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ESTIMATION OF FREQUENCY RESPONSE FUNCTIONS BY RANDOM DECREMENT

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Abstract A method for estimating frequency response functions by the Random Decrement technique is investigated in this paper. The method is based on the auto and cross Random Decrement functions of the input process and the output process of a linear system. The Fourier transformation of these functions is used to calculate the frequency response functions. The Random Decrement functions are obtained by averaging time segments of the processes under given initial conditions. The method will reduce the leakage problem, because of the natural decay of the Random Decrement functions. Also, the influence of noise will be reduced since the FFT is applied to the signatures, where the noise is reduced by averaging. Finally, the proposed technique will typically be faster than the traditional method, where the FFT is applied to every data segment in stead of applying the FFT just one time on the final Random Decrement function. The method is demonstrated by a simulation study.

Nomenclature

a	Value of time series.
C_Y	Trig condition on Y.
D_{YY}	Auto RDD function.
D_{XY}	Crass RDD function.
$h, \mathbf{h}$	Impulse response function/matrix.
$H, \mathbf{H}$	Frequency response function/matrix.
L	RDD function length and length of input time segments to FFT.
M	Number of points in FRF.
N	Number of trig points.
t_i	Discrete time point.
Y, X	Stochastic processes.
y, x	Realizations of Y, X
$\dot{Y}, \dot{X}$	Time derivative of Y , X
v	Value of time-derivative of timeseries.
Z	Fourier transformation of D.
σ_Y	Standard deviation of Y
τ	RDD function length

1 Introduction

This paper deals with the estimation of frequency response functions (FRF) of linear systems using the Random Decrement (RDD) technique. It is known, that the Fourier transformation of the auto- and cross-correlation function of the input and output processes of a linear system can be used to estimate the FRF of the system, see Bendat & Piersol [1]. Vandiver et al. [2] proved that the RDD technique applied with the level crossing trig condition estimates the auto-correlation function of a time series under the assumption of a zero mean Gaussian process. This result was generalized by Brincker et al. [3], [4], who proved under the same assumption, that the RDD technique applied with a general formulated trig condition estimates a weighted sum of the auto- and cross-correlation functions of two time series and their time derivative. This could e.g. be the load process and the response process of a linear system.

In Brincker [3] the idea of using the Fourier transformation of the RDD functions as a basis for estimating FRF's is presented. This new method is the topic of this paper. The method is demonstrated by a simulation study of a linear 3 degree of freedom system loaded by pink noise at the first degree of freedom. The precision of the approach is compared with results obtained from traditional modal analysis based on FFT of the simulated time series. Even though the speed of this method is one of the advantages, compared to traditional modal analysis, this topic is not considered in this paper. Only accuracy is considered.

The influence of the number of points in the FRF estimate and the number of points in the time series is investigated. A quality measure is used to compare the results from RDD estimation combined with FFT (RDD-FFT) with a traditional FFT based technique (FFT). The simulation study shows that RDD-FFT is more reliable than FFT. The accuracy of the FFT estimate of the FRF depends strongly on the number of points in each transformation. Furthermore the RDD-FFT method is less sensitive to noise.

2 Random Decrement Technique

The RDD technique is a method for estimating auto- and cross-correlation functions of Gaussian processes, Vandiver et al. [2], Brincker et al. [3], [4]. The auto and cross RDD functions of the processes X and Y are defined as:

$$D_{YY}(\tau) = E[Y(t+\tau)|C_{Y(t)}] \tag{1}$$

$$D_{XY}(\tau) = E[X(t+\tau)|C_{Y(t)}] \tag{2}$$

i.e. RDD functions are defined as the mean value of a process Y given some trig conditions $C_{Y(t)}$ or $C_{X(t)}$. For a time series the estimates of the auto and cross RDD functions are obtained as the empirical mean.

$$\hat{D}_{YY}(\tau) = \frac{1}{N}\sum_{i=1}^{N} y(\tau + t_i)|C_{Y(t_i)} \qquad (3)$$

$$\hat{D}_{XY}(\tau) = \frac{1}{N}\sum_{i=1}^{N} x(\tau + t_i)|C_{Y(t_i)} \qquad (4)$$

