Bisimulation for Higher-Dimensional Automata. A Geometric Interpretation

Fahrenberg, Ulrich

Publication date:
2005

Document Version
Publisher's PDF, also known as Version of record

Link to publication from Aalborg University

Citation for published version (APA):
Fahrenberg, U. (2005). Bisimulation for Higher-Dimensional Automata. A Geometric Interpretation. Dept. of Mathematical Sciences: Aalborg Universitetsforlag. Research Report Series, No. R-2005-01

AALBORG UNIVERSITY

Bisimulation for Higher-Dimensional Automata. A Geometric Interpretation

by

Ulrich Fahrenberg

R-2005-01

January 2005

Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 96 35 80 80 ▪ Telefax: +45 98 15 81 29
URL: http://www.math.aau.dk

ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

Bisimulation for Higher-Dimensional Automata
A Geometric Interpretation

Ulrich Fahrenberg

Abstract

We show how parallel composition of higher-dimensional automata (HDA) can be expressed categorically in the spirit of Winskel & Nielsen. Employing the notion of computation path introduced by van Glabbeek, we define a new notion of bisimulation of HDA using open maps. We derive a connection between computation paths and carrier sequences of dipaths and show that bisimilarity of HDA can be decided by the use of geometric techniques.

Keywords: Higher-dimensional automata, bisimulation, open maps, directed topology, fibrations.

1 Introduction

In his invited talk at the 2004 Express workshop, van Glabbeek [12] places higher-dimensional automata (HDA) on top of a hierarchy of models for concurrency. In this article we develop a categorical framework for expressing constructions on HDA, building on work by Goubault in [13, 14].

Following up on a concluding remark in [14], we introduce a notion of bisimulation of HDA, both as a relation and using open maps[1] [20]. Our notion differs from the ones introduced by van Glabbeek [11] and Cattani-Sassone [5].

Employing recent developments by Fajstrup [9], we show that bisimilarity of HDA is equivalent to a certain dipath-lifting property, which can be attacked using (directed) homotopy techniques. This confirms a prediction from [14].

This report is the long version of a paper [7] accepted for presentation at the 2005 FOSSACS conference held in Edinburgh.

The author is indebted to Eric Goubault and Emmanuel Haucourt for many valuable discussions during his visit at CEA in Paris, and to Lisbeth Fajstrup and Martin Raussen at the Department of Mathematical Sciences in Aalborg.

[1] Note that there is a clash of terminology here: Openness in the sense of [20] has nothing to do with being open in the topological sense.

2 Cubical Sets

Cubical sets were introduced by Serre in [23] and have a variety of applications in algebraic topology, both in homology, cf. [21], and in homotopy theory, cf. [3, 6, 19]. Compared to the more well-known simplicial sets, they have the distinct advantage that they have a natural sense of (local) direction induced by the order on the unit interval. This makes them well-suited for applications in concurrency theory, cf. [10].

A *precubical set* is a graded set $X = \{X_n\}_{n\in\mathbb{N}}$ together with mappings $\delta^\nu_{i(n)} : X_n \to X_{n-1}$, $i = 1, \dots, n$, $\nu = 0, 1$, satisfying the *precubical identity*

$$\delta^\nu_i \delta^\mu_j = \delta^\mu_{j-1}\delta^\nu_i \quad (i < j) \tag{1}$$

These are called *face maps*, and if $x = \delta^{\nu_1}_{i_1} \cdots \delta^{\nu_n}_{i_n} y$ for some cubes x, y and some (possibly empty) sequences of indices, then x is called a *face* of y. If all $\nu_i = 0$, x is said to be a *lower* face of y; if all $\nu_i = 1$, x is an *upper* face of y.

As above, we shall omit the subscript (n) in $\delta^\nu_{i(n)}$ whenever possible. Elements of X_n are called *n-cubes*.

A *cubical set* is a precubical set X together with mappings $\varepsilon_{i(n)} : X_n \to X_{n+1}$, $i = 1, \dots, n+1$, such that

$$\varepsilon_i\varepsilon_j = \varepsilon_{j+1}\varepsilon_i \quad (i \le j) \qquad\qquad \delta^\nu_i\varepsilon_j = \begin{cases} \varepsilon_{j-1}\delta^\nu_i & (i < j) \\ \varepsilon_j\delta^\nu_{i-1} & (i > j) \\ \mathrm{id} & (i = j) \end{cases} \tag{2}$$

These are called *degeneracies*, and equations (1) and (2) together form the *cubical identities*.

A *cubical set with symmetries* is a cubical set with extra mappings $\tau_{i(n)} : X_n \to X_n$, $n \ge 2$, $i = 1, \dots, n-1$, encoding an action of the symmetric groups $X_n \times S_n \to X_n$ "permuting the coordinates." The following constraints apply:

$$\begin{aligned} \tau_i\tau_i &= \mathrm{id} \\ (\tau_i\tau_{i+1})^3 &= \mathrm{id} \\ \tau_i\tau_j &= \tau_j\tau_i \quad (i \ne j \pm 1) \end{aligned} \qquad \delta^\nu_i\tau_j = \begin{cases} \delta^\nu_i & (i \ne j, j+1) \\ \delta^\nu_{i+1} & (i = j) \\ \delta^\nu_{i-1} & (i = j+1) \end{cases}$$
$$\tau_i\varepsilon_j = \begin{cases} \varepsilon_{i+1} & (j = i) \\ \varepsilon_i & (j = i+1) \\ \varepsilon_j\tau_i & (j \ne i, i+1) \end{cases} \tag{3}$$

The standard example of a cubical set with symmetries is the singular cubical complex of a topological space, cf. [21]: If X is a topological space, let $S_n X = \mathsf{Top}(I^n, X)$, the set of all continuous maps $I^n \to X$, where I is the unit interval. If the maps δ_i^ν, ε_i, τ_i are given by

$$\begin{aligned} \delta_i^\nu f(t_1, \dots, t_{n-1}) &= f(t_1, \dots, t_{i-1}, \nu, t_i, \dots, t_{n-1}) \\ \varepsilon_i f(t_1, \dots, t_n) &= f(t_1, \dots, \hat{t}_i, \dots, t_n) \\ \tau_i f(t_1, \dots, t_n) &= f(t_1, \dots, t_{i-1}, t_{i+1}, t_i, t_{i+2}, \dots, t_n), \end{aligned}$$

(the notation $\hat{t}_i$ means that t_i is omitted) then $SX = \{S_n X\}$ is a cubical set with symmetries.

Morphisms of (pre)cubical sets (with symmetries) are required to commute with the structure maps, i.e. if X, Y are two (pre)cubical sets (with symmetries), then a morphism $f : X \to Y$ is a sequence of mappings $f = \{f_n : X_n \to Y_n\}$ that fulfill the respective subset of the equations

$$\delta_i^\nu f_n = f_{n-1} \delta_i^\nu \qquad \varepsilon_i f_n = f_{n+1} \varepsilon_i \qquad \tau_i f_n = f_n \tau_i$$

This defines three categories, pCub, Cub, and Cub_τ, all of which are presheaf categories over certain small categories of *elementary cubes*, cf. [18], hence they are Cartesian closed, complete, and cocomplete.

The forgetful functors

$$\mathsf{Cub}_\tau \longrightarrow \mathsf{Cub} \longrightarrow \mathsf{pCub}$$

have left adjoints, providing us with "free" functors in the opposite direction, which we in both cases denote by F.

A (pre)cubical set (with symmetries) $X = \{X_n\}$ is said to be k-dimensional if $X_n = \emptyset$ for $n > k$. The full subcategories of k-dimensional objects in our three cubical categories are denoted pCub^k, Cub^k, and Cub_τ^k, respectively. The free-forgetful adjunctions above pass to the k-dimensional categories.

