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# Blahut-Arimoto Algorithm and Code Design for Action-Dependent Source Coding Problems

Kasper Fløe Trillingsgaard, Osvaldo Simeone, Petar Popovski and Torben Larsen

## Abstract

The source coding problem with action-dependent side information at the decoder has recently been introduced to model data acquisition in resource-constrained systems. In this paper, an efficient algorithm for numerical computation of the rate-distortion-cost function for this problem is proposed, and a convergence proof is provided. Moreover, a two-stage code design based on multiplexing is put forth, whereby the first stage encodes the actions and the second stage is composed of an array of classical Wyner-Ziv codes, one for each action. Specific coding/decoding strategies are designed based on LDGM codes and message passing. Through numerical examples, the proposed code design is shown to achieve performance close to the lower bound dictated by the rate-distortion-cost function.

## Index Terms

Rate-distortion theory, side information “vending machine”, Blahut-Arimoto algorithm, code design, LDGM, message passing.

## I. INTRODUCTION

The source coding problem in which the decoder can take actions that affect the availability or quality of the side information at the decoder was introduced in [1]. The problem generalizes the well-known Wyner-Ziv set-up and can be used to model data acquisition in resource-constrained systems, such as sensor networks. In the model studied in [1], each action is associated a cost

and the system design is subject to an average cost constraint. The information-theoretic analysis of the problem was fully addressed in [1]. In this paper, instead, we tackle the practical open issues, namely the computation of the rate-distortion-cost function and code design.

Specifically, the rate-distortion-cost function for the source coding problem with action-dependent side information was derived in [1]. However, no specific algorithm was proposed for its computation. A first contribution of this paper is to propose such an algorithm by generalizing the classical Blahut-Arimoto (BA) approach, which was introduced for the Wyner-Ziv problem in [2]. Convergence of the algorithm is also proved.

Moreover, while the theory in [1] demonstrates the existence of coding and decoding strategies able to achieve the rate-distortion-cost bound, practical code constructions have not been investigated yet. It is recalled that, for classical lossy source coding problems, codes that have been able to achieve rate-distortion bound include Low Density Generator Matrix (LDGM) codes [3], polar codes [4] and trellis-based quantization codes [5]. For the Wyner-Ziv problem, efficient codes include compound LDPC/LDGM codes [6] and polar codes [4]. A second contribution of this paper is hence the study of code design for source coding problems with action-dependent side information. As shown in [1], optimal codes for this problem have a successive refinement structure, in which the first layer produces the action sequence and the refinement layer uses binning to leverage the side information at the decoder. Here, we first observe that a layered code structure in which the refinement layer uses a multiplexing of separate classical Wyner-Ziv codes, one for each action, is optimal. This allows us to simplify the code structure with respect to the successive refinement strategy in [1]. LDGM-based codes with message passing encoding are designed and demonstrated via numerical results to perform close to the rate-distortion-cost function.

The paper is organized as follows. In Section II, the action-dependent source coding problem is described and results from [1] are summarized. In Section III, we describe the proposed algorithm for computation of the rate-distortion-cost function, and in Section IV, a practical code design

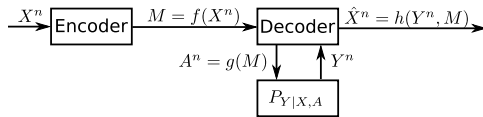


Fig. 1. Source coding with action-dependent side information.

is proposed. Finally, in Section V, we present numerical results for a specific example.

### A. Notation

Throughout this work, we let upper case, lower case and calligraphic letters denote random variables, values and alphabets of the random variables, respectively. For jointly distributed random variables,  $P_X(x)$ ,  $P_{X|Y}(x|y)$  and  $P_{X,Y}(x,y)$  denote the probability mass function (pmf) of  $X$ , the conditional pmf of  $X$  given  $Y$  and the joint pmf of  $X$  and  $Y$ . To simplify notation, the subscripts of the pmfs may be omitted, e.g.,  $P(x|y)$  may be used instead of  $P_{X|Y}(x|y)$ . The notation  $X^n$  represents the tuple  $(X_1, X_2, \dots, X_n)$ , and  $[a, b]$  where  $a, b \in \mathbb{Z}$  with  $a < b$  denotes the set of integers  $\{a, a + 1, \dots, b - 1, b\}$ . Moreover,  $\mathbb{Z}_+ = \{0, 1, \dots\}$ ,  $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$  and  $\mathbb{1}_{\{\text{cond}\}}$  denotes the indicator function, and is one when `cond` is true, and zero otherwise. The notation  $\lfloor \cdot \rfloor$  and  $\lceil \cdot \rceil$  denotes the floor and ceiling operators, respectively.

## II. BACKGROUND

In this section, we recall the definition of source coding problems with action-dependent side information and review the rate-distortion-cost function obtained in [1].

### A. System Model

The source coding problem with action-dependent side information introduced in [1] is illustrated in Fig. 1. In this problem, the source  $X^n \in \mathcal{X}^n$  is memoryless and each sample is distributed according to the pmf  $P_X$ . At the encoder, the encoding function

$$f : \mathcal{X}^n \rightarrow [1, [2^{nR}]], \quad (1)$$

maps the source  $X^n$  into a message  $M \in [1, \lfloor 2^{nR} \rfloor]$ , where  $R$  denotes the rate in bits per sample. At the decoder, an action sequence  $A^n \in \mathcal{A}^n$  is chosen according to an action strategy

$$g : [1, \lfloor 2^{nR} \rfloor] \rightarrow \mathcal{A}^n, \quad (2)$$

which maps the message  $M$  into an action sequence  $A^n$ . Based on  $A^n$ , the side information  $Y^n \in \mathcal{Y}^n$  is conditionally independent and identically distributed (iid) according to the conditional pmf  $P_{Y|X,A}$  so that we have

$$P_{Y^n|X^n, A^n}(y^n|x^n, a^n) = \prod_{i=1}^n P_{Y|X,A}(y_i|x_i, a_i). \quad (3)$$

The decoder makes a reconstruction  $\hat{X}^n \in \hat{\mathcal{X}}^n$  of  $X^n$  according to the decoding function

$$h : [1, \lfloor 2^{nR} \rfloor] \times \mathcal{Y}^n \rightarrow \hat{\mathcal{X}}^n, \quad (4)$$

which maps message  $M$  and side information  $Y^n$  into the estimate  $\hat{X}^n$ .

The action cost function  $\Delta(a) : \mathcal{A} \rightarrow \mathbb{R}_+$  is defined such that  $\Delta(a) = 0$  for some  $a \in \mathcal{A}$  and  $\Delta_{\max} = \max_{a \in \mathcal{A}} \Delta(a) < \infty$ , and the distortion function  $d(x, \hat{x}) : \mathcal{X} \times \hat{\mathcal{X}} \rightarrow \mathbb{R}_+$  is defined such that for each  $x \in \mathcal{X}$  there is an  $\hat{x} \in \hat{\mathcal{X}}$  satisfying  $d(x, \hat{x}) = 0$ . The rate-distortion-cost tuple  $(R, D, C)$  is then said to be achievable if and only if, for all  $\varepsilon > 0$ , there exist an encoding function  $f$ , an action function  $g$  and a decoding function  $h$ , for all sufficiently large  $n \in \mathbb{N}$ , satisfying the distortion constraint

$$\mathbb{E} \left[ \sum_{i=1}^n d(X_i, \hat{X}_i) \right] \leq n(D + \varepsilon) \quad (5)$$

and the action cost constraint

$$\mathbb{E} \left[ \sum_{i=1}^n \Delta(A_i) \right] \leq n(C + \varepsilon). \quad (6)$$

The rate-distortion-cost function, denoted as  $R(D, C)$ , is defined as the infimum of all rates  $R$  such that the tuple  $(R, D, C)$  is achievable.

