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TRACE SPACES IN A PRE-CUBICAL COMPLEX

MARTIN RAUSSEN

ABSTRACT. In directed algebraic topology, (spaces of) directed irreversible (d)-paths are studied from a topological and from a categorical point of view. Motivated by models for concurrent computation, we study in this paper spaces of d-paths in a pre-cubical complex. Such paths are equipped with a natural arc length which moreover is shown to be invariant under directed homotopies. D-paths up to reparametrization (called traces) can thus be represented by arc length parametrized d-paths. Under weak additional conditions, it is shown that trace spaces in a pre-cubical complex are separable metric spaces which are locally contractible and locally compact. Moreover, they have the homotopy type of a CW-complex.

1. INTRODUCTION

1.1. **Background.** With motivations arising originally from concurrency theory within Computer Science, a new field of research, directed algebraic topology, has emerged. Its main characteristic is, that it involves spaces of "directed paths" (or timed paths, executions): these directed paths can be concatenated, but in general *not* reversed; time is not reversable.

A particular model in the investigation of concurrency phenomena leads to Higher Dimensional Automata (HDA); for a recent report describing and assessing those consult e.g. [25]. The underlying space in these models is then – instead of a directed graph – the geometric realization of a pre-cubical set; defined like a pre-simplicial complex, but with cubes as building blocks; cf. e.g. [4, 3]. Every cube carries a natural partial order, and directed paths have to respect the partial orders in their range. In the models, directed paths correspond to executions (calculations); they have two crucial properties:

(1) The reverse of a directed path is, in general, *not* directed;
(2) Directed paths that are *dihomotopic*, i.e., that can be connected through a one-parameter family of directed paths, are equivalent; this means that HDA - calculations along these "schedules" will always lead to the same result.

A nice and flexible framework for directed paths was introduced by Marco Grandis with the notion of d-spaces and, in particular, of d-paths (cf. Definition 2.2) on a topological space X. In [12] and the subsequent paper [11], Grandis developed a framework for directed homotopy theory. The best-studied invariant of such a d-space is its *fundamental category* replacing the fundamental group in ordinary algebraic topology. It is studied in [8] and in [10] and used to decompose the d-space into "components".

A study of other and higher invariants from algebraic topology in the framework of directed topology was initiated in [18]. The general idea is that one ought to study spaces of d-paths with given end points, to organise those in a categorical manner, and, in particular, to find out how the topology of these path spaces changes under variation of the end points.

At present, it is unsatisfactory that only few concrete calculations of algebraic topological invariants of d-spaces (i.e., oftheir path spaces) are known. For general d-spaces, this is probably a hopeless endeavour; as in ordinary algebraic topology, one needs additional structure. This article studies the topology of path spaces (or spaces homotopy equivalent to those) in pre-cubical complexes, i.e., those spaces that are of interest in the applications. It will be followed up in [20] by an attempt to give the path spaces in this case a combinatorial structure that makes calculations of invariants possible; most of the results from this paper will be needed to get going. A rough outline for this program will be sketched in Section 4.

1.2. **Definitions and results from previous work.** Spaces of paths *up to reparametrizations* (both undirected and directed) were studied in [6] - with a few corrections in [19]. For the convenience of the reader, we state important definitions and some of the results:

Definition 1.1. Let X denote a Hausdorff space.

(1) A reparametrization (of the unit interval I) is a *surjective* weakly increasing self-map $\varphi : I \to I$. The space of all reparametrizations (as subspace of the space of all self-maps I^I with the compact-open topology) is called $\mathrm{Rep}_+(I)$. This space is a monoid under composition.

(2) The *strictly* increasing reparametrizations form a sub*group* $\mathrm{Homeo}_+(I) \subset \mathrm{Rep}_+(I)$.

(3) The set X^I of all paths in X is denoted $P(X)$. For $x_0, x_1 \in X$, $P(X)(x_0, x_1)$ is the subspace of all paths p with $p(0) = x_0, p(1) = x_1$. Likewise, for $A_0, A_1 \subset X$, $P(X)(A_0, A_1)$ is the subspace of all paths p with $p(0) \in A_0, p(1) \in A_1$.

(4) Two paths p, q are called *reparametrization equivalent* if and only if there exist reparametrizations $\varphi, \psi \in \mathrm{Rep}_+(I)$ such that $p \circ \varphi = q \circ \psi$.

(5) A path $p \in P(X)$ is called *regular* if it is either constant or if there does not exist any non-trivial (stop)-interval $J = [a, b], 0 \le a < b \le 1$, such that $p|_J$ is constant.

It is shown in [6, Corollary 3.3], that reparametrization equivalence is in fact an equivalence relation. An equivalence class is called a *trace* in X. To divide out the effect of reparametrizations, we form the quotient space $T_P(X) = P(X)/{\mathrm{Rep}_+(I)}$ consisting of the traces in X. It is compared in [6] with the quotient $T_R(X)$ of the action of the group $\mathrm{Homeo}_+(I)$ on the space $R(X)$ of regular paths (regular traces). Both trace spaces fibre over $X \times X$ (end points!) with fibres $T_P(X)(x, y)$, resp. $T_R(X)(x, y)$. Using several non-trivial and even surprising lifting results, it is shown in [6, Theorem 3.6]:

Proposition 1.2. *For all $x, y \in X$, the inclusion map $R(X)(x, y) \hookrightarrow P(X)(x, y)$ induces a homeomorphism $T_R(X)(x, y) \to T_P(X)(x, y)$.*

Counterparts of these results for directed paths (and traces) were shown in [6] in the general framework of d-spaces [12], as well. In particular, we will work in this paper with spaces $\vec{T}(X)$ of directed traces and dipointed subspaces $\vec{T}(X)(x, y)$; we take the liberty to represent individual traces by d-paths or by regular d-paths as is suitable.

Trace spaces have several advantages compared to path spaces: First of all, they form a topological category (with pairs of points as objects). Secondly, they are often compact or at least locally compact.

1.3. **Summary of results.** In this paper, we restrict attention to path and trace spaces in pre-cubical complexes, the geometric realizations of □-sets, cf. [4, 3]. We give a definition for the *arc length* of a d-path in such a space that does not extend to general paths. In a similar manner as for smooth paths in differential geometry, one may reparametrize to obtain an arc length (also called natural) parametrization for such a d-path.