3 Shells and Tori

The *total boundary* of an element $x \in X_n$ of a precubical set X is the $2n$-tuple

$$\partial x = (\delta_1^0 x, \delta_1^1 x, \dots, \delta_n^0 x, \delta_n^1 x)$$

The precubical identity (1) implies some relations between the faces of the elements in ∂x; in general $2n$-tuples of $(n-1)$-cubes satisfying these face relations are called *shells*. So a shell is a $2n$-tuple of $(n-1)$-cubes which could be filled in by an n-cube; if one exists, it is called a *filler* of the shell. To sum up, the set of $(n-1)$-shells in X is

$$\Box X_{n-1} = \big\{ (s_1^0, s_1^1, \dots, s_n^0, s_n^1) \in X_{n-1}^{2n} \mid \\ \delta_i^\nu s_j^\mu = \delta_{j-1}^\mu s_i^\nu \text{ for all } 1 \le i < j \le n-1,\ \nu, \mu = 0, 1 \big\}$$

We can treat shells as (hollow) cubes and define face maps, degeneracies, and symmetries involving $\square X_{n-1}$ and X_{n-1} as follows:

$$\begin{gathered}\tilde{\delta}_i^\nu(s_1^0,\dots,s_n^1) = s_i^\nu \qquad \tilde{\varepsilon}_i x = \partial\varepsilon_i x \\ \tilde{\tau}_i(s_1^0,\dots,s_n^1) = (s_1^0,\dots,s_{i+1}^0,s_{i+1}^1,s_i^0,s_i^1,\dots,s_n^1)\end{gathered} \tag{4}$$

Also, we can extend a mapping $f : X_{n-1} \to Y_{n-1}$ to $\tilde{f} : \square X_{n-1} \to \square Y_{n-1}$ by $\tilde{f}(s_1^0,\dots,s_n^1) = \big(f(s_1^0),\dots,f(s_n^1)\big)$. Noting that the expression $\partial\varepsilon_i x$ above can be resolved using the identities on $\delta_i^\nu\varepsilon_j$ in (2), we have defined functors $\square_n : \mathsf{pCub}^{n-1} \to \mathsf{pCub}^n$, $\square_n : \mathsf{Cub}^{n-1} \to \mathsf{Cub}^n$, $\square_n : \mathsf{Cub}_\tau^{n-1} \to \mathsf{Cub}_\tau^n$ for all $n \in \mathbb{N}_+$. These *coskeleton* functors are in fact right adjoints to the truncation functors $\mathsf{pCub}^n \to \mathsf{pCub}^{n-1}$, $\mathsf{Cub}^n \to \mathsf{Cub}^{n-1}$, respectively $\mathsf{Cub}_\tau^n \to \mathsf{Cub}_\tau^{n-1}$, cf. [3].

We record the following trivial consequence of the definition:

Lemma 1 *The functors $\square_n$ defined above are full and faithful for all $n \in \mathbb{N}_+$.*

Proof: Faithfulness is trivial. To prove fullness, we notice that a shell is uniquely determined by its boundary. Hence if $f : \square_n X \to \square_n Y$, and $s = (s_1^0,\dots,s_n^1) \in \square X_{n-1}$, then

$$f(s) = \big(\delta_1^0 f(s),\dots,\delta_n^1 f(s)\big) = \big(f(\delta_1^0 s),\dots,f(\delta_n^1 s)\big) = \big(f(s_1^0),\dots,f(s_n^1)\big)$$

Thus f is determined by its restriction to a map $X \to Y$. $\square$

Given a precubical set X and $x \in X_n$, say that x is a *torus* if $\delta_i^0 x = \delta_i^1 x$ for all $i = 1,\dots,n$. Applying the definition to shells, we arrive at the set of torus shells $\circledcirc X_{n-1} \subseteq \square X_{n-1}$.

Note that a degenerate shell is not a torus in general, hence the structure $\circledcirc_n X = (\circledcirc X_{n-1}, X_{n-1},\dots,X_0)$ is only precubical. But if $x \in X_{n-1}$ is itself a torus, all $\varepsilon_i x$ are again tori. Hence if we start with a set X_1 of loops, for example, we can iterate the torus shell construction, arriving at the *free ω-torus* on X_1, $\circledcirc^\omega X_1$. Note that there are natural symmetries in $\circledcirc^\omega X_1$, making it an object of Cub_τ.

4 Product and Tensor Product

The *product* of two (pre)cubical sets (with symmetries) is given by

$$(X \times Y)_n = X_n \times Y_n$$

with face maps, degeneracies, and symmetries defined component-wise. This is a product in the categorical sense. A *cubical relation* between cubical sets X, Y is a cubical subset of the product $X \times Y$.

The *tensor product* of two *precubical* sets $Z = X \otimes Y$ is given by

$$Z_n = \bigsqcup_{p+q=n} X_p \times Y_q$$

with face maps

$$\delta_i^\alpha(x,y) = \begin{cases} (\delta_i^\alpha x, y) & (i \le p) \\ (x, \delta_{i-p}^\alpha y) & (i \ge p+1) \end{cases} \qquad (x,y) \in X_p \times Y_q$$

The category Cub inherits this tensor product, however some identifications have to be made to get well-defined degeneracy maps, cf. [4]. The tensor product of two *cubical* sets $Z = X \otimes Y$ is then given by

$$Z_n = \Big(\bigsqcup_{p+q=n} X_p \times Y_q \Big) \Big/ {\sim_n}$$

where $\sim_n$ is the equivalence relation generated by, for all $(x,y) \in X_r \times Y_s$, $r+s = n-1$, letting $(\varepsilon_{r+1} x, y) \sim_n (x, \varepsilon_1 y)$. If $x \otimes y$ denotes the equivalence class of $(x,y) \in X_p \times Y_q$ under $\sim_n$, the face maps and degeneracies of Z are given by

$$\delta_i^\alpha(x \otimes y) = \begin{cases} \delta_i^\alpha x \otimes y & (i \le p) \\ x \otimes \delta_{i-p}^\alpha y & (i \ge p+1) \end{cases} \qquad \varepsilon_i(x \otimes y) = \begin{cases} \varepsilon_i x \otimes y & (i \le p+1) \\ x \otimes \varepsilon_{i-p} y & (i \ge p+1) \end{cases}$$

5 Transition Systems

We shall construct our category of higher-dimensional automata as a special arrow category in Cub. To warm up, we include a section on how *transition systems* can be understood as an arrow category in Cub^1, the category of *digraphs*. Though our exposition differs considerably from the standard one, see e.g. [24], the end result is basically the same.

A *digraph* is a 1-dimensional cubical set, i.e. a pair of sets (X_1, X_0) together with face maps $\delta^0, \delta^1 : X_1 \to X_0$ and a degeneracy mapping $\varepsilon = \varepsilon_1 : X_0 \to X_1$ such that $\delta^0 \varepsilon = \delta^1 \varepsilon = \mathrm{id}$. Morphisms of digraphs (X_1, X_0), (Y_1, Y_0) are thus mappings $f = (f_1, f_0)$ commuting with the face and degeneracy mappings. A *predigraph* is a 1-dimensional precubical set. Note that we allow both loops and multiple edges in our digraphs.

The category of digraphs has a terminal object $*$ consisting of a single vertex and the degeneracy edge on that vertex. A *transition system* is a digraph which is freely generated by a predigraph together with a specified initial point, hence the category of transition systems is $\langle * \downarrow F\mathsf{pCub}^1 \rangle$, the comma category of digraphs freely generated by predigraphs under $*$. In the spirit of [24], passing from a predigraph to the digraph freely generated by it means that we add *idle loops* to each vertex, hence allowing for transition system morphisms which collapse transitions.

As for labeling transition systems, we note that there is an isomorphism between the category of finite sets and the full subcategory of pCub^1 induced by finite one-point predigraphs, given by mapping a finite set Σ to the one-point predigraph with edge set Σ. Identifying finite sets with the digraphs freely generated by their associated predigraphs, we define a *labeled transition system* over Σ to be a digraph morphism $\lambda : \langle * \downarrow F\mathsf{pCub}^1 \rangle \to \Sigma$ which is induced by a predigraph morphism. This last convention is to ensure that idle loops are labeled with the *idle label* $\varepsilon *$.

Say that a morphism $\lambda \in \mathsf{Cub}^1$ is *non-contracting* if $\lambda a = \varepsilon *$ implies $a = \varepsilon \delta^0 a$ for all edges a, and note that if the source and target of λ are freely generated by precubical sets, then λ is non-contracting if and only if it is in the image of the free functor $\mathsf{pCub}^1 \to \mathsf{Cub}^1$.

