Schur rings and non-symmetric association schemes on 64 vertices

Jørgensen, Leif Kjær

Publication date:
2008

Document Version
Publisher's PDF, also known as Version of record

Citation for published version (APA):
Jørgensen, L. K. (2008). *Schur rings and non-symmetric association schemes on 64 vertices*. Department of Mathematical Sciences, Aalborg University. Research Report Series No. R-2008-17

AALBORG UNIVERSITY

Schur rings and non-symmetric association schemes on 64 vertices

by

Leif Kjær Jørgensen

R-2008-17 November 2008

Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 99 40 80 80 ▪ Telefax: +45 98 15 81 29
URL: http://www.math.aau.dk

ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

Schur rings and non-symmetric association schemes on 64 vertices

Leif K. Jørgensen
Department of Mathematical Sciences
Aalborg University
Fr. Bajers Vej 7, 9220 Aalborg Øst, Denmark
leif@math.aau.dk

Abstract

In this paper we enumerate essentially all non-symmetric association schemes with three classes, less than 96 vertices and with a regular group of automorphisms. The enumeration is based on computer search in Schur rings. The most interesting cases have 64 vertices.

In one primitive case and in one imprimitive case where no association scheme was previous known we find several new association schemes. In one other imprimitive case with 64 vertices we find association schemes with an automorphism group of rank 4, which was previous assumed not to be possible.

1 Introduction

1.1 Association schemes with three classes

An association scheme may be viewed as a collection of graphs or binary relations but we give here a definition in terms of matrices.

An association scheme with d classes and on n vertices consists of a list of $n \times n$ matrices $A_0, A_1, \ldots, A_d$ with entries in $\{0, 1\}$ so that the following conditions are satisfied.

- $A_0 + A_1 + \ldots + A_d = J$ and $A_0 = I$, where I is the identity matrix and J is the all ones matrix,

- for every $i = 1, \ldots, d$, there exists i' so that $A_i^* = A_{i'}$ where A^* denotes the transposed (and complex conjugate) matrix of A,
- there exist numbers p_{ij}^k for all $i, j, k \in \{0, \ldots, d\}$ so that

$$A_i A_j = \sum_{k=0}^{d} p_{ij}^k A^k.$$

The matrices $A_1, \ldots, A_d$ are adjacency matrices of (undirected or directed) graphs $R_1, \ldots, R_d$ with common set X of n vertices. These graphs are also known as the relations of the association scheme.

If $i' = i$ for all i then the graphs are undirected the association scheme is said to symmetric, otherwise at least one of the graphs is directed and the association scheme is non-symmetric. If all graphs $R_1, \ldots, R_d$ are connected then the association scheme is said to be primitive, otherwise it is imprimitive.

A strongly regular graph with parameters (v, k, λ, μ) is a k-regular graph on v vertices so that the number of common neighbours of two distinct vertices x and y is λ if x and y are adjacent and μ otherwise. If R_1 is a strongly regular graph and R_2 is the complementary graph then R_1 and R_2 are the relations of a symmetric association scheme with two classes. Conversely, any relation of a symmetric association scheme with two classes is a strongly regular graph.

In this paper we consider non-symmetric association schemes with $d = 3$ classes. We assume that the graphs R_1, R_2, R_3 are enumerated such that $A_1^* = A_2$ and $A_3^* = A_3$, i.e., R_3 is an undirected graph. In this case R_3 will be a strongly regular graph with parameters $(v, k, \lambda, \mu) = (n, p_{33}^0, p_{33}^3, p_{33}^1)$. R_1 and R_2 are orientations of the complementary strongly regular graph.

In the primitive case very few non-symmetric association schemes with three classes are known. Iwasaki [10] found one example on 36 vertices. An infinite family of so-called directed distance regular graphs was constructed by Liebler and Mena [13]. These graphs appear as relation R_1 in a non-symmetric association schemes with three classes and with some specific parameters. The smallest graph in this family has 64 vertices and was first found by Enomoto and Mena [4].

A few feasible parameter sets for primitive non-symmetric association schemes with three classes and up to 50 vertices have been excluded, see [11], so that the first open cases are with 64 vertices, see Table 1.

