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by

Arthur Finbow, Allan Frendrup and  
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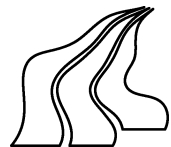
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# Total well dominated trees

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## Abstract

Let  $G = (V, E)$  be a graph with no isolated vertex. A set  $D$  is called total dominating in  $G$  if each vertex in  $G$  is adjacent to a vertex from  $D$ , and  $D$  is a minimal total dominating set if any subset  $D' \subset D$  is not a total dominating set in  $G$ . If all minimal total dominating sets in  $G$  have the same cardinality then  $G$  is a total well dominated graph. In this paper we study composition and decomposition of total well dominated trees. By a reversible process we prove that any total well dominated tree can both be reduced to and constructed from a family of three small trees.

**Keywords:** total domination, decomposition, composition, total well dominated

**AMS subject classification:** 05C69

# 1 Notation

For notation and graph theory terminology we in general follow [4]. Let  $G = (V, E)$  be a graph with vertex set  $V$  and edge set  $E$ . A *dominating set* of  $G$  is a set  $D$  of vertices of  $G$  such that every vertex in  $V \setminus D$  is adjacent to a vertex in  $D$ . Further, if also each vertex in a dominating set  $D$  is adjacent to a vertex from  $D$  then  $D$  is a *total dominating set*. The *total domination number* of  $G$ , denoted by  $\gamma_t(G)$ , is the minimum cardinality of a total dominating set. A total dominating set of minimum cardinality  $\gamma_t(G)$  is called a  $\gamma_t$ -set for  $G$ . If a total dominating set  $D$  satisfies that no proper subset of  $D$  is a total dominating set then  $D$  is a *minimal total dominating set*. The *upper total domination number* of a graph  $G$ , denoted by  $\Gamma_t(G)$ , is the maximum cardinality of a minimal total dominating set. If all minimal total dominating sets in a graph  $G$  have the same cardinality the graph is called *total well dominated* (or just *TWD*). Thus a graph  $G$  is TWD if and only if  $\gamma_t(G) = \Gamma_t(G)$ . In a TWD graph any minimal total dominating set is a  $\gamma_t$ -set. For a set  $D$  and a vertex  $x \in D$  the *private neighbourhood* of  $x$  is defined by  $pn(x, D) := \{y \in N[x] | N[y] \cap D = x\}$ . A vertex of degree one is called a *leaf* and a vertex adjacent to a leaf is called a *stem*.

For two vertices  $x$  and  $y$  in a graph  $G$  we denote the distance between the vertices by  $d_G(x, y)$ . For  $S \subseteq V(G)$  we define  $d_G(x, S) := \min_{s \in S} \{d_G(x, s)\}$ .

If  $G$  is a graph and  $S$  is a vertex set in  $G$ , then the induced subgraph of  $G$  with vertex set  $S$  is denoted  $G[S]$ .

For  $k \geq 1$  let  $A_k$  be the graph with vertex set  $V(A_k) = \{x_1, x_2, \dots, x_{k+1}, x, y\}$  and edge set  $E(A_k) = \{x_1x_2, x_2x_3, \dots, x_kx_{k+1}, x_kx, x_{k+1}y\}$ . Thus  $A_k$  is the graph illustrated in Figure 1,  $A_1 = P_4$  and  $A_2$  is a  $K_{1,3}$  with one edge subdivided.

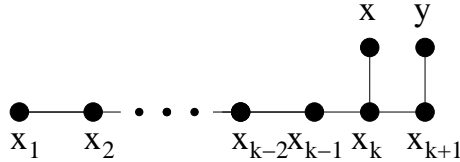


Figure 1: Illustration of  $A_k$ .

Let  $\mathcal{A}_4$  be the family of graphs with the structure illustrated in Figure 2 (a) and let  $\mathcal{A}_5$  be the family of graphs with the structure illustrated in Figure 2(b).

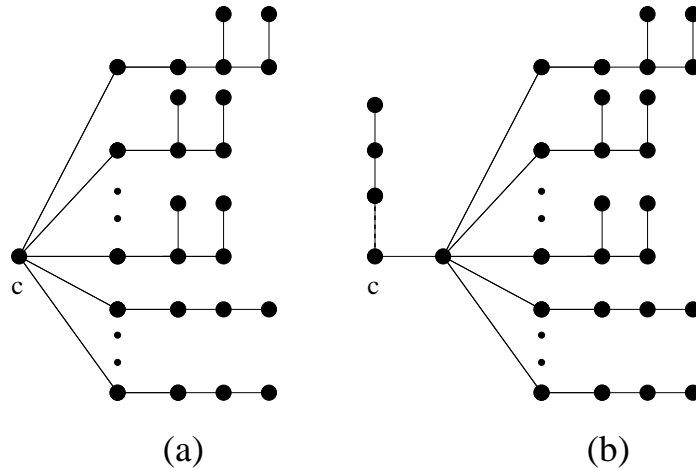


Figure 2: (a) illustrates  $\mathcal{A}_4$  and (b) illustrates  $\mathcal{A}_5$ . In (b) the dotted line indicates that either  $\deg(c) = 1$  or a path  $P_3$  is attached to  $c$ .

In  $\mathcal{A}_i$  we call the vertex  $x_1$  an attachment vertex and in a graph from  $\mathcal{A}_4$  or  $\mathcal{A}_5$  we call the vertex  $c$  an attachment vertex. In a path  $P_n$  a vertex with smallest degree is called an attachment vertex.

For graphs  $H$  with attachment vertex  $a$  and  $G$  with a vertex  $v$  we define the graph obtained by attaching  $H$  to  $v$  in  $G$  as the graph with vertex set  $V(G) \cup V(H)$  and edge set  $E(G) \cup E(H) \cup \{av\}$ . In the obtained graph we say that  $H$  is attached at  $v$ .

