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Fajstrup, Lisbeth

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by

Lisbeth Fajstrup

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Department of Mathematical Sciences
Aalborg University
Fredrik Bajers Vej 7 G ▪ DK - 9220 Aalborg Øst ▪ Denmark
Phone: +45 99 40 80 80 ▪ Telefax: +45 98 15 81 29
URL: http://www.math.aau.dk

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CLASSIFICATION OF DICOVERINGS

LISBETH FAJSTRUP

Abstract. The dicoverings of a "well pointed" d-space are classified as quotients of the universal dicovering space under congruence relations. We prove that the subcategory of d-spaces generated by the subcategory of directed cubes is equal to the category generated by the interval and the directed interval. Similarly, the category of topological spaces generated by simplices may be generated by the interval.

1. Introduction

Dicoverings were introduced in [3] as a tool for investigating d-spaces. A d-space is a topological space X with a subset $\vec{P}(X) \subset X^I$ of the set of paths, denoted the dipaths. A dicovering is a map of d-spaces $p : Y \to X$ satisfying certain lifting properties. We do not require local (di) homeomorphism properties as in non-directed topology, since such properties are not implied by the lifting properties, even in very simple examples, see Ex. 4.7. Our dicoverings are more like fibrations of directed graphs in the sense of [1]. In particular, a dicovering may not be a covering in **Top**. A dicovering $\Pi : \tilde{X} \to X$ is universal, if for all dicoverings, $p : Y \to X$, there is a unique map $\phi : \tilde{X} \to Y$ such that $\Pi = p \circ \phi$. A pointed d-space, (X, x) is well pointed, if all points in X are the target of a dipath with source x. The category of well pointed d-spaces is denoted **wpd-Top**. In Thm. 5.2 we prove the existence $\Pi : (\tilde{X}, \tilde{x}) \to (X, x)$ of a universal dicover of all well pointed d-spaces. In Section 4, we provide a construction of $\tilde{X}$.

The main ingredients in these results are the construction from [3] and the result from [7], where we proved the existence of a universal dicovering of *cubically generated* well pointed d-spaces, i.e., in a subcategory $\textbf{wpd-Top}_{\mathcal{B}} \subset$ **wpd-Top**. A d-space X is cubically generated, if a map $f : X \to Y$ is a d-map whenever $f \circ \phi$ is a d-map for all maps $\phi : B \to X$, and all n-cubes $B = I \times I \times \cdots \times I$ where I is the unit

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interval with either the standard d-structure - dipaths are the increasing paths - or the discrete structure - only constant paths are dipaths. Denote these d-spaces on the interval $\vec{I}$ and I.

In Prop. 3.3 we prove that $\mathbf{d\text{-}Top}_{\mathcal{B}} = \mathbf{d\text{-}Top}_{\mathcal{I}}$ where $\mathcal{I}$ is the full subcategory with two objects, $\vec{I}$ and I. Hence, if continuity of a map $f : X \to Y$ can be established by checking the restriction to cubes, one may check it just by the restriction to paths. A similar result holds for topological spaces generated by simplices $\mathbf{Top}_{\mathcal{D}}$, which is the same subcategory of $\mathbf{Top}$ as $\mathbf{Top}_I$ where I is the full subcategory of $\mathbf{Top}$ with one object, the unit interval.

In order to use the universal dicoverings which we know exist for $\mathbf{wpd\text{-}Top}_{\mathcal{B}}$ for a general wpd-space, we give a boxification functor $\Box : \mathbf{d\text{-}Top} \to \mathbf{d\text{-}Top}_{\mathcal{B}}$. This is right adjoint to the inclusion $\iota : \mathbf{d\text{-}Top}_{\mathcal{B}} \to \mathbf{d\text{-}Top}$. Similarly, we get a simplexification functor $\Delta : \mathbf{Top} \to \mathbf{Top}_{\mathcal{D}}$, which is a right adjoint to the inclusion $\iota : \mathbf{Top}_{\mathcal{D}} \to \mathbf{Top}$. It is well known [7] Prop. 3.5 that such generated subcategories are coreflective. The contribution here is to spell this out in the examples $\mathbf{d\text{-}Top}_{\mathcal{B}}$ and $\mathbf{Top}_{\mathcal{D}}$

The universal dicovering $\Pi : (\tilde{X}, \tilde{x}) \to (X, x)$ supports certain equivalence relations, congruence relations. The dicoverings of X are classified in the following sense: For all congruence relations $\approx$ on $\tilde{X}$, the quotientmap $\psi : (\tilde{X}, \tilde{x}) \to (\tilde{X}/\approx, [\tilde{x}])$ is a dicovering and there is a unique map $p : (\tilde{X}/\approx, [\tilde{x}]) \to (X, x)$ such that p is a dicovering and $p \circ \psi = \Pi$. Moreover, given a dicovering $q : (Y, y) \to (X, x)$, there is a congruence relation $\approx_q$ on $\tilde{X}$ such that the universal map $\phi : (\tilde{X}, \tilde{x}) \to (Y, y)$ factors over $\psi : (\tilde{X}, \tilde{x}) \to (\tilde{X}/\approx_q, [\tilde{x}])$, $\phi = f \circ \psi$ and f is a bijective d-map. Hence (Y, y) and $(\tilde{X}/\approx_q, [\tilde{x}])$ differ only in the topologies in the sense that the quotientspace has more opens than Y.

This is as good as it gets in the sense that we give an example of a $\mathbf{wpd\text{-}Top}_{\mathcal{B}}$ space which is not the quotient of its universal dicovering. Hence, not even the trivial dicovering, the identity map, is a quotient.

It is a pleasure to thank Martin Raussen for very helpful remarks and discussions.

2. d-Spaces and categories generated by a subcategory

We give the definitions and results from [5] and [7] on d-spaces and categories generated by a subcategory. Moreover, we give some examples.

Definition 2.1. A *d*-space is a topological space X with a set of paths $\vec{P}(X) \subset X^I$ such that

- $\vec{P}(X)$ contains all constant paths.
- $\gamma, \mu \in \vec{P}(X)$ implies $\gamma \star \mu \in \vec{P}(X)$, where $\star$ is concatenation.
- If $\phi : I \to I$ is monotone, $t \leq s \Rightarrow \phi(t) \leq \phi(s)$, and $\gamma \in \vec{P}(X)$, then $\gamma \circ \phi \in \vec{P}(X)$. I.e., $\vec{P}(X)$ is closed under taking subpaths and monotone reparametrization.

The *d*-space is saturated if whenever $\phi : I \to I$ a monotone surjection and $\gamma \circ \phi \in \vec{P}(X)$, then $\gamma \in \vec{P}(X)$.

A *d*-map or dimap $f : X \to Y$ is a continuous map, such that if $\alpha \in \vec{P}(X)$ then $f \circ \alpha \in \vec{P}(Y)$.

The set of distinguished paths, $\vec{P}(X)$ are called the *dipaths*. They are *d*-maps from the ordered interval $\vec{I}$ to X.

For $\gamma : I \to X$, we denote $\gamma(0)$ the source and $\gamma(1)$ the target of γ, and we let $\vec{P}(X, A, B)$ denote dipaths with source $\gamma(0) \in A \subseteq X$ and target $\gamma(1) \in B \subseteq X$.

The category of *d*-spaces is denoted **d-Top**

Example 2.2. Let $X = \mathbb{R}^n$ and let a path $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$ be a dipath if $t_1 \leq t_2$ implies $\gamma_i(t_1) \leq \gamma_i(t_2)$ for all i. When we consider $\mathbb{R}^n$ as a d-space, this will be the dipaths, unless we mention otherwise.

Example 2.3. Let X be the geometric realization of a cubical set. In a cube $[0,1]^n \subset \mathbb{R}^n$, the dipaths are all restrictions of dipaths in $\mathbb{R}^n$. Let $\vec{P}(X)$ be generated by concatenation and monotone reparametrization of the dipaths in the cubes.