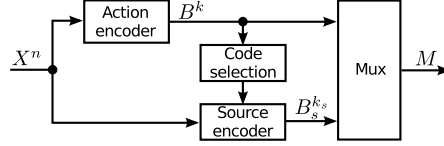


Fig. 2. Optimal encoder for source coding problems with action-dependent side information.

### B. Rate-Distortion-Cost Function

The rate-distortion-cost function  $R(D, C)$  was derived in [1] and is summarized below.

**Lemma 1.** ([1, Theorem 1]) *The rate-distortion-cost function for the source coding problem with action-dependent side information is given as*

$$R(D, C) = \min I(X; A) + I(X; U|Y, A), \quad (7)$$

$$P_{X,Y,A,U}(x, y, a, u) = P_X(x)P_{U|X}(u|x)\mathbb{1}_{\{\eta(u)=a\}}P_{Y|X,A}(y|x, a), \quad (8)$$

and the minimization is over all pmfs  $P_{U|X}$  and deterministic functions  $\eta : \mathcal{U} \rightarrow \mathcal{A}$  under which the conditions

$$\mathbb{E}[d(X, \hat{X}^{opt}(U, Y))] \leq D, \quad (9)$$

and

$$\mathbb{E}[\Delta(A)] \leq C \quad (10)$$

hold. The function  $\hat{X}^{opt} : \mathcal{U} \times \mathcal{Y} \rightarrow \hat{\mathcal{X}}$  denotes the best estimate of  $X$  given  $U$  and  $Y$ , i.e.,

$$\hat{X}^{opt}(u, y) = \arg \min_{\hat{x} \in \hat{\mathcal{X}}} \mathbb{E}[d(X, \hat{x})|U = u, Y = y]. \quad (11)$$

Moreover, the cardinality of the set  $\mathcal{U}$  can be restricted as  $|\mathcal{U}| \leq |\mathcal{X}||\mathcal{A}| + 2$ .

### C. Optimal Coding Strategy

The proof of achievability of the rate-distortion-cost function in [1] shows that an optimal encoder has the structure illustrated in Fig. 2 and consists of the following two steps.

- Action Coding: The source sequence  $X^n$  is mapped to an action sequence  $A^n$ . The action sequence is selected from a codebook  $\mathcal{C}_A$  of about  $2^{nI(X;A)}$  codewords, each type approximately equal to  $P_A$ . The index  $B^k$  identifies the selected codeword  $A^n$ , and hence consists of  $k$ , approximately equal to  $nI(X;A)$ , bits. The selection of  $A^n$  is done with the aim of ensuring that  $A^n$  and  $X^n$  are jointly typical with respect to the joint pmf  $P_{X,A}(x, a) = P_{A|X}(a|x)P_X(x)$ .
- Source Coding: Given the action sequence  $A^n$ , a source codebook is chosen out of a set of around  $2^{nI(X;A)}$  codebooks, one for each codeword in  $\mathcal{C}_A$ . Each codeword  $U^n$  in the selected source codebook has a joint type with  $A^n$  close to  $P_{A,U}$ , and the number of codewords is about  $2^{nI(X;U|A)}$ . The source sequence is mapped to a sequence  $U^n$  taken from the selected codebook with joint type  $P_{A,U}$  and with the objective of ensuring that  $X^n$ ,  $A^n$  and  $U^n$  are jointly typical with respect to the joint pmf  $P_{X,A,U}(x, a, u)$ . Each source codebook is divided into around  $2^{nI(X;U|A,Y)}$  subcodebooks, or bins, in order to leverage the side information at the receiver using Wyner-Ziv decoding.

The message  $M$  is given by the concatenation of the bits  $B^k$  and  $B_s^{k_s}$  and thus the overall rate of the action code and the source codes is given by (7). Upon receiving the message  $M$  from the encoder, the decoder first reconstructs the action sequence  $A^n$ . The action sequence is used to measure the side information  $Y^n$ . As  $A^n$  is known, the decoder also knows the source codebook from which  $U^n$  is selected, and  $U^n$  is then recovered by using Wyner-Ziv decoding based on the side information  $Y^n$ . In the end, the final estimate  $\hat{X}^n$  is obtained as  $\hat{X}_i = \hat{X}^{\text{opt}}(U_i, Y_i)$  for  $i \in [1, n]$ .

### III. COMPUTATION OF THE RATE-DISTORTION-COST FUNCTION

In this section, we first reformulate the problem in (7) by introducing Shannon strategies. This result is then used to propose a BA-type algorithm for the computation of the rate-distortion-cost function (7).

#### A. Shannon Strategies

We first observe that, from Lemma 1, it is sufficient to restrict the minimization to all joint distributions for which  $A$  is a deterministic function  $A = \eta(U)$ . Moreover, the final estimate of  $\hat{X}$  in (11) is a function of both  $U$  and  $Y$ . Based on these facts, we define a Shannon strategy  $T \in \mathcal{T} \subseteq \mathcal{X}^{|\mathcal{Y}|} \times \mathcal{A}$  as a vector of cardinality  $|\mathcal{Y}| + 1$ , in which the first  $|\mathcal{Y}|$  elements are indexed by the elements in  $\mathcal{Y}$  and  $T(y) \in \hat{\mathcal{X}}$  for  $y \in \mathcal{Y}$ , and the last element is denoted  $a(T) \in \mathcal{A}$ . We also define the disjoint sets  $\mathcal{T}^a = \{t \in \mathcal{T} : a(t) = a\}$  for all actions  $a \in \mathcal{A}$ . The rate-distortion-cost function (7) can be restated in terms of the defined Shannon strategies as formalized in the next proposition.

**Proposition 1.** *Let  $T \in \mathcal{T} \subseteq \mathcal{X}^{|\mathcal{Y}|} \times \mathcal{A}$  denote a Shannon strategy vector as defined above. The rate-distortion-cost function in (7) can be expressed as*

$$R(D, C) = \min I(X; a(T)) + I(X; T|Y, a(T)), \quad (12)$$

where the joint pmf  $P_{X,Y,T}$  is of the form

$$P_{X,Y,T}(x, y, t) = P_X(x)P_{T|X}(t|x)P_{Y|A,X}(y|a(t), x), \quad (13)$$

and the minimization is over all pmfs  $P_{T|X}$  under the constraints

$$\mathbb{E}[\Delta(A)] = \sum_{t \in \mathcal{T}, x \in \mathcal{X}} P_X(x)P_{T|X}(t|x)\Delta(a(t)) \leq C \quad (14)$$

and

$$\mathbb{E}[d(X, T(Y))] = \sum_{t \in \mathcal{T}, x \in \mathcal{X}, y \in \mathcal{Y}} P_{X,Y,T}(x, y, t)d(t(y), x) \leq D. \quad (15)$$



Moreover, the cardinality of the alphabet  $\mathcal{T}$  can be restricted as  $|\mathcal{T}| \leq |\mathcal{X}||\mathcal{A}| + 2$ .

