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Mode Interaction in Structures—An Overview

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Abstract Koiter [1] was the first to formulate an asymptotic expansion to investigate postbuckling behavior and imperfection sensitivity of elastic structures. Since then, a large number of analyses of particular structures have appeared as well as some new expansions aimed at specific problems, such as interaction between buckling modes associated with simultaneous or nearly simultaneous buckling modes. In this contribution, various methods of this kind are discussed and compared as regards applicability and ease of use. Focus will be on Koiter's *slowly varying local mode amplitude* [2] and [3], on Byskov & Hutchinson's expansion [4] and on Peek & Kheyrkhahan's method [5], which enlarged the scope of the previous expansions in that it covers nonlinear prebuckling states, also. Other important contributions—a number of which are based on these methods—are also discussed. On the other hand, many important works, e.g. the comprehensive paper by Hunt [6] will not be mentioned in any detail. The accuracy of the methods as well as their mathematical complexity and ease of use are compared. Finally, in view of today's inexpensive and powerful computers, an obvious question is concerned with whether full analyses must always be preferred because asymptotic expansions are obsolete.

Keywords: stability, elastic, mode interaction.

INTRODUCTION

It seems relevant to define the meaning of the term *Mode Interaction* in the present context. Here, mode interaction is to be understood as the phenomenon of erosion of load-carrying capacity caused by interaction between buckling modes associated with the same or nearly the same classical critical load when simultaneity is the result of a design process. Thus, the well-known interaction in spherical and long cylindrical shells, which is not the outcome of an optimization, but a feature inherent in the structure, falls outside this scope. It is equally important to stress that only linear-elastic buckling problems will be addressed below.

As an example, in designing a truss column against buckling it appears that a structure which experiences global buckling of the entire column at the same time as the flanges buckle locally is the optimum one. However, mode interaction in the presence of geometrical imperfections may erode this optimum, although the so-called naive design may still be the best.

One of the first studies of a structure experiencing mode interaction is the one by van der Neut [7], which dealt with interaction in a model column consisting of two flange plates connected by webs that only provide sufficient coupling between the flanges. Here, interaction occurred between an overall, postbuckling neutral Euler buckling mode and postbuckling stable plate buckling of the flanges. Yet, interaction between these modes proved to make the structure imperfection sensitive to certain values of local plate buckling load versus overall Euler buckling load. In his investigation van der Neut used a direct analytic approach, and no general theory of mode interaction was presented. A few years later, Koiter and Kuiken [2] developed the method that is often referred to as the method of the *slowly varying local mode amplitude*, and applied it to the problem of van der Neut. In 1976 Koiter broadened the scope of the method in his *General Theory of Mode Interaction in Stiffened Plates and Shell Structures* [3]. In the monograph by Thompson and Hunt [8] a theory of interaction between coincident modes was developed. Optimization of van der Neut's column and of a truss column is studied, and the very simple and elegant analysis as well as the results for the latter are new. Other early studies are due to Tvergaard, see [9] and [10], who, by use of a method that was

particularly well suited for the problem at hand, found some erosion of the load-carrying capacity of wide, stringer stiffened panels in spite of the fact that the overall mode is a postbuckling neutral Euler-like mode and that the local mode is a postbuckling stable plate mode. In [11] Budiansky describes an asymptotic expansion to handle mode interaction, but does not develop a full set of formulas and does not study any structure. Budiansky suggests that it may be justified to exclude the mixed second order solution, see later, if it proves impossible or very difficult to determine it. In a preliminary version of their paper [4], Byskov and Hutchinson made this exclusion, which prompted Koiter to write his report [3]. From a geometrical viewpoint it is clear that exclusion of the mixed second order field which results in an incorrect value of the coupling term is not justified, because it is equivalent to making a local description of a surface by the exact values of the tangents and of the curvatures along the axes, but using an incorrect value of the mixed second order derivative.[1] In [4] a general theory of mode interaction valid for simultaneous or nearly simultaneous modes under the assumption of linear prebuckling is established. Another condition, which must be met, is that the wavelengths of the interacting modes do not differ by orders of magnitude. Peek and Kheyrkhahan [5] do not assume linear prebuckling and thus extend the work of the previous authors.
The methods mentioned above have been utilized by many authors, whose work will be described in some detail below.

THREE METHODS

The order of the methods mentioned here is not according to chronology, but is chosen to make the presentation natural.