## 2 Introduction

Total domination, treated here, is a follow-up to domination. Graphs with all minimal dominating sets having the same cardinality are called well dominated. They form a subset of well covered graphs, surveyed by M.D. Plummer in 1993 [7] and for girth  $\geq 4$  characterized by Hartnell, Finbow and Nowakowski [2, 3]. For girth  $\geq 6$  the family of well dominated graphs equals the class of well covered graphs ([1]) and for girth  $\geq 5$  well dominated graphs are characterized in [2, 3].

## 3 Decomposition/composition-rules

In this section we give the main decomposition/composition-rules that is used for total well dominated trees. Each rule is of the kind saying that if  $G$  is a graph with some special structure then  $G$  is TWD if and only if certain subgraphs of  $G$  are TWD. We say that the graph is reduced by the rule (or the lemma with the rule).

It will turn out that any TWD tree by application of rules to be described in several

lemmas below can be reduced to a smaller TWD tree of order  $\leq 6$ , namely to  $P_2, P_4$  or  $P_3 \circ K_1$ . Moreover, the rules will be invertible, so starting from these three small graphs we can by application of the rules construct any TWD tree.

For the decomposing/composing we shall use some special vertices and admissible sets, therefore these concepts are defined in the following.

**Definition 1** Let  $G$  be a TWD graph and let  $v \in V(G)$ . If the graph  $G' = (V(G) \cup \{x\}, E(G) \cup \{vx\})$  is TWD and  $\gamma_t(G) = \gamma_t(G')$  then  $v$  is called a special vertex.

Thus, in a TWD graph  $G$  the vertex  $v$  is special if and only if  $G$  has a  $\gamma_t$ -set containing  $v$ . In  $P_6 = x_1x_2 \cdots x_6$  the vertex  $x_3$  is special.

**Definition 2** Let  $G$  be a graph and let  $S \subseteq V(G)$ . The set  $S$  is called admissible if  
(i) neither  $G[S]$  nor  $G - N[S]$  have isolated vertices  
and  
(ii)  $\forall v \in S$ : either  $v$  is a stem in  $G[S]$  or  $pn(v, S \cup (V(G) \setminus N[S])) \neq \emptyset$ .

From the definition of an admissible set it can be seen that a set  $S$  is admissible in  $G$  if and only if  $S \cup (V(G) \setminus N[S])$  is a total dominating set in  $G$  and  $(S \cup (V(G) \setminus N[S])) \setminus \{s\}$  is not a total dominating set in  $G$  for any vertex  $s \in S$ .

**Observation 1** If  $S$  is a admissible set in a graph  $G$ , then  $G$  has no isolated vertex and for each minimal total dominating set  $S'$  in  $G - N[S]$ , the set  $S \cup S'$  is a minimal total dominating set in  $G$ . I.e., an admissible set can be extended to a minimal domination set. Also, if  $G$  is TWD then  $G - N[S]$  is TWD when  $S$  is admissible.

**Lemma 1** A graph  $G$  is TWD if and only if each component of  $G$  is TWD.

**Lemma 2** If a vertex  $v \in V(G)$  is adjacent to two leaves  $l_1$  and  $l_2$  then  $G$  is TWD if and only if  $G - l_1$  is TWD.

**Lemma 3** Let  $G$  be a graph containing two adjacent stems  $s_1$  and  $s_2$ . If  $L$  denotes the leaves adjacent to  $s_1$  or  $s_2$  and  $C_1, \dots, C_k$  is the components of  $G - (\{s_1, s_2\} \cup L)$  then  $G$  is TWD if and only if  $G_i := G - \{C_1, C_2, \dots, C_{i-1}, C_{i+1}, C_{i+2}, \dots, C_k\}$  is TWD for each  $i \in \{1, \dots, k\}$ .

**Proof.** The statement is trivial for  $k \leq 1$ , so assume  $k \geq 2$ . Since  $\delta(G) \geq 1$  if and only if  $\delta(G_i) \geq 1$  for each  $i$ ,  $1 \leq i \leq k$ , we see that  $G$  can be totally dominated if and only if each  $G_i$  can. As  $v_1$  and  $v_2$  are stems they must be contained in any total domination set for any of the graphs  $G, G_i, 1 \leq i \leq k$ . Let  $D_i \subseteq V(G_i), 1 \leq i \leq k$  and  $D = \bigcup_{i=1}^k D_i$ . We note that  $D$  is a total dominating set for  $G$  and, in fact, also that  $D$  is a minimal total

dominating set for  $G$  if and only if each  $D_i$  is a minimal total dominating set for  $G_i, \forall i$ . We have  $|D| = \sum_{i=1}^k |D_i| - 2 \cdot (k-1)$ . That implies that  $G$  is TWD if and only if  $G_i$  is TWD for each  $i, 1 \leq i \leq k$ .  $\square$

**Lemma 4** *Let  $G$  be a graph with a path  $l_1 s_1 v_1 v_2 s_2 l_2$  and assume that  $\deg(l_1) = \deg(l_2) = 1$ . Then  $G$  is TWD if and only if  $G - v_1 v_2$  is TWD.*

**Proof.** Since  $s_1$  and  $s_2$  are stems these vertices must be in all total dominating sets in  $G$  and  $G - v_1 v_2$ . Thus a total dominating set of  $G$  is also a total dominating set in  $G - v_1 v_2$  and a total dominating set of  $G - v_1 v_2$  is trivially a total dominating set of  $G$ . Thus the result follows.  $\square$

**Lemma 5** *Let  $T$  be a TWD tree reduced by Lemma 2, i.e., without multiple leaves, and let  $s$  be a stem of  $T$  having valency at least 3. Then  $T$  contains another stem at distance at most 3 from  $s$ .*

**Proof.** Let  $s$  be a stem in  $T$  and assume that  $\deg(s) \geq 3$  and that no other stem in  $T$  is within distance 3 from  $s$ . Since  $s$  is adjacent to exactly one leaf  $l$  there must exist a path  $P : abcdef$  in  $T$ . For each vertex  $x$  from  $P$  let  $C_x$  denote the union of all components of  $T - x$  not containing vertices from  $P$ .