Now consider the imprimitive case. If R_1 is disconnected then the association scheme essentially reduces to an association scheme with two classes on each connected component of R_1. Thus we will only consider the case where R_1 and R_2 are connected and R_3 is disconnected and then in fact each component of R_3 is a complete graph. We also do not consider association schemes that appear as a wreath product of an association scheme with two classes and an association scheme with one class. In the remaining case exactly half of the edges in R_1 between any two connected components, say U and W, of R_3 are directed from U to W.

In this case Goldbach and Claasen [7] (see also [12]) proved that if the undirected graph R_3 has m components on r vertices then r and m are even numbers so that $r-1$ divides $m-1$.

If $r = 2$ then its known that m is divisible by 4 and that the scheme is related to a non-symmetric association schemes with two classes. The $r = 2$ is not considered in this paper. If $r = m$ then the scheme is related to Bush-type Hadamard matrices, see [11] and [12].

Previously no examples were known with $2 < r < m$. The first open cases are $(r, m) = (4, 10)$ and $(r, m) = (4, 16)$.

The primary goal of this project is to search for non-symmetric association schemes with three classes and 64 vertices. However we enumerate essentially all non-symmetric association schemes with three classes and less than 96 vertices and with a regular group of automorphisms.

1.2 Automorphism groups

For a group G and a set $S \subset G$ the Cayley graph $\mathrm{Cay}(G, S)$ is the graph with vertex set G and with edge set

$$\{(x, y) \mid x^{-1}y \in S\}.$$

A group G is said to be a regular group of automorphisms of the graph R if it is a subgroup of the automorphism group of R and for any two vertices x and y of R there is a unique automorphism in G that maps x to y. It is well-known that a graph is isomorphic to a Cayley graph if and only if it has a regular group of automorphisms.

For a non-symmetric association scheme with three classes enumerated R_1, R_2, R_3 as above, the association scheme is uniquely determined by R_1 and the automorphism group of the association scheme is the automorphism group of R_1. Thus the association scheme has a regular group of automorphisms if and only if R_1 has a regular group of automorphisms.

The rank of an automorphism group G (or permutation group in general) acting transitively on the vertices is the number of orbits of the stabilizer G_x of a vertex x. $\{x\}$ is one of these orbits. For an association scheme with three classes it is particularly interesting if the automorphism group has rank 4, as this would means the automorphism group acts transitively on the vertices and on the edges of each of the relations R_1, R_2 and R_3.

1.3 S-rings

The idea is to search for association schemes on which a group G of order 64 acts as a regular group of automorphisms. This implies that each of the graphs R_i, $i = 1, \ldots, d$, is a Cayley graph $\mathrm{Cay}(G, S_i)$ for some sets $S_i \subset G$.

Let $\mathbb{Z}G$ be the group ring of G. For an arbitrary element $\lambda = \sum_{g \in G} \ell_g g \in \mathbb{Z}G$, where $\ell_g \in \mathbb{Z}$, we use the following notation $\lambda^* = \sum_{g \in G} \ell_g g^{-1}$.

A set $S \subset G$ corresponds to the element

$$\underline{S} = \sum_{g \in S} g$$

in $\mathbb{Z}G$.

From the definition of association schemes we immediately get the following proposition, see Bannai and Ito [1].

Proposition 1 *Let $S_0, S_1, \ldots, S_d$ be subsets of a group G, where $S_0 = \{1\}$ is the set consisting of the group identity. Then the graphs $Cay(G, S_i)$, $i = 1, \ldots, d$ form an association scheme if*

- $\underline{S_0} + \underline{S_1} + \ldots + \underline{S_d} = \underline{G}$ *in* $\mathbb{Z}G$,
- *for every $i = 1, \ldots, d$, there exists i' so that* $\underline{S_i}^* = \underline{S_{i'}}$,
- *there exist numbers p_{ij}^k for all $i, j, k \in \{0, \ldots, d\}$ so that*

$$\underline{S_i} \cdot \underline{S_j} = \sum_{k=0}^{d} p_{ij}^k \underline{S_k}.$$

If these conditions are satisfied then the subring of $\mathbb{Z}G$ spanned by $\underline{S_0}$, $\underline{S_1}$, ..., $\underline{S_d}$ is called an S-ring or Schur ring over G.