*Proof:* Given an alphabet  $\mathcal{U}$ , a pmf  $P_{U|X}$  and a function  $\eta : \mathcal{U} \rightarrow \mathcal{A}$ , the sum of the two mutual informations in (7) can be seen to be equal to the sum of the two mutual informations in (12) and the average distortion and cost in (9) and (10) to be equal to (15) and (14), respectively, by defining  $P_{T|X}$  as follows. For each  $u \in \mathcal{U}$ , define a strategy  $t$  with  $P_{T|X}(t|x) = P_{U|X}(u|x)$  such that  $a(t) = \eta(u)$  and  $t(y) = \hat{X}^{\text{opt}}(u, y)$  for  $y \in \mathcal{Y}$ . ■

*Remark.* The characterization in Proposition 1 generalizes the formulation of the Wyner-Ziv rate-distortion function in terms of Shannon strategies given in [2].

The following lemma extends to the rate-distortion-cost function  $R(D, C)$  some well-known properties for the rate-distortion function (see, e.g. [7], [8]). This will be useful in the next section when discussing the computation of  $R(D, C)$ .

**Lemma 2.** *The following properties hold for the rate distortion cost-function  $R(D, C)$ :*

- 1)  $R(D, C)$  is non-increasing, convex and continuous for  $D \in [0, \infty)$  and  $C \in [0, \infty)$ .
- 2)  $R(D, C)$  is strictly decreasing in  $D \in [0, D_{\max}(C)]$  and  $R(D_{\max}(C), C) = 0$ , where

$$D_{\max}(C) = \min_{P_T} \sum_{t \in \mathcal{T}, x \in \mathcal{X}, y \in \mathcal{Y}} P_{X,Y,T}(x, y, t) d(t(y), x), \quad (16)$$

under the constraint

$$\mathbb{E}[\Delta(a(T))] = \sum_{t \in \mathcal{T}} \Delta(a(t)) P_T(t) \leq C. \quad (17)$$

- 3) For all  $D \in [0, D_{\max}(C)]$ , the minimum in (12) is attained when the distortion inequality (15) is satisfied with equality.

*Proof:* The lemma is proved by the arguments in [8, Lemma 10.4.1]. ■

## B. Computation of the Rate-Distortion-Cost Function

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**Algorithm 1** BA-type Algorithm for Computation of the Rate-Distortion-Cost Function
 

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**input:** Lagrange multipliers  $s \leq 0$  and  $m \leq 0$ .

**output:**  $R(D_{s,m}, C_{s,m})$  with  $C_{s,m}$  and  $D_{s,m}$  as in (19)-(20).

**initialize:**  $P_{T|X}$

**repeat**

    Compute  $Q_A$  as in (25).

    Compute  $Q_{T,Y}$  as in (26).

    Minimize  $F(P_{T|X}, Q_{T,Y}, Q_A)$  with respect to  $P_{T|X}$  using Algorithm 2.

**until** convergence

$P_{T|X}^* \leftarrow P_{T|X}$

---

In order to derive a BA-type algorithm to solve the problem in (12), we introduce Lagrange multipliers  $m$  for the cost constraint in (14) and  $s$  for the distortion constraint (15). The following proposition provides a parametric characterization of the rate-distortion-cost function in terms of the pair  $(s, m)$ .

**Proposition 2.** For each  $s \leq 0$  and  $m \leq 0$ , define the rate-distortion-cost tuple  $(R_{s,m}, D_{s,m}, C_{s,m})$  via the following equations

$$R_{s,m} = sD_{s,m} + mC_{s,m} + \min_{P_{T|X}} \{I(X; A) + I(X; T|Y, \mathbf{a}(T)) - s\mathbb{E}[d(X, T(Y))] - m\mathbb{E}[\Delta(\mathbf{a}(T))]\}, \quad (18)$$

$$C_{s,m} = \sum_{t \in \mathcal{T}, x \in \mathcal{X}} P_X(x) P_{T|X}^*(t|x) \Delta(\mathbf{a}(t)), \quad (19)$$

$$D_{s,m} = \sum_{t \in \mathcal{T}, x \in \mathcal{X}, y \in \mathcal{Y}} P_X(x) P_{T|X}^*(t|x) P_{Y|X,A}(y|x, \mathbf{a}) d(t(y), x), \quad (20)$$

where  $P_{T|X}^*$  denotes a minimizing pmf  $P_{T|X}$  for the optimization problem in (18). Then, the following facts hold

1) The tuple  $(R_{s,m}, D_{s,m}, C_{s,m})$  lies on the rate-distortion-cost function, i.e.,

$$R_{s,m} = R(D_{s,m}, C_{s,m}). \quad (21)$$

2) Every point  $(R, D, C)$  on the rate-distortion-cost function for  $D \in [0, D_{\max}(C)]$  can be written as (18)-(20) for  $s \leq 0$  and  $m \leq 0$ ;

3) The rate-distortion-cost function is given as

$$R(D, C) = \max_{\substack{s \leq 0 \\ m \leq 0}} (R_{s,m} + s(D - D_{s,m}) + m(C - C_{s,m})). \quad (22)$$

*Proof:* The proposition above follows by strong duality as guaranteed by Slater's condition [9, Section 5.2.3], and can also be derived directly as in [7]. ■

Given the proposition above, one can trace the rate-distortion-cost function by solving problem (18) and using (19) and (20) for all  $s \leq 0$  and  $m \leq 0$ . Inspired by the standard BA approach, we now show that problem (18) can be solved by using alternate optimization with respect to  $P_{T|X}$  and appropriately defined auxiliary pmfs  $Q_{T,Y}$  and  $Q_A$ . To do this, we define the function  $F(\cdot)$  of  $P_{T|X}$  and auxiliary pmfs  $Q_{T,Y}$  and  $Q_A$  as in (23),

$$\begin{aligned} F(P_{T|X}, Q_{T,Y}, Q_A) &= D_{KL}(P_{Y,A} || Q_A) - \sum_{x \in \mathcal{X}, y \in \mathcal{Y}, t \in \mathcal{T}} P_{X,Y,T}(x, y, t) \log P_{Y|X,A}(y|x, a(t)) \\ &+ \sum_{x \in \mathcal{X}} P_X(x) D_{KL}(P_{Y,T|X}(\cdot, \cdot | x) || Q_{T,Y}) - s \sum_{t \in \mathcal{T}, x \in \mathcal{X}, y \in \mathcal{Y}} P_{X,Y,T}(x, y, t) d(t(y), x) \\ &- m \sum_{t \in \mathcal{T}, x \in \mathcal{X}} \Delta(a(t)) P_X(x) P_{T|X}(t|x), \end{aligned} \quad (23)$$

where  $D_{KL}(P||Q)$  denotes the Kullback-Leibler (KL) divergence<sup>1</sup> and  $P_{X,Y,T}$ ,  $P_{Y,T|X}$  and  $P_{Y,A}$  are calculated from the joint pmf (13). We then have the following result.

**Proposition 3.** *For any  $s \leq 0$  and  $m \leq 0$ , we have*

$$R(D_{s,m}, C_{s,m}) = sD_{s,m} + mC_{s,m} + \min_{P_{T|X}, Q_{T,Y}, Q_A} F(P_{T|X}, Q_{T,Y}, Q_A), \quad (24)$$

<sup>1</sup>The Kullback-Leibler divergence [8] is defined as  $D_{KL}(P||Q) = \sum_i P(i) \log_2 \frac{P(i)}{Q(i)}$  for pmfs  $P$  and  $Q$ .

with (19)-(20), where the distribution  $P_{T|X}^*$  denotes a minimizing distribution in (24). Moreover, the function  $F(P_{T|X}, Q_{T,Y}, Q_A)$  is jointly convex in the pmfs  $P_{T|X}$ ,  $Q_{T,Y}$  and  $Q_A$ .