Example 2.4. ([3] Ex. 4.7) We define a Hawaiian star for δ an irrational number:

$$\mathcal{S} = \bigcup_{n=1}^{\infty} \{(u\cos(n\pi\delta), u\sin(n\pi\delta)) | [0 \leq u \leq \frac{1}{n}\}$$

with the subspace topology from $\mathbb{R}^2$. The dicone on $\mathcal{S}$ is

$$\mathcal{CS} = \bigcup_{n=1}^{\infty} \{(tu\cos(n\pi\delta), tu\sin(n\pi\delta), t-1) | (u,t) \in [0, 1/n] \times \vec{I}\}$$

with topology induced from $\mathbb{R}^3$ and partial order in terms of the (u,t) coordinates: $(u, t_1) \leq (u, t_2)$ if $t_1 \leq t_2$.

Example 2.5. A space X with $\vec{P}(X) = X^I$ is a d-space with *trivial d-structure*. If $\vec{P}(X)$ is the constant maps, X has the *discrete d-structure*. Note that the d-maps *from* a space with discrete d-structure are the continuous maps. The d-maps *to* a space with trivial d-structure are the continuous maps.

Example 2.6. A subspace $Y \subset X$ of a d-space has an induced d-structure in the obvious way.
The product $X \times Y$ of two d-spaces has a product d-structure: $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is a dipath if both components are.

The *d*-structure on the image of a surjection is defined as in [5], except that we take closure under monotone reparametrization, which [5] forgot. For a thorough discussion of monotone reparametrization, see [2]

Definition 2.7. Let Z be a d-space and let $f : Z \to Y$ be a surjection. Then Y has the *quotient d-structure* if the topology is the quotient topology and $\vec{P}(Y)$ is the closure of the set of dipaths $\{f \circ \gamma | \gamma \in \vec{P}(Z)\}$ under finite concatenation and monotone reparametrization.

Remark 2.8. Let $F : Z \to Y$ be as above with Y the quotient d-space. Let $g : Y \to X$. Suppose $g \circ F$ is a d-map. Then clearly g is continuous. Let $\gamma \in \vec{P}(Y)$, then γ is a monotone reparametrization of $F \circ \eta_1 \star F \circ \eta_2 \cdots F \circ \eta_n$ for a set of $\eta_i \in \vec{P}(Z)$. Hence $g \circ \gamma$ is a monotone reparametrization of $g \circ F \circ \eta_1 \star g \circ F \circ \eta_2 \cdots g \circ F \circ \eta_n$ which is in $\vec{P}(X)$.
So Y is indeed a quotient, i.e., g is a d-map if and only if, $g \circ F$ is a d-map.

Definition 2.9. For a d-space $(X, \vec{P}(X))$, and $x_0, x_1 \in X$ a dihomotopy of dipaths $\gamma_1, \gamma_2 \in \vec{P}(X, x_0, x_1)$ is a d-map $H : \vec{I} \times I \to X$ such that $H(t, 0) = \gamma_1(t)$, $H(t, 1) = \gamma_2(t)$, $H(0, s) = x_0$ and $H(1, s) = x_1$. Here I is the interval with the discrete d-structure and $\vec{I}$ has the subspace structure from $\mathbb{R}$. Similarly, we define dihomotopies with fixed source of dipaths $\gamma_1, \gamma_2 \in \vec{P}(X, x_0, -)$ and dihomotopies of general d-maps.

Remark 2.10. A dihomotopy between dipaths is a d-map from the quotient of $\vec{I} \times I$ under identification of all $(0, s)$ with $(0, 0)$ and all $(1, s)$ with $(1, 0)$.

In [7], we study generated categories in the following sense:

Definition 2.11. Let $\mathbf{D}$ be a full subcategory of a concrete category $\mathbf{C}$. The subcategory generated by $\mathbf{D}$, denoted $\mathbf{C_D}$ is the full subcategory defined by $C \in \mathbf{C}$ is in $\mathbf{C_D}$ if for all $B, K \in \mathbf{C}$, $f : UB \to UK$ lifts to a $\mathbf{C}$ morphism if and only if $f \circ U\phi : UD \to UK$ lifts for all $D \in \mathbf{D}$ and all $\phi : D \to B$. Here U is the forgetful functor to Set.

Remark 2.12. A well known example is the category of k-spaces, which is the subcategory of **Top** generated by compact Hausdorff spaces.

Definition 2.13. Let $\mathcal{B}$ be the full subcategory of **d-Space** with objects all cubes $I_1 \times I_2 \times \ldots \times I_n$ where I_k is the unit interval with subspace topology from $\mathbb{R}$ and either the discrete d-structure or the standard d-structure induced from $\mathbb{R}$. The d-structure on $I_1 \times I_2 \times \ldots \times I_n$ is the product structure.

Example 2.14. The subcategory of **d-Top** generated by $\mathcal{B}$ is denoted $\mathbf{d\text{-}Top}_{\mathcal{B}}$. A d-space X is in $\mathbf{d\text{-}Top}_{\mathcal{B}}$ if $f : X \to Y$ is a d-map whenever $f \circ \phi$ is a d-map for all $\phi : B \to X$ and all $B \in \mathcal{B}$. The requirement $f \circ \gamma$ is a d-map, whenever $\gamma : \vec{I} \to X$ ensures that $f(\vec{P}(X)) \subset \vec{P}(Y)$, so the $X \in \mathbf{d\text{-}Top}_{\mathcal{B}}$ is really a statement about the topology on X.
Since for a directed cube, the identity map from the corresponding discretely ordered cube is continuous and hence a d-map, we may restate the condition: A d-space X is in $\mathbf{d\text{-}Top}_{\mathcal{B}}$ if $f : X \to Y$ is a d-map whenever $f \circ \phi$ is a d-map for all $\phi : B \to X$ and all cubes B with the discrete d-structure (and the standard topology.)

Example 2.15. Let $\mathcal{D}$ be the full subcategory of **Top** with objects the n-simplices Δ_n, $n = 0, 1, \ldots$. Then $\mathbf{Top}_{\mathcal{D}}$ is the category of topological spaces generated by simplices. Since Δ_n is homeomorphic to the n-cube, the same subcategory may be generated by the n-cubes.

3. The boxification functor and the simplexification functor

The inclusion $\iota : \mathbf{d\text{-}Top}_{\mathcal{B}} \to \mathbf{d\text{-}Top}$ has a right adjoint, the boxification functor, which we define here. Similarly, we give a right adjoint, simplexification to the inclusion $\iota : \mathbf{Top}_{\mathcal{D}} \to \mathbf{Top}$. The existence and categorical construction of such adjoints as a U-final lift is well known, [7], and we give the construction in these special cases. Moreover, we prove that the subcategory $\mathbf{d\text{-}Top}_{\mathcal{I}} \subset \mathbf{d\text{-}Top}_{\mathcal{B}}$ generated by the full subcategory $\mathcal{I}$ with two objects, the interval with the discrete structure and the interval with standard d-structure, is in fact the whole $\mathbf{d\text{-}Top}_{\mathcal{B}}$ and similarly $\mathbf{Top}_I \subset \mathbf{Top}_{\mathcal{D}}$ where I is the unit interval, is in fact all of $\mathbf{Top}_{\mathcal{D}}$.

We provide an example, the Hawaiian star, of a compact space, which is not cubically generated.

Definition 3.1. Let $\mathbf{d\text{-}Top}_{\mathcal{B}}$ be the subcategory of **d-Top** generated by $\mathcal{B}$, then we define the Boxification functor $\Box : \mathbf{d\text{-}Top} \to \mathbf{d\text{-}Top}_{\mathcal{B}}$. Let $X \in \mathbf{d\text{-}Top}$, then $\Box X$ is a d-Space on the underlying set UX of X

with topology: $V \subset UX$ is open in $\Box X$ if for all d-maps $\phi : B \to X$, where $B \in \mathcal{B}$, $\phi^{-1}(V)$ is open.