In particular, a Cayley graph $\mathrm{Cay}(G, S_1)$ with $\underline{S_1^*} = \underline{S_1}$ is strongly regular if and only if $\underline{S_1^*}\underline{S_1} = \underline{S_1}^2 = k\underline{S_0} + \lambda\underline{S_1} + \mu\underline{S_2}$, where $S_0 = \{1\}$ and $S_2 = G - S_0 - S_1$. This means that for an element $g \in S_1$ there are exactly λ pairs $s, t \in S_1$ so that $g = s^{-1}t$ and for $g \in S_2$ there are exactly μ such pairs. In this case S_1 is called a partial difference set with parameters $(v = |G|, k = |S_1|, \lambda, \mu)$.

For each group G of order 64, we want to find all partitions of G in sets $S_0 = \{1\}, S_1, S_2, S_3$ so that $S_1^* = S_2$ and $S_3^* = S_3$ and so that these sets span an S-ring.

A group G is called a B-group (or Burnside group) if every primitive association scheme on which G acts as a regular group of automorphisms has only one class. A classical result of Schur and Wielandt (see Wielandt [18]) states that every cyclic group of composite order is a B-group.

In particular, since the unique group of order 85 is cyclic we get the following.

Proposition 2 *There is no association scheme with parameter set no. 13 or 14 in Table 1 with a regular group of automorphisms.*

For primitive association schemes with 64 vertices, i.e., association schemes with parameter set no. 8, 9, or 10, we do not need to consider the case when G is a cyclic group. In fact, also for imprimitive non-symmetric association schemes with three classes (and with 64 vertices) Ma [14] excluded the case of cyclic groups. A few other groups (e.g. dihedral groups and dicyclic groups in the primitive case, see Ma [15]) can also be excluded. However we did not use this fact, because we still need to consider the majority of the 267 groups of order 64 in our computer search.

2 The algorithm

Suppose that a set of feasible parameters for a non-symmetric association scheme with three classes and 64 vertices is given, i.e., the intersection numbers p_{ij}^k are known, and let G be a group of order 64.

If there exists a partition of G in sets $S_0 = \{1\}, S_1, S_2, S_3$ so that $S_1^* = S_2$ and $S_3^* = S_3$ and these sets span an S-ring then $\mathrm{Cay}(G, S_3)$ is a strongly regular graph with parameters $(v, k, \lambda, \mu) = (64, p^0_{33}, p^1_{33}, p^3_{33})$ and thus S_3 is a partial difference set. The first step in the algorithm is to find all possibilities for S_3 using that it is a partial difference set. Since $g \in S_1$ if and only if $g^{-1} \in S_2$ and $S_1 \cap S_2 = \emptyset$, it follows that every involution of G belongs to S_3. If the number of involutions of G is greater than $k = p^0_{33}$ then the requested S-ring over G does not exist. Otherwise we use a backtracking algorithm to search for all possible ways to extend the set of involutions with pairs $\{g, g^{-1}\}$ to a set S_3 of k elements. We backtrack when some element $g \in G$ appears too many times as a quotient $x^{-1}y$ where $x, y \in S_3$.

For every candidate for S_3 we have a candidate for $S_1 \cup S_2$ and then we use a similar backtracking algorithm to find all partitions of this set in S_1 and S_2 that satisfy the condition for an S-ring.

We used the computer algebra system GAP [5] to create the multiplication tables of each of the 267 groups of order 64. The search based on these multiplication tables was done in a C-program.

For each possible set S_1 we then compute the Cayley graph $R_1 = \mathrm{Cay}(G, S_1)$. This list of graphs is then reduced to a set of non-isomorphic graphs. Each of these graphs were then investigated by GAP with share package GRAPE [17] and nauty [16].

3 Results

3.1 The primitive case

In Table 1, we give a list of all feasible parameter sets for primitive non-symmetric association schemes with three classes and at most 96 vertices. We include only those parameter sets that have not been excluded. For a more complete list see [11]. The enumeration of parameter sets are from this complete list.

Goldbach and Claasen [6] proved that the association scheme with parameter set no. 3 is unique. It has a large group of automorphisms (of rank 4) but can not be constructed from an S-ring.

We use the algorithm described in the previous section to enumerate S-rings with parameter sets no. 8, 9, 10 and 12.