Where N is the number of paints fulfilling the trig condition. Alternatively, the RDD functions, D_{XX} and D_{YX} could be estimated. Any trig condition can be constructed from the basic trig condition given by the complete initial conditions.

$$C_{Y(t_i)} = \left[Y(t_i) = a \ \lor \ Y(t_i) = v\right] \qquad (5)$$

Assuming X and Y are stationary zero-mean Gaussian processes, the trig condition eq. (5) applied in eq. (1) and eq. (2) will yield, Brincker et al. [3], [4].

$$D_{YY}(\tau) = \frac{R_{YY}}{\sigma_Y^2}\cdot a - \frac{R'_{YY}}{\sigma_{\dot{Y}}^2}\cdot v \qquad (6)$$

$$D_{XY}(\tau) = \frac{R_{XY}}{\sigma_Y^2}\cdot a - \frac{R'_{XY}}{\sigma_{\dot{Y}}^2}\cdot v \qquad (7)$$

Where σ_Y and $\sigma_{\dot{Y}}$ denotes the standard deviation of Y and the time derivative $\dot{Y}$ of Y, R and R' denotes the auto- and cross-covariance and their time derivatives.

From eq. (6) and eq. (7) several trig conditions can be formulated, which only picks out either the correlation functions or the derivative of the correlation functions, see Brincker et al. [4]. However, in this paper only two different trig conditions are considered: The local extremum trig condition eq. (6) and trigging at zero crossings with positive slope, eq. (9).

$$C_{Y(t)} = \left[Y(t) > 0 " \ \dot{Y}(t) = 0\right] \qquad (8)$$

$$C_{Y(t)} = \left[Y(t) = 0 \ \lor \dot{Y}(t) > 0\right] \qquad (9)$$

Given the previous mentioned assumptions about Y and X, the local extremum trig condition reduces eq. (6) and eq. (7) to:

$$D_{YY}(\tau) = \frac{R_{YY}}{\sigma_Y^2}\cdot a \qquad D_{XY}(\tau) = \frac{R_{XY}}{\sigma_{Y'}^2}\ a \qquad (10)$$

Using zero crossing with positive slope reduces eq. (6) and eq. (7) to:

$$D_{YY}(\tau) = \frac{R'_{YY}}{\sigma_{\dot{Y}}^2}\cdot v \qquad D_{XY}(\tau) = \frac{R'_{XY}}{\sigma_X^2}\cdot v \qquad (11)$$

These trig conditions have the advantage of picking out only the auto/cross-covariance or the derivative of the auto/cross-covariance.

RDD functions are "born" unbiased, some times however, implementation of the trig condition might change it slightly,

and thus, some changes of the functions might take place that in some cases might appear as bias. These problems are not present in this investigation, since the presented technique will work unbiased for any trig condition. The only bias introduced is the leakage bias introduced by the FFT. Because of the decay of the RDD function, the influence of this bias will be smaller than for the traditional FFT.

3 Estimation of FRF

Consider a linear system with n degrees of freedom. The response, Y, of the system to some load X is given by the convolution integral, if the initial conditions are zero or negligible

$$\mathbf{Y}(t) = \int_{-\infty}^{t} \mathbf{h}(t - \eta)\mathbf{X}(\eta)d\eta \qquad (12)$$

Where h is the impulse response matrix. Assuming that any random force has been applied to the i'th degree of freedom only, the response at the j'th degree of freedom is:

$$Y_j(t) = \int_{-\infty}^{t} h_{ij}(t - \eta)X_i(\eta)d\eta \qquad (13)$$

To calculate the conditional mean value, see eq. (1), the time variabels t and η are substituted. Eq. (13) can then be rewritten in the following form:

$$Y_j(t+\tau) = \int_{-\infty}^{\tau} h_{ij}(\tau - \xi)X_i(\xi + t)d\xi \qquad (14)$$