*Proof:* The proof technique for the first part is due to [10], and is based on showing that the pmf  $Q_A$  minimizing  $F(\cdot)$  for fixed  $Q_{T,Y}$  and  $P_{T|X}$  is

$$Q_A(a) = \sum_{x \in \mathcal{X}, t \in \mathcal{T}^a} P_X(x) P_{T|X}(t|x) = P_A(a), \quad (25)$$

and the pmf  $Q_{T,Y}$  minimizing  $F(\cdot)$  for fixed  $Q_A$  and  $P_{T|X}$  is given by

$$Q_{T,Y}(t, y) = \sum_{x \in \mathcal{X}} P_X(x) P_{Y|X,A}(y|x, a(t)) P_{T|X}(t|x) = P_{T,Y}(t, y). \quad (26)$$

The convexity of the function  $F(\cdot)$  follows from the log-sum inequality [8]. ■

Based on Proposition 3, the proposed BA-type algorithm for computation of the rate-distortion-cost function then consists of alternate minimizing (24) with respect to  $P_{T|X}$ ,  $Q_{T,Y}$  and  $Q_A$ . Due to the convexity of (24), the algorithm is known to converge to the optimal point similar to [2]. The proposed algorithm is summarized in Table Algorithm 1. The step of minimizing  $F(P_{T|X}, Q_{T,Y}, Q_A)$  with respect to  $P_{T|X}$  is discussed in the rest of this section.

### C. Minimizing $F$ over $P_{T|X}$

To minimize the function  $F(P_{T|X}, Q_{T,Y}, Q_A)$  with respect to  $P_{T|X}$  for fixed  $Q_A$  and  $Q_{T,Y}$ , we add a Lagrange multipliers  $\lambda_x$  for each equality constraints  $\sum_{t \in \mathcal{T}} P_{T|X}(t|x) = 1$  with  $x \in \mathcal{X}$ , and resort to the KKT conditions as necessary and sufficient conditions for optimality. This property of the KKT conditions follows by strong duality due to the validity of Slater's conditions for the problem [9, Section 5.2.3]. We assume  $P_X(x) > 0$  without loss of generality, since values of  $x$  with  $P_X(x) = 0$  can be removed from the alphabet  $\mathcal{X}$ .

By strong duality, we obtain the following optimization problem

$$\begin{aligned} \min_{\substack{P_{T|X} \geq 0 \\ \sum_{t \in \mathcal{T}} P_{T|X}(t|x) = 1}} F(P_{T|X}, Q_A, Q_{T,Y}) = \\ \max_{\{\lambda_x\} \in \mathbb{R}^{|\mathcal{X}|}} \min_{P_{T|X}} F(P_{T|X}, Q_A, Q_{T,Y}) + \sum_{x \in \mathcal{X}} \lambda_x \left( \sum_{t \in \mathcal{T}} P_{T|X}(t|x) - 1 \right). \end{aligned} \quad (27)$$

In the proposed approach, the outer maximization in (27) is then performed using the standard subgradient method. The inner minimization is instead performed by finding the stationary points of the function. This leads to the system of equalities  $g_{a|x}(P_{A|X}, \mu_x) = P_{A|X}(a|x)$  for  $a \in \mathcal{A}$  and  $x \in \mathcal{X}$ , with

$$g_{a|x}(P_{A|X}, \mu_x) = P_{A|X}(a|x)^\beta \left( \frac{2^{\mu_x} \alpha_{a,x}}{\prod_{y \in \mathcal{Y}} [\sum_{\tilde{x} \in \mathcal{X}} P_X(\tilde{x}) P_{Y|X,A}(y|\tilde{x}, a) P_{A|X}(a|\tilde{x})]^{P_{Y|X,A}(y|x,a)}} \right)^{1-\beta}, \quad (28)$$

where

$$\alpha_{t,x} = Q_A(a(t)) 2^{m\Delta(a(t))} \cdot 2^{\sum_{y \in \mathcal{Y}} P_{Y|X,A}(y|x,a(t)) [sd(t(y),x) + \log Q_{T,Y}(t,y)]}, \quad (29)$$

$$\alpha_{a,x} = \sum_{t \in \mathcal{T}^a} \alpha_{t,x}. \quad (30)$$

and  $\beta \in (0, 1)$  is a parameter of the algorithm (see Appendix A).

**Proposition 4.** *The algorithm in Tables Algorithm 1 and Algorithm 2 converges to the rate-distortion-cost function  $R(D_{s,m}, C_{s,m})$  for all  $s \leq 0$  and  $m \leq 0$ .*

*Proof:* See Appendix A. ■

#### IV. CODE DESIGN

In this section, we consider the design of specific encoders and decoders for the source coding problem with action-dependent side information. The goal is to design codes that perform close the rate-distortion-cost function given in Lemma 1 for some fixed pmf in (8) (or equivalently in Proposition 1 for some fixed pmf  $P_{X,Y,T}$ ).

---

**Algorithm 2** Algorithm for Minimization of  $F$  with respect to  $P_{T|X}$ 


---

**input:**  $Q_{T,Y}$  and  $Q_A$ .

**output:**  $P_{T|X}^*$ .

**parameters:** Subgradient weights  $\theta_i = \frac{1}{i}, i \in \mathbb{Z}_+$  and constant  $\beta \in (0, 1)$ .

**initialization:**  $i = 0$ ;  $\mu_x^{(0)} = 1$  for  $x \in \mathcal{X}$ ;  $P_{A|X}^{(0)}(a|x) = \frac{1}{|\mathcal{T}|}$  for  $t \in \mathcal{T}, x \in \mathcal{X}$ .

**repeat**

Perform fixed-point iterations on the system  $P_{A|X}(a|x) = g_{a|x}(P_{A|X}, \mu_x)$  for  $a \in \mathcal{A}$  and  $x \in \mathcal{X}$  with starting point  $P_{A|X}^{(i)}$  until convergence to obtain  $P_{A|X}^{(i+1)}$ .

Update the subgradients as

$$\mu_x^{(i+1)} = \mu_x^{(i)} + \frac{\theta_i}{P(x)} \left( 1 - \sum_{a \in \mathcal{A}} P_{A|X}^{(i+1)}(a|x) \right) \text{ for } x \in \mathcal{X}.$$

$i \leftarrow i + 1$ .

**until** convergence

Compute  $P_{T|X}^*(t|x) = \frac{\alpha_{t,x}}{\alpha_{a(t),x}} P_{A|X}^{(i)}(a(t)|x)$ .

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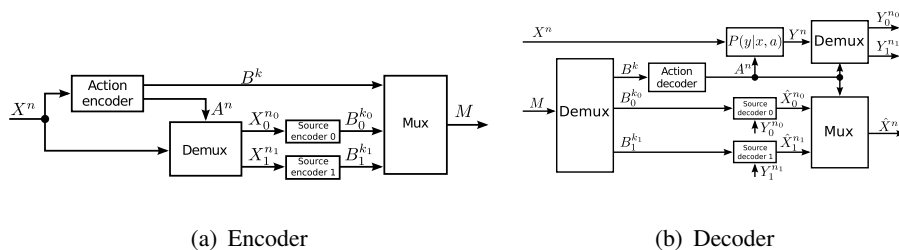


Fig. 3. Code design for source coding problems with action-dependent side information. The illustration is for  $\mathcal{A} = \{0, 1\}$ .

### A. Achievability via Multiplexing

As explained in Section II-C, the achievability proof in [1] is based on an action codebook  $\mathcal{C}_A$  for the action sequences  $A^n$  of about  $2^{nI(X;A)}$  codewords and  $2^{nI(X;A)}$  source codebooks of about  $2^{nI(X;U|A)}$  codewords for the sequences  $U^n$ , where each source codebook corresponds to an action sequence  $A^n$ . We also recall that binning is performed on the source codebooks in

order to reduce the rate.