Table 1: Feasible parameter sets for primitive non-symmetric association schemes with three classes and less than 100 vertices.

No.	R_3 parameters	p^1_{12}	p^3_{12}	scheme exists	S-ring exists	reference
3	$(36, 21, 12, 12)$	0	2	yes, 1	no	Iwasaki [10]
8	$(64, 35, 18, 20)$	4	2	yes	yes	Enomoto and Mena [4]
9	$(64, 27, 10, 12)$	4	6	YES	yes	Theorem 3
10	$(64, 21, 8, 6)$	7	6	?	no	
12	$(81, 30, 9, 12)$	9	5	?	no	
13	$(85, 20, 3, 5)$	13	8	?	no	
14	$(85, 14, 3, 2)$	13	20	?	no	
15	$(96, 38, 10, 18)$	3	4	?	?	
18	$(96, 76, 60, 60)$	16	10	?	?	

3.1.1 Parameter set no. 9

For parameter set no. 9 no association scheme was known previously. We find four such association schemes.

Theorem 3 *There are exactly four association schemes with parameter set no. 9 and with a regular group of automorphisms, see Table 2.*

There may be other association schemes with parameter set no. 9, but then they do not appear from an S-ring.

Each of the four association schemes are represented by a row in Table 2.

The second column in the table shows the order of the automorphism group of the association scheme (i.e., the automorphism group of the graph R_1). The third column is the rank of the action of the automorphism group. The fourth column shows which groups appear as regular subgroups of the automorphism group. G_i denotes group no. i in the GAP catalogue of groups of order 64 ([5]). G_2 is the abelian group $\mathbb{Z}_8 \times \mathbb{Z}_8$. G_3 is nonabelian. Thus each of the association schemes can be constructed from an S-ring over the groups G_2 and G_3, but not from any other groups of order 64. The last column shows whether the graphs R_1 and R_2 are isomorphic.

Table 2: Association schemes with parameter set no. 9 and with regular group of automorphsims.

No.	|Aut|	Rank	Regular Subgroups	$R_1 \cong R_2$
1	256	22	G_2, G_3	yes
2	768	8	G_2, G_3	yes
3	768	8	G_2, G_3	yes
4	256	22	G_2, G_3	yes

Construction. We give a construction of association schemes no. 2 and 3 in Table 2 from S-rings over $\mathbb{Z}_8 \times \mathbb{Z}_8$. The automorphism group of $\mathbb{Z}_8 \times \mathbb{Z}_8$ has order 1536. It has exactly two conjugate classes of cyclic subgroups of order 12, represented by $G = \langle g \rangle$ and $H = \langle h \rangle$ where $(a, b)^g = (a + b, a)$ and $(a, b)^h = (a + b, 5a)$. The orbit under the action of G on $\mathbb{Z}_8 \times \mathbb{Z}_8$ containing $(1, 0)$ is

$$\begin{aligned}(1,0)^G = \ & \{(1,0),(1,1),(2,1),(3,2),(5,3),(0,5),\\ & (5,0),(5,5),(2,5),(7,2),(1,7),(0,1)\}.\end{aligned}$$

And the orbit containing $(6, 0)$ is

$$(6,0)^G = \{(6,0),(6,6),(4,6),(2,4),(6,2),(0,6)\}.$$

The sets $S_1 = (1,0)^G \cup (6,0)^G$, $S_2 = (7,0)^G \cup (2,0)^G$, $S_3 = (1,2)^G \cup (1,3)^G \cup (4,0)^G$ form an S-ring generating association scheme no. 2.

Similarly, the sets $S_1 = (1,0)^H \cup (2,0)^H$, $S_2 = (7,0)^H \cup (6,0)^H$, $S_3 = (1,1)^H \cup (1,6)^H \cup (4,0)^H$ form an S-ring generating association scheme no. 3.

3.1.2 Parameter set no. 8

Theorem 4 *There is a unique association scheme with parameter set no. 8 and with a regular group of automorphisms, see Table 3.*

We give a similar table as above.

G_{55} is the abelian group $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$. This scheme was constructed by Enomoto and Mena [4] from $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$.

Table 3: Association schemes with parameter set no. 8 and with regular group of automorphsims.