The dipaths are given $\vec{P}(\Box X) = \vec{P}(X)$.

Let $f : X \to Y$ be a d-map, then by Lem. 3.9, $f : \Box X \to \Box Y$ is a d-map, so we define $\Box f = f$

Remark 3.2. By Ex. 2.14, the topology on $\Box X$ is actually generated by the cubes with the standard topology and discrete d-structure.
The identity map $i : \Box X \to X$ is a d-map, since $U \subset X$ open implies $\phi^{-1}(U)$ open for all d-maps $\phi : B \to X$ with $B \in \mathcal{B}$, so U is open in $\Box X$.

Proposition 3.3. *Let $\mathcal{I}$ be the full subcategory of* **d-Top** *with objects I and $\vec{I}$. Then* $\mathbf{d\text{-}Top}_{\mathcal{B}} = \mathbf{d\text{-}Top}_{\mathcal{I}}$

Proof. Since $\mathcal{I}$ is a subcategory of $\mathcal{B}$, we have $\mathbf{d\text{-}Top}_{\mathcal{I}} \subset \mathbf{d\text{-}Top}_{\mathcal{B}}$. Now suppose $X \in \mathbf{d\text{-}Top}_{\mathcal{B}}$ and let $Y \in \mathbf{d\text{-}Top}$. Suppose for $f : UX \to UY$ that $f \circ \mu$ is a d-map for all $\mu : I \to X$ and all $\mu : \vec{I} \to X$. The latter ensures, that $f(\vec{P}(X)) \subset \vec{P}(Y)$, so to show that f is a d-map it suffices to study continuity. Let $h : B \to X$, where $B = I^n$ is an n-cube. Then for all paths $\gamma : I \to B$, $f \circ h \circ \gamma$ is continuous.
We prove that $g : I^n \to Y$ is continuous if (and only if) $g \circ \gamma$ is continuous for all paths $\gamma : I \to I^n$. So suppose $g \circ \gamma$ is continuous for all paths $\gamma : I \to I^n$. Since I^n is first countable, it suffices to see, for a convergent sequence $\{x_n\}_{n \in \mathbb{N}}$ in I^n, $\lim_{n\to\infty} x_n = x$ that $g(x_n)$ is convergent to $g(x)$. Let $\gamma : I \to I^n$ be given by $\gamma|_{[1-\frac{1}{k}, 1-\frac{1}{k+1}]}$ is the line segment from x_k to x_{k+1} and $\gamma(1) = x$, then γ is continuous: For a neighborhood V of x, let $B(x, r)$ be an open ball contained in V. There is an N such that for $n \geq N$, $x_n \in B(x, r)$ and by convexity, $t \in]1-\frac{1}{N}, 1]$ implies $\gamma(t) \in B(x, r)$. The sequence $t_k = (1-\frac{1}{k})$ converges to 1 in I, and hence $g \circ \gamma(t_k) = g(x_k)$ converges to $g \circ \gamma(1) = g(x)$.

Hence, $f \circ \mu$ continuous for all μ implies $f \circ h$ continuous for all $h : B \to X$ which implies f is continuous, since $X \in \mathbf{d\text{-}Top}_{\mathcal{B}}$ $\Box$

Corollary 3.4. *Let the intervalization functor $\mathcal{I}$:* $\mathbf{d\text{-}Top} \to \mathbf{d\text{-}Top}_{\mathcal{I}}$ *be defined by $\mathcal{I}(X)$ is a d-Space on the underlying set UX of X with topology: $V \subset UX$ is open in $\mathcal{I}(X)$ if for all d-maps $\phi : I \to X$, and all d-maps $\phi : \vec{I} \to X$ $\phi^{-1}(V)$ is open.*

The dipaths are given $\vec{P}(\mathcal{I}(X)) = \vec{P}(X)$. On morphisms, $\mathcal{I}(f) = f$
Then $\mathcal{I} = \Box$

Proposition 3.5. *Let* **Top** *be the category of topological spaces and continuous maps and $\mathcal{D}$ the full subcategory od simplices as in Ex. 2.15. Then* $\mathbf{Top}_{\mathrm{D}} = \mathbf{Top}_{I}$.

Proof. As above. □

Example 3.6. The Hawaian star, $\mathcal{S}$, Ex. 2.4, is compact and we will see below, that $\mathcal{S} \notin$ **d-Top**$_{\mathcal{B}}$. To see that $\mathcal{S}$ is compact, since it is clearly a bounded subset of $\mathbb{R}^2$, it suffices to see, that it is closed. For $p \in \mathbb{R}^2 \setminus \mathcal{S}$, let $N \geq \frac{2}{|p|}$. The finite union $\mathcal{S}_N = \bigcup_{n=1}^{N} \{(u\cos(n\pi\delta), u\sin(n\pi\delta))|[0 \leq u \leq \frac{1}{n}\}$ is closed, and hence there is an $r > 0$ such that $B(p,r) \cap \mathcal{S}_N = \emptyset$. Since $\bigcup_{n=N+1}^{\infty} \{(u\cos(n\pi\delta), u\sin(n\pi\delta))|[0 \leq u \leq \frac{1}{n}\} \subset B(0, \frac{1}{N}) \subset B(0, \frac{|p|}{2})$ we get $B(p, \min\{r, |p|/2\}) \cap \mathcal{S} = \emptyset$, so $\mathcal{S}$ is closed in $\mathbb{R}^2$. A similar proof shows that the dicone $\mathcal{CS}$ is compact.

Lemma 3.7. *With notation from above, the Hawaiian star $\mathcal{S}$ is not in* **d-Top**$_{\mathcal{B}}$.

Proof. The d-structure is irrelevant - we study the topology. Let $\bigsqcup_{j=1}^{\infty} I_j$ be the disjoint union of countably many copies of the unit interval and for $0 \leq t \leq 1$ let t_j be the point $t \in I_j$. Let $X = \bigsqcup_{j=1}^{\infty} I_j / \{0_j\}$ be the quotient under the relation $0_j \sim 0_k$. X is clearly in **d-Top**$_{\mathcal{B}}$.

Define $\varphi : X \to \mathcal{S}$ by $\varphi(t_n) = \frac{t}{n}(\cos(n\pi\delta), \sin(n\pi\delta))$. Then φ is a continuous bijection, since $\varphi|_{I_n}$ is clearly continuous. However, φ^{-1} is not continuous: With metric on X, $d(t_n, t_l) = |t_n - t_l|$ if $n = l$ and $d(t_n, t_l) = |t_n| + |t_l|$ else, we get $\varphi(B_X(0, 1/2)) = \varphi(\bigsqcup_{j=1}^{\infty}[0, 1/2[/\sim) = \bigcup_{n=0}^{\infty}\{u(\cos(n\pi\delta), \sin(n\pi\delta))|u \in [0, \frac{1}{2n}[\}$. The latter is not open in $\mathcal{S}$, since for all $r > 0$, $\mathcal{S} \cap B_{\mathbb{R}^2}(0, r) \not\subset \varphi(B(0, 1/2))$, so 0 is not an interior point.

Claim: $\varphi : X \to \square(\mathcal{S})$ is a homeomorphism. By 3.9, φ is continuous.

