (1-\theta) f(\tilde{\mathbf{x}}_\mathrm{v}; \mathbf{x}_\mathrm{r}) = \frac{1}{2}\cdot(0^2+0^2) + (1-\frac{1}{2})\cdot(0^2 + |\exp[-1-j\pi]-1|^2) = \frac{1}{2}(1+\exp[-1])^2$ and $f(\theta\mathbf{x}_\mathrm{v}+(1-\theta)\tilde{\mathbf{x}}_\mathrm{v}; \mathbf{x}_\mathrm{r}) = 0^2+(\exp[-\frac{1}{2}-j\frac{1}{2}\pi]-1)^2 = (1+\exp[-1])$. Since $(1+\exp[-1]) > \frac{1}{2}(1+\exp[-1])^2$ the function $f(\mathbf{x}_\mathrm{v}; \mathbf{x}_\mathrm{r})$ is non-convex in $\mathbf{x}_\mathrm{v}$ and the function f is therefore also non-convex. □

We now rewrite the optimization problem by separating the complicated non-convex inducing $\mathbf{x}_\mathrm{v}$ from $\mathbf{x}_\mathrm{r}$ following the idea of (generalized) Benders decomposition [24], [25]. We then obtain

$$\min_{\mathbf{x}} f(\mathbf{x}) = \min_{\mathbf{x}_\mathrm{v};\mathbf{x}_\mathrm{r}} f(\mathbf{x}_\mathrm{v}; \mathbf{x}_\mathrm{r}) = \min_{\mathbf{x}_\mathrm{v}} \min_{\mathbf{x}_\mathrm{r}} f(\mathbf{x}_\mathrm{v}; \mathbf{x}_\mathrm{r}) = \min_{\mathbf{x}_\mathrm{v}} f'(\mathbf{x}_\mathrm{v}) \tag{36}$$

where

$$f'(\mathbf{x}_\mathrm{v}) \;=\; \min_{\mathbf{x}_\mathrm{r}} f(\mathbf{x}_\mathrm{v}; \mathbf{x}_\mathrm{r})\,. \tag{37}$$

From (26) we see that the problem in (37) is an unconstrained linear least squares convex optimization problem with a solution satisfying

$$\boldsymbol{\mathcal{A}}(\mathbf{x}_\mathrm{v})^\mathrm{T}\,\boldsymbol{\mathcal{A}}(\mathbf{x}_\mathrm{v})\,\mathbf{x}_\mathrm{r}^\star(\mathbf{x}_\mathrm{v}) \;=\; \boldsymbol{\mathcal{A}}(\mathbf{x}_\mathrm{v})^\mathrm{T}\,\mathbf{r}_\mathrm{r}\,. \tag{38}$$

C. Optimization strategies

As seen from (18), the problem is to minimize the non-convex real function $f(\mathbf{x}) \geq 0$ depending on six real optimization parameters given by $\mathbf{x}$. With x_3 or x_4 varying we have a non-convex optimization problem. To analyze the problem at hand, we consider a typical example of the function $f'(\mathbf{x}_\mathrm{v})$ shown in Fig. 1. We see that x_4 is very sensitive while even a change of a factor of 30 in x_3 results in almost no change in function values (the lines almost lies on top of each other). The fluctuations also pictures why this function is not convex and warn us that any iterative algorithm can easily get trapped in a local minima.

There are several methods, which can be applied to find a solution to the optimization problem (36). We suggest the following approach although many variants can be developed from this[1].

1) First, make initial estimations $\hat{x}_3$ for x_3, and $\hat{x}_4$ for x_4.
2) In a domain around the initial estimation $\hat{x}_4$, form a grid
$$\tilde{x}_4 \in X_4 = \left\{ \hat{x}_4 + \frac{2lv}{V} \;\middle|\; l = \pm 1, \pm 2, \ldots, \pm\frac{V}{2} \right\} \tag{39}$$
and evaluate the function $f([\hat{x}_3, \tilde{x}_4]^\mathrm{T}, \mathbf{x}_\mathrm{r}^\star([\hat{x}_3, \tilde{x}_4]^\mathrm{T})) = f'([\hat{x}_3, \tilde{x}_4]^\mathrm{T})$ on this grid (we use (38) to obtain the remaining variables). The parameters v and V determine the width and the number of points in the grid, respectively.
3) Since we now have an estimate, which should be fairly close to the global minimum we run an iterative algorithm on all parameters in $\mathbf{x}$ to find the local optimum. The iterative algorithms are started from the U number of points with the smallest objective obtained from step 2). With good initial conditions the local optimum may be the global optimum.