Here, we first observe that the code design can be simplified without loss of optimality by using the encoder and decoder structures in Fig. 3. Accordingly, as in [1], the action encoder selects the action sequence  $A^n$ , and the corresponding index  $B^k$ , from the codebook  $\mathcal{C}_A$  to the decoder, where  $k = \lceil nI(X; A) \rceil$ . However, rather than using  $2^{nI(X; A)}$  source codebooks, we utilize only  $|\mathcal{A}|$  source codebooks  $\mathcal{C}_{s,a}$ ,  $a \in \mathcal{A}$ . Specifically, the source codebook  $\mathcal{C}_{s,a}$  has about  $2^{nP_A(a)I(X; U|A=a)}$  codewords, and each codeword in codebook  $\mathcal{C}_{s,a}$  has a length of  $n_a = \lceil n(P_A(a) + \varepsilon) \rceil$  symbols for some  $\varepsilon > 0$ .

To elaborate, as seen in Fig. 3(a), after action encoding, which takes place as in [1], the source  $X^n$  is demultiplexed into  $|\mathcal{A}|$  subsequences, such that the  $a$ -th subsequence  $X_a^{n_a}$  contains all symbols  $X_i$  for which  $A_i = a$ . Therefore, for sufficiently large  $n$ , by the law of large numbers, the number of symbols in  $X_a^{n_a}$  is less than  $n_a$  with high probability. Appropriate padding is then used to make the length of the sequence exactly  $n_a$  symbols. The  $a$ -th subsequence  $X_a^{n_a}$  is then compressed using the codebook  $\mathcal{C}_{s,a}$  with the objective of ensuring that  $X_a^{n_a}$  and  $U_a^{n_a}$  are jointly typical with respect to the pmf  $P_{X,U|A}(\cdot, \cdot | a)$ . Binning is performed on each source codebook so that the number of bins is  $2^{nP_A(a)I(X; U|Y, A=a)}$ . The bin index  $B_a^{k_a}$  of  $U_a^{n_a}$  is thus of  $k_a = \lceil nP_A(a)I(X; U|Y, A = a) \rceil$  bits. Overall, the rate of the message  $M$ , consisting of the indices  $B^k$  for the action code and  $B_{s,a}^{k_a}$  for the source codes with  $a \in \mathcal{A}$ , is  $I(X; A) + \sum_{a \in \mathcal{A}} P_A(a)I(X; U|Y, A = a) = I(X; A) + I(X; U|A, Y)$  as desired.

At the decoder, as seen in Fig. 3(b), the action sequence  $A^n$  is reconstructed and is used to measure the side information  $Y^n$ . The side information  $Y^n$  is demultiplexed into  $|\mathcal{A}|$  subsequences, such that the  $a$ -th subsequence  $Y_a^{n_a}$  contains all symbols  $Y_i$  for which  $A_i = a$ . Each of the subsequences  $U_a^{n_a}$  are then reconstructed by using Wyner-Ziv decoding based on the message bits  $B_a^{k_a}$  and the side information  $Y_a^{n_a}$ , and the reconstructed source subsequences  $\hat{X}_{a,i}$  are obtained as  $\hat{X}_{a,i} = \hat{X}^{\text{opt}}(U_{a,i}, Y_{a,i})$  for  $i \in [1, n_a]$ , where  $\hat{X}_{a,i}$  denotes the  $i$ -th symbol of the sequence  $X_a^{n_a}$ . Finally, the source reconstruction  $\hat{X}^n$  is obtained by multiplexing the

subsequences  $\hat{X}_a^{n_a}$  for  $a \in \mathcal{A}$ .

*Remark.* The proposed code structure also applies to the classical successive refinement problem [11] and can be used to simplify the code design proposed in [12].

### B. The Action Code

Based on the encoder structure in Fig. 3(a), we discuss the specific design of the action encoder. The action code  $\mathcal{C}_A$  has to ensure that the codewords  $A^n$  approximately have the type  $P_A$ , and the action encoder must obtain a codeword  $A^n$  that is jointly typical with respect to the joint pmf  $P_{X,A}$ . These conditions are satisfied by optimal source codes [4]. Optimal source codes can be designed using LDGM codes or polar codes as shown in [13] and [4], respectively. Here, we adopt LDGM codes as proposed in [13], [14]. Specifically, in the following, we define an encoder based on message passing. This uses ideas from [13] to handle the general alphabet and pmf  $P_A$ , and from [14] to implement message passing and decimation. The key difference with respect to [14] is that there the goal of the encoder is to minimize the Hamming distance, while the aim in this paper is to find an action sequence that is jointly typical with the source.

We use the code described by the factor graph in Fig. 4. The bottom section of the graph is a LDGM code (see, e.g. [13]). The sequence  $B^k$  denotes the message bits with  $k = \lceil nI(X; A) \rceil$  and  $\{g_{\kappa,l} : \kappa \in [1, d], l \in [1, n]\}$  denote the check variables of the LDGM code, where the choice of  $d$  is explained later. The objective of the mappings  $\psi_l : \{0, 1\}^d \times \mathcal{A} \rightarrow \{0, 1\}$  for  $l \in [1, n]$  is to ensure that the types of the codewords, or action variables, are approximately equal to  $P_A$  [13]. Specifically, each mapping  $\psi_l$  applies to the subset of check variables  $\{g_{\kappa,l}\}_{\kappa \in [1,d]}$  and to the symbol  $a_l$  and is defined in terms of a mapping  $\phi : \{0, 1\}^d \rightarrow \mathcal{A}$  as

$$\psi_l(\{g_{\kappa,l}\}_{\kappa \in [1,d]}, a) = \mathbb{1}_{\{\phi(\{g_{\kappa,l}\}_{\kappa \in [1,d]})=a\}}. \quad (31)$$

Following [13], the value of  $d \in \mathbb{Z}_+$  is chosen such that there are integers  $\nu_a$  for  $a \in \mathcal{A}$  satisfying

$$\sum_{a \in \mathcal{A}} \nu_a = 2^d \quad \text{and} \quad P_A(a) \approx \frac{\nu_a}{2^d}. \quad (32)$$



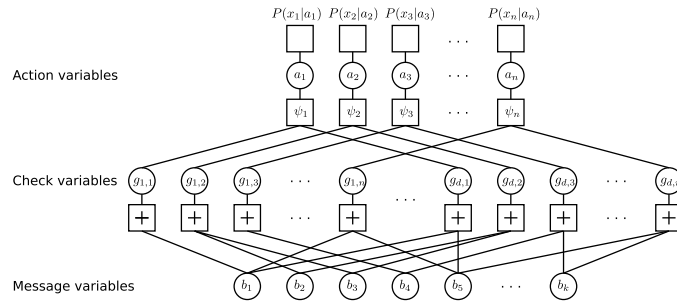


Fig. 4. Factor graph defining the action encoder.

The mapping  $\phi$  is then arbitrarily chosen such that exactly  $v_a$  of the  $2^d$  binary sequences  $\{g_{\kappa,l}\}_{\kappa \in [1,d]}$  map to  $a$ .

Given the source sequence  $X^n$ , the encoder runs the sum-product algorithm with decimation as in [14] in order to obtain the message bits  $B^k$ , and hence the action sequence  $A^n$  (see [4] for a discussion of the role of decimation in source coding problems).

### C. The Source Codes

Based on the proposed encoder structure in Fig. 3(a), the design of each source code  $C_{s,a}$  for  $a \in \mathcal{A}$  is equivalent to optimal codes for classical Wyner-Ziv problems.