This simple idea and its consequences are studied in Section 2: It turns out that length is a *dihomotopy invariant* for d-paths with the same start and end point (unlike for arbitrary paths). Every directed trace (cf. Sect.1.2) has a well-defined arc length parametrized representative, and these are easier to work with than abstract traces. This will be exploited several times in Section 3. From a topological point of view, there is no need to distinguish between spaces of d-paths and traces in a pre-cubical complex: we show in Corollary 2.16, that spaces of d-paths, of regular d-paths and of traces (with given end points) are all homotopy equivalent to each other.

In Section 3, we study topological properties of trace spaces in a pre-cubical complex. We show that trace spaces (with elements viewed as naturally parametrized d-paths) are *metrizable* and in fact homeomorphic to *separable* metric spaces. Next, we show that all subspaces of a trace space consisting of traces of a bounded length are *relatively compact*. In particular, a trace space (in a finite pre-cubical complex) is itself *locally compact*, and every connected component (dihomotopy class) in a trace space with given end points is actually *compact*.

Moreover, we show that spaces of d-paths are *locally contractible*; otherwise it would be hopeless to aim for inductive calculations of algebraic topological invariants. The proof uses techniques introduced by John Milnor in [15], that also allow to show that a trace space in a pre-cubical complex has the homotopy type of a CW-complex. This comes in handy in [20] to prove that certain weak homotopy equivalences are actually honest homotopy equivalences.

In the final Section 4, we give a brief outlook on how the results of this paper will be used in [20] to find "condensed" models of path spaces in pre-cubical complexes up to homotopy equivalence.

2. TRACES IN A PRE-CUBICAL COMPLEX: ARC LENGTH PARAMETRIZATION AND CONSEQUENCES

2.1. **General definitions.** Properties of Higher Dimensional Automata (cf. Section 1.1) are intimately related to the study of directed paths in a pre-cubical set, also called a □-set; this term (cf. [7]) is used in a similar way as a Δ-set – as introduced in [21] –

for a simplicial set without degeneracies . We use $\square_n$ as an abbreviation for the n-cube $I^n = [0, 1]^n$ with the product topology.

Definition 2.1.   (1) A $\square$-set or pre-cubical complex M is a family of disjoint sets $\{M_n | n \geq 0\}$ with face maps $\partial_i^k : M_n \to M_{n-1}$, $n > 0$, $1 \leq i \leq n$, $k = 0, 1$, satisfying the pre-cubical relations $\partial_i^k \partial_j^l = \partial_{j-1}^l \partial_i^k$ for $i < j$.
  (2) A pre-cubical complex M is called *non-self-linked* (cf. [9]) if, for all n, $x \in M_n$ and $0 < i \leq n$, the $2^i \binom{n}{i}$ iterated faces $\partial_{l_1}^{k_1} \cdots \partial_{l_i}^{k_i} x \in M_{n-i}$, $k_i = 0, 1$, $1 \leq l_1 < \cdots < l_i \leq n$, are all different.
  (3) The geometric realization $|M|$ of a pre-cubical set M is given as the quotient space $|M| = (\coprod_n M_n \times \square_n)_{/\equiv}$ under the equivalence relation induced from

$$(\partial_i^k(x), t) \equiv (x, \delta_i^k(t)), \quad x \in M_{n+1}, \quad t = (t_1, \ldots t_n) \in \square_n$$

$$\text{with} \quad \delta_i^k(t) = (t_1, \ldots t_{i-1}, k, t_{i+1}, \ldots, t_n).$$

In a non-self-linked pre-cubical complex, the map $\square_n \simeq \square_n \times e \to |M|$ is *injective* for every n-cell $e \in M_n$. In particular, every element $m \in |M|$ in the image of this map has *uniquely* determined coordinates in $\square_n$, cf. [9]. Moreover, every element $x \in |M|$ has a unique *carrier* cell $e(x) \in M_n, n \geq 0$ such that x comes from an element in the *interior* $\square_n^o$ under the restriction of the quotient map to $\square_n \times e(x)$.

In Section 3, we will make use of particular open sets in $|M|$, the *open stars* of *vertices* in M_0. The open star $St(x, M)$ of $x \in M_0$ consists of the union of the interiors of all cells of which x is a vertex. It was shown in [9], that every such open stars inherits a consistent partial order from the partial orders on individual cells given by their identification with $\square_n \subset \mathbf{R}^n$.

Definition 2.2.   (1) A continuous path $p = (p_1, \ldots, p_k) : I \to \square_k$ is a *d-path* if every component $p_i : I \to I$, $1 \leq i \leq k$, is (not necessarily strictly) *increasing*.
  (2) A continuous path $p : I \to |M|$, M a $\square$-set, is a *d-path* if, for every $J \subseteq I$ such that ps restriction $p_J : J \to |M|$ has a lift $\hat{p}_J^e : J \to e \times \square_k, e \in M_k$, that lift $\hat{p}_J^e$ is a d-path; alternatively, if p can be decomposed as $p = |p_1| * \cdots * |p_k|$ with every p_i a *d-path* in a cell $e_{n_i} \times \square_{n_i}$.
  The set of *all* d-paths in $|M|$ will be denoted $\vec{P}(|M|) \subset |M|^I$. It inherits a topology from the CO-topology on $|M|^I$.

Remark 2.3. In every metric on $|M|$ chosen as in Section 2.2.1, this topology on $\vec{P}(|M|) \subset |M|^I$ corresponds to the uniform convergence topology.

Proposition 2.4. *The pair* $(|M|, \vec{P}(|M|))$ *is a d-space (in the sense of M. Grandis paper* [12]*).*

Proof. It is clear that $\vec{P}(|M|)$ contains the constant paths, that it is closed under concatenation and under increasing reparametrizations. $\square$

From now on, a pre-cubical complex X will be understood as the geometric realization of some non-self-linked $\square$-set M, with the d-space structure introduced above. For a more general discussion of cubical sets, we refer to [13].

2.2. **Arc length and dihomotopies.**

We will in the following also consider paths and in particular, generalizing Definition 2.2, d-paths in a pre-cubical complex X, defined on *arbitrary* intervals $[a, b] \subset [0, \infty[$. Moreover, we need to consider *Moore paths* $p : [0, \infty[\to X$ that are constant after some (arbitrary) parameter $T \geq 0$: $p(t) = p(T)$ for $t \geq T$; $p(T)$ will be called the *end point* for a Moore path. It is clear that any path defined on a finite interval $[0, T]$ can be considered as a Moore path by constant extension from its endpoint $p(T)$. For Moore paths in general compare e.g. [1]. The set $\vec{P}_M(X)$ of *all* Moore d-paths in X inherits a topology from the CO-topology on $X^{[0,\infty[}$.