Let $\mu : I \to \mathcal{S}$. We have to see, that $\phi^{-1} \circ \mu$ is continuous. Let $K = \mu^{-1}(0)$, then $I \setminus K$ is open, and hence a countable disjoint union of open intervals $\bigsqcup_{j \in J}]a_j, b_j[$ where $b_j \leq a_{j+1}$ and $J \subseteq \mathbf{N}$. By continuity, and since $]a_j, b_j[$ is connected, there is an n_j such that $\mu(]a_j, b_j[) \subset S_{n_j}$ where S_{n_j} is the n_j'th strand of $\mathcal{S} \setminus \{0\}$, i.e., $S_{n_j} = \{u(\cos(n_j\pi\delta), \sin(n_j\pi\delta))|u \in]0, \frac{1}{n_j}[\}$; and $\mu(a_j) = \mu(b_j) = 0$. To see that all S_n are open in $\mathcal{S} \setminus \{0\}$ and hence connected components, let $p \in S_n$. Then $|p| > 0$ and the construction from Ex. 3.6 will provide a ball $B_{\mathbb{R}^2}(p, r)$ with $B_{\mathbb{R}^2}(p, r) \cap \mathcal{S} \subset S_n$

The restriction of φ^{-1} to a strand is continuous, since $\phi^{-1} : S_n \to I_n \setminus \{0\}$ is the bijection $\phi^{-1}(u(\cos(n\pi\delta), \sin(n\pi\delta))) = (nu)_n$ and the topology on S_n induced from $\mathbb{R}^2$ is the standard topology on an interval. Hence, $\varphi^{-1} \circ \mu$ is continuous on $]a_j, b_j[$ for all j, and since $\mu(a_j) = \mu(b_j) = 0$, $\varphi^{-1} \circ \mu$ is continuous on $[a_j, b_j]$ with $\varphi^{-1}(a_j) = \varphi^{-1}(b_j) = 0 \in X$ for all a_j, b_j. Moreover, for $t \in [b_j, a_{j+1}]$, $\varphi^{-1}(t) = 0$. □

We have also proved

Proposition 3.8. *The boxification* $\Box\mathcal{S}$ *is* $X = \bigsqcup_{j=1}^{\infty} I_j/\{0_j\}$

Lemma 3.9. *Let* $f : X \to Y$ *be a d-map, then* $f : \Box X \to \Box Y$ *is a d-map. In particular, if* $\gamma : \vec{I} \to X$ *is a dipath, then* $\gamma : \vec{I} \to \Box X$ *is a d-map.*

Proof. Let $U \subset \Box Y$ be open and let $\psi : B \to \Box X$ where $B \in \mathcal{B}$. Then $\psi^{-1}(f^{-1}(U)) = (f \circ \psi)^{-1}(U)$ and the latter is open, since $f \circ \psi$ is a d-map. □

Theorem 3.10. *Let* $X \in$ **d-Top**$_{\mathcal{B}}$ *and* $Y \in$ **d-Top**. *Then a map* $f : X \to Y$ *is a d-map if and only if* $f : X \to \Box Y$ *is a d-map.*

Proof. Since the dipaths of a space and its boxification are the same, we just have to check continuity. Suppose $f : X \to \Box Y$ is continuous. Then, since a subset which is open in Y is also open in $\Box Y$, $f : X \to Y$ is continuous.

If $f : X \to Y$ is continuous, then by Lem. 3.9 $f : \Box X \to \Box Y$ is continuous. But $X = \Box X$. □

Corollary 3.11. *The boxification functor is a right adjoint to the inclusion* ι *of* **d-Top**$_{\mathcal{B}}$ *in* **d-Top**

Proof. Let $X \in$ **d-Top**$_{\mathcal{B}}$ and $Y \in$ **d-Top** and U the forgetful map to **Set**. By Thm. 3.10, a map of sets $f : UX \to UY$, lifts to $f \in Hom_{\textbf{d-Top}_{\mathcal{B}}}(X, \Box Y)$, if and only if it lifts to $f \in Hom_{\textbf{d-Top}}(\iota X, Y)$. Hence $Hom_{\textbf{d-Top}_{\mathcal{B}}}(X, \Box Y) \simeq Hom_{\textbf{d-Top}}(\iota X, Y)$. □

Definition 3.12. The simplexificaton $\Delta :$ **Top** $\to$ **Top**$_{\mathcal{D}}$ is defined on objects $X \in$ **Top**: ΔX is a topological space on the underlying set UX of X. The topology is $V \subset UX$ is open in ΔX if for all continuous maps $f : D \to X$, where $D \in \mathcal{D}$, $f^{-1}(V$ is open. On morphisms, $\Delta g = g$, i.e., the same map on the underlying sets.

Proposition 3.13. *The simplexification functor is right adjoint to the inclusion* $\iota :$ **Top**$_{\mathcal{D}} \to$ **Top**.

Proof. As Cor. 3.11 - Thm. 3.10 and the previous lemmas translate to the non-directed case verbatim. □

4. Dicoverings and universal dicoverings. Definitions and existence

In [7] we prove that **d-Top**$_{\mathcal{B}}$ has universal dicoverings. We change the setup here to be in accordance with the recent amendment to [3]:

Definition 4.1. Let $X, Y \in$ **d-Top** and $p : Y \to X$. Then p is a dicovering of X if

- for each dipath $\gamma : \vec{I} \to X$ with $\gamma(0) = x$ and for each $y \in p^{-1}(x)$, there is a unique lift $\hat{\gamma} : \vec{I} \to Y$ with $\hat{\gamma}(0) = y$, i.e., there is a unique filler $\hat{\gamma}$ of the following commutative diagram

$$\begin{array}{ccc} \{0\} & \longrightarrow & Y \\ \big\downarrow & \overset{\hat{\gamma}}{\nearrow} & \big\downarrow p \\ \vec{I} & \xrightarrow{\gamma} & X \end{array}$$

- Let J be the coequalizer of $f_1, f_2 : I \to \vec{I} \times I$ where $f_1(s) = (0, s)$ and $f_2(s) = (0, 0)$. Then there is a unique filler $\hat{H}$ of this commutative diagram.

$$\begin{array}{ccc} (0,0) & \longrightarrow & Y \\ \big\downarrow & \overset{\hat{H}}{\nearrow} & \big\downarrow p \\ J & \xrightarrow{H} & X \end{array}$$

  I.e., dihomotopies with fixed source lift uniquely.
- Let K be the coequalizer of $g_1, g_2 : I \sqcup I \to \vec{I} \times I$, $g_1(s_1) = (0, s_1), g_1(s_2) = (1, s_2)$ and $g_2(s_1) = (0, 0), g_2(s_2) = (1, 0)$, then we require a unique filler of

$$\begin{array}{ccc} (0,0) & \longrightarrow & Y \\ \big\downarrow & \overset{\hat{H}}{\nearrow} & \big\downarrow p \\ K & \xrightarrow{H} & X \end{array}$$

  i.e., the unique lift of a dihomotopy with both source and target fixed is a dihomotopy with fixed source and target.

A *pointed d-space* is a pair (X, x) consisting of a d-space X and a point $x \in X$. A morphism of pointed d-spaces $(X, x) \to (Y, y)$ is a d-map $f : X \to Y$ such that $f(x) = y$.

A *pointed dicovering*, a p-dicovering, is a pointed d-map which is a dicovering.

The category of pointed d-spaces and pointed d-maps, pd-maps, is denoted **pd-Top** and the category of pointed d-spaces (X, x) s.t. $X \in$ **d-Top**$_{\mathcal{B}}$ is denoted **pd-Top**$_{\mathcal{B}}$. Dicoverings in subcategories of **d-Top** are defined as above - requiring the same lifting properites.

Definition 4.2. ([7]) A *universal p-dicovering* of $(X, x) \in$ **pd-Top**$_{\mathcal{B}}$ is a p-dicovering $\pi : (\tilde{X}, \tilde{x}) \to (X, x)$ in **pd-Top**$_{\mathcal{B}}$ such that for any p-dicovering $g : (Y, y) \to (X, x)$ in **pd-Top**$_{\mathcal{B}}$ there is a unique pd-map $\phi : (\tilde{X}, \tilde{x}) \to (Y, y)$ such that ϕ is a dicovering and $\pi = g \circ \phi$.

Remark 4.3. If a d-map ϕ exists,s.t. $\Pi = g \circ \phi$, then ϕ is a dicovering, which one can see by examining the diagram with Π, φ and g, where g and Π are known to satisfy the lifting properties.