Here we rely on the less sensitive estimate of x_3 and only grid x_4, in which case a finer grid is possible at the same computational cost. The use of Benders decomposition and the observation that x_3 is less sensitive, have brought the grid method from six dimensions to one dimension which gives a reduction in computational complexity from $\mathcal{O}(V^6)$ to $\mathcal{O}(V)$. This is an immense reduction in computation and makes the proposed method computational feasible. Note that since we are dealing with a non-convex problem there are no general tractable algorithm for obtaining an optimal solution or certifying a candidate solution. Hence we have no way to

[1]Note that when we use the Bender decomposition strategy to obtain f', there are a great number of different global optimization strategies we can follow to find a minimizer of f' [31]–[33].

certify the correctness of E_{rms} in problem (6). We can only provide a an approximate solution $\mathbf{x}_{\mathrm{p}}$ with $E(\mathbf{x}_{\mathrm{p}}) \geq E_{\mathrm{rms}}$ where the inequality follows from the fact that (6) is an unconstrained minimization problem. The signals $\mathbf{r}, \mathbf{t}$ and parameters $\mathbf{x}_{\mathrm{p}}$ then form a *certificate* (read: proof) of the obtained value $E(\mathbf{x}_{\mathrm{p}}) \geq E_{\mathrm{rms}}$.

One iterative algorithm which can be applied to refine a candidate solution is the Newtons method, which gives the parameter vector for iteration $i+1 = 1, 2, \ldots, I$ as

$$\mathbf{x}^{(i+1)} \;=\; \mathbf{x}^{(i)} \;-\; \lambda_i \, \mathbf{d}^{(i)}, \quad \text{where } \mathbf{H}(\mathbf{x}^{(i)})\mathbf{d}^{(i)} = \mathbf{g}(\mathbf{x}^{(i)}) \tag{40}$$

and $\mathbf{g}(\mathbf{x}^{(i)})$ is the gradient vector containing the first-order derivatives around the point $\mathbf{x}^{(i)}$, and $\mathbf{H}(\mathbf{x}^{(i)})$ is the Hessian matrix containing the second-order derivatives around the point $\mathbf{x}^{(i)}$, and λ_i is a step size. To determine the step size λ_i we use the Armijo rule [26]. As stopping criteria we use the squared norm of the gradient $\|\mathbf{g}(\mathbf{x}^{(i)})\|_2^2 \leq \epsilon$ with $\epsilon = 10^{-3}$.

V. Initial Conditions

When performing the optimization, several things can be done to speed up the process — and help to find correct results as well. One thing is to carefully select the initial point $\mathbf{x}^{(0)}$ instead of relying on the static selection $\mathbf{x}^{(0)} = [1,\, 0,\, 0,\, 0,\, 0,\, 0]^{\mathrm{T}}$ [20].

If we again consider that the problem is convex provided x_3 and x_4 can be treated as constants (when optimizing for the other parameters), it is only necessary to derive initial conditions for $\mathbf{x}_{\mathrm{v}} = [x_3, x_4]$ and then directly determine $\mathbf{x}_{\mathrm{r}} = [x_1, x_2, x_5, x_6]^{\mathrm{T}}$ by use of (38).

A. Loss factor, x_3

With an estimate $\delta \simeq \hat{\delta}$, we can evaluate $x_3(\hat{\delta}; n; \Delta_n)$ in (11) and use $\Lambda(\mathbf{t}', \hat{\delta}, n) \simeq t[n]$. We then obtain an estimate $\hat{x}_3$ by averaging over all possible n and Δ_n

$$\hat{x}_3 = \frac{-2}{N\,(N+1)} \sum_{n=1}^{N-1} \sum_{\Delta_n=1}^{N-n} \frac{1}{\Delta_n} \ln \left\{ \left| \frac{t[n]}{r[n]} \right| \, \left| \frac{r[n+\Delta_n]}{t[n+\Delta_n]} \right| \right\} . \tag{41}$$

B. Frequency offset, x_4

We will obtain two different frequency offset estimators. A direct approach and an iterative PLL based approach.