Since no stem  $s' \neq s$  is within distance 3 from  $s$  the set  $V(C_x) \setminus N[x]$  is a total dominating set for  $C_x$  when  $x \in \{b, e\}$ . Thus there is a minimal total dominating set  $D_x$  of  $C_x$  not containing a vertex adjacent to  $x$ .

For  $x \in \{c, d\}$  let  $D_x^+$  be a minimal total dominating set of  $C_x$  and  $D_x^-$  be a minimal total dominating set of  $C_x - N[x]$ .

For  $x \in \{a, f\}$  let  $y$  be the vertex from  $P$  adjacent to  $x$ . Further let  $D_x^+$  be a minimal total dominating set of  $G[V(C_x) \cup \{x, y\}]$  not containing  $y$  and  $D_x^-$  be a minimal total dominating set of  $G[V(C_x) \cup \{x\}]$  not containing  $x$ .

Further let  $D_s$  be a minimal total dominating set of  $C_s - N[s]$  and let  $S := D_s \cup D_b \cup D_e$ .

Now let  $A := \{c, s, d\} \cup D_a^- \cup D_c^- \cup D_d^- \cup D_f^- \cup S$ ,  $B := \{s, d\} \cup D_a^+ \cup D_c^+ \cup D_d^- \cup D_f^- \cup S$ ,  $C := \{e, s\} \cup D_a^- \cup D_c^- \cup D_d^+ \cup D_f^+ \cup S$  and  $D := \{s, l\} \cup D_a^+ \cup D_c^+ \cup D_d^+ \cup D_f^+ \cup S$ . By construction all of these sets are minimal total dominating sets, and since  $T$  is TWD  $|A| = |B| = |C| = |D|$ . Since  $|A| = |B|$  we obtain  $|D_a^-| + |D_c^-| + 1 = |D_a^+| + |D_c^+|$  and since  $|A| = |C|$  we obtain  $|D_d^-| + |D_f^-| + 1 = |D_d^+| + |D_f^+|$ . But then

$$|D| = 2 + |D_a^+| + |D_c^+| + |D_d^+| + |D_f^+| + |S| = 4 + |D_a^-| + |D_c^-| + |D_d^-| + |D_f^-| + |S| = |A| + 1.$$

Thus we obtain a contradiction.  $\square$

**Lemma 6** *Let  $G$  be a graph with a vertex  $v$  adjacent to two stems  $s_1$  and  $s_2$ . If  $G$  is reduced by Lemma 2, 3 and 4 then  $G \cong P_3 \circ K_1$  or  $G$  is not TWD.*

**Proof.** Assume  $G$  is TWD and reduced by Lemma 2, 3 and 4. Let  $l_i$  be a leaf adjacent to  $s_i$  for  $i \in \{1, 2\}$ . First assume that  $A := \{s_1, s_2, l_1, l_2\}$  is an admissible set. Then there is a minimal total dominating set  $D$  such that  $A \subseteq D$ . But then  $(D \setminus \{l_1, l_2\}) \cup \{v\}$  is a total dominating set of smaller cardinality. Thus it can be assumed that  $A$  is not admissible and thus  $s_1$  or  $s_2$  must be adjacent to a stem. Since  $G$  is reduced by Lemma 4 that stem must be  $v$ , and since  $G$  is reduced by Lemma 2 and Lemma 3 we obtain that  $G \cong P_3 \circ K_1$ .  $\square$

Removing all but one attached  $P_3$  from a vertex does not change the property of being TWD.

**Lemma 7** *Let  $G$  be a graph and let  $v \in V(G)$ . If  $G_i$  is the graph obtained from  $G$  by attaching  $i$   $P_3$ 's to the vertex  $v$  then  $G_1$  is TWD if and only if  $G_2$  is TWD.*

**Proof.** Let  $v_1v_2v_3$  and  $u_1u_2u_3$  be the two  $P_3$ 's added to  $G$  to obtain  $G_2$  and assume that  $\{v_1v, u_1v\} \subseteq E(G_2)$ . Since  $\{u_2, u_3\}$  is an admissible set in  $G_2$  it follows that  $G_1 = G_2 - N[\{u_2, u_3\}]$  is TWD if  $G_2$  is TWD. Now assume  $G_1$  is TWD and let  $D$  be any minimal total dominating set in  $G_2$ . Assume WLOG that  $d_{G_2}(v, \{v_1, v_2, v_3\} \cap D) \leq d_{G_2}(v, \{u_1, u_2, u_3\} \cap D)$  then, if  $u_1 \in D$  we can replace it by  $u_3$ , so we may assume  $\{u_2, u_3\} \subseteq D$ ,  $u_1 \notin D$ . Then  $D \setminus \{u_2, u_3\}$  is a minimal total dominating set in  $G_2 - N_{G_2}[\{u_2, u_3\}] \cong G_1$ . Since  $|D \cap \{u_1, u_2, u_3\}| = 2$  and  $G_1$  is TWD it follows that  $|D| = \gamma_t(G_1) + 2$  and thus  $G_2$  is TWD.  $\square$

**Lemma 8** *Let  $G$  be a connected graph with a  $P_4$  attached at a vertex  $v$  and let  $H$  be the graph obtained by removing the attached  $P_4$ . If  $v$  is adjacent to a stem in  $G$  then  $G$  is TWD if and only if  $H$  is TWD and  $v$  is special in  $H$ .*