In the special case where  $\hat{\mathcal{X}} = \{0, 1\}$ , and the distortion metric is Hamming, the coding problem reduces to the binary Wyner-Ziv problem with Hamming distortion which was studied in [15], [4].

## V. NUMERICAL EXAMPLES

To exemplify the problems of interest and to demonstrate the tools developed in this paper, we consider the source coding problem with action-dependent side information depicted in Fig. 5

and described in the following. Let  $X \in \mathcal{X} = [1, K + 1]$  be a random variable with pmf

$$P_X(x) = \begin{cases} \frac{1-q}{K} & \text{if } x \in [1, K] \\ q & \text{if } x = K + 1 \end{cases}, \quad (33)$$

for  $q \in [0, 1]$ . The letters  $1, \dots, K$  denote source outcomes that are relevant for the decoder, and thus should ideally be distinguishable by the latter, while the letter  $x = K + 1$  represents a source outcome that is irrelevant for the decoder. Examples where this situation arises includes monitoring systems in which the decoder wishes to recover the values of a physical quantity only when above, or below, a certain pre-determined threshold. To account for this requirement, the distortion function is given by

$$d(x, \hat{x}) = \mathbb{1}_{\{x \neq \hat{x} \text{ and } x \in [1, K]\}} \quad (34)$$

i.e., the decoder is only penalized if it makes an error when  $x$  is a relevant letter.

At each time  $i$ , the decoder can choose an action  $A_i \in \{0, 1\}$ , such that, if  $A_i = 0$ , the side information is given by  $Y_i = e$ , where  $e$  denotes an erasure symbol, and if  $A_i = 1$ , the side information is given by  $Y_i = \tilde{Y}_i$ , where  $\tilde{Y}_i$  is the output of an erasure channel in which  $\tilde{\mathcal{Y}} = \mathcal{X} \cup \{e\}$  and

$$P_{\tilde{Y}|X}(\tilde{y}|x) = \begin{cases} p & \text{for } \tilde{y} = e \\ 1 - p & \text{for } \tilde{y} = x \\ 0 & \text{otherwise} \end{cases}, \quad (35)$$

where  $p \in (0, 1)$  is the erasure probability. The action cost function  $\Delta(\cdot)$  is given by  $\Delta(a) = \mathbb{1}_{\{a=1\}}$ , which implies that the cost constraint with  $0 \leq C \leq 1$  enforces that no more than  $nC$  samples of the side information  $\tilde{Y}^n$  can be measured by the receiver.

#### A. Computation of the Rate-Distortion-Cost Function

We apply the proposed BA-type algorithm to the described scenario in order to compute the rate-distortion-cost function. For reference, we also consider the simplified strategy, in which the

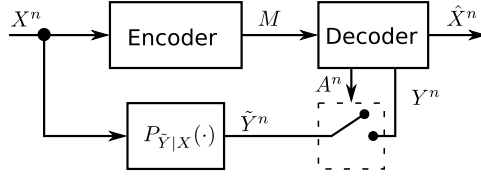


Fig. 5. The action-dependent source coding problem.

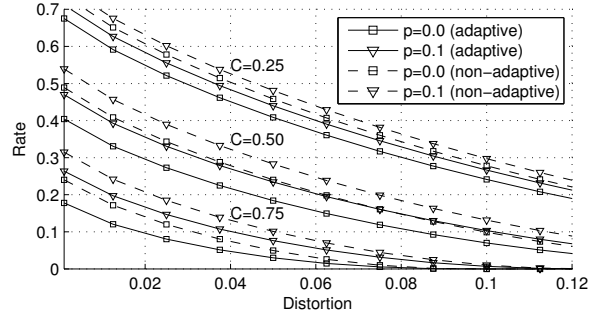


Fig. 6. Computed rate-distortion-cost function  $R(D, C)$  for  $K = 4$ , erasure probability  $p \in \{0.0, 0.1\}$  and  $q = \frac{1}{2}$ .

actions are chosen independently of the message  $M$ . We refer to the optimal approach discussed thus far as “adaptive actions”, while labeling as “non-adaptive actions” the simplified class of strategies in which the actions are selected independently of the encoder’s message (see [1]). The performance with non-adaptive actions can be obtained from Proposition 1 by imposing that  $A$  and  $X$  are independent.

Fig. 6 shows  $R(D, C)$  for  $K = 4$ ,  $q = \frac{1}{2}$  and  $p \in \{0, 0.1\}$  with both adaptive actions and non-adaptive actions. We see that for the given scenario, we achieve significant gains using adaptive actions in comparison to non-adaptive actions. Moreover, the effect of the erasures decreases as the action cost decreases due to the reduced availability of the side information at the decoder.

### B. Code Design

We now turn to the issue of code design for the scenario. We consider the case in which  $p = 0$ , so that, the measured side information is noiseless and we adopt the code design proposed in

Section IV. We start with some analytical considerations of the rate-distortion-cost function that will be useful for designing the codes. By symmetry, the pmf  $P_{A|X}$  can be written as

$$P_{A|X}(a|x) = \begin{cases} \frac{C-q\gamma}{1-q} & \text{if } a = 1 \wedge x \in [1, K] \\ \frac{1-q-C+q\gamma}{1-q} & \text{if } a = 0 \wedge x \in [1, K] \\ \gamma & \text{if } a = 1 \wedge x = K + 1 \\ 1 - \gamma & \text{if } a = 0 \wedge x = K + 1 \end{cases}, \quad (36)$$

where  $\gamma \in \left[0, \min\left(1, \frac{C}{q}\right)\right]$  is a parameter to be determined. The mutual information  $I(X; A)$  can thus be computed in terms of  $P_{A|X}$  and  $P_X$ , and the rate-distortion-cost function in (7) is then obtained via the following optimization problem

$$R(D, C) = \min_{\gamma \in [0, \min(1, \frac{C}{q})]} I(X; A) + (1 - C)\bar{R}\left(\frac{D}{1 - C}, P_{X|A=0}\right), \quad (37)$$

where  $\bar{R}(D, P_X)$  is the classical rate-distortion function of a memoryless source with pmf  $P_X$ . Note that we have used the fact that  $I(X; U|Y, A = 1) = 0$  since  $Y = X$  for  $A = 1$ .

From (37), it is seen that we only need to design an action code and the source code  $\mathcal{C}_{s,0}$ , where the latter is a classical rate-distortion code. For the action code, we use the approach proposed in Section IV and for the source code we use the related LDGM scheme proposed in [13].

We consider the case where  $q = \frac{1}{2}$ ,  $K = 4$ , which yields  $d = 2$  for both the action code  $\mathcal{C}_A$  and the source code  $\mathcal{C}_{s,0}$ . We fix a blocklength of  $n = 10\,000$ , yielding LDGM codes of blocklength, 20 000. Each point is averaged over 50 source realizations and LDGM codes. For both codes, we use the sum-product algorithm with decimation in [14]. As in [14], we use damping after 30 iterations and the maximum number of iterations is set to 100. Nodes are decimated if their log-likelihood ratios are larger than 2. Suitable irregular degree distributions optimized for the AWGN channel are obtained from [16]. The results are shown in Fig. 7. It is seen that the resulting distortions are close the lower bounds for both the adaptive and non-adaptive actions

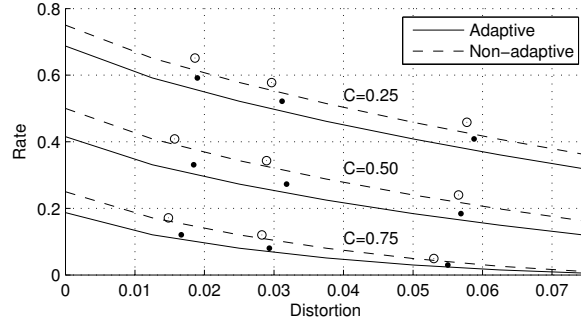


Fig. 7. Rate-distortion-cost function (lines) compared to the performance of the proposed code design (markers) with both adaptive and non-adaptive actions.

strategies. Moreover, the theoretical gains of the adaptive action strategy versus the non-adaptive one are confirmed by the practical implementation.