Assuming that the impulse response matrix is time invariant, the conditional mean value, eq. (1), of eq. (14) can be calculated as:

$$E[Y_j(t+\tau)|C_{Y_i(t)}] =$$
$$\int_{-\infty}^{\tau} h(\tau - \xi)E[X_i(t+\xi)|C_{Y_i(t)}]d\xi \qquad (15)$$

or

$$D_{Y_j Y_i}(t+\tau) = \int_{-\infty}^{\tau} h(\tau - \xi)D_{X_i Y_i}(t+\tau)d\xi \qquad (16)$$

The Fourier transformation, Z(w), of the RDD function $D(r)$ is defined as:

$$Z(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} D(\tau)e^{-\omega\tau}d\tau \qquad (17)$$

Applying this definition to the time domain formulation eq. (16) together with the convolution theorem yields:

$$Z_{Y_j Y_i}(\omega) = H_{ij}(\omega)Z_{X_i Y_i}(\omega) \qquad (18)$$

If the trig condition is applied on the load process of the system, an alternative formulation is obtained.

$$Z_{Y_j X_i}(\omega) = H_{ij}(\omega)Z_{X_i X_i}(\omega) \qquad (19)$$

Eq. (18) and eq. (19) shows that the Fourier transformation of the RDD functions can be used for estimation of the

frequency response matrix of a linear system. The method is very alike traditional modal analysis, were the frequency response matrix is estimated from the Fourier transformation of the measured load X and the measured response Y. The method has several advantages compared to traditional modal analysis. Since the RDD technique averages out the random errors before the Fourier transformation, the technique is ecpected to be less sensitive to noise. Furthermore if the length of the RDD function is chosen long enough, the decay will reduce leakage. Dependent of the length of the RDD functions the number of trig points and the length of the measured time series, this method migth be more or less faster than traditional modal analysis.

4 Simulation of 3DOF System

The purpose of this simulation study is to illustrate the application of the method for estimating FRF's of a multi degree of freedom system. The system is linear with the following mass and stiffness matrices.

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{K} = \begin{bmatrix} 400 & -300 & 0 \\ -300 & 600 & -300 \\ 0 & -300 & 350 \end{bmatrix} (20)$$

The system is Raleigh damped with the damping matrix:

$$C = 0.4 \cdot M + 0.0004 \cdot K \qquad (21)$$

The modal parameters from this system are:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} 1.095 \\ 3.085 \\ 4.845 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \begin{bmatrix} 0.030 \\ 0.014 \\ 0.013 \end{bmatrix} \qquad (22)$$

$$\Phi = \begin{bmatrix} 1.000 & 1.000 & 1.000 \\ 1.175 & 0.081 & -1.756 \\ 1.165 & -0.940 & 0.913 \end{bmatrix} \qquad (23)$$

The system was loaded by a pink noise process at the first mass. The pink noise process was simulated by an ARMA(2,1)-model, Pandit [6]. The response of the system to this load is simulated using standard routines from the MATLAB CONTROL TOOLBOX, [7]. The sampling frequency was chosen to 15 Hz and 30000 points are simulated. All investigations in this paper were performed using the same time series, allthough mostly only the first part of the time series was used. Figure 1 shows the first part of the load process and the corresponding response at the first mass.
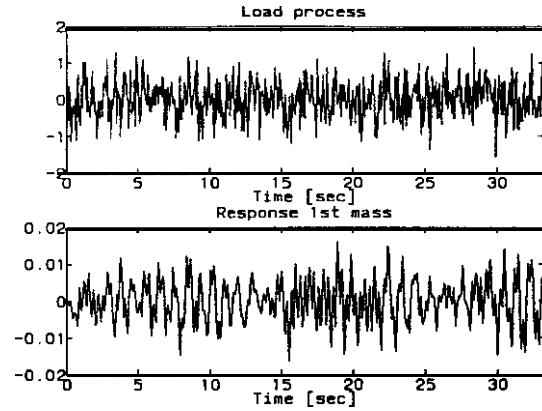

Figure 1: First part of simulated load process and the response at the first mass.