For morphisms between labeled transition systems we need to allow functions that map labels to "nothing," i.e. *partial* alphabet functions. The category of finite sets with partial mappings is isomorphic to the full subcategory $\mathsf{\Sigma}$ of Cub^1 induced by digraphs freely generated by finite one-point predigraphs. Hence we can define the category of labeled transition systems to be the *non-contracting comma-arrow category* $\langle * \downarrow F\mathsf{pCub}^1 \rightrightarrows \mathsf{\Sigma} \rangle$, with objects pairs of morphisms—the second one non-contracting

$$* \longrightarrow X \Longrightarrow \Sigma$$

and morphisms pairs of arrows making the following square commute:

$$\begin{array}{ccc} * & = & * \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 \\ \Downarrow & & \Downarrow \\ \Sigma_1 & \longrightarrow & \Sigma_2 \end{array}$$

We shall always visualise non-contracting morphisms by double arrows.

Note that our transition systems have the special feature that there can be more than one transition with a given label between a pair of edges; in the terminology of [24] they are not *extensional.* Except for that, our definition is in accordance with the standards.

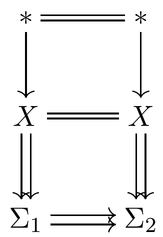
To express parallel composition of transition systems, we follow the approach of [24] and use a combination of product, relabeling and restriction. In our context, the product of two transition systems $* \to X_1 \to \Sigma_1$, $* \to X_2 \to \Sigma_2$ is the transition system $* \longrightarrow X_1 \times X_2 \xrightarrow{\lambda} \Sigma_1 \times \Sigma_2$, where the arrow λ is given by the universal property of the product $\Sigma_1 \times \Sigma_2$. We note that, indeed, the product of two one-point digraphs with edge sets Σ_1

respectively Σ_2 is again a one-point digraph, with edge set

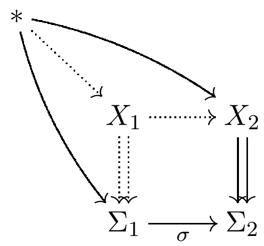
$$\{(a,b),(a,\varepsilon*),(\varepsilon*,b) \mid a \in \Sigma_1, b \in \Sigma_2\}$$

One easily shows λ to be non-contracting, and the so-defined product is in fact the categorical product in the category $\langle * \downarrow F\mathsf{pCub}^1 \rightrightarrows \Sigma\rangle$.

A relabeling of a transition system is a non-contracting alphabet morphism under the identity, i.e. an arrow in $\langle * \downarrow F\mathsf{pCub}^1 \rightrightarrows \Sigma\rangle$ of the form



Restriction of transition systems is defined using pullbacks; given a transition system $* \to X_2 \to \Sigma_2$ and a mapping $\sigma : \Sigma_1 \to \Sigma_2$, we define the restriction of X_2 to Σ_1 by the pullback



where the mapping $* \to \Sigma_1$ is uniquely determined as Σ_1 is a one-point digraph. It is not difficult to show that the so-defined morphism $X_1 \to \Sigma_1$ is in fact non-contracting.

6 Higher-Dimensional Automata

The category Cub has a terminal object $*$ consisting of a single point and all its higher-dimensional degeneracies. The category of higher-dimensional automata is the comma category $\langle * \downarrow F\mathsf{pCub}\rangle$, with objects cubical sets freely generated by precubical sets with a specified initial 0-cube.

For labeling HDA, we follow the approach laid out in [13, 14]. We assume the finite set Σ of labels to be *totally ordered* and define a precubical set $!\Sigma'$ as follows: $!\Sigma'_0 = \{*\}$, $!\Sigma'_n$ is the set of (not necessarily strictly) increasing sequences of length n of elements of Σ, and

$$\delta^\alpha_{i(n)}(x_1,\dots,x_n) = (x_1,\dots,\hat{x}_i,\dots,x_n)$$

Then we let $!\Sigma$ be the free cubical set on $!\Sigma'$.

Lemma 2 *Given a finite set Σ, then the free cubical set with symmetries on $!\Sigma$ is isomorphic to the free ω-torus $\circledcirc^\omega\Sigma$.*

Proof: Passing from the precubical set $!\Sigma'$ to the cubical set $!\Sigma$ amounts to introducing sequences of symbols $(x_1, \ldots, x_n)$ where some of the x_i have been replaced by the special symbol $\perp$, and to define

$$\varepsilon_i(x_1, \ldots, x_n) = (x_1, \ldots, x_{i-1}, \perp, x_i, \ldots, x_n)$$

Passing to the free cubical set *with symmetries* amounts to allow all permutations of such sequences $(x_1, \ldots, x_n)$, with τ_i being the permutation $(\ldots, x_i, x_{i+1}, \ldots) \mapsto (\ldots, x_{i+1}, x_i, \ldots)$.

Mapping sequences $(x_1, \ldots, x_n)$ to $(x_1, x_1, \ldots, x_n, x_n)$, with the obvious operations on the mappings δ_i^ν, ε_i, τ_i, we have the claimed isomorphism. $\square$

Define a morphism $f : X \to Y$ of cubical sets to be *non-contracting* if $f(x) = \varepsilon_i \delta_i^0 f(x)$ implies $x = \varepsilon_i \delta_i^0 x$ for all $x \in X_n$, $n \in \mathbb{N}$, $i = 1, \ldots, n$. Note again that if the cubical sets X, Y are freely generated by precubical sets, then a morphism $f : X \to Y$ is non-contracting if and only if it is the image of a precubical morphism under the free functor.

Let $\mathbf{!\Sigma}$ be the full subcategory of Cub induced by the cubical sets $!\Sigma$ as above. By Lemmas 2 and 1, $\mathbf{!\Sigma}$ is isomorphic to the category $\mathbf{\Sigma}$ of the preceding section, hence $\mathbf{!\Sigma}$ is isomorphic to the category of finite sets and partial (and not necessarily order-preserving) mappings.

The category of *labeled higher-dimensional automata* is then defined to be $\langle * \downarrow F\mathsf{pCub} \rightrightarrows \mathbf{!\Sigma} \rangle$, with objects $* \longrightarrow X \Longrightarrow !\Sigma$ and morphisms commutative diagrams

$$\begin{array}{ccc} * & = & * \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_2 \\ \Downarrow & & \Downarrow \\ !\Sigma_1 & \longrightarrow & !\Sigma_2 \end{array}$$

Note that by this construction, the label of an n-cube is the ordered n-tuple of the labels of all its 1-faces.

7 Constructions on HDA

As in [13], we replace the product of transition systems by the *tensor product* of higher-dimensional automata. The tensor product of two HDA $* \to X_1 \xrightarrow{\lambda} !\Sigma_1$, $* \to X_2 \xrightarrow{\mu} !\Sigma_2$ is defined to be

$$* \longrightarrow X_1 \otimes X_2 \xrightarrow{\lambda \otimes \mu} !\Sigma_1 \otimes !\Sigma_2$$

The following lemma, where $\Sigma_1 \uplus \Sigma_2$ denotes the disjoint union of Σ_1 and Σ_2 with the order induced by declaring $\Sigma_1 < \Sigma_2$, ensures that this in in fact a HDA:

Lemma 3 *Given alphabets Σ_1, Σ_2, then $!\Sigma_1 \otimes !\Sigma_2 = !(\Sigma_1 \uplus \Sigma_2)$.*

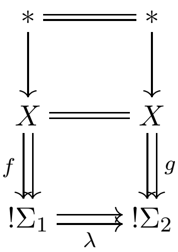
Proof: Starting with the precubical sets, we have

$$!(\Sigma_1 \uplus \Sigma_2)'_n = \{(x_1, \dots, x_n) \mid \exists\, k : x_1 < \dots < x_k \in \Sigma_1, x_{k+1} < \dots < x_n \in \Sigma_2\}$$
$$(!\Sigma'_1 \otimes !\Sigma'_2)_n = \bigsqcup_{k=0}^{n} \{(x_1, \dots, x_n) \mid x_1 < \dots < x_k \in \Sigma_1, x_{k+1} < \dots < x_n \in \Sigma_2\}$$

so these sets are the same. Also the face maps are easily seen to be identical, hence the precubical sets $!(\Sigma_1 \uplus \Sigma_2)'$ and $!\Sigma'_1 \otimes !\Sigma'_2$ are isomorphic. To see that equality also holds for their *cubical* cousins, we only have to note that the relation $(\varepsilon_{r+1}x, y) \sim_n (x, \varepsilon_1 y)$ in defining the tensor product does not actually *identify* anything in this case, as

$$(\varepsilon_{r+1}x, y) = (x, \varepsilon_1 y) = (x_1, \dots, x_r, \bot, y_1, \dots, y_s) \qquad \Box$$

For relabeling HDA we use non-contracting morphisms under the identity, and we note that if g is defined by the diagram

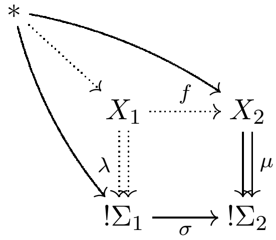


then non-contract ability of g follows from f and λ being non-contracting.