1) Direct approach: Lets assume that the origin offset is negligible and we aim for $e_{\mathbf{x}}[n] = 0$, we have from (5) that

$$r[n] \;\simeq\; (x_1 + j\,x_2)t[n]\,\exp[-(x_3 + j\,\kappa\,x_4)\,(n-1)]\,. \tag{42}$$

Taking the ratio $r[n+\Delta_n]/r[n]$ where $n = 1, 2, \ldots, N$ and $\Delta_n = 1, 2, \ldots, N-n$ gives

$$\frac{r[n+\Delta_n]}{r[n]} \;\simeq\; \frac{t[n+\Delta_n]}{t[n]}\,\frac{\exp[-x_3\,\Delta_n]}{\exp\,[j\,\kappa\,x_4\,\Delta_n]}\,. \tag{43}$$

We need to consider that zero crossings symbols of reference or test signals (should that unexpectedly happen) can not be used and must be omitted in (43). Now, knowing an estimate $\hat{x}_3 \simeq x_3$ we can rearrange (43), and average over the entire data set for n with $\Delta_n = 1$ to avoid the Δ_n influence on x_4. This leads to the relation

$$A\,\exp[j\,\psi] \;=\; \exp\,[-j\,\kappa\,x_4] \tag{44}$$

where

$$A\,\exp[j\,\psi] \;=\; \frac{\exp[x_3]}{N-1}\,\sum_{n=1}^{N-1}\frac{r[n+1]}{t[n+1]}\,\frac{t[n]}{r[n]}, \quad |\psi| < \pi\,. \tag{45}$$

So we compute the right hand side of (45) and put this to magnitude/phase form to identify the phase ψ. Then a method to estimate the frequency offset $x_4 \simeq \hat{x}_4$ is

$$\hat{x}_4 \;=\; \frac{-\psi}{\kappa}\,. \tag{46}$$

2) Iterative approach: Another approach is to obtain the estimate through a Phase-Locked Loop (PLL) on the test signal where we track the instantaneous phase. Note that since we are calculating the EVM we also have the reference signal and can hence exploit this in the PLL design. The test signal is modelled as

$$t[n] = \frac{1}{x_1 + jx_2}\,\exp\big[(x_3 + j\kappa\,x_4)(n-1)\big]\,\big(r[n] + e[n] + (x_5 + jx_6)\big)\,. \tag{47}$$

Using the estimate $\hat{x}_3$ and an instantaneous phase $\hat{\phi}[n]$ we then form the following signal

$$g_{\mathrm{PLL}}[n] = t[n]\,\exp\big[-\hat{x}_3\cdot(n-1) - j\hat{\phi}[n]\big]\,\frac{1}{r[n]}\,. \tag{48}$$

Note that if $\hat{x}_3 = x_3$, and we aim for $e[n] = 0$ and $(x_5 + jx_6)$ negligible, we have the signal

$$g_{\mathrm{PLL}}[n] \simeq \frac{1}{x_1 + jx_2}\,\exp\big[j(\kappa\,x_4\cdot(n-1) - \hat{\phi}[n])\big]\,. \tag{49}$$

We then low-pass filter $g_{\mathrm{PLL}}[n]$ to $g'_{\mathrm{PLL}}[n]$ and obtain $\hat{\phi}[n+1] = K \sum_{k=1}^{n} \angle g'_{\mathrm{PLL}}[k]$. In this setup we are not interested in the synthesized frequency offset compensated signal but the frequency offset estimation. We can then run the iterative process from $n = 1, \ldots, N$ and finally obtain the frequency offset estimation $\hat{x}_4$ via the first-order linear least-squares regression problem

$$\underset{\phi_{\mathrm{o}}, \hat{x}_4}{\text{minimize}} \sum_{n=N'}^{N} \left\{ \phi_{\mathrm{o}} + \kappa \, \hat{x}_4 \cdot (n-1) - \hat{\phi}[n] \right\}^2 \tag{50}$$

where ϕ_{o} is a constant offset representing an estimate of the steady state error. The frequency offset estimate is then obtained as a solution to the above problem $\hat{x}_4 = \hat{x}_4^{\star}$. The parameter N' is a time offset to avoid fitting in a time period where the PLL is not locked. We find experimentially that $N' = 15$ is a reasonable choice. We also find experimentially that the iterative approach is the most accurate approach. However, we will use the direct approach for $N \geq 5 \cdot 10^4$ and the iterative approach otherwise since the direct approach is faster and as accurate for large N.