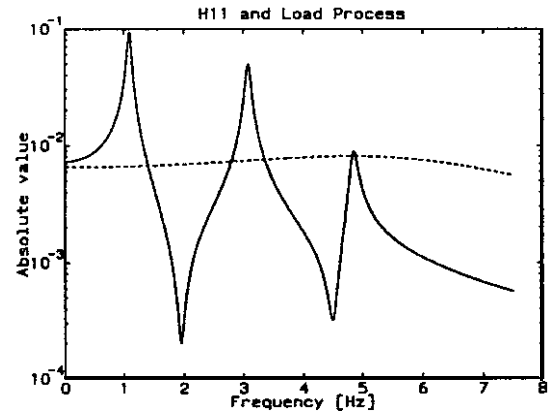Figure 2 shows the absolute value of the theoretical FRF (H_{11}) of the system and the load process

Figure 2: Theoretical absolute value of FRF, H_{11}, and theoretical absolute value of load applied to mass 1.

The purpose of the investigations is to compare the accuracy of the RDD-FFT technique and the FFT. To have a measure for the accuracy of the different methods compared to the theoretical value of the FRF's, the following error function is defined:

$$error = \frac{1}{M} \sum_{i=1}^{M} |H_i - \hat{H}_i| \qquad (24)$$

Where M is the number of points in the FRF's, H_i is the i'th value of the theoretical FRF and $\hat{H}_i$ is the i'th value of the estimated FRF. The error function is independent of the number of points in the FRF's. The influence of the length of the time series and the length of the RDD functions and the length of the time segments of the *time* series used by FFT in the estimation of the FRF's will be investigated. In the graphical presentation of the results L denotes the length of the RDD functions, which always is equal to the length of the time segments used as input for FFT. The following relation between L and the number of paints in the FRF's exists: $L = 2 * M + 1$. To illustrate the influence of L and the length

of the time series the following quality measure is defined and used.

$$\text{quality} = \frac{1}{error} \tag{25}$$

Furthermore the influence of noise is investigated by adding a white noise sequence to the time series. The noise level is described by the signal to noise ratio defined as $\frac{\sigma_{noise}}{\sigma_{signal}}$. In order to use all information from the time series the RDD functions are estimated using a two step method. First the signatures are calculated as described in section 2, then the sign of the time series is changed. A new function is calculated, again according to section 2 and the average of these two functions are used as the resulting functions. Using this method all information from the time series are extracted.

5 Local Extremum Trigging

Figure 3 and figure 4 shows the auto and 3 cross RDD functions estimated using the local extremum trig condition, eq. (5). The number of points in the time series 14000 paints, and L has a size of 450 points. The trig condition is applied at the response of the first mass.
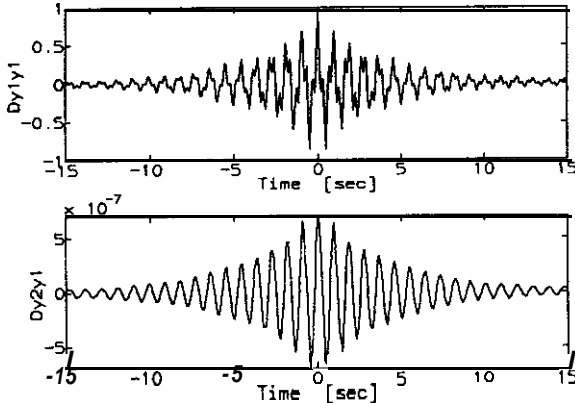
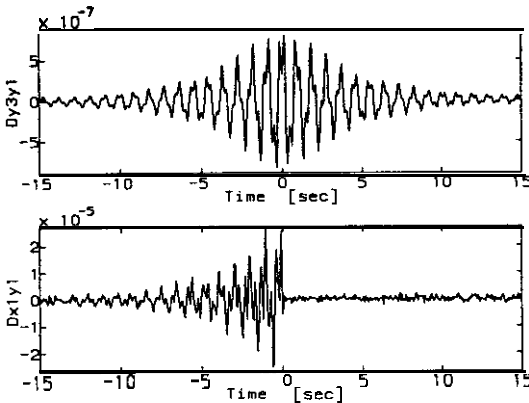Figure 3: Auto, $D_{y_1 y_1}$, and cross, $D_{y_2 y_1}$ RDD signatures.