**Proof.** First assume that  $G$  is TWD and let  $v_1v_2v_3v_4$  be the  $P_4$  attached at  $v$  such that  $vv_1 \in E(G)$ . Since  $\{v_2, v_3\}$  and  $\{v_3, v_4\}$  are admissible sets  $H = G - N[\{v_2, v_3\}]$  is TWD, and  $G - N[\{v_3, v_4\}]$  is TWD which proves that  $v$  is special in  $H$ . Now assume conversely that  $H$  is TWD and  $v$  is a special vertex in  $H$  and consider a minimal total dominating set  $D$  in  $G$ . If  $\{v_3, v_4\} \subseteq D$  then  $D \setminus \{v_3, v_4\}$  is just a minimal total dominating set in  $G - \{v_2, v_3, v_4\}$  and since  $v$  is special we obtain  $|D| = 2 + \gamma_t(H)$  in this case. Otherwise  $D \cap \{v_1, v_2, v_3, v_4\} = \{v_2, v_3\}$  since  $v$  is adjacent to a stem in  $G$ . We see that  $D \setminus \{v_2, v_3\}$  is a minimal total dominating set in  $H$  and, as  $H$  is TWD, that  $|D \setminus \{v_2, v_3\}| = \gamma_t(H)$ , implying that  $|D| = 2 + \gamma_t(H)$ . It follows that  $G$  is TWD.  $\square$

**Lemma 9** *Let  $G$  be a graph with a  $P_2 : v_1v_2$  attached at a nonstem  $v \in V(G)$  such that  $vv_1 \in E(G)$  and  $\deg(v) \geq 3$ . Let  $C_1, \dots, C_k$  be the components of  $G - \{v, v_1, v_2\}$ . If  $G_i$  denotes the graph  $G[V(C_i) \cup \{v, v_1, v_2\}]$  for  $i \in \{1, \dots, k\}$  then  $G$  is TWD if and only if  $G_1, \dots, G_k$  is TWD.*

**Proof.** Let  $D$  be a minimal total dominating set in  $G$ . If  $v \in D$  then  $v$  is the only vertex that dominates  $v_1$  and it follows that  $D \cap G_i$  is a minimal total dominating set in  $G_i$ . Since



$|D \cap \{v, v_1, v_2\}| = 2$  it follows that  $G$  is TWD if  $G_1, \dots, G_k$  is TWD. Assume that  $G$  is TWD, we choose an index  $i$  and we shall prove that  $G_i$  is TWD. Let  $D_i$  be a minimal total dominating set in  $G_i$ . Since  $v$  is not a stem  $G$  has a minimal total dominating set  $D$  such that  $D \cap G_i = D_i$ . If  $v_2 \in D_i$  then  $(D \cup \{v\}) \setminus \{v_2\}$  must be a minimal total dominating set in  $G$  since  $G$  is TWD and therefore  $(D_i \cup \{v\}) \setminus \{v_2\}$  must be a minimal total dominating set in  $G_i$ . Thus it follows that  $G_i$  is TWD if all of the minimal total dominating sets in  $G_i$  not containing  $v_2$  have the same cardinality. Assume that  $D_i$  and  $D'_i$  are minimal total dominating sets in  $G_i$  not containing  $v_2$ . Since  $G$  is TWD and the sets  $D$  and  $(D \setminus D_i) \cup D'_i$  are minimal total dominating sets in  $G$  it follows that  $|D_i| = |D'_i|$ . Thus  $G_i$  is TWD and consequently  $G_1, \dots, G_k$  are TWD if  $G$  is TWD.  $\square$

**Lemma 10** *Let  $T$  be a tree reduced by Lemma 4 and 8 with a vertex  $v$  such that all components of  $T - v$  except one namely  $C$ ,  $C \not\cong P_1$ , are components isomorphic to  $P_4$  or  $A_2$  attached at  $v$  and at least one  $P_4$  is attached at  $v$ . Let  $x$  denote the vertex from  $V(C) \cap N[v]$ , let  $C_1, \dots, C_k$  be the components of  $C - N[x]$  and let  $v_i$  denote the vertex in  $C_i$  adjacent to a vertex from  $C - C_i$ . Then  $T$  is TWD if and only if each of  $C_1, \dots, C_k$  are TWD,  $v_i$  is a special vertex in  $C_i$  adjacent to a stem in  $C_i$  and  $C_i \not\cong P_2$ .*

**Proof.** First assume that  $T$  is TWD. Let  $x_1x_2x_3x_4x_5 = vx_6 = xx_7 \dots x_a$  be a path in  $T$  such that  $x_1x_2x_3x_4$  is a  $P_4$  attached at  $v$  and  $x_4v \in E(G)$ . We have  $x_8 = v_i$  for some  $i$ . Since  $G$  is reduced by Lemma 8 the vertex  $x$  cannot be a stem. Neither is  $x_7$  a stem, for assume otherwise that  $x_7$  is a stem and let  $D$  be a minimal total dominating set in the component of  $T - xx_7$  containing  $x_7$ . Since  $x_7$  is a stem,  $D$  is an admissible set in  $T$  containing  $x_7$  and by Observation 1 all components of  $T - N[D]$  are TWD. But the component of  $T - N[D]$  containing  $v$  is not TWD so we have a contradiction. Thus we may assume that  $x_7$  is not a stem and therefore  $\{v, x\}$  is an admissible set. Since  $T - N[\{v, x\}]$  contains  $C_1, \dots, C_k$  as components each of these is TWD. Let  $C_i$  be the component containing  $x_8 = v_i$ , we shall show that  $C_i$  has a stem adjacent to  $x_8$  and that  $C_i \not\cong P_2$ . Assume otherwise that either  $C_i \cong P_2$  or  $C_i$  does not have a stem adjacent to  $x_8$ . Let  $C'$  be the component of  $T - xx_7$  containing  $x_7$ . By the assumptions the set  $D'' := V(C') \setminus (N(x_8) \cap V(C_i))$  is a total dominating set for  $C'$ . Let  $D'$  be a minimal total dominating set of  $C'$  such that  $D' \subseteq D''$ . Since  $N(x_8) \cap D'' = \{x_7\}$  the vertex  $x_7$  must be in  $D'$  and  $D'$  is an admissible set in  $T$ . But the component of  $T - N[D']$  containing  $v$  is not TWD. This contradiction proves that in  $C_i$  there is a stem adjacent to  $x_8$  and that  $C_i \not\cong P_2$ . Thus it just remains to prove that  $x_8$  is a special vertex in  $C_i$ . Since  $T$  is reduced by Lemma 4 and  $x_8$  is adjacent to a stem in  $C_i$  the vertex  $x_7$  cannot be adjacent to a stem in  $G - C_i$ . Since  $\{x_1, x_2, v, x\}$  and  $\{x_1, x_2, x_4, v\}$  are admissible sets all components of  $C - x$  must be TWD and for each such component  $H$  the graph obtained by removing the vertex adjacent to  $x$  must be TWD and have the same total domination number as  $H$ .