VI. Numerical Examples

A toolbox named `EVMBOX` has been developed and is is freely distributed under the Apache license and can be downloaded at http://www.sparsesampling.com to support the reproducible research paradigm [27]. To support standard specific error vector magnitude (such as EDGE or WLAN) a front-end must be made, which defines certain standard depending settings such as a measurement filter and which parameters $x_1 \cdots x_6$ that are allowed. En example for EDGE is provided in the toolbox.

A. Validation

To validate the procedure a number of simple errors were introduced, which allows for closed form solution of E_{rms}. It obviously does not prove that the method is correct but it does provide some indications for correct behavior.

Gain imbalance and phase error: Suppose the only error present is an imbalance for the in-phase and quadrature-phase gains and we have a the following continuous time RF model

$$s(t) = c_{\mathrm{I}} \, r_{\mathrm{I}}(t) \cos(\omega_0 t) + c_{\mathrm{Q}} \, r_{\mathrm{Q}}(t) \sin(\omega_0 t + \phi) \, . \tag{51}$$

Ideally, $c_{\mathrm{I}} = c_{\mathrm{Q}}$ and $\phi = 0$ but if this is not the case it leads to an EVM larger than zero. In complex baseband and discrete symbol time, (51) corresponds to the test signals

$$\Re\{t[n]\} = c_{\mathrm{I}} \, \Re\{r[n]\} + c_{\mathrm{Q}} \, \sin(\phi) \, \Im\{r[n]\}, \quad \Im\{t[n]\} = c_{\mathrm{Q}} \, \cos(\phi) \, \Im\{r[n]\} \, . \tag{52}$$

Manual derivation of the optimum x_1 and x_2 variables (the rest are zero) leads to

$$x_1^\star = \frac{c_\mathrm{I} + c_\mathrm{Q}\cos(\phi)}{c_\mathrm{I}^2 + c_\mathrm{Q}^2}\,,\; x_2^\star = \frac{c_\mathrm{Q}\sin(\phi)}{c_\mathrm{I}^2 + c_\mathrm{Q}^2}\,. \tag{53}$$

Leading to an error vector magnitude as

$$E_\mathrm{rms} = \sqrt{\frac{(\mu-1)^2 + 4\mu\sin^2(\frac{\phi}{2})}{2(\mu^2+1)}}\,,\; \mu = \frac{c_\mathrm{I}}{c_\mathrm{Q}}\,. \tag{54}$$

The results from applying the proposed technique and (54) are shown in Fig. 2. As observed the agreement between the two computation techniques is excellent and as expected the model can fully compensate for the case $\phi = 0$, $\mu = 1$ where we correctly observe $E_\mathrm{rms} \simeq 0$.

We have not found a setting of v, V from (39), which generates provable good results. We instead rely on certain assumptions and heuristics to try ensuring that $\max_{x_4 \in X_4} |x_4^\star - x_4|$ is sufficiently small. Similar interval conditions are also required in [28]. Generally we observe that if N increases then v can be decreased because the initial estimation becomes more reliable but we need to sample finer since the dip in f' becomes more narrow. However, the decrease in v can usually compensate for the finer sampling in which case V can be kept almost constant.

In the following we assume we have a correctly identified the offset δ, such that we can eliminate the influence from a wrongly estimated $\hat{\delta} \neq \delta$ and then focus on comparison and design choices for the optimization procedures for $\min f(\mathbf{x})$. We generate the test signal using

$$t[n] = \frac{\exp[(\bar{x}_3 + j\kappa\,\bar{x}_4)(n-1)]}{(\bar{x}_1 + j\bar{x}_2)}\big(r[n] + w[n] + (\bar{x}_5 + j\bar{x}_6)\big) \tag{55}$$

for $n = 1, \cdots, N$ where $r[n]$ is a correlated reference signal generated with a moving average model of order 3, MA(3) [29], using a Butterworth LP filter with cuf-off digital frequency 0.05π, $\bar{\mathbf{x}} = [\bar{x}_1, \cdots, \bar{x}_6]$ is the true latent parameters and $\mathbf{w} = [w[1], \cdots, w[N]]^\mathrm{T} \sim \mathcal{N}(0, \sigma^2 I)$.

All timings are conducted with 4x AMD Phenom(tm) II X4 965 Processors, $3400\,\mathrm{Mhz}$, $16\,\mathrm{GB}$ RAM, `MATLAB` version 7.11.0.584 (R2010b) and with a single worker if nothing else is noted.