If we want to express the tensor product of two HDA $* \to X \to !\Sigma_1$, $* \to Y \to !\Sigma_2$ with *non-disjoint* alphabets Σ_1, Σ_2, we can do so by following the tensor product above with a relabeling $!\Sigma_1 \otimes !\Sigma_2 \to !(\Sigma_1 \cup \Sigma_2)$ induced by the natural projection $\Sigma_1 \uplus \Sigma_2 \to \Sigma_1 \cup \Sigma_2$ (which is not necessarily order-preserving). This projection is a *total* alphabet morphism, hence the relabeling map is indeed non-contracting.

For restrictions we again use pullbacks:

Proposition 4 *Given a higher-dimensional automaton $* \to X_2 \to !\Sigma_2$ and an injective mapping $!\Sigma_1 \to !\Sigma_2$, then $* \to X_1 \to !\Sigma_1$ as defined by the* *pullback diagram*



is again a higher-dimensional automaton.

The arrow $* \to !\Sigma_1$ is uniquely determined as $!\Sigma_1$ has only one cube in dimension zero. We will need the injectivity of σ later, to show that our to-be-defined notion of bisimilarity is respected by restrictions.

Proof: We need to show that λ is non-contracting. Let $x \in X'_n \in X'$ and $i \in \{0, \dots, n\}$, and assume that $\lambda x = \varepsilon_i \delta_i^0 \lambda x$. Then

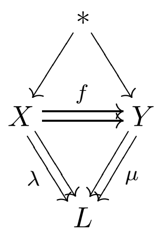
$$\mu f x = \sigma \lambda x = \varepsilon_i \delta_i^0 \sigma \lambda x = \varepsilon_i \delta_i^0 \mu f x$$

and as μ is non-contracting, this implies $f(x) = \varepsilon_i \delta_i^0 f(x)$.

Let Z be the cubical set generated by a single n-cube $Z_n = \{z\}$, and let $h_1, h_2 : Z \to X'$ be the cubical morphisms induced by $h_1 z = x$, $h_2 z = \varepsilon_i \delta_i^0 x$. Then $\lambda h_1 z = \lambda h_2 z = \varepsilon_i \delta_i^0 \lambda x$ and $f h_1 z = f h_2 z = \varepsilon_i \delta_i^0 f x$, hence by the universal property of the pullback, $h_1 = h_2$, whence $x = \varepsilon_i \delta_i^0 x$. □

8 Bisimulation

In this section we fix a labeling cubical set L and work in the non-contracting double comma category $\langle * \downarrow F\mathsf{pCub} \downdownarrows L \rangle$ of HDA over L. The morphisms



in this category respect labelings, hence they are non-contracting themselves: If $f(x) = \varepsilon_i \delta_i^0 f(x)$ for some $x \in X$ and some i, then $\lambda(x) = \mu(f(x)) = \varepsilon_i \delta_i^0 \lambda(x)$ and thus $x = \varepsilon_i \delta_i^0 x$.

A *computation path*, cf. [11], in a precubical set X is a finite sequence $(x_1, \dots, x_n)$ of cubes of X such that for each $k = 1, \dots, n-1$, either $x_k = \delta_i^0 x_{k+1}$ or $x_{k+1} = \delta_i^1 x_k$ for some i. A computation path $(x_1, \dots, x_n)$ is said to be *acyclic* if there are no other relations between the x_i than the ones above. A *rooted* computation path in a HDA $* \xrightarrow{i} X$ is a computation path $(i*, \ldots, x_n)$, and a cube x of the HDA is said to be *reachable* if there is a rooted computation path $(i*, \ldots, x)$. Figure 1 shows an example of an acyclic rooted computation path.

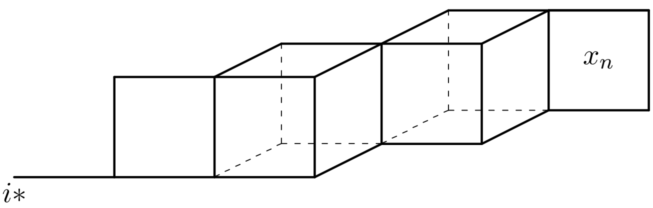


Figure 1: An acyclic rooted computation path which ends in a 2-cube x_n

We say that a precubical set X *is* a computation path if there is a computation path $(x_1, \ldots, x_n)$ of cubes in X such that all other cubes in X are faces of one of the x_i, and similarly for acyclic computation paths. An *elementary computation step* is an inclusion $(x_1, \ldots, x_n) \hookrightarrow (x_1, \ldots, x_n, x_{n+1})$ of computation paths.

Let CPath be the full subcategory of the category of HDA induced by the acyclic rooted computation paths, then it is not difficult to see that any morphism in CPath is a finite composite of elementary computation steps and isomorphisms.

Following the terminology of [20], we say that a morphism $f : X \to Y$ is CPath-open if it has the right-lifting property with respect to morphisms in CPath. That is, we require that for any morphism $m : P \to Q \in \mathsf{CPath}$ and any commutative diagram as below, there exists a morphism r filling in the diagram

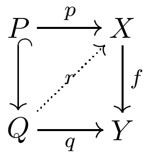
$$\begin{array}{ccc} P & \longrightarrow & X \\ {\scriptstyle m}\downarrow & \overset{r}{\nearrow} & \downarrow{\scriptstyle f} \\ Q & \longrightarrow & Y \end{array}$$

Lemma 5 *A morphism $f : X \to Y$ is CPath-open if and only if it satisfies the property that for any reachable $x \in X$ and for any $z' \in Y$ such that $f(x) = \delta_i^0 z'$ for some i, there is a $z \in X$ such that $x = \delta_i^0 z$ and $z' = f(z)$.*

Following established terminology, this could be called a "higher-dimensional zig-zag property."

Proof: Assume first f to be CPath-open. Let $x \in X$ be reachable and $z' \in Y$ such that $f(x) = \delta_i^0 z'$. Let $P' = (i*, \ldots, x)$ be a computation path in X, and let $P = (i*, \ldots, x')$ be an acyclic computation path with a surjective mapping $p : P \to P'$. Let α be a cube of the same dimension as z', put $\delta_i^0 \alpha = x'$, and let $Q = (i*, \ldots, x', \alpha)$.

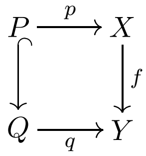
Let $q : Q \to Y$ be the morphism defined by $q(w) = f(p(w))$ for $w \in (i*, \dots, x')$, and $q(\alpha) = z'$. We have a commutative diagram



and as f is CPath-open, we can fill it in with a morphism $r : Q \to X$. Let $z = r(\alpha)$, then $f(z) = z'$, and $\delta_i^0 z = r(\delta_i^0 \alpha) = r(x') = p(x') = x$.

For the other direction, assume f to satisfy the property of the lemma. By induction it is sufficient to show that f is open with respect to elementary computation steps.