Let  $D$  be a minimal total dominating set in  $C' \cap (C_1 \cup C_2 \cup \dots \cup C_{i-1} \cup C_{i+1} \cup C_{i+2} \cup \dots \cup C_k)$  not containing any of the vertices  $v_1, \dots, v_k$ . Now  $D \cup \{v, x\}$  and  $D \cup \{x_4, v\}$  are admissible sets and thus all components of  $T - N[D \cup \{v, x\}]$  and  $T - N[D \cup \{x_4, v\}]$  is TWD. Since  $|D \cup \{v, x\}| = |D \cup \{x_4, v\}|$  we have  $\gamma_t(T - N[D \cup \{x_4, v\}]) = \gamma_t(T - N[D \cup \{v, x\}])$  and by considering the components of these graphs we obtain that  $C_i$  and the graph obtained by

attaching a  $P_1$  to  $x_8$  in  $C_i$  must be TWD and have the same total domination number, i.e.  $x_8$  is special in  $C_i$ .

Now assume conversely that  $T$  can be constructed as described in the lemma, we shall prove that  $T$  is TWD. Let  $D$  be a minimal total dominating set in  $T$ . Since  $v_1, \dots, v_k$  is adjacent to a stem in  $C_1, \dots, C_k$  the set  $D \cap C_i$  is a minimal total dominating set in  $C_i$  and if  $v_i \in D$  then also in the graph obtained from  $C_i$  by attaching a  $P_1$  to  $v_i$ . Now consider the set  $D' := D \setminus \{V(C_1) \cup \dots \cup V(C_k)\}$ . If  $D'$  does not contain isolated vertices it is a minimal total dominating set in  $T - V(C_1) - \dots - V(C_k)$ . Otherwise  $D'$  contains exactly one vertex  $y$  from  $N(x) \setminus \{v\}$ ,  $\{x, v\} \cap D' = \emptyset$  and  $D' \setminus \{y\}$  is a minimal total dominating set in the component of  $T - x$  containing  $v$ . By considering minimal total dominating sets not containing  $v$  in this component it can be observed that they all have cardinality  $\gamma_t(T - C_1 - \dots - C_k) - 1$  and thus we obtain that  $G$  is TWD.  $\square$

**Corollary 1** *Let  $T$  be a TWD tree reduced by Lemma 2, 3, 4, 8, 9 and 10. For any leaf  $v$  in  $T$  we have that either  $T \in \{P_2, P_4, P_3 \circ K_1\}$  or  $T$  has the structure as one of the graphs from Figure 3 where  $v$  is a leaf in  $T - T'$ .*

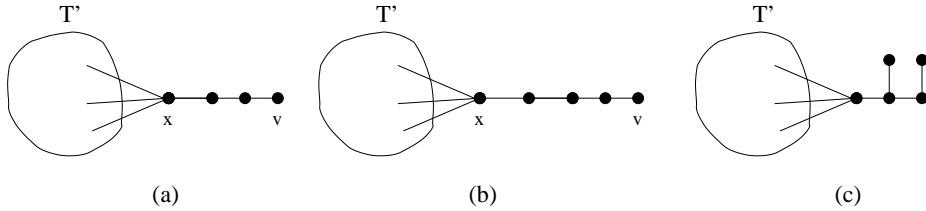


Figure 3: Illustration of structure near leaf.  $\deg(x) \geq 3$  in (a) and (b) .

**Remark.** Let  $T$  be a reduced TWD tree. The only such trees with  $\leq 4$  vertices are  $P_2$  and  $P_4$ . Let  $v$  be a leaf in  $T$ . Assume  $v$  is attached to a stem  $x_2$  with  $\deg_T(x_2) \geq 3$ . Then  $T$  by Lemma 5 contains another stem at distance at most 3 from  $x_2$ . By Lemma 4 it cannot be at distance 3 from  $x_2$ , so the distance is 1 or 2. If the other stem is at distance 2 from  $x_2$  then  $T = P_3 \circ K_1$  by Lemma 6. So otherwise  $T$  has two adjacent stems, one of which is  $x_2$  with  $\deg_T(x_2) \geq 3$ . But then we obtain from Lemma 3 that  $T$  has a subtree  $A_2$  as described on Figure 1. Here, in Figure 3(c) we have  $V(A_2) = \{x_1, x_2, x_3, v, y\}$  where  $v$  is a leaf with stem  $x_3$ ,  $\deg_T(x_3) = 2$ ;  $y$  is a leaf with stem  $x_2$ ,  $\deg_T(x_2) \geq 3$ ; and  $x_1$  is the attachment vertex in  $A_2$  to the rest of  $T$ ,  $\deg_T(x_1) \geq 2$ .

For the cases where  $v$ 's stem  $v_1$  has degree 2 we have by Lemma 9 that the nonleaf neighbour to  $v_1$  has degree 2, so we obtain one of Figure 3(a) or Figure 3(b) both with  $\deg(x) \geq 3$  because  $T$  has no attached  $P_5$ .