The Byskov-Hutchinson Method

In order to establish a common frame of reference, a very short account of the asymptotic method developed by Byskov and Hutchinson [4] is given below. Following [4] the displacement field $\boldsymbol{u}(\boldsymbol{x};\lambda)$ is expanded according to:

$$\boldsymbol{u}(\boldsymbol{x};\lambda) = \lambda\boldsymbol{u}_0(\boldsymbol{x}) + \xi_j(\lambda)\boldsymbol{u}_j(\boldsymbol{x}) + \xi_j(\lambda)\xi_k(\lambda)\boldsymbol{u}_{jk}(\boldsymbol{x}) + \cdots\,, \text{ sum over } (j,k) = (1,\ldots,M) \tag{1}$$

where $\boldsymbol{u}_0$ denotes the prebuckling displacement field, $\boldsymbol{x}$ the spatial coordinates, ξ_j is the amplitude of buckling mode $\boldsymbol{u}_j$, λ is a scalar load parameter, M is the number of interacting modes, and $\boldsymbol{u}_{jk}$ designates the second order field associated with both $\boldsymbol{u}_j$ and $\boldsymbol{u}_k$. The expansion given by (1) and similar expansions for strains $\boldsymbol{\varepsilon}$ and stresses $\boldsymbol{\sigma}$ are introduced in the potential energy or the principle of virtual displacements with the result:

Prebuckling: $$\boldsymbol{\sigma}_0 \cdot \mathbf{l}_1(\delta\boldsymbol{u}) = \overline{\boldsymbol{T}} \cdot \delta\boldsymbol{u} \tag{2}$$

Buckling: $$\boldsymbol{\sigma}_J \cdot \mathbf{l}_1(\delta\boldsymbol{u}) + \lambda_J\boldsymbol{\sigma}_0 \cdot \mathbf{l}_{11}(\boldsymbol{u}_J,\delta\boldsymbol{u}) = 0\,, \text{no sum over uppercase indices} \tag{3}$$

Postbuckling: $$\boldsymbol{\sigma}_{jk} \cdot \mathbf{l}_1(\delta\boldsymbol{u}) + \lambda_c\boldsymbol{\sigma}_0 \cdot \mathbf{l}_{11}(\boldsymbol{u}_{jk},\delta\boldsymbol{u}) = \tfrac{1}{2}\big(\boldsymbol{\sigma}_j \cdot \mathbf{l}_{11}(\boldsymbol{u}_k,\delta\boldsymbol{u}) + \boldsymbol{\sigma}_k \cdot \mathbf{l}_{11}(\boldsymbol{u}_j,\delta\boldsymbol{u})\big) \tag{4}$$

where $\mathbf{l}_1(\,)$ and $\mathbf{l}_{11}(\,,\,)$ are linear and bilinear operators associated with the strain definition, respectively, a dot $(\cdot)$ indicates integration over the entire structure, $\overline{\boldsymbol{T}}$ designates the load distribution, δ signifies variations, λ_J is the classical critical load associated with mode J, $\lambda_c = \min(\lambda_J)$, and:

$$\boldsymbol{\sigma}_0 = \mathbf{H}(\boldsymbol{\varepsilon})\,, \quad \boldsymbol{\sigma}_j = \mathbf{H}(\boldsymbol{\varepsilon}_j)\,, \quad \boldsymbol{\sigma}_{jk} = \mathbf{H}(\boldsymbol{\varepsilon}_{jk}) \tag{5}$$

where $\mathbf{H}(\,)$ is a linear, i.e. Hooke, constitutive operator.
Based on the expressions (1)–(5) the value of λ_c may be determined from:

$$\left(1 - \frac{\lambda}{\lambda_J}\right)\xi_J + a_{jkJ}\xi_j\xi_k + b_{jkmJ}\xi_j\xi_k\xi_m = \left(1 - \frac{\lambda}{\lambda_J}\right)\bar{\xi}_J\,, \quad J = (1,\ldots,M) \tag{6}$$

where a_{jkJ} and b_{jkmJ} are the first and second order postbuckling constants, respectively, and $\bar{\xi}_J$ is the amplitude of the imperfection in the shape of buckling mode J. The value of the a_{jkJ} may be determined once the

[1] The reason why Byskov and Hutchinson in their preliminary version of [4] made this exclusion was a combination of lack of time and severe difficulties in obtaining a sound estimate of the mixed second order field.

buckling problems are solved, while computation of b_{jkmJ} requires solution of the postbuckling problems, too.
Comparison between the buckling and postbuckling problems shows that the postbuckling problem, as it stands, is singular if $\lambda_c \approx \lambda_c$. On the other hand, the orthogonality conditions:

$$\boldsymbol{\sigma}_0 \cdot \mathbf{l}_{11}(\boldsymbol{u}_j, \boldsymbol{u}_{km}) = 0\,, \quad (j,k,m) = (1,\ldots,M) \tag{7}$$

ensure solvability. However, if the wavelength of one mode, say j, is much smaller than that of another, say k, then the mixed second order problem for $\boldsymbol{u}_{jk}$, see (4), becomes ill-conditioned, which is a main problem concerning this method.
One advantage of the Byskov-Hutchinson method is that it entails a sequence of linear boundary value and eigenvalue problems which lend themselves to implementation into finite element programs as well as other numerical and analytical methods.