2.2.1. *Trace spaces are metrizable.* It is well-known, that the topology of the geometric realization $|X|$ of a $\square$-set X is metrizable: Choose compatible metrics d_n on $\square_n$ each inducing the standard topology and such that $d_{n+1}((\mathbf{x}, t), (\mathbf{y}, t)) = d_n(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \square_n$ and $t \in I$. The inner pseudometric on the identification space $|X|$ given by the infimum of the length of chains in $|X|$ is then in fact a metric (on each connected component; cf. [2, 17]).

The compact-open topology on $\vec{P}(X)$ is induced from the supremum metric given by $d(p, q) = \max_{t \in I} d(p(t), q(t))$. Likewise, the compact-open topology on $\vec{P}_M(X)$ is inherited from the supremum metric given by $d(p, q) = \max_{s \in [0,\infty[} d(p(s), q(s))$. In the following, we will in particular use that the topology can be seen as inherited from the l^1- (also called Manhattan-) metric.

2.2.2. *Arc length.* We define the *length* of a d-path p in a non-self linked pre-cubial complex X as follows: For a d-path $p : J = [a, b] \to \square_k$ and $p(a) = (x_1^0, \ldots, x_n^0)$, $p(b) = (x_1^1, \ldots, x_n^1)$, we define $l_e(p) = \sum_1^n x_i^1 - x_i^0$ as the l^1-(or Manhattan) distance between the end points. Remark that arc length within a cell only depends on end points. It is non-negative, and positive for non-constant d-paths – at this point, it is essential that we restrict to d-paths. Moreover, it is additive under concatenation; this is why we choose the l^1-distance.

This definition of path length extends to d-paths on cells in a pre-cubical set, and for $e = \partial_i^k f$, $p \in \vec{P}(|e|)$, we get $l_e(p) = l_f(\delta_i^k \circ p)$. Hence the length of a general d-path $p \in \vec{P}(X)$ in a pre-cubical complex X may be defined as follows:

By Definition 2.2, every d-path $p \in \vec{P}(X)$ is a finite concatenation $p = |p_1| * \cdots * |p_k|$ of d-paths with p_i contained in *one* cell e_{n_i} for every i. The length $l(p) = \sum_1^k l_{e_{n_i}}(p_i)$ is well-defined and additive under concatenation. Moreover, it is invariant under reparametrization (by non-decreasing reparametrizations $\varphi \in Rep_+(I) \subset I^I$, cf. [6]) and hence an invariant of *traces* in $\vec{T}(X)$ (cf. Sect. 1.2).

Remark 2.5.   (1) The property $l(p) = d_1(x_1, x_0)$ defining arc length l for a path p with
$p(0) = x_0$, $p(1) = x_1$ *within a cell* holds actually for a path p *within the open star
of a vertex* $v \in X_0$.
   (2) Two d-paths in a pre-cubical complex with the *same* end points can have *different
arc lengths*. This happens certainly if there exists a non-trivial directed loop in the
complex, e.g., for an oriented circle (geometric realization of a $\square$-set with one cell
of dimensions 0 and 1 each).
   (3) Arc length increases by a factor 2 under cubical barycentric subdivision.

Altogether, for a given pre-cubical complex, arc length defines functions $l : \vec{P}(X) \rightarrow$
$\mathbf{R}_{\geq 0}$ on the space of d-paths and $l_M : \vec{P}_M(X) \rightarrow \mathbf{R}_{\geq 0}$ on the space of Moore d-paths in
X.

Proposition 2.6. *For a pre-cubical complex X, the length functions l and l_M are* continuous.

Proof. For a given d-path $p \in \vec{P}(X)$, there is a decomposition of the unit inverval $I =$
$[0, t_1] \cup [t_1, t_2] \cup \cdots \cup [t_{k-1}, t_k]$ such that $p([t_{i-1}, t_i]) \subseteq S_i$ for $0 < i \leq k$ with S_i denoting
the open star of some vertex v_i. Every d-path $q \in \vec{P}(X)$ close to p (in the compact-open
topology) satisfies likewise $q([t_{i-1}, t_i]) \subseteq S_i$ for $0 < i \leq k$.
   Extending the l^1-metric from each individual cell to a metric d_1 on all of X (cf. Sect. 2.2.1
or [2, 17]), we may ask that $d_1(p(t_i), q(t_i)) < \varepsilon$ for a given $\varepsilon > 0$ and $0 \leq i \leq k$.
With $p_i = p_{|[t_{i-1}, t_i]}$ and $q_i = q_{|[t_{i-1}, t_i]}$, we get: $|l(p_i) - l(q_i)| = |d_1(p(t_i), p(t_{i-1})) -$
$d_1(q(t_i), q(t_{i-1}))| \leq d_1(p(t_{i-1}), q(t_{i-1})) + d(p(t_i), q(t_i)) < 2\varepsilon$ and hence $|l(p) - l(q)| <$
$2n\varepsilon$. $\square$

Remark 2.7. It is *not* possible to extend the length function to a continuous function on
the space of all paths that are l^1-rectifiable on cells. Consider for instance a sequence of
square paths with the same central point (in a square in the Euclidean plane) with side
length $\frac{1}{n}$ and winding number n; each path has arc length 4, but the sequence converges
to a constant path of arc length 0 in the compact-open topology.

2.2.3. *Dihomotopies preserve arc length.* A one-parameter family of d-paths in a d-space
is called a *dihomotopy* (cf. [9]); formally:

Definition 2.8. Let X denote a d-space.
   (1) A continuous map $H : \vec{I} \times I \rightarrow X$ is called a dihomotopy if every map $H_t : \vec{I} \rightarrow$
X, $t \in I$, is a d-path.
   (2) If, moreover, $H(0, t)$ and $H(1, t)$ are *fixed* under the dihomotopy, then H is a
dihomotopy rel end points.

Remark 2.9. There are both more general and more special dihomotopies. D-homotopies
$H : X \times I \rightarrow Y$ of d-maps (preserving d-paths) between arbitrary d-spaces are inves-
tigated in e.g. [12, 18]. In order to obtain a van-Kampen theorem for the fundamental
category of a d-space, Grandis introduces a d-homotopy between d-paths in which the
paths $H(s, t)$ for given s are concatenations of paths that are either d-paths themselves

or reverses of d-paths ("zig-zags"). Lisbeth Fajstrup showed in [7] that the classification of d-paths in a (geometric) pre-cubical complex yields the same result whatever one considers classification up to ordinary dihomotopy (as in Definition 2.8) or up to zig-zag d-homotopy (as above).