We shall later see that for trees with sufficiently large diameter the structure from a leaf as a refinement of Figure 3 can be described by Figure 7.

**Lemma 11** *Let  $G$  be a graph with two  $A_2$ 's attached at a vertex  $v$  and let  $H$  be the graph obtained by removing one of the attached  $A_2$  graphs. Then  $G$  is TWD if and only if  $H$  is TWD.*

**Proof.** Let  $vv_1v_2v_3v_4$  and  $vu_1u_2u_3u_4$  be paths in  $G$  such that  $v_1$  and  $u_1$  are contained in different  $A_2$  graphs attached at  $v$ . Since  $\{v_2, v_3\}$  is a admissible set in  $G$  we obtain that the graph  $H = G - N[\{v_2, v_3\}]$  is TWD if  $G$  is TWD. Conversely, let  $D$  be a minimal total dominating set  $D$  in  $G$ . Since  $v_1$  and  $u_1$  cannot both be in  $D$  we may assume that  $v_1 \notin D$ . Thus the  $A_2$  attached at  $v$  containing  $v_1$  has exactly two vertices  $v_2, v_3$  from  $D$ . The set  $D \setminus \{v_2, v_3\}$  is then a minimal total dominating set of  $H = G - N[\{v_2, v_3\}]$ . Thus if  $H$  is TWD then  $G$  must also be TWD.  $\square$

**Lemma 12** *Let  $G$  be a graph with two  $A_3$ 's attached at a nonstem  $v$  with  $\deg(v) \geq 3$ . Then  $G$  is not TWD.*

**Proof.** Let  $vv_1v_2v_3v_4v_5$  and  $vu_1u_2u_3u_4u_5$  be paths in  $G$  such that  $v_1$  and  $u_1$  is contained in different  $A_2$  graphs attached at  $v$ . Let  $D := V(G) \setminus \{v, v_1, u_1\}$  then  $D$  is a total dominating set of  $G$  since  $v$  is not a stem and  $\deg(v) \geq 3$ . Let  $D'$  be a minimal total dominating set such that  $D' \subset D$ . Since  $D'' := (D \setminus \{v_2, u_2\}) \cup \{v\}$  is a total dominating set and  $|D''| < |D'|$  the graph cannot be TWD.  $\square$

**Lemma 13** *Let  $H$  be a graph with a path  $P : v_1v_2v_3$  such that  $\deg(v_1) \geq 2, \deg(v_2) = 2, \deg(v_3) = 1$ . If  $G$  is the graph obtained from  $H$  by attaching an  $A_1$  to each of  $v_1$  and  $v_3$  then  $G$  is TWD if and only if  $H$  is TWD.*

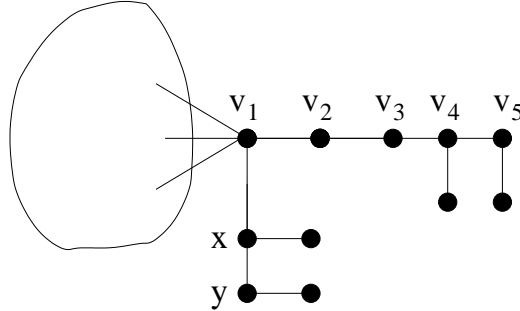


Figure 4: Illustration of  $G$ .

**Proof.** In the following notation is as illustrated in Figure 4. Let  $D$  be a minimal total dominating set in  $G$ . It follows that  $v_2 \notin D$  and  $(D \cap V(H)) \cup \{v_2\}$  is a minimal total dominating set for  $H$  and  $|(D \cap V(H)) \cup \{v_2\}| = |D| - 3$ . Thus  $G$  is TWD if  $H$  is TWD.

Conversely let  $D$  be a minimal total dominating set in  $H$ . Since  $v_2$  is a stem in  $H$  we have  $v_2 \in D$ . Consider the set  $D' := (D \setminus \{v_2\}) \cup \{x, y, v_4, v_5\}$ . This set is a minimal total dominating set and  $|D'| = |D| + 3$ . Thus  $H$  is TWD if  $G$  is TWD.  $\square$

**Lemma 14** *Let  $H$  be a graph and let  $v \in V(H)$ . Now let  $G$  be a graph obtained from  $H$  by attaching  $A_1$  and a graph from  $\mathcal{A}_4$  to  $v$  and let  $G'$  be the graph obtained from  $H$  by attaching  $P_2$  to  $v$ . Then  $G$  is TWD if and only if  $G'$  is TWD.*

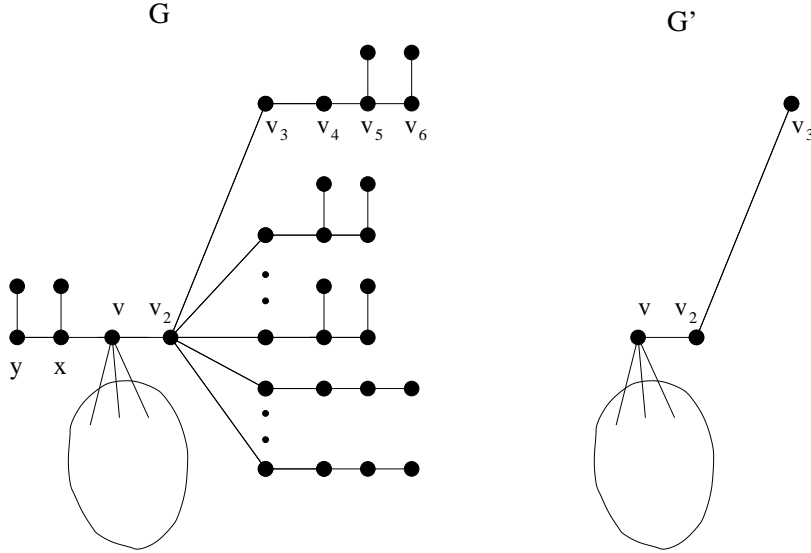


Figure 5: Illustration of  $G$  and  $G'$  from Lemma 14.