Let $P = (i*, \dots, x)$, $Q = (i*, \dots, x, \alpha)$ be two acyclic rooted computation paths with $\delta_i^0 \alpha = x$ for some i, and with morphisms $p : P \to X$, $q : Q \to Y$ such that the diagram



commutes. Then $f(p(x)) = q(x) = q(\delta_i^0 \alpha) = \delta_i^0 q(\alpha)$, hence there exists $z \in X$ such that $f(z) = q(\alpha)$ and $p(x) = \delta_i^0 z$. We can now define a morphism $r : Q \to X$ filling in the diagram by $r(w) = p(w)$ for $w \in (i*, \dots, x)$, and $r(\alpha) = z$. □

This suggests the following definition of bisimulation of HDA: Given two HDA $* \xrightarrow{i} X \xrightarrow{\lambda} L$, $* \xrightarrow{j} Y \xrightarrow{\mu} L$ over the same alphabet, then a bisimulation of X and Y is a cubical relation $R \subseteq X \times Y$ which respects initial states and labelings, i.e. $(i*, j*) \in R_0$, and if $(x, y) \in R$ then $\lambda x = \mu y$; and for all reachable $x \in X$, $y \in Y$ such that $(x, y) \in R$,

- if $x = \delta_i^0 z$ for some z, then $y = \delta_i^0 z'$ for some z' so that $(z, z') \in R$,
- if $y = \delta_i^0 z'$ for some z', then $x = \delta_i^0 z$ for some z so that $(z, z') \in R$.

Note that bisimilarity is indeed an equivalence relation.

Proposition 6 *Two HDA Y, Z are bisimilar if and only if there is a span of* CPath*-open maps $Y \leftarrow X \to Z$.*

Proof: If $f : X \to Y$ is CPath-open, then the cubical relation $R = \{(x, f(x)) \mid x \in X\}$ is a bisimulation by Lemma 5.

Conversely, if $R \subseteq Y \times Z$ is a bisimulation, then as R respects labelings, it is itself a HDA over the same alphabet as Y and Z, with maps $Y \xleftarrow{f} R \xrightarrow{g} Z$ induced by the projections of the product. These maps respect labelings, and they are CPath-open by Lemma 5. □

Note that when restricted to labeled transition systems, bisimulation of HDA is equivalent to strong bisimulation [22], the only difference being that strong bisimulation requires the *existence* of corresponding transitions, whereas HDA-bisimulation actually *specifies* a correspondence.

9 Bisimulation is a Congruence

We show that bisimulation is a congruence with respect to the constructions on HDA introduced in Section 7. For relabelings this is clear, and for tensor product we have the following lemma.

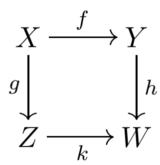
Lemma 7 *Given* CPath*-open morphisms* $f \in \langle * \downarrow F\mathsf{pCub} \downdownarrows L\rangle$*,* $g \in \langle * \downarrow F\mathsf{pCub} \downdownarrows M\rangle$*, then* $f \otimes g \in \langle * \downarrow F\mathsf{pCub} \downdownarrows L \otimes M\rangle$ *is again* CPath*-open.*

Proof: Let $f : X_1 \to Y_1$, $g : X_2 \to Y_2$, and let $x_1 \otimes x_2 \in X_1 \otimes X_2$ be reachable. Let $z_1' \otimes z_2' \in Y_1 \otimes Y_2$ such that $f \otimes g(x_1 \otimes x_2) = f(x_1) \otimes g(x_2) = \delta_i^0(z_1' \otimes z_2')$ for some i. Assume that $\delta_i^0(z_1' \otimes z_2') = \delta_i^0 z_1' \otimes z_2'$, then $f(x_1) = \delta_i^0 z_1'$, $g(x_2) = z_2'$. By CPath-openness of f we have $z_1 \in X_1$ such that $x_1 = \delta_i^0 z_1$ and $z_1' = f(z_1)$, hence $f \otimes g(z_1 \otimes x_2) = z_1' \otimes z_2'$ and $\delta_i^0(z_1 \otimes x_2) = x_1 \otimes x_2$. The proof for the case $\delta_i^0(z_1' \otimes z_2') = z_1' \otimes \delta_{i-p}^0 z_2'$ is similar, using CPath-openness of g. □

Hence if we have spans of CPath-open morphisms $Y_1 \xleftarrow{f_1} X_1 \xrightarrow{g_1} Z_1$, $Y_2 \xleftarrow{f_2} X_2 \xrightarrow{g_2} Z_2$, then $Y_1 \otimes Y_2$ and $Z_1 \otimes Z_2$ are bisimilar via the span of CPath-open morphisms $Y_1 \otimes Y_2 \xleftarrow{f_1 \otimes f_2} X_1 \otimes X_2 \xrightarrow{g_1 \otimes g_2} Z_1 \otimes Z_2$.

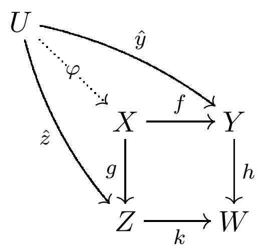
For restriction, we need the following lemma first.

Lemma 8 *Let*



be a pullback square in Cub*, and let* $y \in Y$*,* $z \in Z$ *such that* $h(y) = k(z)$*. Then there exists* $x \in X$ *fulfilling* $f(x) = y$*,* $g(x) = z$*.*

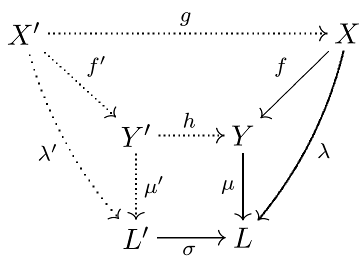
Proof: Assume that $h(y) = k(z) \in W_n$, then $y \in Y_n$, $z \in Z_n$. Let U be the cubical set generated by a single n-cube $U_n = \{u\}$, and consider the diagram



where $\hat{y} : U \to Y$, $\hat{z} : U \to Z$ are the morphisms induced by $u \mapsto y$ and $u \mapsto z$, respectively, and φ is defined by the universal property of the pullback. Let $x = \varphi(u)$, then by commutativity, $f(x) = y$ and $g(x) = z$. $\square$

Congruency of bisimilarity with respect to restriction is then implied by the next lemma.

Lemma 9 *Given a* CPath*-open morphism* $f : X \to Y \in \langle * \downarrow F\mathsf{pCub} \downdownarrows L\rangle$ *and a non-contracting injective morphism* $\sigma : L' \to L$*, then the unique morphism* $f' : X' \to Y'$ *defined by the double pullback diagram*



is again CPath*-open.*

Proof: Note [1, Prop. 2.5.3] that injectivity of σ implies that also g and h are injective.

Let $x' \in X'$, $y' \in Y'$, and $i \in \mathbb{N}_+$ such that $f'(x') = \delta_i^0 y'$. Then $f(g(x')) = h(f'(x')) = \delta_i^0 h(y')$, hence by CPath-openness of f we have $z \in X$ such that $g(x') = \delta_i^0 z$ and $f(z) = h(y')$.

Now $\lambda(z) = \mu(f(z)) = \mu(h(y')) = \sigma(\mu'(y'))$, hence by Lemma 8 we have $z' \in X'$ such that $g(z') = z$. Then $g(\delta_i^0 z') = \delta_i^0 z = g(x')$ and $h(f'(z') = f(g(z')) = f(z) = h(y')$, hence by injectivity, $\delta_i^0 z' = x'$ and $f'(z') = y'$. $\square$

Hence if $Y, Z \in \langle * \downarrow F\mathsf{pCub} \downdownarrows L\rangle$ are bisimilar via a span of CPath-open maps $Y \leftarrow X \to Z$, the above lemma yields a span of CPath-open maps $Y' \leftarrow X' \to Z'$ of their restrictions to L'.

10 Geometric Realisation of Precubical Sets

We want to relate CPath-openness of a morphism of higher-dimensional automata to a *geometric* property of the underlying precubical sets. In order to do that, we need to recall some of the technical apparatus developed in [10, 9].

The *geometric realisation* of a precubical set X is the topological space

$$|X| = \bigsqcup_{n \in \mathbb{N}} X_n \times [0,1]^n / \equiv$$

where the equivalence relation $\equiv$ is induced by identifying

$$(\delta_i^\nu x; t_1, \ldots, t_{n-1}) \equiv (x; t_1, \ldots, t_{i-1}, \nu, t_i, \ldots, t_{n-1})$$

for all $x \in X_n$, $n \in \mathbb{N}$, $i = 1, \ldots, n$, $\nu = 0, 1$, $t_i \in [0,1]$. Geometric realisation is turned into a functor from pCub to Top by mapping $f : X \to Y \in \mathsf{pCub}$ to the function $|f| : |X| \to |Y|$ defined by

$$|f|(x; t_1, \ldots, t_n) = (f(x); t_1, \ldots, t_n)$$

This is similar to the well-known geometric realisation functor from *simplicial* sets to topological spaces, cf. [2].