Koiter's Slowly Varying Local Mode Method

Koiter's method, see [2] and [3], is particularly aimed at structures whose buckling modes differ substantially in wavelength. Typically, one of the interacting modes is a long-wave global mode $\boldsymbol{u}_G$, while the other, $\boldsymbol{u}_L$, is a short-wave local mode. Using this notation (1) becomes:

$$\begin{aligned}\boldsymbol{u}(\boldsymbol{x};\lambda) = {}& \lambda\boldsymbol{u}_0(\boldsymbol{x}) + \xi_G(\lambda)\boldsymbol{u}_G(\boldsymbol{x}) + \xi_L(\lambda)\boldsymbol{u}_L(\boldsymbol{x}) \\ & + \big(\xi_G(\lambda)\big)^2\boldsymbol{u}_{GG}(\boldsymbol{x}) + \xi_G(\lambda)\xi_L(\lambda)\boldsymbol{u}_{GL}(\boldsymbol{x}) + \big(\xi_L(\lambda)\big)^2\boldsymbol{u}_{LL}(\boldsymbol{x}) + \cdots\end{aligned} \tag{8}$$

A simple way of viewing Koiter's idea is to envision a structure which suffers local imperfections in the shape of a short-wave local buckling mode, while its global imperfections are small. A truss column, such as the one treated by Thompson and Hunt [8] and by Byskov [12] or the van der Neut column [7], [2], [13] and [14], are a good examples. In these structures, when the load is increased from 0, then the column bends in a shape very similar to the global mode. At the same time the local buckles on the concave side of the column grow in amplitude, while the buckles on the convex side decrease in size. Of course, this effect is largest where the global curvature is the largest, e.g. at the midpoint of a simply supported realization of the column. Thus, the local buckles grow or decrease at different rates depending on the position along the column axis. The term $\xi_G\xi_L\boldsymbol{u}_{GL}(\boldsymbol{x})$ in (8) is aimed at describing this phenomenon. Note, however, that the mixed second order field $\boldsymbol{u}_{GL}(\boldsymbol{x})$ does not change shape with increased load. In view of this, Koiter suggested that this term may be omitted, provided that the local mode amplitude ξ_L is no longer only a function of λ, but also of the spatial coordinate $\boldsymbol{x}$. Thus, Koiter's series is:

$$\boldsymbol{u}(\boldsymbol{x};\lambda) = \ \lambda\boldsymbol{u}_0(\boldsymbol{x}) + \xi_G(\lambda)\boldsymbol{u}_G(\boldsymbol{x}) + \xi_L(\boldsymbol{x};\lambda)\boldsymbol{u}_L(\boldsymbol{x}) + \big(\xi_G(\lambda)\big)^2\boldsymbol{u}_{GG}(\boldsymbol{x}) + \big(\xi_L(\boldsymbol{x};\lambda)\big)^2\boldsymbol{u}_{LL}(\boldsymbol{x}) + \cdots \tag{9}$$

where $\xi_L(\boldsymbol{x};\lambda)$ is assumed to vary slowly with $\boldsymbol{x}$.
The idea behind Koiter's approach is intuitively clear, but in reality not that easy to apply to specific structural problems, because it requires a good physical understanding of the mechanics of the problem and a fair amount of mathematical skills, mainly associated with the consequences of the slow variation and the terms that may be omitted because of that. Moreover, implementation of the method in terms of finite elements is not a straightforward task—it is difficult to tell a computer that a function is slowly varying.

The Peek-Kheyrkhahan Method

While the previous two methods expand all fields about one of the classical critical loads, typically the minimum one, the Peek-Kheyrkhahan method, see [5], relaxes this constraint and allows for expansions about almost any point on the prebuckling path, albeit it is recommended to choose the point associated with the smallest classical critical load $\lambda_{\min}$, see Fig. 1. Although [5] contains a detailed recipe for application of the method it seems to have been utilized by few other authors, see later.
One major idea is thus to choose another reference point than the bifurcation point associated with the lowest buckling load—or any other bifurcation point for that matter. In the same spirit the space of admissible displacements A is decomposed into a subspace A_0, which is spanned by a finite number of modes $\tilde{\boldsymbol{u}}_j(\boldsymbol{x})$,

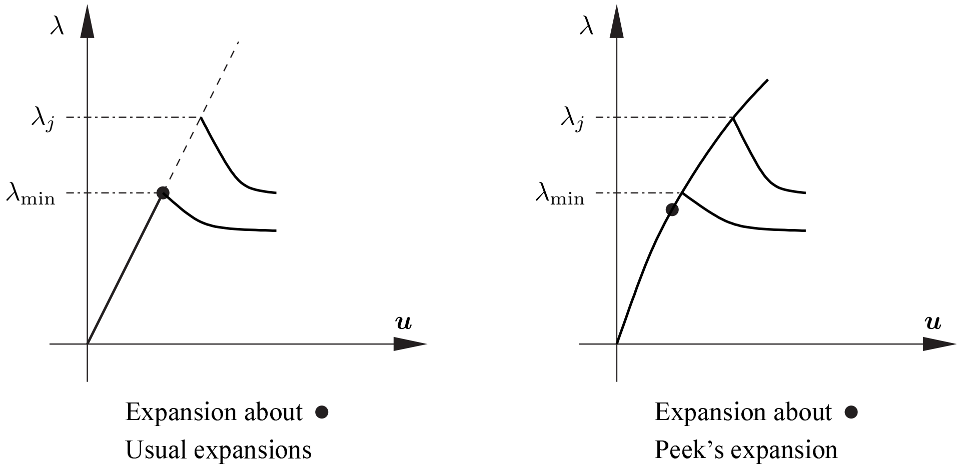

Fig. 1 Buckling and postbuckling with linear and nonlinear prebuckling.