Corollary 4.4. *([7] Corollary 5.6) A universal p-dicovering exists for every pd-space in* **pd-Top**$_{\mathcal{B}}$

Definition 4.5. A pd-Space (X, x) is *well pointed*, if for all $z \in X$, there is a dipath $\gamma : \vec{I} \to X$ with $\gamma(0) = x$ end $\gamma(1) = z$. We denote this condition $z \succeq x$.

A dicovering $p : (Y, y) \to (X, x)$ of a well pointed space (X, x) is surjective, since dipaths lift. Moreover, if we restrict to $\uparrow_Y y = \{z \in Y | z \succeq y\}$, then $p :\uparrow_Y y \to X$ is a surjective dicovering in the category **wpd-Top** of well pointed spaces.

Corollary 4.6. *If* $\pi : (\tilde{X}, \tilde{x}) \to (X, x)$ *is the universal dicovering in* **pd-Top**$_{\mathcal{B}}$ *and* $(X, x) \in$ **wpd-Top**$_{\mathcal{B}}$*, then* $\uparrow_{\tilde{X}} \tilde{x} \to X$ *is the universal dicovering in* **wpd-Top**$_{\mathcal{B}}$*, i.e., for well pointed dicoverings.*

Example 4.7. The unit circle $S^1 = \{(\cos(t), \sin(t)) | 0 \leq t \leq 2\pi\}$ with dipaths running counterclockwise and basepoint $x_0 = (-1, 0)$ has the positive reals as its universal dicovering.
If instead the dipaths are the paths which increase in the first coordinate, then the universal dicover is $\vec{I} \sqcup \vec{I} / 0_1 \sim 0_2$ where coordinates t_1 are in the first copy of $\vec{I}$ and similarly for t_2. The dicoveringmap is $\Pi(t_1) = (\cos(\pi(1 - t_1)), \sin(\pi(1 - t_1)))$ and $\Pi(t_2) = (\cos(\pi(1 + t_2)), \sin(\pi(1 + t_2)))$. Notice that this dicovering har fiberdimension one over all points except at $(0, 0)$, where $\Pi^{-1}((0, 0)) = \{1_1, 1_2\}$

5. Dicoverings and boxification

For a well pointed d-space $(X, x) \in$ **wpd-Top**, the universal dicovering of the boxification of (X, x), $\Pi : (\widetilde{\Box X}, \tilde{x}) \to (\Box X, x)$ is also a universal dicovering for (X, x) in the sense that for a dicovering $p : (Y, y) \to (X, x)$ in **wpd-Top**, there is a unique pd-map $\phi : (\widetilde{\Box X}, \tilde{x}) \to (Y, y)$, s.t. ϕ is a dicovering and $p \circ \phi = i \circ \Pi$, where $i : \Box X \to X$ is the identity.

Lemma 5.1. *Let* $p : (Y, y) \to (X, x)$ *be a dicovering in* **pd-Top**. *Then the boxification* $\Box p : (\Box Y, y) \to (\Box X, x)$ *is a dicovering in* **pd-Top**$_{\mathcal{B}}$

Proof. We have a commutative diagram in **pd-Top**

$$\begin{array}{ccc} (\Box Y, y) & \xrightarrow{i} & (Y, y) \\ {\scriptstyle \Box p}\downarrow & & \downarrow{\scriptstyle p} \\ (\Box X, x) & \xrightarrow{i} & (X, x) \end{array}$$

Here i is the identity map in **pd-Top**. $\Box p$ is a d-map by Lem. 3.9 and it is a dicovering: Let $H : J \to \Box X$ where J is as in Def. 4.1. Then $i \circ H$ lifts to $\hat{H} : J \to Y$, and $J \in \mathcal{B}$, so $\hat{H} : J \to \Box Y$ is a d-map by Thm. 3.10 and this is the unique lift. The same argument works for the other lifting requirements. $\Box$

Theorem 5.2. *Let* $(X, x) \in$ **wpd-Top** *and let* $\Pi : (\widetilde{\Box X}, \tilde{x}) \to (\Box X, x)$ *be the universal dicovering in* $\mathbf{wpd\text{-}Top}_{\mathcal{B}}$*. Then* $i \circ \Pi : (\widetilde{\Box X}, \tilde{x}) \to (X, x)$ *is universal for dicoverings in* **wpd-Top***.*

Proof. Let $p : (Y, y) \to (X, x)$ be a dicovering in **wpd-Top**. In the commutative diagram in **wpd-Top**:

$$\begin{array}{ccccc} (\widetilde{\Box X}, \tilde{x}) & \xrightarrow{\phi} & (\Box Y, y) & \xrightarrow{i} & (Y, y) \\ & {\scriptstyle \Pi}\searrow & {\scriptstyle \Box p}\downarrow & & \downarrow{\scriptstyle p} \\ & & (\Box X, x) & \xrightarrow{i} & (X, x) \end{array}$$

the maps Π, ϕ, $\Box p$ and p are dicoverings. Since $\vec{I}, K$ and J are in $\mathcal{B}$, lifting properties for $i \circ \phi$ and $i \circ \Pi$ may be proven as in the proof of Lem. 5.1 to give that $i \circ \phi$ and $i \circ \Pi$ are dicoverings.

Uniqueness of $i \circ \phi$: Let $z \in \widetilde{\Box X}$ and let $\gamma : (\vec{I}, 0, 1) \to (\widetilde{\Box X}, \tilde{x}, z)$ be a dipath from $\tilde{x}$ to z. Let $i \circ \hat{\Pi} \circ \gamma$ be the unique lift of $i \circ \Pi \circ \gamma$ to (Y, y_0). Then $i \circ \phi(z) = \hat{\gamma}(1)$.

Uniqueness: Suppose $P : (\hat{X}, \hat{x}) \to (X, x)$ in **wpd-Top** is universal for dicoverings in **wpd-Top**. Then $\Box P : (\Box \hat{X}, \hat{x}) \to (\Box X, x)$ is a dicovering in $\mathbf{wpd\text{-}Top}_{\mathcal{B}}$ and $i \circ \Box P : (\Box \hat{X}, \hat{x}) \to (X, x)$ is a dicovering in **wpd-Top**. Hence there is a diagram

$$\begin{array}{ccc} (\hat{X}, \hat{x}) & \xrightarrow{\psi} & (\Box \hat{X}, \hat{x}) \\ & {\scriptstyle P}\searrow & \downarrow{\scriptstyle i \circ \Box P} \\ & & (X, x) \end{array}$$

Use that $(\hat{X}, \hat{x})$ is well pointed to see that ψ is the identity map on the underlying sets. Hence, $i : (\Box\hat{X}, \hat{x}) \to (\hat{X}, \hat{x})$ is a d-equivalence with inverse ψ, so $(\hat{X}, \hat{x}) \in \mathbf{wpd\text{-}Top}_{\mathcal{B}}$.

Hence $(\hat{X}, \hat{x})$ is dihomeomorphic to $(\widetilde{\Box X}, \tilde{x})$, the dihomeomorphism being the universal maps.

$\Box$

6. Construction of the universal dicovering

In [3] we constructed a candidate $(\hat{X}, \hat{x})$ for a universal dicovering of a well pointed space (X, x) satisfying certain diconnectivity conditions. The topology on $\hat{X}$ was not right, in the sense that the "universal" maps to other dicoverings failed to be continuous in various examples. We give the construction here again in a more general setting and prove that for a wpd-space, the two constructions are related via a bijective d-map. They may have different topology.

Definition 6.1. For a wpd-space (X, x), let $(\hat{X}, \hat{x}) = \{[\gamma] | \gamma : \vec{I} \to X, \gamma(0) = x\} \in \mathbf{Set}$ where $[\gamma]$ is the dihomotopy class of γ with fixed endpoints. $\hat{x}$ is the dihomotopy class of the constant dipath to x. Let $\hat{\pi} : (\hat{X}, \hat{x}) \to (X, x)$ be the endpoint map $\hat{\pi}([\gamma]) = \gamma(1)$.