B. Comparing sampling and an iterative based method

A global optimization method for locating a minimum of a non-convex function such as $f'(\mathbf{x}_\mathrm{v})$ is the Nelder-Mead method [30]. To test this method, we used the `fminsearch` implementation of the Nelder-Mead algorithm in `MATLAB` with the initial conditions $\mathbf{x}_\mathrm{v} = [\hat{x}_3; \hat{x}_4]$ and increased the maximum number of iterations and function evaluations to 10^4. We generated $M = 200$

reference and test signals of length $N = 147$. We denote $\mathbf{x}_{\mathrm{p}}[m]$ and $\mathbf{x}_{\mathrm{NM}}[m]$ the solution for burst m of the proposed and the Nelder-Mead method, respectively. We use 4 `workers` and run both the Nelder-Mead method and the sampling based method parallel across the M bursts using the `parfor` loop in `MATLAB`. We collect the bursts with $\{\mathbf{x}[m]\}_{m=1\cdots M} = \{\mathbf{x}[m]\}$, where we will use the latter notation for convenience. We also define the function $C\big(\{\mathbf{x}[m]\}\big) = |\{m \mid E(\mathbf{x}[m]) \leq E(\bar{\mathbf{x}}[m])\}|$, that is the number of bursts the obtained EVM is better than using the true $\bar{\mathbf{x}}$ which generated the test signal. The reason it is possible to locate an $\mathbf{x}$ with $f(\mathbf{x}) \leq f(\bar{\mathbf{x}})$, is because it is possible to slightly model the noise for finite length signals. The result is shown in Table I, where we also use the joint EVM measure

$$\bar{E}_{\mathrm{rms}}\big(\{\mathbf{x}[m]\}\big) = 100\sqrt{\frac{1}{M}\sum_{k=1}^{M} E^2\big(\mathbf{x}[k]\big)} \quad (\%)\,. \tag{56}$$

Table I shows that the Nelder-Mead method is slightly slower and does not provide as accurate results. Indeed, the Newton method is good for obtaining accurate results. The grid based method followed by Newton optimization locates a point with a better objective in $C(\{\mathbf{x}_{\mathrm{p}}[m]\}) = 200$ bursts out of $M = 200$ compared to the EVM of the parameters that generated the test signal.

C. Comparison w.r.t. parameter sensitivity

A related approach is described in [20], which uses the gradient method with independent step size for respectively $[x_1; x_2]$, $[x_3; x_4]$, and $[x_5; x_6]$, as well as fixed initial conditions $\mathbf{x}_0 = [1,\, 0,\, 0,\, 0,\, 0,\, 0]^{\mathrm{T}}$. We compare this reference approach with the method proposed in the present paper. The difficult part of the optimization problem seems to be estimation of the parameter x_4. To exemplify this we set the true $\bar{\mathbf{x}} = [1,\, -0.5,\, 6.53 \cdot 10^{-4},\, \bar{x}_4,\, 0,\, 0]^{\mathrm{T}}$, $N = 250$ and investigate the obtained E_{rms} values for different $\bar{x}_4$. The result is presented in Fig. 3. We see that the reference gradient method with independent step sizes achieves the same E_{rms} value for some $0 \leq \bar{x}_4 \leq 0.4$. But for $\bar{x}_4 \geq 0.5$ and $\bar{x}_4 \leq -0.1$, the gradient method with fixed initial conditions gets trapped in a local minima with high E_{rms} (since the proposed method finds an estimate with significantly lower EVM). We used $I = 10^5$ iterations for the reference gradient method to ensure full convergence.

D. Comparison w.r.t. accuracy and efficiency

Continuing with the same example as in VI-C, we now fix $x_4 = 0.2$, and analyse the objective versus iteration number. We investigate the relative difference between the objective from the algorithm in [20] at iteration i using the gradient approach and the solution found by the proposed method given by

$$F(i) \;=\; \frac{f(\mathbf{x}_{\mathrm{g}}^{(i)}) - f(\mathbf{x}_{\mathrm{p}})}{f(\mathbf{x}_{\mathrm{p}})} \tag{57}$$