and a complementary space $\hat{A}$, which is not necessarily the orthogonal complement of A_0. That this could be feasible was originally pointed out by Thompson and Hunt [8]. The expansion established by Peek and Kheyrkhahan is more complicated than the ones of Byskov and Hutchinson and of Koiter. In the notation used above, the expansion is:

$$\boldsymbol{u}(\xi_k;\lambda;\varepsilon) = \boldsymbol{u}_0(\boldsymbol{x};\lambda) + \xi_j\tilde{\boldsymbol{u}}_j(\boldsymbol{x}) + \hat{\boldsymbol{u}}(\xi_k;\lambda;\varepsilon)\ ,\ \text{sum over } j=(1,\ldots,M) \tag{10}$$

where ε is the imperfection amplitude and $\tilde{\boldsymbol{u}}_j(\boldsymbol{x})$ does not necessarily denote buckling modes, as mentioned above. Since Peek and Kheyrkhahan do not assume linear prebuckling the derivations become more involved than the ones performed by Koiter [2] and [3] and by Byskov and Hutchinson [4]. Expanding about the reference point presents further complications because of the nonlinear prebuckling state, and a number of new derivatives about this point with respect to the load parameter λ must be included in the analysis. Using the notation of [5] the potential energy is $\phi(u,\lambda,\bar{u})$, where u denotes the displacements and $\bar{u}$ is the imperfection. In order to arrive at a set of algebraic equations with the amplitudes ξ_i of the modes a reduced potential ψ is introduced by:

$$\psi(\xi_i,\lambda,\varepsilon) \equiv \phi(u(\xi_i,\lambda,\varepsilon),\lambda,\varepsilon\bar{\bar{u}}) \tag{11}$$

where $\bar{\bar{u}}$ denotes the shape of the imperfection and ε is the amplitude. The Frechet derivatives $\psi_{,i}$ of the reduced potential with respect to ξ_i are then expanded to give:

$$\begin{aligned}\psi_{,i} = {} & \phi_{i\varepsilon} + \phi_{ij}\xi_j + \phi_{ij\lambda}\xi_j\Delta\lambda + \tfrac{1}{2}\phi_{ijk}\xi_j\xi_k + \phi_{ij\varepsilon}\xi_j\varepsilon + \phi_{i\varepsilon\lambda}\varepsilon\Delta\lambda + \tfrac{1}{6}\phi_{ijkl}\xi_j\xi_k\xi_l + \tfrac{1}{2}\phi_{ijk\lambda}\xi_j\xi_k\Delta\lambda \\ & + \tfrac{1}{2}\phi_{ij\lambda\lambda}\xi_j(\Delta\lambda)^2 + \cdots = 0\ ,\ \text{sum over } (j,k,l)\end{aligned} \tag{12}$$

where the terms stem from Frechet derivatives of ϕ. Because of space limitations, it is not possible to show all terms, but, as an indication of their type, the simplest is given here:

$$\phi_{ij} \equiv \phi^c_{uu}u_iu_j\ ,\ \text{no sum over } (i,j) \tag{13}$$

where superscript ^{c} indicates that the term is evaluated at $\lambda=\lambda_c$. In their paper [4], Byskov and Hutchinson write a similar, but much simpler expansion for their potential energy ϕ:

$$\begin{aligned}\phi = {} & \tfrac{1}{2}\sum_J(\lambda-\lambda_J)(\xi_J)^2\boldsymbol{\sigma}_0\cdot\mathbf{l}_2(\boldsymbol{u}_J) + \tfrac{1}{2}\xi_i\xi_j\xi_k\boldsymbol{\sigma}_i\cdot\mathbf{l}_{11}(\boldsymbol{u}_j,\boldsymbol{u}_k) \\ & + \tfrac{1}{4}\xi_i\xi_j\xi_k\xi_l\big(\boldsymbol{\sigma}_{ij}\cdot\mathbf{l}_{11}(\boldsymbol{u}_k,\boldsymbol{u}_l) + 2\boldsymbol{\sigma}_i\cdot\mathbf{l}_{11}(\boldsymbol{u}_j,\boldsymbol{u}_{kl})\big) + \lambda\sum_J\xi_J\bar{\xi}_J\boldsymbol{\sigma}_0\cdot\mathbf{l}_2(\boldsymbol{u}_J)\ , \\ & \text{sum over } (i,j,k,l),\ \text{sum over } J \text{ as indicated}\end{aligned} \tag{14}$$

where $\mathbf{l}_2(\ )$ is the quadratic operator associated with $\mathbf{l}_{11}(\ ,\ )$, i.e. $\mathbf{l}_2(\boldsymbol{u}) = \mathbf{l}_{11}(\boldsymbol{u}, \boldsymbol{u})$. By taking derivatives of ϕ, as given by (14), with respect to ξ_K, it is possible to identify a number of the terms from (12), but, clearly, (12) must contain many more terms because of the less restrictive assumptions behind it.