Definition 6.2. For a basis $\mathcal{U}$ for the topology on X, we get a subbasis for a topology on $\hat{X}$ - all sets

$$U_{[\gamma]} = \{[\mu] | \mu \in \vec{P}(X, x_0, U), \mu \sim_U \gamma\}$$

for $U \in \mathcal{U}$ and $\gamma \in \vec{P}(X, x_0, U)$. The relation $\sim_U$ is defined by $\mu \sim_U \gamma$, if there is $H : (J, (0,0)) \to (X, x)$ s.t. $H(t, 0) = \gamma(t)$, $H(t, 1) = \mu(t)$ and $\eta_{s_0}(t) = H(t, s_0) \in \vec{P}(X, x_0, U)$ for all $s_0 \in I$

The d-structure is $\vec{P}(\tilde{X}, [\gamma], -) = \{\eta(t) = [\gamma \star \mu|_{[0, \frac{t+1}{2}]}]$, where $\mu \in \vec{P}(X, \gamma(1), -)\}$, $\gamma \star \mu|_{[0, \frac{t+1}{2}]}(s) = \gamma(2s)$ for $[0 \leq s \leq 1/2]$ and $\gamma \star \mu|_{[0, \frac{t+1}{2}]}(s) = \mu((2s-1)\frac{t+1}{2})$ for $1/2 \leq s \leq 1$.

And subpaths and monotone reparametrizations of such.

Remark 6.3. The strange choice of parameter value $\eta(t) = [\gamma \star \mu|_{[0, \frac{t+1}{2}]}]$ gives $\eta(0) = [\gamma]$ and $\eta(1) = [\gamma \star \mu]$ by definition of concatenation of paths. For a proof of continuity see [3]. Since we close off under subpath and monotone reparametrization, clearly we have a d-structure.

Lemma 6.4. *With the above d-structures, the projection map, $\hat{\pi} : (\hat{X}, \hat{x}) \to (X, x)$ is a d-map. And a dicovering.*

Proof. See the proof of 3.9 and 3.11 in [3]. We sketch the proof in this more general context:
Continuity: Let $U \in \mathcal{U}$, then $\hat{\pi}^{-1}(U) = \cup_{\{[\gamma]|\gamma\in\vec{P}(X,x,U)\}} U_{[\gamma]}$, a union of opens.

The image of a dipath is clearly a dipath.

Since (X,x) is a wpd-space, it suffices to establish lifting properties for dipaths and dihomotopies initiating in x: A dipath $\gamma : \vec{I} \to X$, $\gamma(0) = x$ lifts to $\Gamma(t) = [\gamma|_{[0,t]}]$, where by $\gamma|_{[0,t]}$ we understand the linear monotone reparametrization of the restriction - our paths should be defined on $[0,1]$. For uniqueness, notice that all dipaths $\Gamma_i : \vec{I} \to \hat{X}$ with $\Gamma_i(0) = \hat{x}$ are of the form $\Gamma_i(t) = [\gamma_i(t)]$ for some $\gamma_i : \vec{I} \to X$. So if $\hat{\pi} \circ \Gamma_1 = \hat{\pi} \circ \Gamma_2$, $\gamma_1(t) = \gamma_2(t)$ and hence $\Gamma_1 = \Gamma_2$. Unique lift of dihomotopies follows the proof in [3] verbatim. There is a lift of the dipaths, and one proves that it is continuous in the homotopy parameter. If $H : \vec{I} \times I \to X$ has fixed endpoints x and x_1, $H(t,s) = \gamma_s(t)$, then H lifts to $\hat{H}(t,s) = \Gamma_s(t)$, and $\Gamma_s(1) = [\gamma_s]$, and since H provides a dihomotopy with fixed endpoints of all γ_s, $\hat{H}(1,s)$ is constant.

□

Theorem 6.5. *Let $\hat{\pi} : (\hat{X}, \hat{x}) \to (X,x)$ be the dicovering of a wpd-space $(X,x) \in$ **wpd-Top** defined in 6.2. Then the induced d-map $\phi : (\widetilde{\square X}, \tilde{x}) \to (\square\hat{X}, \hat{x}_0)$ from the universal dicovering $(\widetilde{\square X}, \tilde{x})$ is a bijective d-map. The composition $i \circ \phi : (\widetilde{\square X}, \tilde{x}) \to (\hat{X}, x)$ with the identity map $i : (\square\hat{X}, \hat{x}) \to (\hat{X}, x)$ is a bijective d-map. Both ϕ and $i \circ \phi$ are dicoverings.*

Proof. We have a diagram

$$\begin{array}{ccccc}
(\widetilde{\square X}, \tilde{x}) & \xrightarrow{\phi} & (\square\hat{X}, \hat{x}) & \xrightarrow{i} & (\hat{X}, \hat{x}) \\
& \searrow^{\Pi} & \downarrow{\scriptstyle \square\hat{\Pi}} & & \downarrow{\scriptstyle \hat{\Pi}} \\
& & (\square X, x) & \xrightarrow{i} & (X, x)
\end{array}$$

The maps ϕ and hence $i \circ \phi$ is a dicovering of wpd-spaces, hence surjective and a d-map. For injectivity: Suppose $i \circ \phi(z_1) = i \circ \phi(z_2) = [\gamma]$. Let $\mu_i : \vec{I} \to \widetilde{\square X}$, $\mu_i(0) = x$, $\mu_i(1) = z_i$. Then $i \circ \Pi \circ \mu_i$ lifts along $\hat{\Pi}$ to $\eta_i(t) = [(i \circ \Pi \circ \mu_i)_{[0,\frac{t+1}{2}]}]$. This is a unique lift, so $i \circ \phi \circ \mu_i(t) = \eta_i(t)$. Hence $\eta_1(1) = i \circ \phi(z_1) = i \circ \phi(z_2) = \eta_2(1)$, i.e., $[i \circ \Pi \circ \mu_1] = [i \circ \Pi \circ \mu_2]$. A dihomotopy with fixed endpoints, $H : K \to X$ between $i \circ \Pi \circ \mu_1$ and $i \circ \Pi \circ \mu_1$ lifts uniquely to a dihomotopy with fixed endpoints between μ_1 and μ_2. In particular, $\mu_1(1) = \mu_2(1)$ □

7. Dicoverings as quotients

Given a dicovering of a wpd-Space $p : (Y, y) \to (X, x)$ there is a *congruence relation* $\approx$, Def. 7.1, on the universal dicovering space $(\widetilde{\Box X}, \tilde{x})$ of the boxification of X, such that the universal map $\phi : (\widetilde{\Box X}, \tilde{x}) \to (\Box Y, y)$ factors over the quotient map $\psi : (\widetilde{\Box X}, \tilde{x}) \to (\widetilde{\Box X}/\approx, [\tilde{x}])$, $\phi = f \circ \psi$ and f is a bijective d-map. Composing with the bijective d-map $i : \Box Y \to Y$, $i \circ f$ is a bijective d-map from the quotient to Y. The topology on $\Box Y$ may not be the quotient topology (See Ex. 7.7), and neither is the topology on Y, but the dipath structure is the quotient structure.
Vice versa: Quotients of the universal dicovering space under congruence relations are dicoverings. In fact, for a congruence relation, the quotient map $\psi : (\widetilde{\Box X}, \tilde{x}) \to (\widetilde{\Box X}/\approx, [\tilde{x}])$ is a dicovering and there is a dicovering map $q : (\widetilde{\Box X}/\approx, [\tilde{x}]) \to (\Box X, x)$ s.t. $q \circ \psi = \Pi$, the universal map. Compose with $i : \Box X \to X$ to get a dicovering of (X, x).