Given $x \in X_n \in X$, we denote its image in the geometric realisation by $|x| = \{(x; t_1, \ldots, t_n) \mid t_i \in [0,1]\} \subseteq |X|$. The *carrier*, $\operatorname{carr} z$, of a point $z \in |X|$ is z itself if $z \in X_0$, or else the unique cube $x \in X$ such that $z \in \operatorname{int}|x|$, the interior of $|x|$. The *star* of z is the open set

$$\operatorname{St} z = \left\{ z' \in |X| \;\middle|\; \operatorname{carr} z \triangleleft \operatorname{carr} z' \right\}$$

There is a natural order on the cubes $[0,1]^n$ which is "forgotten" in the transition $\mathsf{pCub} \longrightarrow \mathsf{Top}$. One can recover some of this structure by instead defining functors from pCub to the *d-spaces* or the *spaces with distinguished cubes* of M. Grandis [15, 16, 17], however here we take a different approach as laid out in [10].

Given a precubical set X and $x, y \in X$, we write $x \triangleleft y$ if x is a face of y. This defines a preorder $\triangleleft$ on X. If x is a lower face of y we write $x \triangleleft^- y$, if it is an upper face we write $x \triangleleft^+ y$. The precubical set X is said to be *locally finite* if the set $\{y \in X \mid x \triangleleft y\}$ is finite for all $x \in X_0$.

Define a precubical set X to be *non-selflinked* if $\delta_i^\nu x = \delta_j^\mu x$ implies $i = j$, $\nu = \mu$ for all $x \in X$, $i, j \in \mathbb{N}_+$, $\nu, \mu \in \{0,1\}$. Note [10, Lemma 6.16]: If $x \triangleleft y$ in a non-selflinked precubical set, then there are *unique* sequences $\nu_1, \ldots, \nu_\ell$, $i_1 < \cdots < i_\ell$ such that $x = \delta_{i_1}^{\nu_1} \cdots \delta_{i_\ell}^{\nu_\ell} y$.

The geometric realisation of a non-selflinked precubical set contains no self-intersections; if $(x, s_1, \ldots, s_n) \equiv (x, t_1, \ldots, t_n)$, then $s_i = t_i$ for all $i = 1, \ldots, n$. By [10, Thm. 6.27], the geometric realisation of a non-selflinked precubical set is a *local po-space*; a Hausdorff topological space with a relation $\leq$ which is reflexive, antisymmetric, and *locally* transitive, i.e. transitive in each U_α for some collection $\mathcal{U} = \{U_\alpha\}$ of open sets covering X. In our case, the relation $\leq$ is induced by the natural partial orders on the unit cubes $[0,1]^n$, and a covering $\mathcal{U}$ is given by the stars $\operatorname{St}|x|$ of all vertices $x \in X_0$.

A *dimap* between local po-spaces $(X, \leq_X)$, $(Y, \leq_Y)$ is a continuous mapping $f : X \to Y$ which is *locally increasing*: for any $x \in X$ there is an open neighbourhood $U \ni x$ such that for all $x_1 \leq_X x_2 \in U$, $f(x_1) \leq_Y f(x_2)$. Local po-spaces and dimaps form a category lpoTop, and by [10, Prop. 6.38], geometric realisation is a functor from non-selflinked precubical sets to local po-spaces.

Let $\vec{I}$ denote the unit interval $[0,1]$ with the natural (total) order, and define a *dipath* in a local po-space $(S,\leq)$ to be a dimap $p:\vec{I}\to S$. We recall [9, Def. 2.17]: Given a locally finite precubical set X and a dipath $p:\vec{I}\to |X|$, then there exists a partition of the unit interval $0=t_1\leq\cdots\leq t_{k+1}=1$ and a unique sequence $x_1,\ldots,x_k\in X$ such that

- $x_i\neq x_{i+1}$
- $t\in[t_i,t_{i+1}]$ implies $p(t)\in|x_i|$
- $t\in\,]t_i,t_{i+1}[$ implies $\operatorname{carr}p(t)=x_i$
- $\operatorname{carr}p(t_i)\in\{x_{i-1},x_i\}$

The sequence $(x_1,\ldots,x_k)$ is called the *carrier sequence* of the dipath p, and we shall denote it by $\operatorname{carrs}p$. It can be shown, cf. [9, Lemma 3.2], that for all $i=2,\ldots,n$, either $x_{i-1}\lhd^- x_i$ or $x_i\lhd^+ x_{i-1}$. Note that the definition in [9] makes an extra assumption on X which, in fact, is not necessary. Figure 2 shows an example of a carrier sequence.

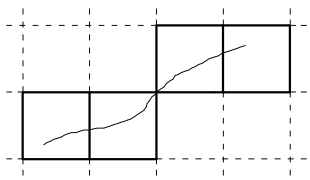


Figure 2: A dipath and its carrier sequence

In general we call a sequence of cubes $(x_1,\ldots,x_n)$ a carrier sequence if $x_{i-1}\lhd^- x_i$ or $x_i\lhd^+ x_{i-1}$ for all $i=2,\ldots,n$. Note that computation paths are carrier sequences, and conversely, that carrier sequences can be turned into computation paths by adding in some intermediate cubes. The next lemma shows that any carrier sequence actually is the carrier sequence of a dipath.

Lemma 10 *Given a carrier sequence $(x_1,\ldots,x_n)$ in a locally finite non-selflinked precubical set X and $z\in\operatorname{int}|x_n|$, there exists a dipath $p:\vec{I}\to|X|$ such that* $\operatorname{carrs}p=(x_1,\ldots,x_n)$ *and $p(1)=z$.*

Proof: We shall inductively construct a sequence of dipaths $(p_n,\ldots,p_1)$ such that $p_i(1)=z$ and $\operatorname{carrs}p_i=(x_i,\ldots,x_n)$ for all i. Let p_n be the constant dipath $p_n(t)=z$.

Assume that we have a dipath p_{i+1} such that $p_{i+1}(1)=z$ and $\operatorname{carrs}p_{i+1}=(x_{i+1},\ldots,x_n)$. Then $p_{i+1}(0)\in\operatorname{int}|x_{i+1}|$. We treat the case $x_i\lhd^- x_{i+1}$ first, then $x_i=\delta^0_{j_1}\cdots\delta^0_{j_\ell}x_{i+1}$ for a unique sequence $j_1<\cdots<j_\ell$. Let $J=\{j_1,\ldots,j_\ell\}$, then $(x_i;t'_1,\ldots,t'_k)\equiv(x_{i+1};t_1,\ldots,t_{k+\ell})$, where $t_\alpha=0$ for

$\alpha \in J$, and the sequence $(t'_1, \ldots, t'_k)$ is constructed from $(t_1, \ldots, t_{k+\ell})$ by deleting all 0s.

Let $p_{i+1}(0) = (x_{i+1}; s_1, \ldots, s_{k+\ell})$, and define $y = (x_{i+1}; \tilde{s}_1, \ldots, \tilde{s}_{k+\ell})$ by $\tilde{s}_\alpha = 0$ if $\alpha \in J$, $\tilde{s}_\alpha = s_\alpha$ if $\alpha \notin J$. Then $y \leq p_{i+1}(0)$, $y \in |x_i| \cap |x_{i+1}|$, and $y \in \operatorname{int}|x_i|$. We can concatenate p_{i+1} (on the left) with a dipath from y to $p_{i+1}(0)$ to get p_i.

Now for the case $x_{i+1} \lhd^+ x_i$. Let again $J = \{j_1, \ldots, j_\ell\}$ such that $x_{i+1} = \delta^1_{j_1} \cdots \delta^1_{j_\ell} x_i$ for a unique sequence $j_1 < \cdots < j_\ell$, then $(x_{i+1}; t'_1, \ldots, t'_k) \equiv (x_i; t_1, \ldots, t_{k+\ell})$, where $t_\alpha = 1$ for $\alpha \in J$, and the sequence $(t'_1, \ldots, t'_k)$ is constructed from $(t_1, \ldots, t_{k+\ell})$ by deleting all 1s.