No.	$\lvert\mathrm{Aut}\rvert$	Rank	Regular Subgroups	$R_1 \cong R_2$
1	896	6	G_{55}, G_{68}	yes

3.1.3 Parameter set no. 10

Theorem 5 *There is no association scheme with parameter set no. 10 and with a regular group of automorphisms.*

3.1.4 Parameter set no. 12

We find that there are exactly two strongly regular graphs with parameters $(81, 30, 9, 12)$ and with a regular group of automorphisms. But a non-symmetric association scheme with three classes and with a regular group of automorphisms can not be constructed from these graphs.

Theorem 6 *There is no association scheme with parameter set no. 12 and with a regular group of automorphisms.*

In this case abelian groups are easy to exclude. Since the Krein parameters are not integers, the dual of an S-ring over an abelian group can not exist and thus an S-ring over an abelian can not exist, see Bannai and Ito [1].

3.2 The imprimitive case

We consider imprimitive non-symmetric association schemes with three classes where R_3 is isomorphic to m disjoint copies of the complete graph K_r, and a vertex in one K_r has exactly $\frac{r}{2}$ out-neighbours in R_1 in any other K_r. We know that r and m are even and that $r-1$ divides $m-1$. For $r=2$ we know that m is divisible by 4 and that existence of such association schemes are equivalent to existence of a non-symmetric association scheme with two classes and with $m-1$ vertices, see [12].

In this paper we consider the case $r > 2$. Table 4 is a list of all posibilities with less than 100 vertices. It is known that for $r = m = 4$ there are exactly two association schemes. One of them has an intransitive automorphism

Table 4: Feasible imprimitive non-symmetric association schemes with three classes where R_3 consists of m components isomorphic to K_r, $r > 2$, $rm < 100$. In the case $r = m = 4$ there are exactly two association schemes.

r	m	p^1_{12}	p^3_{12}	scheme exists	S-ring exists	reference
4	4	2	2	yes, 2	yes	
6	6	6	6	yes	no	
4	10	8	6	?	no	
4	16	14	10	yes	yes	Theorem 9
8	8	12	12	yes	yes	Theorem 7
4	22	20	14	?	no	
6	16	21	18	?	?	

group. The other association scheme can be constructed from an S-ring over $\mathbb{Z}_4 \times \mathbb{Z}_4$ and from an S-ring over the non-abelian group which is no. 4 in the GAP catalogue of groups of order 16. This association scheme has automorphism group of order 96 with rank 4.

In the cases $(r, m) = (4, 10)$ and $(r, m) = (6, 6)$ our search shows that there are no S-rings. In the first of these cases no association schemes are known but in the second case we know of 4 association schemes, all with trivial automorphism group, see [11].

In the case where $r = m$ is a multiple of 4, Ionin and Kharaghani [8] constructed association schemes from Hadamard matrices of order m.

3.2.1 $r = m = 8$

We give here a complete list of imprimitive non-symmetric association schemes with three classes and with $r = m = 8$ that can be constructed from S-rings.

Theorem 7 *There are exactly 46 imprimitive association schemes with $r = m = 8$ and with a regular group of automorphisms.*

The set $S_3 \cup \{1\}$ is a subgroup of order 8. This subgroup contains all involutions of the regular group.

Table 5 shows those association schemes that are constructed from an S-ring over a group of order 64 where the involutions generate a subgroup of order 8.

Table 6 shows those association schemes that are constructed from an S-ring over a group where the involutions generate a subgroup of order 4.

No association schemes appear in both ways.

$G_2 \cong \mathbb{Z}_8 \times \mathbb{Z}_8$ *and* $G_{55} \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ *are abelian. All other regular groups are non-abelian.*

From Table 5 we see that some of these association schemes have a large automorphism group.

Corollary 8 *There are at least four imprimitive association schemes with* $r = m = 8$ *and with automorphism group of rank 4, i.e., the automorphism group acts transitively on the vertices and on the edges of each of the graphs* R_1, R_2, R_3.

Ito [9] claimed to prove that if an imprivitive association scheme with $r = m$ has automorphism group of rank 4 then $r = m = 4$. Clearly, this is not true. The association scheme no. 14 in Table 5 was actually constructed by Ito [9]. The association scheme no. 11 and no. 12 in Table 5 can be constructed from S-rings over $(\mathbb{Z}_4)^3$. Davis and Polhill [3] showed that these two association schemes belong to an infinite family of imprimitive association schemes with $r = m = 4^s$ constructed from S-rings over $(\mathbb{Z}_4)^s$. These association schemes all have rank 4 automorphism group.