In the following, we will write $\Pi : (\tilde{X}, \tilde{x}) \to (X, x)$ for the universal dicovering of a wpd-space. That is, the space $(\tilde{X}, x) = (\widetilde{\Box X}, \tilde{x})$ where we think of the points as in the construction Def. 6.2, i.e., dihomotopy classes $[\gamma]$ of dipaths initiating in x, and the topology is the topology from Cor. 4.6. The map Π is tacitly composed with $i : \Box X \to X$ when $X \notin$ **wpd-Top**$_{\mathcal{B}}$

Definition 7.1. Let $\Pi : (\tilde{X}, \tilde{x}) \to (X, x)$ be the universal dicovering of a wpd-space. An equivalence relation on $\tilde{X}$ is a congruence if $[\gamma_1] \approx [\gamma_2]$ implies $\gamma_1(1) = \gamma_2(1)$ and $[\gamma_1 \star \mu] \approx [\gamma_2 \star \mu]$ for all dipaths μ initiating in $\gamma_i(1)$.

Example 7.2. The relation $[\gamma_1] \approx [\gamma_2]$ if $\gamma_1(1) = \gamma_2(1)$ is a congruence, and there is a bijective d-map from $\tilde{X}_{x_0}/\approx$ with the quotient d-structure to X. (By Thm. 7.5)

Example 7.3. Let $f : X \to Y$ be an injective d-map, then the relation $[\gamma] \approx_f [\eta]$ if $[f \circ \gamma] = [f \circ \eta]$ is a congruence, since $f \circ (\gamma \star \mu) = f \circ \gamma \star f \circ \mu$ and $[f \circ \gamma] = [f \circ \eta]$ implies $f \circ \gamma(1) = f \circ \eta(1)$, so $\gamma(1) = \eta(1)$, as f is injective.

In particular, the inclusion of $X = \vec{I} \times \vec{I} \setminus \{(1/2, 1/2)\}$ into $Y = \vec{I} \times \vec{I}$ induces relation on $\tilde{X}$, making the two dihomotopy classes of dipaths fom $(0, 0)$ to $(1, 1)$ equivalent.

Lemma 7.4. *Let $f : Y \to X$ be a surjection and suppose $Y \in$ **d-Top**$_{\mathcal{B}}$ and X has the quotient d-structure. Then X is in **d-Top**$_{\mathcal{B}}$. If $(Y, y_0) \in$ **wpd-Top**, then $(X, f(y_0)) \in$ **wpd-Top***

Proof. $f : Y \to X$ is a d-map and hence $f : Y \to \Box(X)$ is a d-map by Lem. 3.10. Boxification adds more opens, but the quotient topology is the maximal topology on X with $f : Y \to X$ continuous. Hence $X = \Box(X)$. Suppose $(Y, y_0) \in$ **wpd-Top** and $x \in X$, $x = f(y)$. Let $\gamma : (\vec{I}, 0, 1) \to (Y, y_0, y)$. Then $f \circ \gamma : (\vec{I}, 0, 1) \to (X, f(y_0), x)$, so $(X, f(x)) \in$ **wpd-Top**. $\Box$

Theorem 7.5. *Let $p : (Y, y_0) \to (X, x_0)$ be a dicovering in* **wpd-Top***, let $i \circ \Pi : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ be the universal dicovering of (X, x_0) in* **wpd-Top** *and let $i \circ \phi : (\tilde{X}, \tilde{x}_0) \to (Y, y_0)$ be the induced dicovering map.*

Define a congruence relation on $\tilde{X}$ as follows: $[\gamma_1] \approx_p [\gamma_2]$ if $\hat{\gamma}_1(1) = \hat{\gamma}_2(1)$, where $\hat{\gamma}_i$ is the unique lift of γ_i to Y with initial point y_0. Then ϕ factors over the quotient $\psi : \tilde{X} \to \tilde{X}/\approx_p$, $\phi = f \circ \psi$ and $f : \tilde{X}/\approx_p \to \Box Y$ is a bijective d-map. Similarly $i \circ f : \tilde{X}/\approx_p \to Y$ is a bijective d-map.

Proof. The relation is a congruence relation, since $\hat{\gamma}_1(1) = \hat{\gamma}_2(1)$ implies $\gamma_1(1) = (p \circ \hat{\gamma}_1)(1) = (p \circ \hat{\gamma}_2)(1) = \gamma_2(1)$ and $\widehat{\gamma_i \star \mu}$ is $\hat{\gamma}_i$ composed with a lift of μ initiating in $\gamma_i(1)$.

The dipath in $\tilde{X}$ initiating in $\tilde{x}_0$, $\Gamma_i(t) = [\gamma_i|_{[0,t]}]$ is the unique lift of $\hat{\gamma}_i(t)$ along ϕ. Hence, $\phi([\gamma_i]) = \phi(\Gamma_i(1)) = \hat{\gamma}_i(1)$, so $[\gamma_1] \approx_p [\gamma_2]$ implies $\phi([\gamma_1]) = \phi([\gamma_2])$ and ϕ then factors over the quotient. By the same argument, $\phi([\gamma_1]) = \phi([\gamma_2])$ implies $[\gamma_1] \approx_p [\gamma_2]$, so the map f is a bijection. $\Box$

Remark 7.6. Hence all dicoverings are quotients of the universal dicovering. But the topology may not be the quotient topology. The following example illustrates this problem.

Example 7.7. Let I_{disc} be the interval with the discrete topology and $f_1, f_2 : I_{disc} \to \vec{I} \times I_{disc}$ be $f_1(s) = (0, s)$, $f_2(s) = (0, 0)$. Let A be the coequalizer in **d-Top** of f_1, f_2 and let $x_0 = (0, 0) \in A$ then $(A, x_0) \in$ **wpd-Top**$_{\mathcal{B}}$. Let $B = \vec{I} \times I$ and let X be the coequalizer in **d-Top** of $g_1, g_2 : I_{disc} \to A \cup B$, $g_1(s) = (1, s) \in A$, $g_2(s) = (0, s) \in B$. Then $(X, x_0) \in$ **wpd-Top**$_{\mathcal{B}}$.

The only dipaths in X are paths with constant s. Hence the image of a dihomotopy with fixed initial point is a set with s constant, and dipaths are dihomotopic with fixed endpoints if and only if one is a monotone reparametrization of the other.

As a consequence, $\pi : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a bijective d-map, but it is not a dihomeomorphism:

We define a dicovering $\bar{X} \in$ **wpd-Top** is the coequalizer $f_1, f_2 : I_{disc} \to \vec{[0,2]} \times I_{disc}$ f_1 and f_2 as above - it is a bouquet of directed intervals. The basepoint is $(0,0)$. $p : \bar{X} \to X$ is $p(t,s) = (t,s) \in A$ for $0 \leq t \leq 1$ and $p(t,s) = (t-1,s) \in B$ for $1 \leq t \leq 2$. It is easy to check that p is a dicovering and a bijective d-map. Hence, the universal dicovering induces a bijective d-map $(\tilde{X}, \tilde{x}_0) \to (\bar{X}, \bar{x}_0)$, which is a dicovering. Since all dipaths lift and the topology on $\bar{X}$ is generated by the full subcategory of **d-Top** with objects $\vec{I}$, the d-map is a d-homeomorphism, so the bijection $p : \bar{X} \to X$ is in fact the universal dicovering. The induced congruence relation on $\bar{X}$ is trivial, and p is not a d-homeomorphism, so X is not a quotient of the universal dicover.

Theorem 7.8. *Let* $\Pi : (\tilde{X}, \tilde{x}) \to (X, x)$ *be the universal dicovering of* $(X, x) \in$ **wpd-Top**$_{\mathcal{B}}$ *and let* $\approx$ *be a congruence relation on* $\tilde{X}$*. Then the quotient map* $\psi : (\tilde{X}, \tilde{x}) \to (\tilde{X}/\approx, \psi(\tilde{x}))$ *and the map* $q : (\tilde{X}/\approx, \psi(\tilde{x})) \to (X, x)$ *defined by* $q(\psi([\gamma])) = \gamma(1)$ *are dicoverings in* **wpd-Top**$_{\mathcal{B}}$

Proof. ψ is a d-map by definition of the quotient structure and $(\tilde{X}/\approx, \psi(\tilde{x}_0)) \in$ **wpd-Top**$_{\mathcal{B}}$ by Lem. 7.4.

q is a d-map, since $q \circ \psi = \Pi$, Π is a d-map and ψ is a quotient of d-spaces.