## VI. CONCLUSION

In this paper, we have considered computation of the rate-distortion-code function and code design for source coding problems with action-dependent side information. We have formulated the problem using Shannon strategies and proposed a BA-type algorithm that efficiently computes the rate-distortion function. Convergence of this algorithm was proved. Moreover, we proposed a code design based on multiplexing that was shown, via numerical results, to perform close to the rate-distortion bound.

## APPENDIX A

### PROOF FOR LEMMA 4

The BA-type algorithm detailed in Tables Algorithm 1 and Algorithm 2 is based on alternatively optimizing  $F(\cdot)$  in (23) with respect to  $P_{T|X}$ ,  $Q_A$  and  $Q_{T,Y}$ . Given the convexity of this function, shown in Proposition 3, this procedure is known to converge [17]. The optimization with respect to  $Q_A$  for fixed  $P_{T|X}$  and  $Q_{T,Y}$  and with respect to  $Q_{T,Y}$  for fixed  $P_{T|X}$  and  $Q_A$

are performed as in the proof of Proposition 3. Therefore, the proof is concluded once it is demonstrated that the procedure of Table Algorithm 2 converges to an optimum  $P_{T|X}$  for fixed  $Q_A$  and  $Q_{T,Y}$ . This is discussed next. The procedure in Table Algorithm 2 for the optimization with respect to  $P_{T|X}$  for fixed  $Q_A$  and  $Q_{T,Y}$  is based on the dual minimization (27) via an outer loop that performs subgradient iterations and an inner loop that performs fixed-point iterations to obtain a stationary point of the Lagrangian function (38) (see below). We first show that this nested loop procedure obtains an optimal dual solution  $P_{T|X}$  of the dual problem and then argue that this is also a solution for the original primal problem.

Convergence of the outer loop follows immediately by the well-known properties of the subgradient approach for weights that are selected as  $\Theta_i = \frac{1}{i}$  [17]. Note that the constraints  $1 - \sum_{a \in \mathcal{A}} P^{(i)}(a|x)$  for  $x \in \mathcal{X}$  are the subgradients with respect to  $\lambda_x$  of the dual function given by the minimization in (27) [18]. Therefore, by defining  $\mu_x = -\frac{\lambda_x}{P(x)} + 2$ , the updates of the variables  $\mu_x^{(i)}$  in Table Algorithm 2 can be seen to correspond to the classical subgradient updates. Given the known convergence properties of the subgradient method with the weights as in Table Algorithm 2, the outer maximization converges [17].

Next, we need to show that we can solve the inner minimization in (27) by using the fixed-point iterations in (47) (see below). It is first shown that we can solve the minimization problem by solving a system of stationarity equations for  $P(a|x)$ ,  $a \in \mathcal{A}$ ,  $x \in \mathcal{X}$ . Then, we conclude the proof using Banach fixed-point theorem [19].

The Lagrangian to be minimized is given by (cf. (27))

$$\mathcal{L}(P_{T|X}, \{\lambda_x\}) = F(P_{T|X}, Q_A, Q_{T,Y}) + \sum_{x \in \mathcal{X}} \lambda_x \left( \sum_{t \in \mathcal{T}} P_{T|X}(t|x) - 1 \right). \quad (38)$$

It is noted that the function  $\mathcal{L}$  is coercive in  $P_{T|X}$ , and hence from Weierstrass theorem [20] a minimizer of  $\mathcal{L}$  exists. The minimizer must be a stationary point, i.e., it must satisfy the KKT conditions [9, Section 5.5.3]. We obtain the following stationarity conditions by differentiating

(38) with respect to  $P(t|x)$  and equating to zero, leading to

$$\begin{aligned} \log P(t|x) + \sum_{y \in \mathcal{Y}} P(y|x, a(t)) \log [P(y, a(t))] = \\ m\Delta(a(t)) + \sum_{y \in \mathcal{Y}} P(y|x, a(t)) [sd(t(y), \tilde{x}) + \log Q(t, y) + \log Q(a(t))] + \mu_x, \end{aligned} \quad (39)$$

$$\log P(t|x) + \sum_{y \in \mathcal{Y}} P(y|x, a(t)) \log [P(y, a(t))] = \log \alpha_{t,x} + \mu_x \quad (40)$$

where  $\alpha_{t,x}$  is given in (29) and  $P(y, a)$  is calculated from the joint pmf in (13). We can then rewrite (40) by applying the exponential function to both sides and solving for  $P(t|x)$

$$P(t|x) = \frac{2^{\mu_x} \alpha_{t,x}}{\prod_{y \in \mathcal{Y}} [\sum_{\tilde{x} \in \mathcal{X}} P(\tilde{x}) P(a(t)|\tilde{x}) P(y|\tilde{x}, a(t))]^{P(y|x, a(t))}} \quad (41)$$

where  $\alpha_{t,x}$  is given in (29). Note that the right-hand side only depends on  $P_{T|X}$  through  $P_{A|X}$ , and hence by computing  $P(a|x)$  for  $a \in \mathcal{A}$  and  $x \in \mathcal{X}$ ,  $P(t|x)$  can be calculated. By summing (41) over  $t \in \mathcal{T}^a$ , we obtain

$$P(a|x) = \frac{2^{\mu_x} \alpha_{a,x}}{\prod_{y \in \mathcal{Y}} [\sum_{\tilde{x} \in \mathcal{X}} P(\tilde{x}) P(y|\tilde{x}, a) P(a|\tilde{x})]^{P(y|x, a)}}, \quad (42)$$

where  $\alpha_{a,x}$  is given in (30). Given  $\{\mu_x\}$ , the equalities in (42) for  $a \in \mathcal{A}$  and  $x \in \mathcal{X}$  form a system of  $|\mathcal{A}||\mathcal{X}|$  nonlinear equation with  $|\mathcal{A}||\mathcal{X}|$  unknowns, namely the  $|\mathcal{A}||\mathcal{X}|$  values  $P(a|x)$  for  $a \in \mathcal{A}$  and  $x \in \mathcal{X}$ . By solving for  $P(a|x)$ , we can compute  $P(t|x)$  as in (41). Note that the constants  $\alpha_{a,x}$  are sums of exponential functions, and hence  $P(a|x)$  in (42) are strictly positive for  $a \in \mathcal{A}$  and  $x \in \mathcal{X}$ . Now, define

$$h_{a|x}(P_{A|X}, \mu_x) = \frac{2^{\mu_x} \alpha_{a,x}}{\prod_{y \in \mathcal{Y}} [\sum_{\tilde{x} \in \mathcal{X}} P_{X,A,Y}(\tilde{x}, a, y)]^{P_{Y|X,A}(y|x, a)}}, \quad (43)$$

$$H_{a|x}(\mathbf{q}, \mu_x) = \log h(2^{\mathbf{q}}, \mu_x), \quad (44)$$

$$\begin{aligned} \text{and } G_{a|x}(\mathbf{q}, \mu_x) &= \log g_{a|x}(2^{\mathbf{q}}, \mu_x) \\ &= \beta \mathbf{q} + (1 - \beta) H_{a|x}(\mathbf{q}, \mu_x) \end{aligned} \quad (45)$$