Let $p_{i+1}(0) = (x_i; s_1, \ldots, s_{k+\ell}) = (x_{i+1}; s'_1, \ldots, s'_k)$, and define $y = (x_i; \tilde{s}_1, \ldots, \tilde{s}_{k+\ell})$ by $\tilde{s}_\alpha = \frac{1}{2}$ if $\alpha \in J$, $\tilde{s}_\alpha = s_\alpha$ if $\alpha \notin J$. Then $y \leq p_{i+1}(0)$ and $y \in \operatorname{int}|x_i|$, hence we can again concatenate p_{i+1} with a dipath from y to $p_{i+1}(0)$ to get p_i. □

We can similarly fix $z \in \operatorname{int}|x_1|$ and get a dipath p with $p(0) = z$, but we will only need the former case. We shall also need the following two technical lemmas.

Lemma 11 *Given locally finite non-selflinked precubical sets X, Y, a morphism $f : X \to Y$, and a dipath $p : \vec{I} \to |X|$, then* $\operatorname{carrs}(|f| \circ p) = f(\operatorname{carrs} p)$.

Proof: Let $\operatorname{carrs} p = (x_1, \ldots, x_n)$, $\operatorname{carrs}(|f| \circ p) = (y_1, \ldots, y_m)$. We can assume that $x_1 = \operatorname{carr} p(0) \in X_0$, then $\operatorname{carr} |f|(p(0)) = y_1 \in Y_0$. We proceed by induction:

Let $k \in \{1, \ldots, n\}$ such that $y_i = f(x_i)$ for all $i = 1, \ldots, k-1$, and let $s_k \in [0, 1]$ such that $\operatorname{carr} p(s_k) = x_k$ and $\operatorname{carr} p(s) \neq x_k$ for $s < s_k$. Note that $\operatorname{carr} |f|(p(s_k)) = y_k$.

Assume that $x_k \lhd^+ x_{k-1}$, then $x_k = \delta^1_{i_1} \cdots \delta^1_{i_\ell} x_{k-1}$ for some sequence of indices $I = (i_1, \ldots, i_\ell)$. Write $p(s_k) = (x_{k-1}; t_1, \ldots, t_q)$, then $t_j = 1$ if $j \in I$, and $0 < t_j < 1$ otherwise. Also, $p(s_k) = (x_k; t_{j_1}, \ldots, t_{j_\alpha})$, where the j_i are exactly those indices not in I. Hence

$$|f|(p(s_k)) = (f(x_{k-1}); t_1, \ldots, t_q) = (f(x_k); t_{j_1}, \ldots, t_{j_\alpha})$$

and as $0 < t_{j_i} < 1$ for all j_i, this implies $|f|(p(s_k)) \in \operatorname{int} f(x_k)$, and thus $y_k = \operatorname{carr} |f|(p(s_k)) = f(x_k)$. The proof for $x_{k-1} \lhd^- x_k$ is similar. □

Note that, taking p to be a constant dipath, the lemma implies that $\operatorname{carr} |f|(z) = f(\operatorname{carr} z)$ for any $z \in |X|$.

Lemma 12 *Given locally finite non-selflinked precubical sets X, Y, a morphism $f : X \to Y$, a dipath $q : \vec{I} \to |Y|$, and a carrier sequence $(x_1, \ldots, x_n)$ in X such that* $\operatorname{carrs} q = (f(x_1), \ldots, f(x_n))$, *then there exists a dipath $p : \vec{I} \to |X|$ such that* $\operatorname{carrs} p = (x_1, \ldots, x_n)$ *and* $q = |f| \circ p$.

Proof: Denote $f(x_i) = y_i$, and let $0 = t_1 \leq \cdots \leq t_{n+1} \leq 1$ be the partition of the unit interval associated with $\operatorname{carrs} q = (y_1, \dots, y_n)$. Let $i \in \{1, \dots, n\}$, and assume $y_i \in Y_k$.

For $t \in [t_i, t_{i+1}]$ we have $q(t) \in |y_i|$, hence there exists a dimap $q_i : \vec{I} \to \vec{I}^k$ such that

$$q\,|_{[t_i, t_{i+1}]}(t) = (y_i, q_i(t))$$

and we can define p on $[t_i, t_{i+1}]$ by $p\,|_{[t_i, t_{i+1}]}(t) = (x_i, q_i(t))$. Then $p(t) \in \operatorname{int}|x_i|$ for $t \in \,]t_i, t_{i+1}[$, hence $\operatorname{carrs} p = (x_1, \dots, x_n)$. We have $q = |f| \circ p$ by definition, and p is continuous because the cubes x_i are glued together in the same way as the cubes y_i: Y being non-selflinked implies that if $y_i = \delta^0_{j_1} \cdots \delta^0_{j_\ell} y_{i+1}$, then also $x_i = \delta^0_{j_1} \cdots \delta^0_{j_\ell} x_{i+1}$, and similarly for $y_{i+1} \lhd^+ y_i$.$\square$

Note again the implication of the lemma for constant dipaths: If $x \in X$ and $z' \in |Y|$ are such that $\operatorname{carr} z' = f(x)$, then there exists $z \in |X|$ such that $\operatorname{carr} z = x$ and $z' = |f|(z)$.

11 Bisimulation and Dipaths

In this final section we again fix a labeling cubical set L and work in the category of higher-dimensional automata over L. Recall that in this category, all morphisms are non-contracting.

First we note the following stronger variant of Lemma 5:

Lemma 13 *A morphism $f : X \to Y$ is* CPath*-open if and only if it satisfies the property that for any reachable $x_1 \in X$ and for any computation path $(y_1, \dots, y_n)$ in Y with $y_1 = f(x_1)$, there is a computation path $(x_1, \dots, x_n)$ in X such that $y_i = f(x_i)$ for all $i = 1, \dots, n$.*

Proof: Inductively apply Lemma 5 to the computation path $(y_1, \dots, y_n)$.$\square$

We call a HDA $* \longrightarrow X$ *special* if the cubical set X is freely generated by a locally finite non-selflinked precubical set, and for the rest of this section we assume our HDA to be special. Note that this is not a severe restriction: Local finiteness is hardly an issue, and the requirement on a precubical set to be non-selflinked is a natural one which is quite standard in algebraic topology, cf. [2, Def. IV.21.1].

A point $z \in |X|$ in the geometric realisation of a HDA $* \overset{i}{\longrightarrow} X$ is said to be *reachable* if there exists a dipath $p : \vec{I} \to |X|$ with $p(0) = |i\,*|$ and $p(1) = z$. This notion of "geometric" reachability is closely related to the one of computation path reachability defined in Section 8:

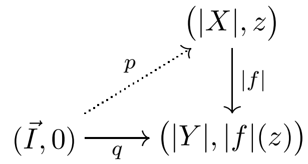
Proposition 14 *A point $z \in |X|$ in the geometric realisation of a special HDA $* \overset{i}{\longrightarrow} X$ is reachable if and only if* $\operatorname{carr} z$ *is reachable.*

Proof: Assume first z to be reachable, and let $p : \vec{I} \to |X|$ be a dipath with $p(0) = |i*|$, $p(1) = z$. Let $(x_1, \dots, x_n) = \operatorname{carrs} p$, then $x_n = \operatorname{carr} z$, so by turning $\operatorname{carrs} p$ into a computation path, we have a computation path from $i*$ to $\operatorname{carr} z$.

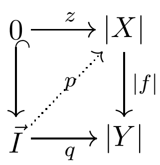
Now assume $\operatorname{carr} z$ to be reachable, and let $(i*, \dots, \operatorname{carr} z)$ be a computation path. Let $p : \vec{I} \to |X|$ be its associated dipath as given by Lemma 10. Then $\operatorname{carrs} p = (i*, \dots, \operatorname{carr} z)$, hence $p(0) = |i*|$, and $p(1) = z$. □

We can now prove the main result of this paper, linking bisimulation of HDA with a dipath-lifting property of their geometric realisations:

Theorem 15 *Given a morphism $f : X \to Y$ of two special HDA, then f is* CPath*-open if and only if, for any reachable $z \in |X|$ and for any dipath $q : \vec{I} \to |Y|$ such that $q(0) = |f|(z)$, there is a dipath $p : \vec{I} \to |X|$ filling in the diagram*



If we identify z with the mapping $z : 0 \mapsto z \in |X|$, we can draw the above diagram in a more familiar fashion as



That is, if we make the assumption that all cubes in the HDA X are reachable, then f is CPath-open if and only if $|f|$ has the right-lifting property with respect to the inclusion $0 \hookrightarrow \vec{I}$.