Figure 4: Crass, $D_{y_3 y_1}$, and cross, $D_{x_1 y_1}$ RDD signatures.

From these RDD functions it is expected to have a leakage-free estimate of the FRF's, because the functions are decaying to zero. Figure 5 and figure 6 shows the absolute value and the phase of the theoretical FRF H_{11}, $\hat{H}_{11}$ estimated using RDD-FFT and pure FFT.
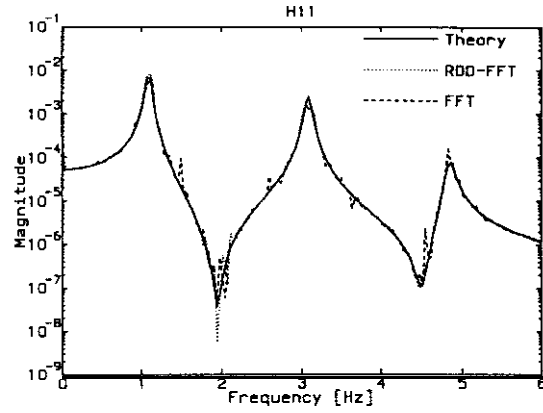

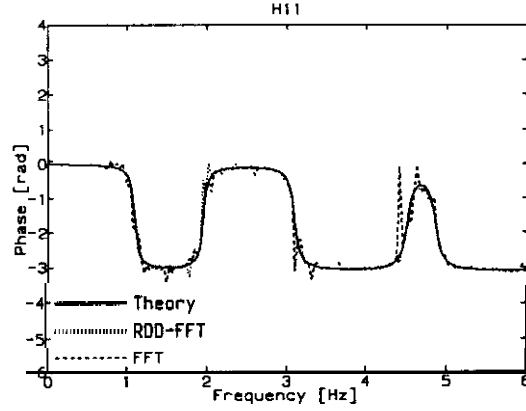Figure 5: Magnitude of frequency response function, H_{11}

Figure 6: Phase of frequency response function, H_{11}

Figure 7 • figure 12 shows the quality calculuted as in eq. (25) of RDD-FFT and FFT with the size of the time series and L as variabels. The size of the time series is varied from 2000 points to 14000 points with steps of 1000 points. L is varied from 120 points to 512. No noise is added.
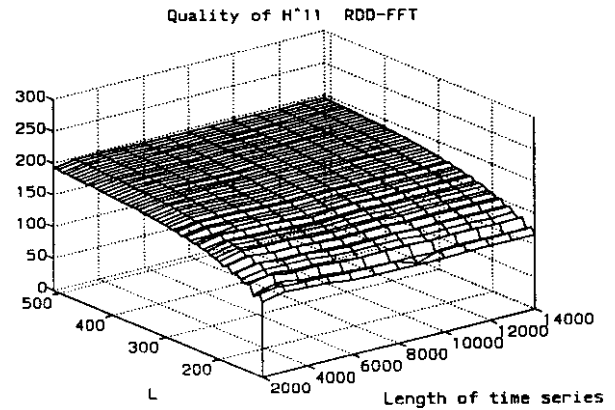
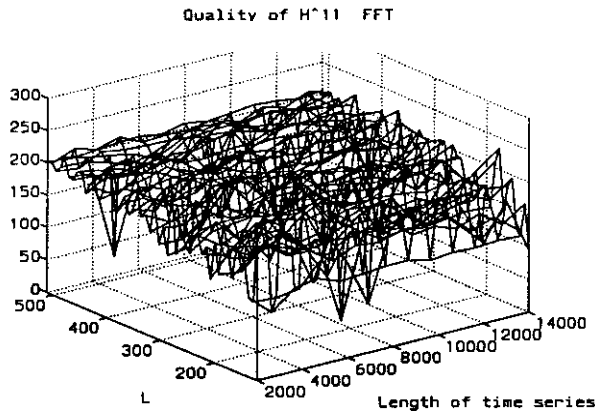