Corollary 2.10. *For a dihomotopy* $H : \vec{I} \times I \to X$ *of d-paths in a pre-cubical complex, the length map* $l(H) : I \to \mathbf{R}_{\geq 0}$, $t \mapsto l(H_t)$ *is continuous.*

Proof. This follows from Proposition 2.6 since the map $H^* : I \to \vec{P}(X)$, $t \mapsto H_t$ is continuous. $\qquad\square$

In ordinary topology, the length of paths can grow arbitrarily in a homotopy class with fixed end points; just concatenate with nulhomotopic zig-zag paths! But for dihomotopies of *d-paths* in precubical complexes this is not at all the case:

Proposition 2.11. *D-paths in a precubical complex X with the same start and end point that are dihomotopic rel end points have the* same arc length.

Proof. First, we restrict attention to the case in which both start point x and end point y are *vertices* in the 0-skeleton X_0 of X. It is easy to see that the arc length $l(p)$ of every d-path $p \in \vec{P}(X)(x, y)$ is then an *integer*. By Corollary 2.10, arc length varies continuously along a dihomotopy; hence it needs to be constant.

In the general case, let x_0 denote the minimal vertex in the carrier cell $e(x)$ containing the start point x and y_0 the maximal vertex in the carrier cell $e(y)$ containing the end point y (cf. Sect. 1.2). Choose d-paths σ from x_0 to x and τ from y to y_0 that are contained in these carrier cells; their length is independent of the particular choice. If $p, q \in \vec{P}(X)(x, y)$ are d-homotopic, then $\tau * p * \sigma$, $\tau * q * \sigma \in \vec{P}(X)(x_0, y_0)$ are d-homotopic, as well. Hence they have the same length $l(\tau) + l(p) + l(\sigma) = l(\tau) + l(q) + l(\sigma)$, whence $l(p) = l(q)$. $\qquad\square$

Remark 2.12. As an immediate corollary of the proof above, remark that the fractional parts of the lengths of all dipaths with given start and end point agree; all length differences are integers!

2.3. **Natural parametrization of a d-path in a pre-cubial complex.** For a d-path $p : [0, l] \to X$ or $p : [0, \infty[\to X$, let $p_t : [0, t] \to X$, $t \leq l$, denote the sub-d-path obtained by restriction of the domain. Then the function $l_p : [0, l] \to [0, l(p)]$, $t \mapsto l(p_t)$, is contained in the space $\vec{P}(\mathbf{R}_{\geq 0})(0)$ consisting of weakly increasing paths sending 0 into 0 and equipped with the compact-open topology. For a Moore d-path defined on $[0, \infty[$, $l(p_t) = l(p)$ for $t \geq T$ if p is constant on $[T, \infty[$. Furthermore, we define length functionals $L : \vec{P}(X) \to \vec{P}_M(\mathbf{R}_{\geq 0})(0)$ and $L : \vec{P}_M(X) \to \vec{P}_M(\mathbf{R}_{\geq 0})(0)$ by $L(p) = l_p$. From Proposition 2.6 we deduce:

Lemma 2.13. *The length functionals L on* $\vec{P}(X)$ *and on* $\vec{P}_M(X)$ *are continuous.* $\qquad\square$

Definition 2.14.   (1) A d-path $p : [0, l] \rightarrow [0, l]$ d-path is called *natural* if $l(p_t) = t$
     for all $t \in [0, l]$; in particular, $l = l(p)$. A Moore d-path $p : [0, \infty[\rightarrow X$ is called
     natural if, for some finite T: $l(p_t) = t$ for all $t \leq T$ and $p(t) = p(T)$ for $t \geq T$.
  (2) The space of all natural d-paths in X (with the topology inherited from the compact-
     open topology on the space of Moore d-paths as explained above) is denoted
     $\vec{N}(X) \subset \vec{P}_M(X)$ – with subspaces $\vec{N}(X)(A_0, A_1)$, resp. $\vec{N}(X)(x_0, x_1)$ of d-paths
     starting in $A_0 \subseteq X$, resp. $x_0 \in X$, and ending in $A_1 \subseteq X$, resp. $x_1 \in X$.

To compare $\vec{P}(X)$ and $\vec{N}(X)$, we consider the following two maps:

   normalization: *norm* $: \vec{P}_M(X) \rightarrow \vec{P}(X)$, $norm(p)(t) = p(l(p) \cdot t)$.
   Normalization restricts to a map *norm* $: \vec{N}(X) \subset \vec{R}_M(X) \rightarrow \vec{R}(X)$ on the space
   $\vec{R}_M(X)$ of *regular* Moore d-paths.
   naturalization: *nat* $: \vec{P}(X) \rightarrow \vec{N}(X)$, $nat(q)(s) = q(l_q^{-1}(s))$.

In general, $l_q^{-1}(s)$ may be a non-trivial interval; nevertheless, $q(l_q^{-1}(s))$ is well-defined,
and $nat(q)$ is continuous; compare Proposition 3.7 in [6] and Proposition 2.2 in [19].

Proposition 2.15.   (1) *The map nat is invariant under the action of the monoid $Rep_+(I)$ of*
     non-decreasing reparametrizations on $\vec{P}(X)$.
  (2) *The maps norm and nat are continuous.*

Proof.   (1) Arc length is invariant under this action.
  (2) The continuity of *norm* follows from Proposition 2.6.
     To prove the continiuity of *nat* at $p \in \vec{P}(X)$, consider an ε-neighborhood U of p
     in the metric inherited from the l_1-metric on the individual cells (cf. Sect. 2.2.1).
     As in the proof of Proposition 2.6, there is an integer n such that $|l(p_t) - l(q_t)| <$
     $2n\varepsilon$ for all $t \in I$ and $q \in U$. Given $0 \leq s \leq l(p)$, choose $t_1, t_2 \in I$ such that
     $nat(p)(s) = p(t_1)$ and $nat(q)(s) = q(t_2)$; in particular, $l(p_{t_1}) = l(q_{t_2}) = s$. Then
     $d_1(nat(p)(s), nat(q)(s)) = d_1(p(t_1), q(t_2)) \leq d_1(p(t_1), p(t_2)) + d_1(p(t_2), q(t_2)) \leq$
     $|l(p_{t_2}) - l(p_{t_1})| + \varepsilon < (2n + 1)\varepsilon$.
                                                                              $\square$

2.4. **Homeomorphisms and homotopy equivalences.** We use the two maps *norm* and
nat to show that the spaces of d-paths considered so far (for definitions, we refer to
Section 1.2) are all homotopy equivalent. It will be particularly useful (in Section 3) that
traces in non-self-linked pre-cubical complexes can be represented by *natural* d-paths up
to homeomorphism. Remark that the topology of path spaces first becomes interesting
after restriction to both end points – the space $\vec{P}(X)$ itself is homotopy equivalent to X!