where $\mathbf{x}_{\mathrm{g}}^{(i)}$ is the $\mathbf{x}$-vector for iteration i generated by the gradient based reference method [20], and $f(\mathbf{x}_{\mathrm{p}})$ is the objective of the solution provided by the proposed method. The result is shown in Fig. 4, which indicates that the gradient method with fixed initial conditions requires approximately 3200 iterations to obtain an iterate within 1% accuracy of the proposed method. In our implementation, 3200 iterations of the reference method requires $2.4\,\mathrm{s}$ while the proposed method found a solution in $0.07\,\mathrm{s}$. We see that the gradient based reference method has trouble finding a solution within an accuracy of 0.1%, and never obtains an iterate with a lower objective than that of the proposed method since the relative difference is positive. Note that the solution $\mathbf{x}_{\mathrm{p}}$ provided by the proposed method is not necessarily the solution $\mathbf{x}^{\star}$ to the optimization problem. But we do have $f(\mathbf{x}_{\mathrm{p}}) < f(\bar{\mathbf{x}})$ meaning that not even knowing the true $\bar{\mathbf{x}}$-vector, which generated the test data, yields a lower objective than the proposed method.

E. EDGE

Using a cascode power amplifier, a nonlinear AM-AM/AM-PM model was developed for this device using PNA-X measurements [34]. Only the forward $S21$ transmission parameter was included in the model for simplicity (input reflection was below -10 dB). The response to an EDGE input signal was then computed based on a cubic spline interpolation of the AM-AM/AM-PM model data. It was verified that the signal was in-bound of the model. The EDGE standard calls for post filtering [3], but we also calculate the pre filtering case for comparison. The EDGE standard requirement was implemented: 1) compensation using all parameters $x_1 \cdots x_6$ is allowed 2) the required measurement filter and 3) the linear function $t[n] = \Lambda(\mathbf{t}', \delta, n) = t'[(n-1)\Theta + \delta + 1]$. The reference and test signals were then passed to `EVMBOX`, which computed the 10 EVM points for different P_{in} across $M = 200$ burst using $V = 10$ grid points and refines $U = 2$ grid points using Newtons method. This gives in total $10M = 2 \cdot 10^3$ time

offset estimations, $10VM = 2 \cdot 10^4$ least-squares problems and $10MU = 4 \cdot 10^3$ refinements of the solution using Newtons methods. The post filtering solution was obtained in $\simeq 8$ minutes where we used 4 `workers` and a `parfor` loop in `MATLAB` to calculate the grid in (39). The pre filtering solution was obtained in $\simeq 3$ minutes and is faster since the measurement filter is applied significantly fewer times. For pre filtering it is just necessary to filter the test and reference signal once for each burst before optimization, such that the measurement filter is applied $\mathcal{O}(M)$ times. For post filtering, it is necessary to apply the measurement filter every time we form the matrix $\boldsymbol{\mathcal{A}}(\mathbf{x}_{\mathrm{v}})$, see (32), such that every function evaluation on the grid requires filtering with the measurement filter. Also, it is necessary to apply the measurement filter for calculating the gradient and Hessian because of (32). Post and pre filtering then requires $\mathcal{O}(M(V+U))$ and $\mathcal{O}(M)$ filtering operations, respectively. We observe that there is a reasonable agreement between the measured and `EVMBOX` simulated EVM measures. Some disagreement due to only including forward transmission in the model must be expected.

VII. Conclusions

This paper presents an approach to determine error vector magnitude (EVM) as described in several standards. The EVM metric is used to describe signal quality in communication systems. The paper analyzes the underlying optimization problem of EVM and proves the problem to be non-convex, which has consequences for suitable choices of optimization techniques. A possible frequency offset and/or power reduction over time is the cause of non-convexity. The optimization problem is tricky as in particular the frequency offset between reference and test signals means that lots of local minima exist with a deep and narrow global minimum. Without caution this global minimum may easily be overlooked leading to incorrect result. Examples also show that the error vector magnitude may be significantly different when comparing local minima and the global minima. We present an optimization procedure where we first carefully estimate the two non-convex causing optimization variables based on the full data set available. Based on these initial estimates we sample the domain of the non-convex optimization variables in a region around the estimates. For each point in this grid we can solve the remaining convex problem by standard techniques leading to an efficient and robust procedure. The final step is to refine the best points by using the Newton method based on analytically derived gradient vector and Hessian matrix. Several examples and measurements of a nonlinear amplifier are shown to illustrate the

proposed method and comparison with other techniques are also included.