Figure 7: Quality of FRF H_{11} estimated by RDD-FFT,

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Figure 8: Quality of FRF H_{11} estimated by FFT.
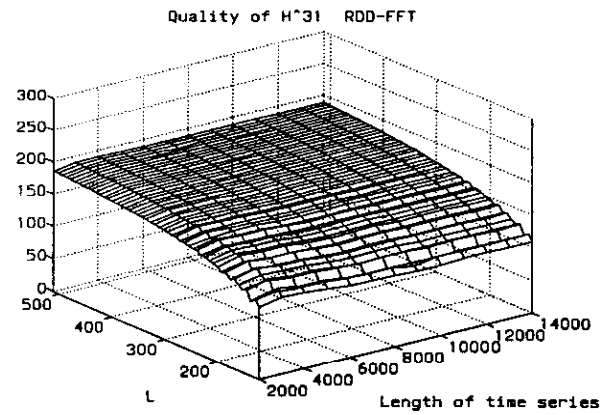

Figure 11: Quality of FRF H_{31} estimated by RDD-FFT
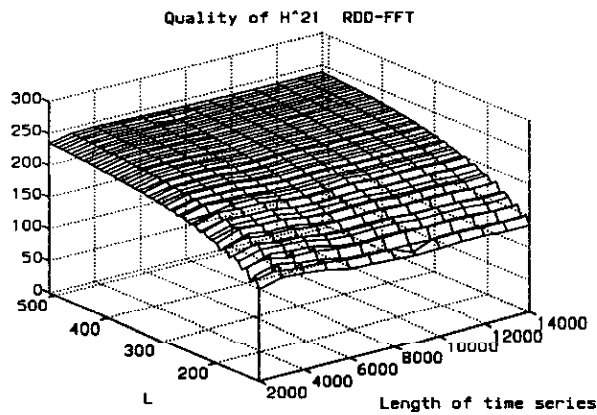

Figure 9: Quality of FRF H_{21} estimated by RDD-FFT.
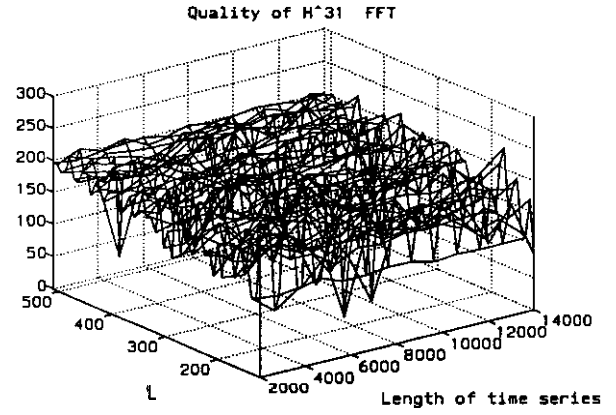

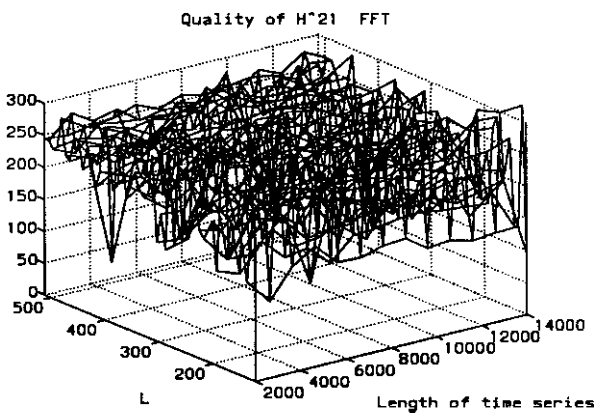Figure 12: Quality of FRF H_{31} estimated by FFT

Figure 10: Quality of FRF H_{21} estimated by FFT.

The above figures shows that, the estimations using RDD-FFT and FFT are different. The RDD-FFT gives a smooth curve, while the FFT results in curves with a lot of pitfalls. This shows that the RDD-FFT is more reliable. In average the quality of the two methods is very alike. After adding 1% and 3% noise, respectively, the quality of the estimates were calculated again. The results of this analysis showed the same tendency as in the above figures, see Asmussen et al. [8].