Without going into further details, it should be obvious from the above that the Peek-Kheyrkhahan method [5] covers a larger field of problems than that of Byskov and Hutchinson [4] at the expense of a much more complicated set of formulas.

APPLICATIONS

Over the years, the three methods have been utilized by several authors interested in problems of a more or less practical nature. Already at this point it is worth mentioning that in the interest of space it is not possible to cover *all*—not even all *important*—contributions.

Thin-Walled Beams

A large portion of the work, which has utilized one or the other of the available methods or has applied a different approach, has been concerned with compression or bending of thin-walled beams or beam-like structures.

Truss Columns

Probably the simplest example of a structure exhibiting mode interaction is the truss column, see Fig. 2, and, consequently, many studies regarding mode interaction in this have been performed. The first of these seems

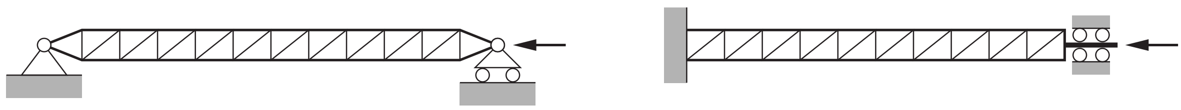

Fig. 2 Truss columns.

to be the one by Thompson and Hunt [8], followed by Crawford and Hedgepeth [15]. In both cases an ad hoc procedure, which considered a simply supported truss column, see the left-hand sketch in Fig. 2, with locally imperfect flanges, but straight overall column axis. Under these assumptions it is a fairly straightforward procedure to determine bifurcation from the straight configuration into a sinusoidal overall buckling mode. Although both modes are Euler buckling modes and, as such imperfection *insensitive* by themselves, interaction makes the truss column imperfection sensitive, especially for designs with simultaneous classical critical loads and when the local classical critical local is the lower. These analyses disregarded continuity of the flanges and, therefore, overestimated the severity of the mode interaction. This issue was addressed by Byskov in [12] who applied the Byskov-Hutchinson as well as the Koiter approach. Since one of the central assumptions in this case is that the bay length is much smaller than the column length, the Byskov-Hutchinson expansion should not provide reliable results. It turned out that both methods furnished results which indicated less imperfection sensitivity than the ad hoc procedure and that their predictions are close. One reason for this is that in application of the Byskov-Hutchinson expansion to this problem only the buckling modes must be retained, while the postbuckling fields all vanish with the consequence that only the terms a_{jkJ} of (6) enter. Thus, ill-conditioning of the postbuckling problems caused by different orders of magnitude of the wavelengths does not enter. Since the analysis by use of the Byskov-Hutchinson method is much simpler than that of the Koiter approach, this result might tempt one to always apply the former method rather than the latter. However, as shown by Byskov [16], if the asymptotic expansion of Byskov and Hutchinson is applied to a clamped-clamped realization of the truss column, see the right sketch in Fig. 2, *no* imperfection sensitivity is predicted because all a_{jkJ} vanish. This is because the only ones of these constants that may be non-vanishing are found as the integral of the axial force of the overall mode multiplied by the square of the derivative of the transverse displacement component of the local mode and, here, the axial force of the overall mode varies as a cosine along the length of the flanges, i.e. the contributions from the parts with negative axial force cancel the contributions from the rest. This error does not occur if Koiter's method is applied.

The van der Neut Column

Several authors have used the van der Neut Column as a test example of their own method. The original work by van der Neut [7] and the later study by Gilbert and Calladine [17] did not rely on asymptotic methods, but attacked the problem more directly, while Koiter and Kuiken [2] and Byskov, see [13] and [14], apply their asymptotic methods. The results obtained by Koiter and Kuiken [2] must be judged very accurate and valid for relatively large as well as small imperfections. Therefore, in his papers [13] and [14], Byskov used these results as a basis for assessing the range of applicability of the Byskov and Hutchinson asymptotic expansion [4]. Except for differences in numerical values, the above-mentioned analyses all demonstrate the original findings of van der Neut that his column is imperfection sensitive, neutral or even imperfection insensitive depending on the ratio between local and overall classical critical loads.

Other Columns

The van der Neut Column must be considered a model structure and, thus the numerical values obtained in the above-mentioned investigations are not important for practical purposes. On the other hand, a number of other, more realistic thin-walled columns or column-like structures have been studied by many authors. An integrally stiffened panel, which may be viewed as a wide column, was the subject of an early study by Tvergaard [9] who applied a series expansion which was tailored to the specific problem.