We have to prove lifting properties. First dipaths:

Let $\gamma : \vec{I} \to X$ be a dipath and $y \in q^{-1}(\gamma(0))$. Let $\tilde{y} \in \psi^{-1}(y) \subset \Pi^{-1}(\gamma(0))$ then γ lifts uniquely to $\hat{\gamma} : (\vec{I}, 0) \to (\tilde{X}, \tilde{y})$ and $\psi \circ \hat{\gamma} : (\vec{I}, 0) \to \tilde{X}/\approx, y)$ is then a lift of γ along q.
For uniqueness suppose $\beta_1, \beta_2 : (\vec{I}, 0) \to (\tilde{X}/\approx, y)$ are lifts of γ with a common source y and $\beta_1 \neq \beta_2$. Let $\hat{\beta}_i : (\vec{I}, 0) \to (\tilde{X}, \tilde{y})$ be the lifts of β_i provided by Lem. 7.10. Clearly $\hat{\beta}_1 \neq \hat{\beta}_2$, and $\Pi \circ \hat{\beta}_1 = \gamma = \Pi \circ \hat{\beta}_2$ which contradicts the unique lifting along Π.

Let $H : (J, 0) \to (X, x)$ and let $\hat{H}$ be the lift to $(\tilde{X}, \tilde{y})$. Then $\psi \circ \hat{H}$ is a lift of H to $(\tilde{X}/\approx, y)$; it is a composition of d-maps, and by unique lifting of dipaths, it is unique. For a dihomotopy with fixed endpoints, the same construction works.

Let $H : (J, 0) \to (\tilde{X}/\approx, y)$. The unique lift of $q \circ H$ along Π with initial point z provides a unique lift of H along ψ.

□

Corollary 7.9. *Let* $i \circ \Pi : (\tilde{X}, \tilde{x}) \to (X, x)$ *be the universal dicovering of* $(X, x) \in$ **wpd-Top** *and let* $\approx$ *be a congruence relation on* $\tilde{X}$*. Then the quotient map* $\psi : (\tilde{X}, \tilde{x}) \to (\tilde{X}/\approx, \psi(\tilde{x}))$ *and the map*

$q : (\tilde{X}/\approx, \psi(\tilde{x})) \to (X, x)$ defined by $q(\psi([\gamma])) = \gamma(1)$ are dicoverings in **wpd-Top**

Lemma 7.10. *Let $\approx$ be a congruence relation on the universal dicovering $\Pi : (\tilde{X}, \tilde{x}) \to (X, x)$ of a* **wpd-Top**$_{\mathcal{B}}$*-Space. Let $\psi : (\tilde{X}, \tilde{x}) \to (\tilde{X}/\approx, [\tilde{x}])$ be the projection to the quotient. Then $\vec{P}(\tilde{X}/\approx) = \psi(\vec{P}(\tilde{X}))$ and if, for dipaths $\mu, \eta \in \vec{P}(\tilde{X})$, $\psi \circ \eta = \psi \circ \mu$, then μ is the unique lift of $\Pi \circ \eta$ with source $\mu(0)$.*

Proof. By definition, $\vec{P}(\tilde{X}/\approx) \supseteq \psi(\vec{P}(\tilde{X}))$. Let $\gamma \in \vec{P}(\tilde{X}/\approx)$, then $\gamma = \psi \circ \eta_1 \star \psi \circ \eta_2 \cdots \psi \circ \eta_n$ for $\eta_i \in \vec{P}(\tilde{X})$. We construct a dipath $\bar{\eta} \in \vec{P}(\tilde{X})$ s.t. $\gamma = \psi \circ \bar{\eta}$ iteratively as follows:

Let $\eta_1(0) = [\alpha_0]$ then, since dipaths lift uniquely along Π, $\eta_1(t) = [\alpha \star \Pi \circ \eta_1](\frac{t+1}{2})$, where we define $[\mu](t) = [\mu_{[0,t]}]$ for $[\mu] \in \tilde{X}$. Let $\bar{\eta}_1(t) = \eta_1(t)$.

For $1 \leq k \leq n-1$, let $[\alpha_k] = \bar{\eta}_k(1)$ and $\bar{\eta}_{k+1}(t) = [\alpha_k \star \Pi \circ \eta_{k+1}](\frac{t+1}{2})$.

Claim: $\eta_k(t) \approx \bar{\eta}_k(t)$ for $k = 1, \ldots, n$.

Induction: $\eta_1 = \bar{\eta}_1$. Suppose $\eta_k(t) \approx \bar{\eta}_k(t)$ for $k < i$. Let $\eta_i(0) = [\beta_i]$. By unique lifting, $\eta_i(t) = [\beta_i \star \Pi \circ \eta_i](\frac{t+1}{2})$. Since $\psi \circ \eta_{i-1}$ composes with $\psi \circ \eta_i$, $\eta_{i-1}(1) \approx \eta_i(0)$. Combine with the induction hypothesis and get $[\alpha_{i-1}] = \bar{\eta}_{i-1}(1) \approx \eta_i(0) = [\beta_i]$. As $\eta_i(t) = [\beta_i \star \Pi \circ \eta_i](\frac{t+1}{2})$ and $\bar{\eta}_i(t) = [\alpha_{i-1} \star \Pi \circ \eta_i](\frac{t+1}{2})$ and $\approx$ is a congruence, we are done.

Now $\gamma = \psi \circ (\bar{\eta}_1 \star \bar{\eta}_2 \cdots \star \bar{\eta}_n$

If $\psi \circ \mu = \psi \circ \eta$, then $\Pi \circ \mu = \Pi \circ \eta$. By unique lifting, μ is the unique lift of $\Pi \circ \mu$ and hence of $\Pi \circ \eta$ initiating in $\mu(0)$ ☐

8. Conclusions and outlook

Dicoverings in **wpd-Top** are now classified. For $(X, x) \in$ **pd-Top**$_{\mathcal{B}}$, we proved existence of a universal dicovering in [7]. In a subsequent paper, we will study a d-space X by restricting to the wpd-spaces $\uparrow_X x$ for varying basepoint x. In particular, the maps between the universal dicoverings of these subspaces induced by dipaths between the basepoints will carry information about the d-structure of the whole space - a representation of the fundamental category.

In [4], dicoverings and universal coverings are defined for *streams* in the sense of [6]. In that approach, the restriction to the underlying spaces are coverings in **Top**, and thus the requirements on the underlying spaces are more restrictive than here. In the motivating examples from computer science, geometric realizations of precubical sets, both approaches provide unversal dicoverings.

The existence of universal coverings in [7] can be restated in a non-directed setting, and it would be interesting to see, whether this gives

new information, in particular for spaces not satisfying the restrictions on local connectedness required in the usual setting of covering theory. The new description of $\mathbf{Top}_{\mathcal{D}}$ and $\mathbf{d\text{-}Top}_{\mathcal{B}}$ as generated by paths (and dipaths) should give a better understanding of these categories. In particular in the (di)covering setting, since lifting properties for paths is a very strong property in these categories, where continuity is decided by studying the restriction to paths. Some consequences, lifting properties for maps of *dicones*, $\vec{I} \times Z/\{0\} \times Z$ and disuspensions $\vec{I} \times Z/\{0\} \times Z, \{1\} \times Z$ for $Z \in \mathbf{d\text{-}Top}_{\mathcal{B}}$ are studied in a subsequent paper.
The original motivation for studying dicoverings comes from computer science, where a dicovering is a *functional bisimulation* in the sense that geometric representations of programs are d-spaces where dipaths represent executions of programs, dihomotopies are equivalences of such executions and dicoverings are then equivalences of programs. Hence the classification provided here should give information on bisimilarity.

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Department of Mathematical Sciences, University of Aalborg
E-mail address: fajstrup@math.aau.dk