Figure 13 figure 15 shows a comparison of RDD-FFT and FFT using 5000 points from the timeseries. The figures shows the quality with 0%, 1% and 3% noise added.
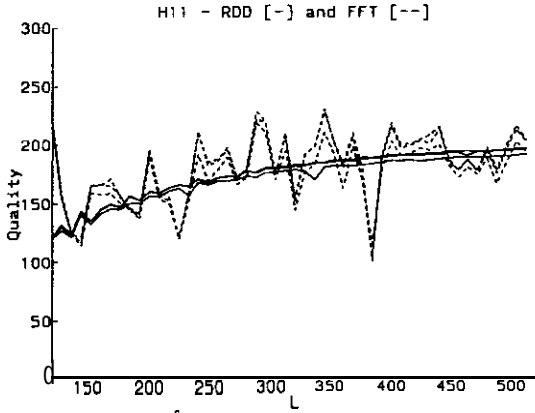
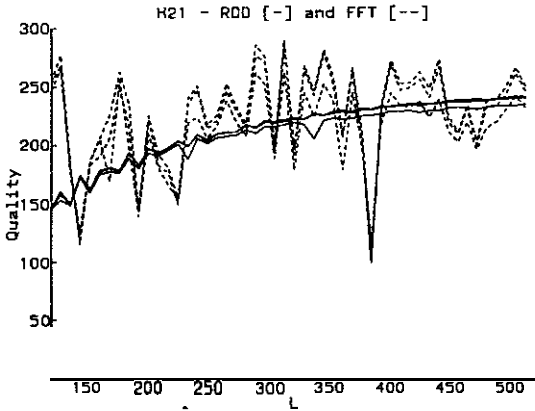Figure 13: Quality of $\hat{H}_{11}$ estimated by RDD-FFT and FFT. 5000 points used.

Figure 14: Quality of $\hat{H}_{12}$ estimated by RDD-FFT and FFT. 5000 points used.
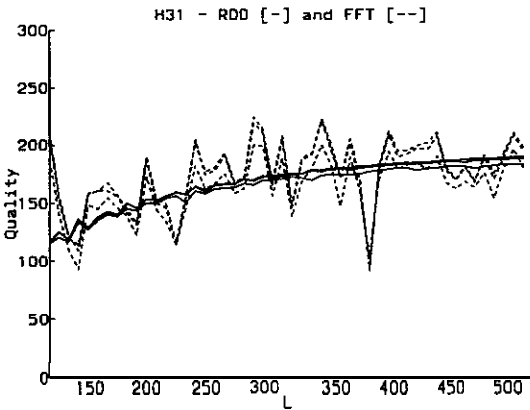

Figure 15: Quality of $\hat{H}_{13}$ estimated by RDD-FFT and FFT. *5000* points used.

Again it appears, that RDD-FFT is more reliable than FFT. The quality of the FFT is depending in an unpredictable way on the number of paints, L, used in the transformations. The average quality, however, is about equal. Furthermore FFT seems more sensitive to noise.

Figure 16 figure 18 shows a comparison of RDD-FFT and FFT with 30000 points used from the timeseries. The figures shows the quality with 0%, 1% and 3% noise added.
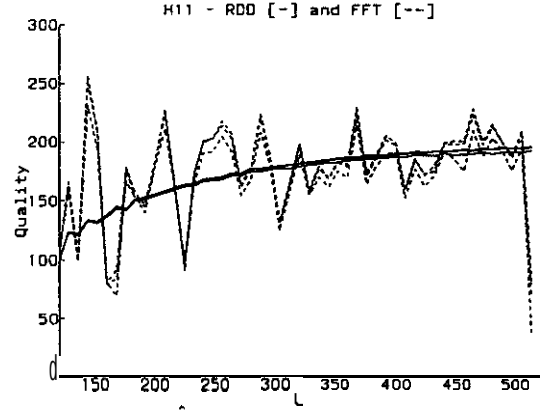