3.2.2 $r = 4, m = 16$

We also give a complete list of imprimitive non-symmetric association schemes with three classes and with $(r, m) = (4, 16)$ that can be constructed from S-rings. This is the first time association schemes with $2 < r < m$ have been constructed.

Theorem 9 *There are exactly 40 imprimitive association schemes with* $r = 4$, $m = 16$ *and with a regular group of automorphisms.*

For 36 of these association schemes the full automorphism group has order 64, so that automorphism group is the unique regular group. There are four such schemes for each of the groups $G_{11}, G_{160}, G_{172}, G_{179}, G_{182}, G_{238}$ *and 12 for the group* G_{156}. *For each of these 36 association schemes,* R_1 *and* R_2 *are non-isomorphic.*

The remaining four association schemes are shown in Table 7.

All regular groups are non-abelian.

Table 5: Imprimitive association schemes with $r = m = 8$ and with a regular group of automorphisms. First case: involutions generate a subgroup of order 8.

No.	|Aut|	Rank	i: G_i regular subgroup	$R_1 \cong R_2$
1	512	16	9, 59, 63, 64, 68, 70, 72, 76, 79, 81	yes
2	10752	4	9, 20	yes
3	1536	10	9, 20	yes
4	192	24	20	no
5	192	24	20	no
6	512	16	20, 59, 63, 68, 70, 79, 81, 82	yes
7	64	64	20	no
8	64	64	20	no
9	768	12	55, 57, 59, 63, 65, 68, 72, 81, 82	yes
10	1536	10	55, 57, 59, 63, 64, 65, 68, 70, 72, 76, 79, 81, 82	yes
11	1792	4	55, 63, 68, 70, 79, 81	yes
12	5376	4	55, 64, 68, 72, 76, 82	yes
13	768	12	57, 59, 64, 65, 68, 70, 79, 81, 82	yes
14	10752	4	57, 59, 64, 65, 68, 70, 79, 81, 82	yes
15	256	22	57, 59, 63, 64, 68, 70, 72, 76, 79, 81, 82	yes
16	512	16	57, 59, 63, 68, 70, 72, 76, 79, 81, 82	yes
17	768	10	57, 59, 63, 64, 68, 70, 72, 76, 79, 81	yes
18	512	16	59, 64, 68, 70, 72, 79, 81	yes

Table 6: Imprimitive association schemes with $r = m = 8$ and with a regular group of automorphisms. Second case: involutions generate a subgroup of order 4.

No.	$\|\text{Aut}\|$	Rank	i: G_i regular subgroup	$R_1 \cong R_2$
19	256	22	2, 3, 28, 45	yes
20	256	22	2, 3, 28, 45	yes
21	128	40	11	yes
22	128	40	11	yes
23	128	40	13, 14	yes
24	128	40	13, 14	yes
25	128	40	15, 16	yes
26	128	40	15, 16	yes
27	256	22	28, 45, 122, 168, 175, 179	yes
28	256	22	28, 45, 122, 168, 175, 179	yes
29	128	40	43, 46	no
30	256	22	43, 46, 143, 156	no
31	64	64	120	no
32	64	64	120	no
33	64	64	122	no
34	64	64	122	no
35	64	64	143	no
36	64	64	143	no
37	64	64	143	no
38	64	64	143	no
39	64	64	156	no
40	64	64	156	no
41	64	64	160	no
42	64	64	160	no
43	64	64	168	no
44	64	64	168	no
45	64	64	172	no
46	64	64	172	no

Table 7: Imprimitive association schemes with $r = 4, m = 16$ and with a regular group of automorphisms.

No.	$\|\mathrm{Aut}\|$	Rank	i: G_i regular subgroup	$R_1 \cong R_2$
5	256	22	156, 158, 172,182	no
16	768	8	156, 158, 172,182	no
35	128	40	238, 245	yes
36	384	14	238, 245	yes

Figure 1: Two possibilities for the edges of R_1 directed from U to V.