Sridharan and coworkers have contributed a number of papers, see e.g. [18] and [19], that are concerned with

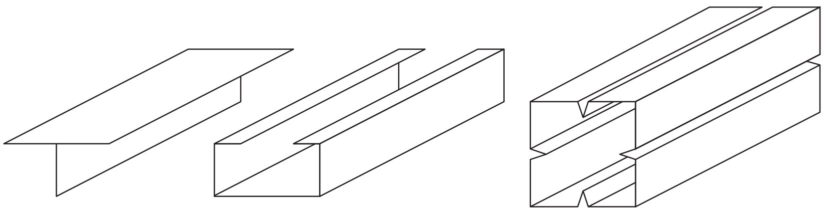

Fig. 3 Various column cross-sections.

columns and column-like structures like some of the ones shown in Fig. 3. In much of his work, Sridharan has utilized the Byskov-Hutchinson asymptotic expansion, sometimes with variations, see later. In addition to valuable results aimed at practice, Sridharan found that Tvergaard in his paper [9] overestimates the imperfection sensitivity of his panel. According to Sridharan, the main reason for this is that in Tvergaard's analysis disregards the term which is quartic in the local mode amplitude ξ_L from the potential energy, a term which furnishes a stabilizing effect. At the time of Tvergaard's study a generally applicable asymptotic method was not available, and, therefore, his analysis was a more specialized one which was aimed at the problem at hand.

In a series of papers Kolakowski and coworkers have studied the same kind of structures as Sridharan, see e.g. [20], [21] and [22], and provided a wealth of useful results. In most of his work, Kolakowski uses the Byskov-Hutchinson asymptotic expansion, often implemented in a finite strip analysis.

In their paper [23], Menken, Kouhia and Groot performed a finite element analysis of mode interaction in T-beams, which utilized the Byskov-Hutchinson expansion [4]. Interaction between more than one local mode and an overall mode was studied. Comparison with experimental results showed very good agreement. A so-called "simple" model having 6 degrees of freedom was also used to capture the essentials of the behavior.

Shells

The asymptotic method by Byskov and Hutchinson [4] was established in a paper concerned with mode interaction in stringer stiffened shells. By changing the number of stringers while keeping the stringer and skin material constant it was possible for a particular family of shells to cover cases with postbuckling stable, neutral and unstable local panel modes. For all designs the overall shell mode was postbuckling unstable. It was found that, independently of the postbuckling characteristics of the local mode, the shells exhibited increased imperfection sensitivity due to mode interaction. The study was semi-analytic and carried out using a smearing-unsmearing technique to describe the effect of the stringers. Later Byskov and Hansen [24] employed a finite element method entailing the transverse displacement component and an Airy-type stress function to carry out a computation where the stringers are discrete entities and found that the results by Byskov and Hutchinson were quite accurate. Hui [25] applied Koiter's method of the slowly varying local mode amplitude to the shells of Byskov and Hutchinson and, at the same time, extended the analysis to include the effect of torsional stringer stiffness. Although the results found by Hui are valuable they suffer from the same problem as the analysis of the same shells by Koiter [3] in that the number of axial half-waves of the local mode cannot be considered very large compared with that of the overall mode, which is a crucial assumption of Koiter's method. Three-mode interaction was considered by Byskov, Damkilde and Jensen [26] with the outcome that interaction between two overall shell modes and a local panel mode may predict an increased imperfection sensitivity over a two-mode analysis.

Kasagi and Sridharan [27] studied composite shells using the Byskov-Hutchinson expansion combined with central features from Koiter's method. In spite of the application of the concept of a slowly varying local mode they kept the $\boldsymbol{u}_{12}$-term, and thus, their expansion becomes:

$$\begin{aligned}\boldsymbol{u}(x,\theta) = \ & \xi_1\boldsymbol{u}_1(x,\theta) + \xi_{ij}f_i(\theta)\phi_j(x)\boldsymbol{u}_2(x,\theta) + \xi_1^2\boldsymbol{u}_{11}(x,\theta) + \xi_1\xi_{ij}f_i(\theta)\phi_j(x)\boldsymbol{u}_{12}(x,\theta) \\ & + \xi_{ij}\xi_{kl}f_i(\theta)\phi_j(x)f_k(\theta)\phi_l(x)\boldsymbol{u}_{22}(x,\theta) + \cdots , \qquad \text{sum over } i \text{ and } j \qquad (15)\end{aligned}$$

where the notation complies with the one used by Byskov and Hutchinson. In (15) x and θ denote the axial and the circumferential coordinate, respectively, while the ξ's are coefficients. The rationale behind the presence of the $\boldsymbol{u}_{12}$-term is that there are many other modes, other than the primary ones $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$, which are hidden in the functions f_i and ϕ_j. The analyses show that the shells treated by Kasagi and Sridharan exhibit strong imperfection sensitivity due to mode interaction.