Proposition 2.16. *Let X be a non-self-linked pre-cubical complex and let $A_0, A_1 \subset X$.*
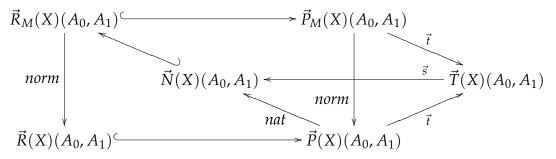
(1) *The diagrams*

$$
\begin{array}{ccc}
\vec{P}(X)(A_0, A_1) & \xrightarrow{\ L\ } & \vec{P}(\mathbf{R})(0) \\
{\scriptstyle nat}\downarrow & & \downarrow{\scriptstyle end\ point} \\
\vec{N}(X)(A_0, A_1) & \xrightarrow{\ l\ } & \mathbf{R}_{\geq 0}
\end{array}
\qquad
\begin{array}{ccc}
\vec{R}(X)(A_0, A_1) & \xrightarrow{\ L\ } & \vec{R}(\mathbf{R})(0) \\
{\scriptstyle nat}\downarrow & & \downarrow{\scriptstyle end\ point} \\
\vec{N}(X)(A_0, A_1) & \xrightarrow{\ l\ } & \mathbf{R}_{\geq 0}
\end{array}
$$

are pullback diagrams of topological spaces.

(2) *The restriction* $t_N : \vec{N}(X)(A_0, A_1) \to \vec{T}(X)(A_0, A_1)$ *of the trace map* $\vec{t} : \vec{P}_M(X)(A_0, A_1) \to \vec{T}(X)(A_0, A_1)$ *is a* homeomorphism *(with inverse* $\vec{s} : \vec{T}(X)(A_0, A_1) \to \vec{N}(X)(A_0, A_1)$*).*

(3) *All maps in the diagram*

are homotopy equivalences.

Proof.   (1) The inverse map from the fibered product into $\vec{P}(X)(A_0, A_1)$ is given by $(q, \varphi) \mapsto norm(q \circ \varphi)$.

(2) We define a section $\vec{s} : \vec{T}(X)(A_0, A_1) \to \vec{N}(X)(A_0, A_1)$ by $\vec{s}(\vec{t}(p)) = nat(p)$ for $p \in \vec{P}(X)(A_0, A_1)$. It is well-defined by Proposition 2.15; it is continous since *nat* is so, and since $\vec{T}(X)(A_0, A_1)$ has the quotient topology. It is trivial to check that the maps $\vec{t}_N$ and $\vec{s}$ are inverse to each other.

(3) By definition, *nat* ∘ *norm* is the identity on $\vec{N}(X)(A_0, A_1)$. Using the pullback diagram (1), the self map *norm* ∘ *nat* on $\vec{N}(X)(A_0, A_1) \times_{\mathbf{R}_{\geq 0}} \vec{P}(\mathbf{R}_{\geq 0})(0)$ maps (q, φ) into $(q, L(id_{|[0, l(q)]}))$. A homotopy to the identity is given by $(q, \varphi; t) \mapsto (q, (1 - t) \cdot \varphi + t \cdot L(id_{|[0, l(q)]}))$.

$\square$

Remark 2.17. The only tool used heavily in this section is that of a continuous additive length functional on the space of d-paths. It should be possible to recover the results of this section for d-spaces with such a length function in greater generality.

3. GENERAL PROPERTIES OF TRACE SPACES

3.1. **Trace spaces as metric spaces.** In Section 2.2.1 we referred to that the compact-open topology on d-path spaces $\vec{P}(X)$ and $\vec{P}_M(X)$ and their subspaces is the uniform convergence topology with respect to a metric on X.

3.1.1. *Paracompactness.*

Corollary 3.1. *For a pre-cubical complex X and for $A_0, A_1 \subset X$, the spaces $\vec{N}(X)(A_0, A_1)$, $\vec{T}(X)(A_0, A_1)$, $\vec{R}(X)(A_0, A_1)$ and $\vec{P}(X)(A_0, A_1)$ are all Hausdorff and paracompact.*

Proof. A metric space is Hausdorff (obvious) and paracompact [24, 22].   □

3.1.2. *Separability.*

Proposition 3.2. *For a finite non-self-linked pre-cubical complex X, the spaces $\vec{N}(X) \subset \vec{R}(X) \subset \vec{P}(X)$ and $\vec{T}(X)$ are all separable metric spaces. Likewise subspaces such as $\vec{N}(X)(A_0, A_1)$, $\vec{R}(X)(A_0, A_1)$, $\vec{P}(X)(A_0, A_1)$ and $\vec{T}(X)(A_0, A_1)$.*

In the proof, we make use of

Lemma 3.3. *Given two d-paths $p, q \in \vec{P}(X)$ in a finite dimensional pre-cubical complex X sharing start point and carrier sequence. Then $|l(p) - l(q)| < \dim X$. If they share the same end point, they have equal length.*

Proof. First, we arrive at d-paths p^* and q^* with the same end point by concatenating with linear d-paths to the supremum of the end points within the (same) final cell. Denote the sequence of carriers by $e_1, \ldots, e_k$ and choose $0 \le t_1 \le \cdots \le t_k$ such that $p(t_i) \in e_i$, $1 \le i \le k$. After a reparametrization, we may assume that $q(t_i) \in e_i$, as well (cf. [6]). Since $e_i \subset e_{i+1}$ or $e_i \supset e_{i+1}$, the restrictions of both p^* and of q^* to $[t_i, t_{i+1}]$ are contained in the same cell, and the two paths are therefore dihomotopic by a cellwise linear dihomotopy. By Proposition 2.11, $l(p^*) = l(q^*)$, and hence $|l(p) - l(q)| \le d_1(p(1), q(1)) < \dim X$. If p and q share the same end point, then $p = p^*$ and $q = q^*$.   □