where  $\mathbf{q} \in \mathcal{R}^{|\mathcal{A}||\mathcal{X}|}$  and  $2^{\mathbf{q}} \in \mathcal{R}_+^{|\mathcal{A}||\mathcal{X}|}$  are the vectors corresponding to the elements  $q_{a|x} = \log P(a|x)$  and  $P(a|x)$ , respectively, and  $\beta \in (0, 1)$ . Moreover, let  $\mathbf{G}(\mathbf{q}, \{\mu_x\}) \in \mathcal{R}^{|\mathcal{A}||\mathcal{X}|}$  denote the vectors collecting the functions  $G_{a|x}$  for  $a \in \mathcal{A}, x \in \mathcal{X}$ . With these definitions it is now evident that (42) is equivalent to the following equation

$$q_{a|x} = H_{a|x}(\mathbf{q}, \{\mu_x\}). \quad (46)$$

We now show that the fixed-point iteration of the form

$$\mathbf{q}^{(k+1)} = \mathbf{G}(\mathbf{q}^{(k)}, \{\mu_x\}). \quad (47)$$

converges towards a fixed-point  $\mathbf{q}^*$ , which is a unique fixed-point of (46) for any  $\beta \in (0, 1)$ .

Recall that the existence of a fixed-point  $\mathbf{q}^*$  is guaranteed by the necessity of the KKT conditions and by Weierstrass theorem. In the following, we apply Banach fixed-point theorem. To this end, we have to demonstrate that there is a closed subset  $\Omega \in \mathbb{R}^{|\mathcal{A}||\mathcal{X}|}$ , such that the vector function  $\mathbf{G}$  maps from vectors  $\mathbf{q} \in \Omega$  into  $\Omega$ , and is a contraction in  $\Omega$ . By the existence of a fixed-point  $\mathbf{q}^*$ , we define the subset  $\Omega$  as the closed ball

$$\Omega = B_r(\mathbf{q}^*) = \{\mathbf{q} \in \mathbb{R}^{|\mathcal{A}||\mathcal{X}|} \mid \|\mathbf{q} - \mathbf{q}^*\|_\infty \leq r\}, \quad (48)$$

for some  $r > \|\mathbf{q}^{(0)} - \mathbf{q}^*\|_\infty$ . In order to show that  $\mathbf{G}$  maps from  $\Omega$  into  $\Omega$  and is a contraction, we compute the partial derivatives of  $H_{a|x}(\mathbf{q})$  and  $G_{a|x}(\mathbf{q})$  as following

$$\frac{\partial H_{\tilde{a}|\tilde{x}}(\mathbf{q})}{\partial q_{a'|x'}} = -\mathbb{1}_{\{\tilde{a}=a'\}} \sum_{y \in \mathcal{Y}} P(y|\tilde{x}, \tilde{a}) \frac{P(x')P(y|x', a')2^{q_{a'|x'}}}{\sum_{x \in \mathcal{X}} P(x)P(y|x, \tilde{a})2^{q_{\tilde{a}|x}}} \quad (49)$$

$$\text{and } \frac{\partial G_{\tilde{a}|\tilde{x}}(\mathbf{q})}{\partial q_{a'|x'}} = \beta \mathbb{1}_{\{\tilde{a}=a' \text{ and } \tilde{x}=x'\}} + (1 - \beta) \frac{\partial H_{\tilde{a}|\tilde{x}}(\mathbf{q})}{\partial q_{a'|x'}}. \quad (50)$$

It is clear that the derivative  $\frac{\partial H_{\tilde{a}|\tilde{x}}(\mathbf{q})}{\partial q_{\tilde{a}|\tilde{x}}}$  is strictly negative for  $\mathbf{q} \in \mathbb{R}^{|\mathcal{A}||\mathcal{X}|}$  since  $P(x) > 0$ , and it can be seen that

$$\sum_{a' \in \mathcal{A}, x' \in \mathcal{X}} \frac{\partial H_{\tilde{a}|\tilde{x}}(\mathbf{q})}{\partial q_{a'|x'}} = -1. \quad (51)$$



Therefore, for  $\beta \in (0, 1)$ , we must have that

$$\sum_{a' \in \mathcal{A}, x' \in \mathcal{X}} \left| \frac{\partial G_{\tilde{a}|\tilde{x}}(\mathbf{q})}{\partial q_{a'|x'}} \right| < 1. \quad (52)$$

It follows that we can bound the  $l_\infty$ -norm of the Jacobian for  $\mathbf{G}(\mathbf{q})$ ,  $J_{\mathbf{G}}(\mathbf{q})$ , as

$$\|J_{\mathbf{G}}(\mathbf{q})\|_\infty < 1. \quad (53)$$

By the definition of the  $l_\infty$ -norm and by the mean value theorem [19], there exist values  $\tilde{a} \in \mathcal{A}$ ,  $\tilde{x} \in \mathcal{X}$  and  $\zeta \in (0, 1)$  such that

$$\|\mathbf{G}(\mathbf{q}^1) - \mathbf{G}(\mathbf{q}^2)\|_\infty = |G_{\tilde{a}|\tilde{x}}(\mathbf{q}^1) - G_{\tilde{a}|\tilde{x}}(\mathbf{q}^2)| \quad (54a)$$

$$\leq \|\mathbf{q}^1 - \mathbf{q}^2\|_\infty \sum_{a \in \mathcal{A}, x \in \mathcal{X}} \left| \frac{\partial G_{\tilde{a}|\tilde{x}}}{\partial q_{a|x}}(\zeta \mathbf{q}^1 + (1 - \zeta) \mathbf{q}^2) \right| \quad (54b)$$

$$\leq \|\mathbf{q}^1 - \mathbf{q}^2\|_\infty \max_{\mathbf{q} \in \Omega} \|J_{\mathbf{G}}(\mathbf{q})\|_\infty \quad (54c)$$

$$\leq K \|\mathbf{q}^1 - \mathbf{q}^2\|_\infty \quad (54d)$$

for  $\mathbf{q}^1, \mathbf{q}^2 \in \Omega$ , where the last inequality follows by the fact that  $\|J_{\mathbf{G}}(\mathbf{q})\|_\infty$  must attain a maximum value  $K < 1$  when  $\mathbf{q} \in \Omega$ , since  $\Omega$  is closed and bounded, by Weierstrass theorem. The chain of inequalities in (54) demonstrates that  $\mathbf{G}$  is a contraction mapping. To show that  $\mathbf{G}$  maps from  $\Omega$  into  $\Omega$ , suppose  $\mathbf{q} \in \Omega$ . Since  $\Omega$  contains the fixed-point  $\mathbf{q}^*$ , it is then seen that

$$\|\mathbf{G}(\mathbf{q}) - \mathbf{q}^*\|_\infty = \|\mathbf{G}(\mathbf{q}) - \mathbf{G}(\mathbf{q}^*)\|_\infty \quad (55)$$

$$< \|\mathbf{q} - \mathbf{q}^*\|_\infty < r, \quad (56)$$

and hence  $\mathbf{G}(\mathbf{q}) \in \Omega$ . By invoking the Banach fixed-point theorem, the fixed-point iteration defined by (47) converges to a unique fixed-point  $\mathbf{q}^*$ .

We finally observe that, since the fixed-point is unique, the minimizer of the Lagrangian function  $\mathcal{L}$  is unique, and hence the optimal  $P_{T|X}$  of the primal and the dual optimization problem coincide, thus concluding the proof.

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