COMPARISON OF THE THREE METHODS

From the literature it appears that the Byskov-Hutchinson expansion [4] is the more popular of the three methods mentioned above. This may be somewhat surprising in that it has some inherent weaknesses, as mentioned above. One reason is probably that both Koiter's papers [2] and [3] as well as the paper by Peek and Kheyrkhahan [5] utilize a notation which may be alien to most engineers. While most formulas in [4] are expressed in terms of principles of virtual work [2], [3] and [5] base their developments on application of Frechet or Gateaux derivatives of potential energy, which, at least to most engineers, tend to obscure the physical contents of the formulas in question. As an example, (7) of [5]:

$$\dot{\phi}_{,uu} \equiv \left[\frac{d}{d\lambda}\phi_{,uu}(\overset{0}{u}(\lambda),\lambda,0)\right]_{\lambda=\lambda_c} = \phi^{\,c}_{,uuu}\dot{u} + \phi^{\,c}_{,uu\lambda} \ , \quad \dot{u} = \left[\frac{d\overset{0}{u}}{d\lambda}\right]_{\lambda=\lambda_c}$$

is not expressed in the language of most structural engineers. Moreover, when Peek and Kheyrkhahan speak of decomposing the space of admissible displacements A "into a subspace A_0, which is spanned by by a finite number of modes $\overset{i}{u}$, and a complementary space $\hat{A}$ such that:

$a = A_0 \oplus \hat{A}$, $\quad A_0 \cap \hat{A} = \{0\}$"

the terminology lies far from the day-to-day language of the structural engineer. This is a pity, especially because Peek and Kheyrkhahan provide a detailed recipe for implementation of his method.

Although Koiter [3] does not use terms such as "admissible space," his level of abstraction is also rather high. For instance (2.1) in [3]:[2]

$$\mathbf{P}[\boldsymbol{u}] = \mathbf{P}_2^0[\boldsymbol{u}] - \lambda\mathbf{P}_2'[\boldsymbol{u}] + \mathbf{P}_3[\boldsymbol{u}] + \mathbf{P}_4[\boldsymbol{u}]$$

and (2.9):

$$\mathbf{P}_3[\boldsymbol{u}_1] = \mathbf{P}_{21}[\boldsymbol{u}_1,\boldsymbol{u}_2] = \mathbf{P}_{12}[\boldsymbol{u}_1,\boldsymbol{u}_2] = \mathbf{P}_3[\boldsymbol{u}_2] = 0$$

do not immediately lend themselves to interpretation, unless one is familiar with Koiter's nomenclature. In addition to this, as mentioned above, the concept of a "slowly varying function" is difficult to implement in a

[2] The fonts used by Koiter [3] are different from the ones used here.

computer program and, by the way, special attention must be paid to this idea during derivation of the analysis of a particular structure.

In conclusion, it may be said that, although the method of Byskov and Hutchinson [4] must be applied with care, it is the easiest of the three methods to utilize and program.

PROBLEMS AND POSSIBLE IMPROVEMENTS OF THE SERIES

Like most other asymptotic methods, the above-mentioned ones center on the immediate neighborhood of the bifurcation point. Therefore, it may not be a justified demand that the expansions hold for large values of the perturbation parameter—nobody would use a MacLaurin expansion to determine the values of, say $\sin(x)$, for large values of x. Thus, we must expect poor performance of the series if no special precautions are taken, as was done by Koiter in his method of the slowly varying local mode amplitude. Otherwise, much of the effort may be wasted on improving the description of the behavior in the immediate vicinity of the bifurcation point. In his study [14] of the van der Neut Column Byskov applied the Byskov-Hutchinson expansion with the overall and the (primary) local modes, as well as the overall mode and an infinity of local modes, see Fig. 4. As mentioned above, the results obtained by Koiter and Kuiken [2] may be considered accurate, in

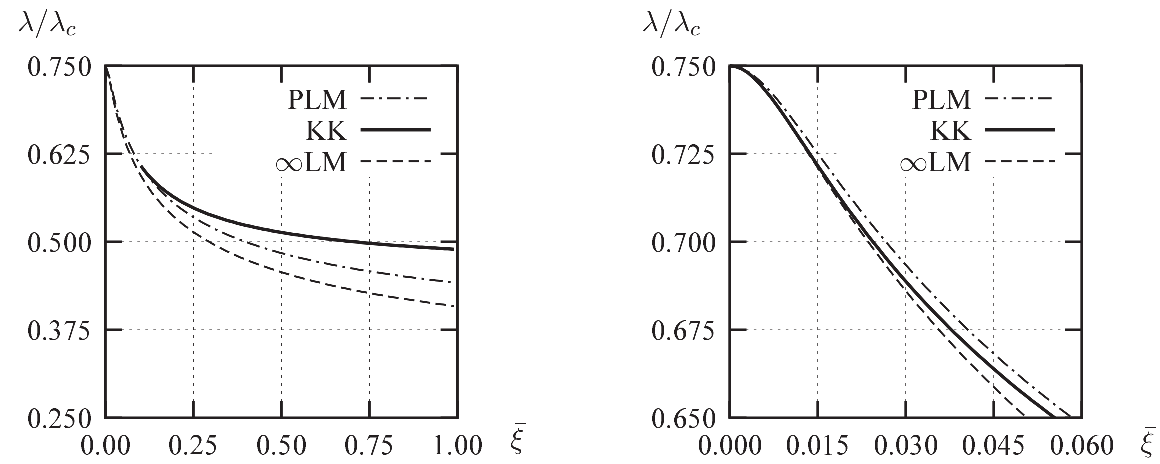

Fig. 4 Imperfection sensitivity of the van der Neut Column. Fairly thick flanges $\lambda_L/\lambda_G = 0.75$. *PLM: The Byskov-Hutchinson method with the primary local mode. KK: Koiter's method.* ∞*LM: The Byskov-Hutchinson method with infinitely many local modes.*