**Proof.** In the proof of this lemma we use the notation from Figure 5. First assume that  $G$  is TWD and let  $D$  be a minimal total dominating set of  $G'$ . Let  $A$  be the vertices at distance 2 and 3 from  $v_2$  in  $G - vv_2$  that are not leaves. Then  $D' := (D \cup A \cup \{v_6, x, y\}) \setminus \{v_2\}$  is a minimal total dominating set in  $G$ . Since  $|D'| - |D|$  does not depend on the choice of  $D$  the graph  $G'$  is TWD if  $G$  is TWD.

Next, let  $D$  be a minimal total dominating set in  $G$ . It can be observed that the cardinality of  $D \cap (V(G) \setminus (V(G') \setminus \{v, v_2, v_3\}))$  does not depend on the choice of  $D$ . If  $v \in D$  then  $v_3 \notin D$ . Thus if  $D' := (D \cap V(G')) \cup \{v_2\}$  when  $v \in D$  and  $D' := (D \cap V(G')) \cup \{v_2, v_3\}$  when  $v \notin D$  then  $|D'| - |D|$  does not depend on  $D$ . Since  $D'$  is a minimal total dominating set in  $G'$  the graph  $G$  is TWD if  $G'$  is TWD.  $\square$

**Lemma 15** *Let  $G$  be a graph with a nonstem  $v \in V(G)$  adjacent to a stem  $x$ . Assume that a graph from  $\mathcal{A}_5$  is attached at  $v$ . If  $H$  is the graph obtained by removing the attached  $\mathcal{A}_5$ -graph and attaching  $P_7$  to  $v$  then  $G$  is TWD if and only if  $H$  is TWD.*

**Proof.** Assume that  $G$  and  $H$  are as described in the lemma. For both graphs let  $P$  denote a path  $v_1 v_2 \dots v_7$  contained in the graph attached at  $v$  such that  $vv_1$  is an edge. Let  $G'$  be the component of  $G - vv_1$  (or  $H - vv_1$ ) containing  $v$ . Since  $G$  contains a admissible set  $A$  such that  $G' = G - N[A]$  and  $H$  contains a admissible set  $B$  such that  $G' = H - N[B]$  the graph  $G'$  is TWD if either  $G$  or  $H$  is TWD.

Assume  $H$  is TWD and let  $D$  be a minimal total dominating set in  $G$ . It can easily be observed that  $|D \cap (V(G) \setminus V(G'))|$  does not depend on the choice of  $D$ . Further  $D' := D \cap V(G')$  must be a minimal total dominating set of  $G'$ . If this is not the case then  $v$  is in  $D$  and its sole purpose is to dominate  $v_1$ , so that the sets  $D' \cup \{v_3, v_4, v_6, v_7\}$  and  $(D' \cup \{v_2, v_3, v_6, v_7\}) \setminus \{v\}$  are both minimal total dominating sets in  $H$ . But since  $H$  is TWD this is a contradiction. Thus it follows that  $G$  is TWD if  $H$  is TWD. By similar arguments it can be proven that  $H$  is TWD if  $G$  is TWD.  $\square$

**Lemma 16** *Let  $G$  be a graph. If a graph from  $\mathcal{A}_4$  and a graph from  $\mathcal{A}_5$  are attached at a vertex  $v \in V(G)$  then  $G$  is TWD if and only if the graph  $H$  obtained by removing the attached graph from  $\mathcal{A}_5$  is TWD.*

**Proof.** First assume that  $G$  is TWD. It can be observed that all graphs from  $\mathcal{A}_5$  have a total dominating set  $D$  such that its attachment vertex is not contained in  $D$  and each vertex from  $D$  is adjacent to exactly one vertex from  $D$ . If  $D$  is such a set in the graph from  $\mathcal{A}_5$  attached to  $v$  then  $D$  is an admissible set in  $G$  and thus  $H = G - N[D]$  is TWD.

Now assume that  $H$  is TWD and let  $D$  be a minimal total dominating set in  $G$ . Let  $G'$  be the graph from  $\mathcal{A}_5$  attached at  $v$ . By considering  $G'$  it can be observed that  $|D \cap V(G')| = \gamma_t(G')$ . Assume first that  $D'' := D \cap H$  is not a minimal total dominating set in  $H$ . Let  $v_1 \dots v_6$  be a path in the attached graph from  $\mathcal{A}_4$  such that  $vv_1 \in E(G)$ .

If  $v \in D$  then  $D''$  must dominate  $H$  but the only neighbour to  $v$  contained in  $D$  is the vertex from  $N[v] \cap V(G')$ . Thus  $v$  must be an isolated vertex in  $H[D'']$ ,  $v_3 \in D''$  and  $(D'' \setminus \{v_3\}) \cup \{v_1\}$  is a minimal total dominating set in  $H$ . If  $v \notin D$  then  $v$  is not dominated by  $D''$  and  $v_3 \in D$ . Thus  $(D'' \setminus \{v_3\}) \cup \{v_1\}$  is a minimal total dominating set in  $H$ . In all cases  $|D''| = \gamma_t(H)$  and we obtain that  $G$  is TWD if  $H$  is TWD.  $\square$

**Lemma 17** *Let  $G$  be a graph with the structure illustrated in Figure 6 and assume  $N[x]$  does not contain any stems and all vertices at distance two from  $x$  in  $G - vx$  is adjacent to a stem. Let  $H$  be the component of  $G - xv$  containing  $x$ . Then  $G$  is TWD if and only if  $H$  is TWD and  $x$  is special in  $H$ .*