Proof: The morphism f is non-contracting, hence it is the image of a precubical morphism, also denoted f, under the free functor. Assume first f to be CPath-open, let $z \in |X|$ be reachable and $q : \vec{I} \to |Y|$ a dipath with $q(0) = |f|(z)$. Turn $\operatorname{carrs} q$ into a computation path $(y_1, \dots, y_n)$. Let $x_1 = \operatorname{carr} z$, then x_1 is reachable, and $y_1 = \operatorname{carr} |f|(z) = f(x_1)$.

We can invoke Lemma 13 to get a computation path $(x_1, \dots, x_n)$ in X such that $(y_1, \dots, y_n) = f(x_1, \dots, x_n)$. Lemma 12 then provides a dipath $p : \vec{I} \to |X|$ such that $q = |f| \circ p$. The construction in the proof of Lemma 12 implies that $p(0) = z$.

For the other direction, assume $|f|$ to have the dipath lifting property of the theorem, let $x_1 \in X$ be reachable, $y_1 = f(x_1) \in Y$, and let $(y_1, \dots, y_n)$ be a computation path in Y.

Let $q : \vec{I} \to |Y|$ be a dipath associated with $(y_1, \dots, y_n)$ as given by Lemma 10. Then $\operatorname{carr} q(0) = f(x_1)$, thus we have $z \in |X|$ such that $\operatorname{carr} z = x_1$ and $q(0) = |f|(z)$. By Proposition 14 the point z is reachable. The dipath-lifting property then implies that we have a dipath $p : \vec{I} \to X$ such that $q = |f| \circ p$ and $p(0) = z$.

Let $(x_1, \dots, x_n) = \operatorname{carrs} p$, then $y_i = f(x_i)$ by Lemma 11. We show that $(x_1, \dots, x_n)$ is actually a computation path; this will finish the proof. Assume $x_i \lhd^- x_{i+1}$, i.e. $x_i = \delta^0_{j_1} \cdots \delta^0_{j_\ell} x_{i+1}$ for some sequence of indices, then $y_i = \delta^0_{j_1} \cdots \delta^0_{j_\ell} y_{i+1}$. Since $(y_1, \dots, y_n)$ is a computation path, and Y is non-selflinked, the sequence of indices contains only one element j_ℓ, and $x_i = \delta^0_{j_\ell} x_{i+1}$. Similar arguments apply to the case $x_{i+1} \lhd^+ x_i$. □

12 Conclusion and Future Work

We have in this article introduced some synchronisation operations for higher-dimensional automata, notably tensor product, relabeling, and restriction. Whether these operations capture the full flavour of HDA synchronisation remains to be seen; some other primitives might be needed. Recent work by Worytkiewicz [25] suggests some directions.

We have also defined a notion of bisimulation for HDA which is closely related to van Glabbeek's [11] computation paths. The notion of bisimulation also defined in [11] appears to be weaker than ours, and their relation should be worked out in detail.

The notions of computation paths defined in Cattani-Sassone's [5] and in [25] differ considerably from van Glabbeek's, and as a consequence they arrive at different concepts of bisimulation and even simulation. These differences need to be worked out, and also the apparent similarities between [5] and [25].

We have shown that our notion of bisimulation has an interpretation as a dipath-lifting property of morphisms, making the problem of deciding bisimilarity susceptible to some machinery from algebraic topology. In topological language, a dipath-lifting morphism is a weak kind of *fibration*, hinting that fibrations (well-studied in algebraic topology) could have applications, as well. This also suggests that a general theory of directed fibrations should be developed.

We believe that our bisimulation notion should be weakened, also taking equivalence of computation paths [11] into account. We plan to elaborate on this in a future paper, and we conjecture that this bisimulation-up-to-equivalence has a topological interpretation as a property of lifting dipaths *up to directed homotopy*. This weaker bisimulation appears to be closely related to van Glabbeek's, and there appears to be a strong connection between his unfoldings of HDA and directed coverings of local po-spaces [8].

References

[1] Francis Borceux. *Handbook of Categorical Algebra 1.* Cambridge University Press, 1994.

[2] Glen E. Bredon. *Topology and Geometry.* Springer-Verlag, 1993.

[3] Ronald Brown and Philip J. Higgins. On the algebra of cubes. *Journal of Pure and Applied Algebra*, 21:233–260, 1981.

[4] Ronald Brown and Philip J. Higgins. Tensor products and homotopies for ω-groupoids and crossed complexes. *Journal of Pure and Applied Algebra*, 47:1–33, 1987.

[5] Gian Luca Cattani and Vladimiro Sassone. Higher dimensional transition systems. In *Proc. LICS'96*, pages 55–62. IEEE Press, 1996.

[6] Sjoerd Crans. *On Combinatorial Models for Higher Dimensional Homotopies.* PhD thesis, Utrecht University, 1995.

[7] Ulrich Fahrenberg. A category of higher-dimensional automata. In *Proc. FOSSACS'05*, Lecture Notes in Computer Science. Springer-Verlag, 2005. To be published.

[8] Lisbeth Fajstrup. Dicovering spaces. *Homology, Homotopy and Applications*, 5(2):1–17, 2003.

[9] Lisbeth Fajstrup. Dipaths and dihomotopies in a cubical complex. Report R-2003-22, Department of Mathematical Sciences, Aalborg University, 2003. `http://www.math.aau.dk/research/reports/R-2003-22.ps`. Submitted to *Advances in Applied Mathematics.*

[10] Lisbeth Fajstrup, Eric Goubault, and Martin Raussen. Algebraic topology and concurrency. Report R-99-2008, Department of Mathematical Sciences, Aalborg University, 1999. `http://www.math.aau.dk/research/reports/R-99-2008.ps`. Conditionally accepted for publication in *Theoretical Computer Science.*

[11] Robert Jan van Glabbeek. Bisimulations for higher dimensional automata. Email message, 1991. `http://theory.stanford.edu/~rvg/hda`.

[12] Robert Jan van Glabbeek. On the expressiveness of higher dimensional automata. Preprint, 2004. `http://www.cse.unsw.edu.au/~rvg/hda.pdf`.

[13] Eric Goubault. *The Geometry of Concurrency.* PhD thesis, Ecole Normale Supérieure, Paris, 1995. `http://www.di.ens.fr/~goubault/papers/these.ps.gz`.

[14] Eric Goubault. Labelled cubical sets and asynchronous transition systems: an adjunction. In *Preliminary Proceedings CMCIM'02*, 2002. http://www.di.ens.fr/~goubault/papers/cmcim02.ps.gz.

[15] Marco Grandis. Directed homotopy theory I. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 44:281–316, 2003. Preprint available as http://arxiv.org/abs/math.AT/0111048.

[16] Marco Grandis. Directed homotopy theory II. *Theory and Applications of Categories*, 14:369–391, 2002.

[17] Marco Grandis. Directed combinatorial homology and noncommutative tori. *Math. Proc. Cambridge Philos. Soc.*, 2004. to appear. Preprint available as http://www.dima.unige.it/~grandis/Bsy.pdf.

[18] Marco Grandis and Luca Mauri. Cubical sets and their site. *Theory and Applications of Categories*, 11(8):185–211, 2003.

[19] J. F. Jardine. Cubical homotopy theory: a beginning. Preprint, 2002.

[20] André Joyal, Mogens Nielsen, and Glynn Winskel. Bisimulation from open maps. *Information and Computation*, 127(2):164–185, 1996.

[21] William S. Massey. *A Basic Course in Algebraic Topology*, volume 127 of *Graduate Texts in Mathematics*. Springer-Verlag, 1991.

[22] Robin Milner. *Communication and Concurrency*. Prentice Hall, 1989.

[23] Jean-Pierre Serre. *Homologie singulière des espaces fibrés*. PhD thesis, Ecole Normale Supérieure, 1951.

[24] Glynn Winskel and Mogens Nielsen. Models for concurrency. In Samson Abramsky, Dov M. Gabbay, and Thomas S.E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 4, pages 1–148. Clarendon Press, Oxford, 1995.

[25] Krzysztof Worytkiewicz. Synchronization from a categorical perspective. Preprint, 2004. http://arxiv.org/abs/cs.PL/0411001.