particular in an asymptotic sense. Judging from Fig. 4, left, inclusion of more than one local mode in the Byskov-Hutchinson expansion makes the error larger compared with the case of including the primary local mode, only. If, however, focus is on very small imperfections, see Fig. 4, right, it is clear that inclusion of the infinitely many local modes improves the accuracy locally. It may be shown, see [14], that to lowest order the results obtained by inclusion of the infinitely many local modes are identical with those from the Koiter-Kuiken analysis. For columns with more slender flanges, i.e. when $\lambda_G \geq \lambda_L$ inclusion of more than the primary local mode does not improve results, neither for small nor larger imperfections. The reason is that, once the postbuckling stiffness of the flanges comes into play, the asymptotic expansion by Byskov and Hutchinson becomes less reliable.

It does not seem a reasonable demand that a series expansion should be correct, not even in the asymptotic sense, for extremely small imperfections at the expense of the accuracy for realistic imperfection levels. Therefore, one might seek for other series than the polynomial ones, preferably series in $\bar{\xi}_j$ which result in a finite, positive value of λ for large values of $\bar{\xi}$.

FINITE ELEMENT IMPLEMENTATION

Today, most analyses of important structures are performed by use of a finite element method and one might ask, why bother use an asymptotic method instead of performing a full nonlinear computation of the structure with its geometric imperfections? One reason is that by applying one or the other of the asymptotic methods you get simple algebraic equations that cover "all" imperfection levels and such equations are much faster and safer to solve than thousands and thousands of finite element equations. Another reason is that it is very easy to overlook the some of the basic features of the behavior of a geometrically imperfect structure if a full nonlinear analysis is applied. Finally, by use of an asymptotic method, possibly combined with finite element solution of the associated set of linear problems, see (2)–(4), it is very fast to get an estimate of whether the structure is imperfection sensitive or not. This information may then be exploited to find a reasonable design. Then, the actual structural design with a number of imperfection levels may be investigated closer by use of a full nonlinear analysis.

Special Finite Element Problems

By now, it is a well-known fact that in kinematically nonlinear finite element computations you should pay attention to the issue of nonlinear membrane “locking,” i.e. the internal mismatch between different terms of the axial strain in a beam or arch, or of the membrane strain of a plate or shell. This problem is even worse in asymptotic finite element analyses, as pointed out by Olesen and Byskov [28], who suggested an approximate method to deal with membrane locking. In asymptotic postbuckling analyses the postbuckling strain $\boldsymbol{\varepsilon}_{jk}$ may be written:

$$\boldsymbol{\varepsilon}_{jk} = \mathbf{l}_1(\boldsymbol{u}_{jk}) + \tfrac{1}{2}\mathbf{l}_{11}(\boldsymbol{u}_j, \boldsymbol{u}_k) \tag{16}$$

To be specific, consider a straight beam, then the axial strain ε_{jk} is:

$$\varepsilon_{jk} = u'_{jk} + \tfrac{1}{2} w'_j w'_k \tag{17}$$

Typically, the first term on the right-hand side is a polynomial of degree 0 or 1, depending on whether there is no or one internal axial degree of freedom, while the second term is usually of degree 4. To make the problem worse, the second term is given once the buckling problems have been solved. When the cross-sectional area of the beam is constant, the postbuckling axial strain is constant, too. This, however, may not be described by the finite element unless the number of axial degrees of freedom is increased to 5. The fact that the second term may be considered a driving term in the postbuckling boundary value problem means that the two terms, loosely speaking, do not have the ability to accommodate each other.

In computations the more rigorous and consistent method by Byskov [29], which makes use of *Lagrange Multipliers* in a straightforward way, must be preferred over [28]. Application of the “direct” approach by Noe Poulsen and Damkilde [30] seems unnecessarily complicated without providing better results than [29]. In fact, for the cases where [30] applies it gives the same results as [29], while it is very doubtful whether [30] can be extended to cover curved structures.

Commercial Finite Element Codes

Today, the idea of performing an asymptotic mode interaction analysis has spread to commercial programs, such as DIANA [31], which utilizes the Byskov-Hutchinson expansion for the finite element analysis.

CONCLUDING REMARKS

Although general purpose finite element codes exist there is still a need for asymptotic methods to reveal the fundamentals of mode interaction in a structure. Which of the above mentioned asymptotic methods one prefers is, to some extent, a matter of taste.

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