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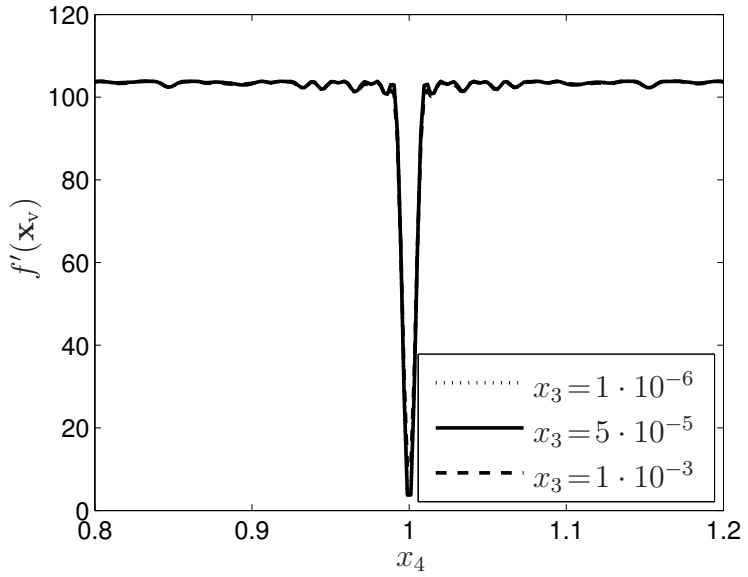

Fig. 1. Typical example of $f'(\mathbf{x}_\mathrm{v})$ versus x_4 for 3 values of x_3. $N = 10^3$. We show a wide range of x_4 values to visualize the behaviour of the function $f'(\mathbf{x}_\mathrm{v})$.

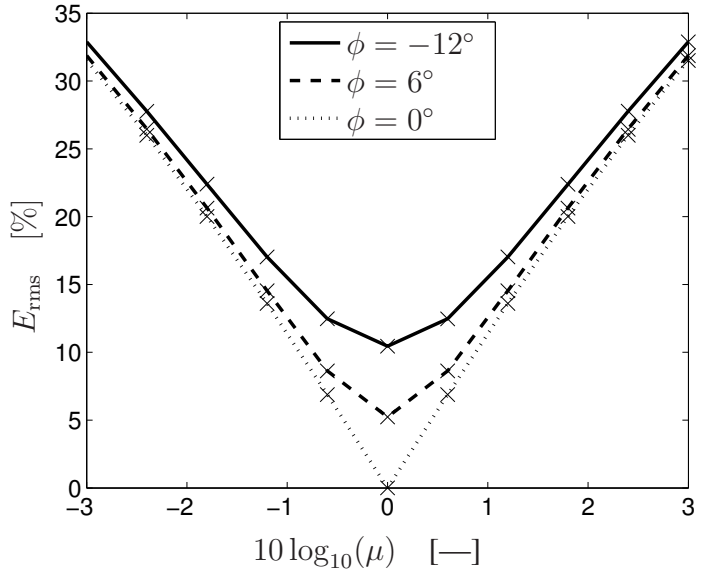

Fig. 2. ($\times$) Simulated E_rms obtained from the proposed method and (lines) theoretically evaluated using (54). $N = 10^5, M = 10$.

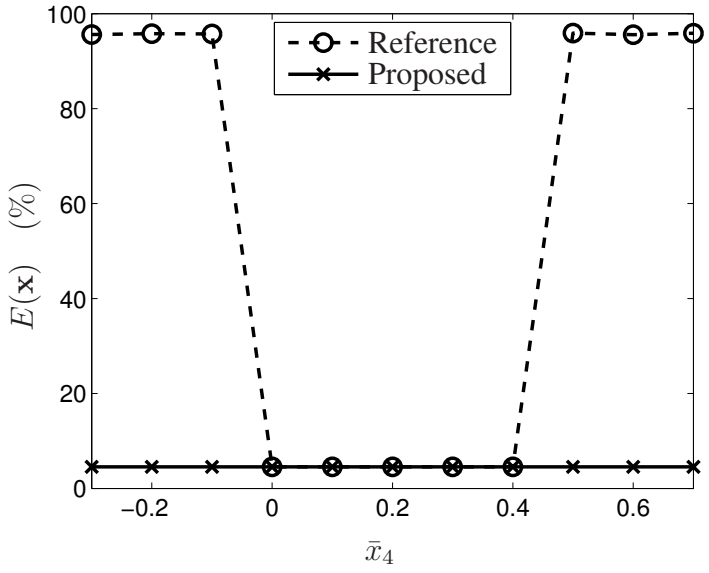

Fig. 3. $E(\mathbf{x})$ versus $\bar{x}_4$ obtained by use of the reference method (see [20]) and the proposed method. $N = 250$.