Proof of Proposition 3.2: At first, we concentrate on the space $\vec{N}(X)$. From Proposition 2.16(2), we then get separability of the space $\vec{T}(X)$ for free. In [7], L. Fajstrup described an explicit method to approximate a given d-path p in a pre-cubical complex X by a (dihomotopic) d-path q on the *1-skeleton* of that complex; this approximation may be given a natural parametrization, as well. Such an approximation is not well-determined, in general; on the other hand, every such approximation has the property that it shares the carrier sequence with the original d-path. We will now show, that p and q cannot be too far apart in the l^1-metric:

Choose $0 \le t_1 \le \cdots \le t_k$ and $0 \le u_i \le t_i$ such that $p(t_i) \in e_i \cap e_{i+1}, 1 \le i < k$, and such that $q(u_i)$ is the minimal vertex in $e_i \cap e_{i+1}$. Let p_i, resp. q_i, denote the restrictions of p to $[0, t_i]$, resp. q to $[0, u_i]$ and $q_i^* : [0, t_i] \to X$ the concatenation of q_i with the arc length parametrized linear path connecting $q(u_i)$ and $p(t_i)$. The d-paths p_i and q_i^* share start and end point and carrier sequence; by Lemma 3.3, $u_i \le t_i = l(p_i) = l(q_i^*) = u_i + d_1(q(u_i), p(t_i)) < u_i + \dim X$. Hence $d_1(p(t_i), q(t_i)) \le d_1(p(t_i), q(u_i))) + l(q_{|[u_i, t_i]}) \le 2 \dim X$. Finally, for $t \in [t_i, t_{i+1}]$, we have: $d_1(p(t), q(t)) \le l(p_{[t_i, t]}) + d_1(p(t_i), q(t_i)) + l(q_{[t_i, t]}) < 4 \dim X$.

Next, we apply this argument to the N-th cubical subdivision $X^{(N)}$ of the original complex X. We conclude – with q now on the 1-skeleton of the *subdivided* complex – from Remark 2.5(3) that $d_1(p(t), q(t)) \leq 4\frac{\dim X}{2^N}$ (with respect to the original distance function *before* subdivison).

For every pair of non-negative integers N and L, there is a *finite* set $\vec{N}_L(X_1^{(N)})$ of natural d-paths of length bounded by $L \in \mathbf{N}$ on the 1-skeleton of the N-th cubical subdivision of a finite pre-cubical complex X; these are in fact determined by the (finitely many) vertices that may occur as values $p(t)$ of such a path p at $t = \frac{k}{2^N}$, $0 \leq k \leq 2^N L$. The union $\bigcup_{N,L \in \mathbf{N}} \vec{N}_L(X_1^{(N)})$ of these path sets is countable and dense in $\vec{N}(X)$; hence, $\vec{N}(X)$ is separable.

By Proposition 2.16(1), the space $\vec{P}(X)$ can be viewed as a subspace of the product space $\vec{N}(X) \times \vec{P}(\mathbf{R}_{\geq 0})(0)$. The second factor is a subspace of the metric space $\mathbf{R}^I$ of continuous functions on the unit interval. This metric space is separable by the Weierstrass approximation theorem - polynomials with rational coefficients form a dense countable subset.

Finally, products of separable metric spaces are separable, and subspaces of metric spaces are separable. $\square$

3.2. **Trace spaces are (locally) compact.** A space of d-paths is never compact – unless it only contains constant paths. This is so since the space of reparametrizations $Rep_+(I) = \vec{P}(I)(0,1)$ is not compact; it is not even equicontinuous, a necessary condition for compactness by the Arzela-Ascoli theorem (cf. e.g. [5, 16]).

Trace spaces are in general not compact either. If the d-space X contains a non-trivial loop based at $x_0 \in X$, then the closed subspace $\vec{T}(X)(x_0, x_0)$ has d-paths of infinitely many lengths and thus by Proposition 2.11 infinitely many connected components whence it cannot be compact. But compactness results are available if one bounds the lengths of d-paths:

Let $\vec{N}_L(X) \subset \vec{N}(X)$ consist of all natural d-paths of length less than or equal to L introduced in Proposition 3.2. A subset $H \subseteq \vec{T}(X)$ is called *of bounded length* if there exists $L \geq 0$ such that $H \subseteq \vec{t}(\vec{N}_L(X))$.

In the following Proposition 3.4 and its corollaries, X will always denote a *finite* – hence compact – non-self-linked pre-cubical complex:

Proposition 3.4. *A subset $H \subseteq \vec{T}(X)$ of bounded length is relatively compact.*

Corollary 3.5. *Trace space $\vec{T}(X)$ is locally compact.*

Corollary 3.6. *For $x_0, x_1 \in X$, every d-homotopy class (connected component) in $\vec{T}(X)(x_0, x_1)$ is compact.*

Proof of Proposition 3.4: Via the homeomorphism $t_{\vec{N}}$, we regard H as a subspace of $\vec{N}_L(X) = \{p = p_1 * p_2 : [0, L] \to X| \ p_1 \text{ natural}, p_2 \text{ constant}\}$ and apply the Arzela-Ascoli theorem

(e.g. [5, 16]). The conditions are satisfied since X is compact and since $\vec{N}(X)$ – consisting of distance preserving paths – is clearly equicontinuous. □

3.3. **Trace spaces have the homotopy type of a CW complex.**

John Milnor investigated in [15] conditions on spaces that ensure that certain mapping spaces have the homotopy type of a CW-complex:[1] We check that these criteria can be applied to spaces of traces in a pre-cubical complex and conclude that those spaces have the homotopy type of a CW-complex. This allows us to conclude that a weak homotopy equivalence between trace spaces actually is a (strong) homotopy equivalence; this will be used several times in [20].

Definition 3.7. [15] A topological space A is called ELCX (equi locally convex) if there exists

   (1) a neighborhood U of the diagonal $\Delta A \subset A \times A$ and a map $\lambda : U \times I \to A$ satisfying
       $\lambda(a, b, 0) = a, \lambda(a, b, 1) = b$ for all $(a, b) \in U$, and $\lambda(a, a, t) = a$ for all $a \in A, t \in I$;
   (2) an open covering of A by sets V_β such that $V_\beta \times V_\beta \subset U$ and $\lambda(V_\beta \times V_\beta \times I) = V_\beta$.

Lemma 3.8. (a special case of [15], Lemma 4)
Every paracompact ELCX space has the homotopy type of a CW-complex. □

In fact, Milnor shows that a paracompact ELCX space is dominated by a simplicial complex and thus (see e.g. [14], Appendix, Proposition A.11) homotopy equivalenct to a CW-complex.