3.2.3 $r = 4, m = 22$

Theorem 10 *There is no imprimitive non-symmetric association scheme with three classes with $r = 4$, $m = 22$ and with a regular group of automorphisms.*

3.2.4 $r = 4$, general properties

Suppose that U and V are two connected components of the graph R_3 of an imprimitive association scheme with $r = 4$. Then there are two possible isomorphism types for the edges of R_1 directed from U to V, see Figure 1. In one of the two association schemes with $r = m = 4$, both of these isomorphism types appear. This association scheme is intransitive. The other association scheme with $r = m = 4$ satisfies the assumption of the next theorem.

Theorem 11 *Suppose that an imprimitive association scheme with $r = 4$ has a regular group of automorphisms G in which the involutions generate a subgroup H of order 4.*

Then the edges of R_1 between any two components of R_3 are as shown in the left half of Figure 1.

Proof Since H is generated by the involutions of G, it is a normal subgroup. Let $s \in S_1$, where $R_1 = \mathrm{Cay}(G, S_1)$. Since there are no edges of R_1 joining two vertices of H, there are no edges of R_1 joining two vertices of the coset sH. Thus sH is a block of imprimitivity. Let $t \in sH \cup S_1$ be the other out-neighbour of 1 in sH. As $t \in sH = Hs$ there exists $h \in H$ so that $t = hs$. As $h^2 = 1$, $ht = s$. Thus are edges from h to $hs = t$ and to $ht = s$. The theorem follows by vertex transitivity. □

References

[1] E. Bannai and T. Ito, Algebraic Combinatorics. I. Benjamin/Cumming, Menlo Park, 1984.

[2] W. G. Bridges and R. A. Mena, Rational circulants with rational spectra and cyclic strongly regular graphs, *Ars Combin.* **8** (1979), 143–161.

[3] J. A. Davis and J. Polhill, Difference set constructions of DRADs and Association Schemes.

[4] H. Enomoto and R. A. Mena, Distance-regular digraphs of girth 4, *J. Combin. Th., Ser. B* **43** (1987), 293–302.

[5] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.10; 2007. (http://www.gap-system.org)

[6] R. W. Goldbach and H. L. Claasen, A primitive non-symmetric 3-class association scheme on 36 elements with $p^1_{11} = 0$ exists and is unique, *Europ. J. Combin.* **15** (1994), 519–524.

[7] R. W. Goldbach and H. L. Claasen, The structure of imprimitive non-symmetric 3-class association schemes, *Europ. J. Combin.* **17** (1996), 23–37.

[8] Y. J. Ionin and H. Kharaghani, Doubly regular digraphs and symmetric designs, *J. Combin. Th., Ser. A* **101** (2003) 35–48.

[9] N. Ito, Automorphism groups of DRADs, *Group Theory (Singapore, 1987)*, de Gruyter, Berlin (1989), 151–170.

[10] S. Iwasaki, A characterization of $\mathrm{PSU}(3, 3^2)$ as a permutation group of rank 4, *Hokkaido Math. J.* **2** (1973) 231–235.

[11] L. K. Jørgensen, Algorithmic Approach to Non-symmetric 3-class Association Schemes

[12] L. K. Jørgensen, G. A. Jones, M. H. Klin and S. Y. Song, Normally regular digraphs, association schemes and related combinatorial structures. In preparation.

[13] R. A. Liebler and R. A. Mena, Certain Distance regular digraphs and related rings of characteristic 4, *J. Combin. Th., Ser B* **47** (1988), 111–123.

[14] S. L. Ma, Schur rings and cyclic association schemes of class three, *Graphs Combin.* **5** (1989), 355–361.

[15] S. L. Ma, A survey of partial difference sets, *Designs, Codes and Cryptography* **4** (1994), 221–261.

[16] B. D. McKay, nauty user's guide (version 1.5), Technical report TR-CS-90-02. Australian National University, Computer Science Department, 1990.

[17] L. H. Soicher, GRAPE: a system for computing with graphs and groups. In: L. Finkelstein and W. M. Kantor, eds., Groups and Computation, *DIMACS series in Discrete Mathematics and Theoretical Computer Science* **11** (1993), 287–291.

[18] H. Wielandt, Finite permutation groups, Academic Press (1964).