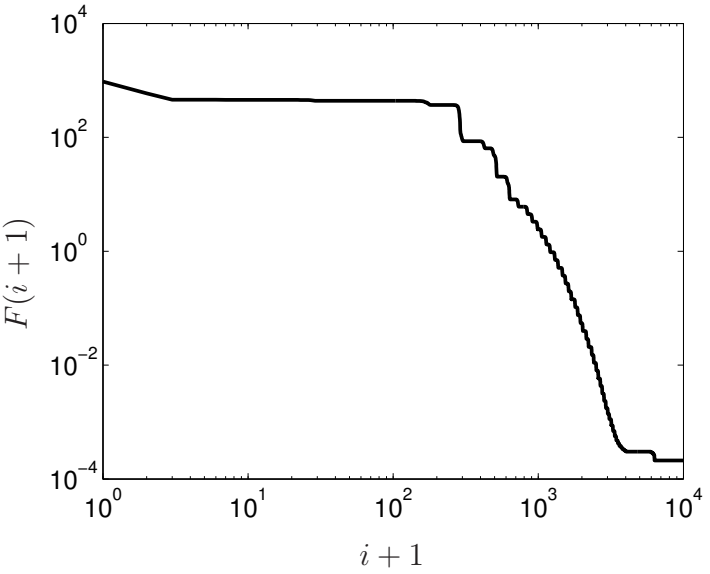

Fig. 4. $F(i)$ from (57), showing the relative difference between the gradient method [20] and the proposed method, versus iteration number i for $\bar{x}_4 = 0.2$, $N = 250$. The iterates are shifted one iteration to ensure proper plotting in the log domain.

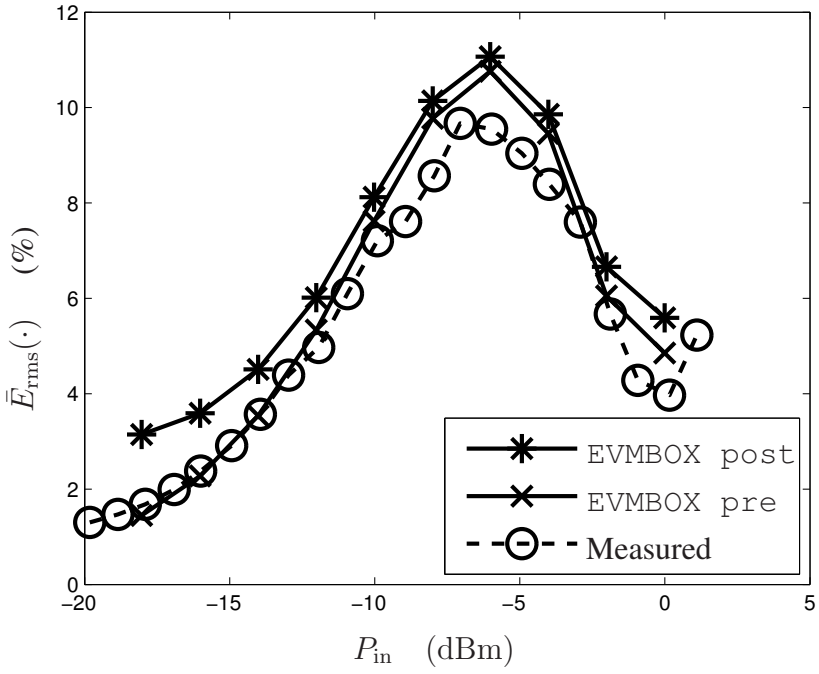

Fig. 5. Measured and simulated (EVMBOX) RMS EVM measures versus average input power P_{in}.

—	Time [s]	$\bar{E}_{\mathrm{rms}}\big(\{\mathbf{x}[m]\}\big)$	$C\big(\{\mathbf{x}[m]\}\big)$
Sampled $\mathbf{x}_{\mathrm{p}}[m]$	2.26	8.75	200
Nelder-Mead $\mathbf{x}_{\mathrm{NM}}[m]$	4.47	8.84	194

TABLE I

EXECUTION TIME, JOINT EVM, AND COMPARISON MEASURE FOR THE PROPOSED SAMPLED AND THE NELDER-MEAD METHOD. THE GENERATED DATA GIVES $\bar{E}_{\mathrm{rms}}(\{\bar{x}[m]\}) = 8.88$.