3.3.1. *Pre-cubical complexes are ELCX.*

Proposition 3.9. *A non-self-linked pre-cubical complex X is ELCX.*

In the proof, we need the following

Lemma 3.10. *There exists a continuous "average" map $m : \{(x, y) \in I^2 | \; | y - x | \neq 1\} \to I$ preserving (partial) orders[2] that satisfies for all (x, y) in the domain:*

   (1) *for $\alpha = 0, 1, \; x = \alpha$ or $y = \alpha$ implies $m(x, y) = \alpha$;*
   (2) *$m(x, x) = x$;*
   (3) *$\min(x, y) \leq m(x, y) \leq \max(x, y)$.*

Proof of Lemma 3.10: It is easy to check that the map $m(x, y) = \frac{\min(x, y)}{1 - |y - x|}$ (increasing linearly from 0 to 1 on parallels to the diagonal from the lower to the upper boundary of I^2) has properties (1) – (3). To check that it preserves partial orders, use either the heuristic description above or calculate partial derivatives. □

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[1] I am grateful to W. Lück (Univ. Münster) for drawing my attention to [15].
[2] a d-map in the terminology of [12]

Proof of Proposition 3.9: As in the proof of Lemma 2 in [15], let V_β denote the open star neighborhood (cf. Section 2.1) of a vertex β in the cubical complex and let $U = \bigcup_\beta V_\beta \times V_\beta$. For every $x \in X$, let $e(x)$ denote the carrier cell containing x in its interior. Below, we define an "average" map $\mu : U \to X$ with the property that $\mu(x, y) \in \overline{e(x)} \cap \overline{e(y)}$ for all $(x, y) \in U$ and $\mu(x, x) = x$ for all $x \in X$. Then, as in [15], the path $\lambda(x, y, t), t \in I$, is given as the concatenation of the canonical line parametrizations connecting first x with $\mu(x, y)$ within $e(x)$ and then $\mu(x, y)$ with y within $e(y)$.

For $(x, y) \in V_\beta \times V_\beta \subseteq U$, consider the nonempty cell $\overline{e(x)} \cap \overline{e(y)}$ (containing β) as the closure of an iterated face of both $e(x)$ and $e(y)$. Since X is non-self linked, we may assume (after reordering the coordinates) that $\overline{e(x)} = I^k \times I^l$, $\overline{e(y)} = I^k \times I^m$, that $(x_1, \ldots, x_k; x_1', \ldots, x_l') \in \overline{e(x)} \cap \overline{e(y)} \subset \overline{e(x)}$ if $x_i' = \alpha_i, 1 \leq i \leq l$, and that $(y_1, \ldots, y_k; y_1', \ldots, y_m') \in \overline{e(x)} \cap \overline{e(y)} \subset \overline{e(y)}$ if $y_j' = \beta_j, 1 \leq j \leq m$, for certain $\alpha_i, \beta_j \in \{0, 1\}$. Using the map m from Lemma 3.10, represent $\mu(x, y)$ by $(m(x_1, y_1), \ldots, m(x_k, y_k); \alpha_1, \ldots, \alpha_l) \in \overline{e(x)} \cap \overline{e(y)} \subset \overline{e(x)}$ (or by $(m(x_1, y_1), \ldots, m(x_k, y_k); \beta_1, \ldots, \beta_m) \in \overline{e(x)} \cap \overline{e(y)} \subset \overline{e(y)}$). Remark that $m(x_i, y_i)$ is defined, since x and y both belong to the *open* star of the common vertex β; in particular, $|x_i - y_i| < 1$. Property (1) of m in Lemma 3.10 makes sure that μ factors over the face relations and thus defines a continuous map on U.   □

Remark 3.11.   (1) Since the map m from Lemma 3.10 preserves partial orders, the map μ and therefore also λ in the proof of Proposition 3.9 are *directed* in the following sense: If $x \leq x'$ within $e(x)$ and $y \leq y'$ within $e(y)$, then $\mu(x, y) \leq \mu(x', y')$ within $\overline{e(x)} \cap \overline{e(y)}$ and thus $\lambda(x, y, t) \leq \lambda(x', y', t)$ within $\overline{e(x)}$, resp. $\overline{e(y)}$. This will be essential in the next Section 3.3.2.

(2) μ cell-preserving implies: $\lambda(x, y, t) \in \overline{e(x)}$ for $x \leq 0.5$ and $\lambda(x, y, t) \in \overline{e(y)}$ for $y \geq 0.5$. In particular, both maps preserve open stars of vertices.

Lemma 3.12. *With respect to the metric d_1 on X induced by the l^1-metric d_1 on each cell (cf. Sect. 2.2.1), the maps μ and λ satisfy for $(x, y) \in U, t \in I$:*

- $d_1(x, \mu(x, y)), d_1(y, \mu(x, y)) \leq d(x, y).$
- $d_1(x, \lambda(x, y, t)), d_1(y, \lambda(x, y, t)) \leq d(x, y).$

Proof. If x and y are contained in one cell, these properties follow immediately from Lemma 3.10(3). If not, represent $e(x)$ and $e(y)$ as in the proof of Proposition 3.9. Then $d(x, y) = \sum |x_i - y_i| + \sum |x_j' - \alpha_j| + \sum |y_k' - \beta_k| \geq \sum |x_i - m(x_i, y_i)| + \sum |x_j' - \alpha_j| = d(x, m(x, y))$ by Lemma 3.10(3); similarly for $d(y, m(x, y))$ and for the distances to $\lambda(x, y, t)$.   □

3.3.2. *Spaces of d-paths are ECLX.*

Proposition 3.13. *For every non-self-linked pre-cubical complex X and elements $x_0, x_1 \in X$, the spaces $\vec{P}(X)$ and $\vec{P}(X)(x_0, x_1)$ are ELCX.*

Proof. For a partition $I = I_1 \cup \cdots \cup I_k$ into finitely many closed intervals and a sequence $\beta_1, \ldots \beta_k$ of vertices in X, let $\vec{P}(X)(I_1, \ldots, I_k; \beta_1, \ldots \beta_k)$ denote the subspace of all d-paths p such that $p(I_j)$ is contained in the open star of β_j for $1 \le j \le k$; it is an open subspace in the topology induced on $\vec{P}(X)$ from the compact-open topology. Those subsets play the role of the V_β in Definition 3.7.