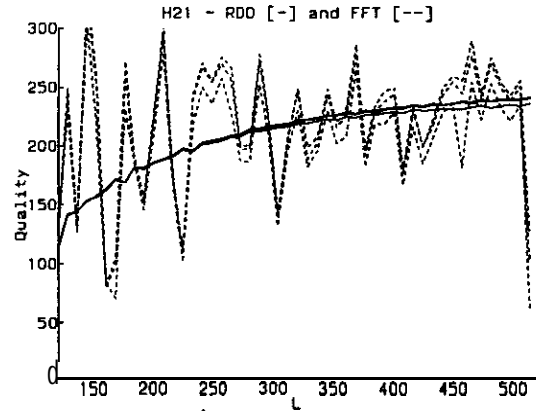Figure 16: Quality of $\hat{H}_{11}$ estimated by RDD-FFT and FFT. 30000 points used.

Figure 17: Quality of $\hat{H}_{21}$ estimated by RDD-FFT and FFT. 30000 paints used.
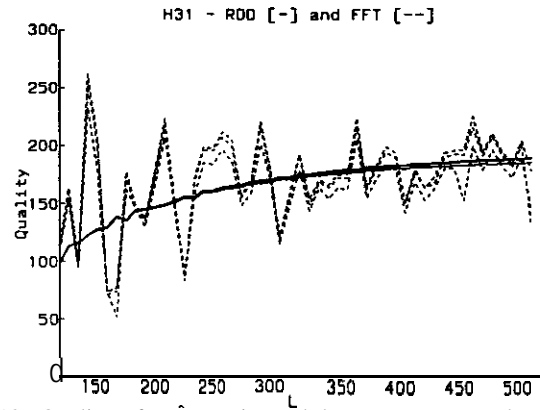

Figure 18: Quality of $\hat{H}_{33}$ estimated by RDD-FFT and FFT. 30000 points used.

Also for larger time series, it is seen that FFT fluctuates un-

predictable. The influence of noise by the RDD-FFT method has decreased by using 30000 points compared to the results obtained using 5000 points. This indicates as expected, that more trig points averages out noise.

6 Zero Crossing Trig Condition

The analysis made in section 5 is repeated with the zero crossing trig condition. The results of this analysis can be seen in Asmussen et al. [8]. This analysis shows that the zero crossing trig condition cannot be compared with the results from the local extremum trig condition. The quality is more dependent on L than the local extremum trig condition. This is illustrated in figure 19.
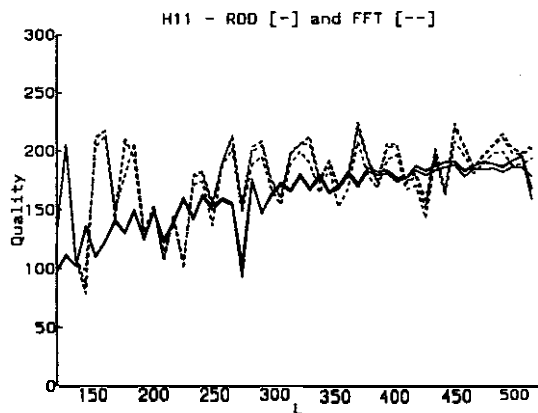

Figure 19: Quality of H_{21} estimated by RDD-FFT and FFT. 14000 points used.

The figure illustrates that quality curve of the RDD-FFT method is not as smooth as in figure 13 - figure 18.

7 Conclusion

A new method for performing modal analysis has been investigated. The method is based on FFT of RDD functions. Two different trig conditions: Local extremum and zero crossing has been compared. The results shows that the local extremum trig condition is the most reliable trig condition. The comparison is based an variations of the length of the analysed time series and the length of the RDD functions equal to the length of the time segments used by FFT. The precision of the trig conditions is investigated by defining a quality measure. No attention is given to the estimation time, but the RDD-FFT technique will be faster than pure FFT in most applications.

The RDD-FFT method is also compared with traditional modal analysis based on FFT. From the quality of the estimations, it is concluded that RDD-FFT is more reliable than FFT. Even though the average quality of the two methods is very alike. Furthermore it seems that RDD-FFT is less sensitive to noise. Especially if a long time series is used, RDD-FFT averages out noise. To have more information about RDD-FFT several other trig conditions should be investigated. The estimation time of both methods should also be investigated as a function of the quality.

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