Let $U \subset \vec{P}(X) \times \vec{P}(X)$ denote the union of all "squares" of these open subspaces; obviously an open neighborhood of the diagonal in $\vec{P}(X)$. A continuous map $\Lambda : U \times I \to X^I$ is given by $\Lambda((p,q),t)(s) = \lambda(p(s), q(s), t)$. By Remark 3.11(1), the image of Λ is in fact contained in $\vec{P}(X)$. Since λ, by Remark 3.11(2), preserves open stars, Λ maps $\vec{P}(X)(I_1, \ldots, I_k; \beta_1, \ldots \beta_k) \times \vec{P}(X)(I_1, \ldots, I_k; \beta_1, \ldots \beta_k) \times I$ into $\vec{P}(X)(I_1, \ldots, I_k; \beta_1, \ldots \beta_k)$.

By Lemma 3.10(2), the map λ is constant on the diagonal. Hence the map Λ above preserves end points of d-paths along the parameter interval I, whence $\vec{P}(X)(x_0, x_1)$ is ELCX, as well. $\qquad\square$

3.3.3. *Spaces of d-paths and trace spaces have the homotopy type of a CW-complex.* A combination of Corollary 3.1, Proposition 3.13, Lemma 3.8 and Proposition 2.16 yields:

Proposition 3.14. *For every non-self-linked pre-cubical complex X and for all elements $x_0, x_1 \in X$, the spaces*

    (1) $\vec{P}(X)$ *and* $\vec{P}(X)(x_0, x_1)$
    (2) $\vec{T}(X)$ *and* $\vec{T}(X)(x_0, x_1)$

have the homotopy type of a CW-complex. $\qquad\square$

The interest here is in the di-pointed versions, since, as remarked in Section 2.4, $\vec{P}(X) \simeq \vec{T}(X) \simeq X$.

3.4. **Trace spaces are locally contractible.** Using similar techniques, we prove local contractibility of path spaces and of trace spaces; this property is necessary and gives hope for inductive calculations of algebraic topological invariants of such spaces. Note that there are two versions of local contractibility:

In a strongly locally contractible space Y, for every $y \in Y$ and every open neighborhood U of y, there exists a neighborhood of $V \subseteq U$ of y that contracts to y in V. In a weakly locally contractible space Z, for every open neighborhood U of z, there exists a neighborhood of $V \subseteq U$ of z that contracts to z in U.

Proposition 3.15. *For every non-self-linked pre-cubical complex X and for all elements $x_0, x_1 \in X$, the spaces $\vec{P}(X)$ and $\vec{P}(X)(x_0, x_1)$, are strongly locally contractible.*

Proof. For every d-path $p \in \vec{P}(X)$, there exist a partition $I_1, \ldots, I_k$ of the unit interval I and an open star sequence $V_{\beta_1}, \ldots, V_{\beta_k}$ such that the open set $\vec{P}(X)(I_1, \ldots, I_k; \beta_1, \ldots, \beta_k) \subset \vec{P}(X)$ contains p; see the proof of Proposition 3.13. Choose $\varepsilon > 0$ such that $U_\varepsilon(p) \subseteq \vec{P}(X)(I_1, \ldots, I_k; \beta_1, \ldots, \beta_k)$ (with respect to the metric induced by the infinity metric on

X and thus $\vec{P}(X)$; cf. Section 2.2.1). By Lemma 3.12, the map Λ from the proof of Proposition 3.13 restricts to a contraction $U_\varepsilon(p) \times \{p\} \times I$ to p *within* $U_\varepsilon(p)$.

The same proof applies to the relative case, since Λ preserves end points. □

Proposition 3.16. *For every non-self-linked pre-cubical complex X and for all elements $x_0, x_1 \in X$, the spaces $\vec{T}(X)$ and $\vec{T}(X)(x_0, x_1)$ are weakly locally contractible.*

Proof. As earlier, Proposition 2.16(2) allows us to represent $\vec{T}(X)$ by the homeomorphic space $\vec{N}(X)$ of arc length parametrized d-paths. The trouble is that the map Λ, defined on a neighborhood of the diagonal in $\vec{N}(X)$, will in general leave $\vec{N}(X)$. We have to replace it by $\Lambda_N : (U \cap \vec{N}(X)) \times I \to \vec{N}(X)$ with $\Lambda_N(p, q, t) = nat(\Lambda(p, q, t))$ with naturalization *nat* as defined in Section 2.3.

With $\varepsilon > 0$ chosen as in the proof of Proposition 3.15, we prove that Λ_N contracts $U_{\frac{\varepsilon}{2}}(p)$ to $p \in \vec{N}(X)$ within $U_\varepsilon(p)$: By Proposition 3.15, the contraction Λ itself preserves $U_{\frac{\varepsilon}{2}}(p)$. Using Lemma 3.12 and the argument comparing length functions of close dihomotopic d-paths from the proof of Proposition 3.2, we conclude for a given $q \in U_{\frac{\varepsilon}{2}}(p)$ for the lengths of all intermediate d-paths $\Lambda((q, p), t)$ at s:
$|L(\Lambda((q, p), t)(s)) - s| < \frac{\varepsilon}{2}$. Thus, the reparametrization $nat(\Lambda((q, p), t))$ differs only slightly from the original $\Lambda((q, p), t)$: There is a function $s(t)$ with $|s(t) - s| < \frac{\varepsilon}{2}$ for all $t \in I$ such that $nat(\Lambda((q, p), t))(s) = \Lambda((q, p), t)(s(t))$. As a result,
$d(p(s), nat(\Lambda((q, p), t))(s)) < d(p(s), p(s(t))) + d(p(s(t)), \Lambda((q, p), t)(s(t))) < \varepsilon$ for all $s, t \in I$. □

4. CONCLUSION AND FURTHER WORK

In Section 2, we have established that traces in pre-cubical complexes have a nice and useful representation by *natural* d-paths. Making use of this represenation, it is shown in Section 3 that trace spaces have nice topological properties: they can be viewed as separable metric spaces, they are locally contractible and locally compact, and they have the homotopy type of a CW-complex.

All these properties are applied in [20] to compare trace spaces with subspaces of particular traces that give rise to additional combinatorial structure; e.g., traces that are *piecewise linear*, i.e., linear on each individual cube in the range. Weak homotopy equivalences between several of these trace spaces will be established using Smale's version [23] of the Vietoris-Begle theorem; the properties of trace spaces granted by the results of this paper are needed as conditions to apply this theorem. In many cases, Proposition 3.14 will allow us to conclude moreover, that the spaces involved are actually homotopy equivalent.

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