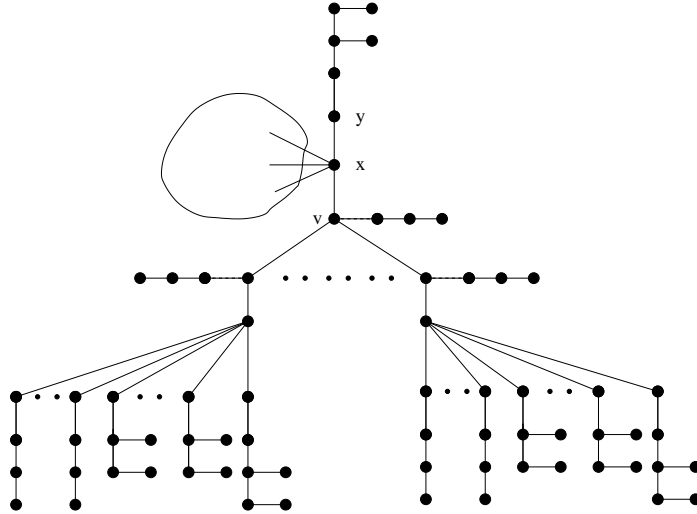


Figure 6: Illustration of  $G$ .

**Proof.** First assume that  $G$  is TWD. Let  $G'$  be the union of all components in  $G - v$  not containing  $x$ . By considering  $G'$  it can be seen that it has minimal total dominating sets  $A$  and  $B$  such that  $N[A] \cap v \neq \emptyset$ ,  $N[B] \cap v = \emptyset$ ,  $|A| = |B|$  and for  $X \in \{A, B\}$  each vertex from  $X$  is adjacent to exactly one vertex from  $X$ . Each of the sets  $A$  and  $B$  are admissible in  $G$ . Thus  $H = G - N[A]$  is TWD and  $G - N[B]$  is TWD proving that  $x$  is special in  $H$ .

Now assume that  $H$  is TWD and let  $D$  be a minimal total dominating set of  $G$ . Let  $G'$  be the union of all the graphs from  $\mathcal{A}_5$  attached at  $v$  (not containing  $x$ ) and let  $G'' := G - V(G')$ . By considering  $G'$  it can be observed that  $|D \cap V(G')| = \gamma_t(G')$ . In the following we prove that  $|D \cap V(G'')|$  does not depend on the choice of  $D$ . If  $v \notin D$  then let  $D' := D \cap V(H)$  and otherwise let  $D' := ((D \cap V(H)) \setminus \{v\}) \cup \{y\}$ . Since  $N[x]$  does not contain any stems and all vertices at distance two from  $x$  in  $H$  is adjacent to a stem the set  $D'$  must be a minimal total dominating set for  $H$  or the graph obtained by attaching a  $P_1$  to  $x$  in  $H$ . Since we assume that  $x$  is special in  $H$  we have  $|D'| = \gamma_t(H)$ . Thus  $|D \cap V(G'')|$  does not depend on the choice of  $D$  since  $|D \cap V(G'')| = |D'|$  if no  $P_3$  is attached at  $v$  and  $|D \cap V(G'')| = |D'| + 2$  if a  $P_3$  is attached at  $v$ . Thus the graph  $G$  is TWD if  $H$  is TWD.  $\square$

## 4 Main Result

In this section we consider TWD trees that cannot be reduced by any of the decomposition/composition rules. Such a tree is called *reduced* and the following theorem proves that  $\{P_2, P_4, P_3 \circ K_1\}$  is the family of reduced trees.

The idea of the proof will be to traverse a diametrical path from an end towards its center and examine which subtrees it can, or rather cannot, have attached.

**Theorem 1** *Let  $T$  be a TWD tree. Then  $T$  is reduced if and only if  $T \in \{P_2, P_4, P_3 \circ K_1\}$ .*

**Proof.** Assume that  $T$  is a reduced TWD and  $T \notin \{P_2, P_4, P_3 \circ K_1\}$ . In the following we consider a path  $P : x_1 \dots x_k$  in  $T$  such that

1.  $x_1$  is a leaf.
2. Any path  $x_k x_{k-1} u_1 u_2 \dots u_l$  in  $T$  has length at most  $k - 1$ .
3. If  $C$  is the center-vertices in  $T$  and  $P'$  is a path between  $C$  and  $x_1$ , then  $V(P) \subseteq V(C) \cup V(P')$ .
4. No path  $x_1 \dots x_k x_{k+1}$  satisfy conditions 1-3.

In the following we only say that a graph  $H$  with attachment vertex  $a$  is attached at  $x_i$  if a longest path  $x_i a u_1 \dots u_l$  where  $\{a, u_1, \dots, u_l\} \subseteq V(H)$  has length at most  $i - 1$  and  $a \neq x_{i-1}$ .

The only vertex from  $P$  to which a  $P_1$  can be attached is  $x_3$ . Since  $T$  cannot be reduced it follows from Corollary 1 that a  $P_1$  is not attached at  $x_i$  for  $i \geq 5$ , because  $x_i$  should then have degree two or be adjacent to a stem of degree two as on Figure 3(c), and that is not the case. Since  $T$  cannot be reduced Lemma 2 implies that a  $P_1$  is not attached at  $x_2$ . If a  $P_1$  is attached at  $x_4$  then  $T \cong P_3 \circ K_1$  by Lemma 6. No vertex of  $P$  can have a  $P_2$  attached, since it follows from Corollary 1 that a  $P_2$  cannot be attached at  $x_i$  for  $i \geq 5$  because  $x_i$  should then either have degree two or have a leaf attached, neither of which can occur. Since  $T$  is reduced Lemma 6 implies that a  $P_2$  is not attached at  $x_3$  and Lemma 4 implies that a  $P_2$  is not attached at  $x_4$ .

Now consider the vertex  $x_4$ . Since  $T$  is reduced only a  $P_3$  or the graph  $A_1$  can be attached at  $x_4$ . By Lemma 4 the graph  $A_1$  cannot be attached at  $x_4$ . If  $x_3$  is a stem then it follows from Lemma 4 that a  $P_3$  is not attached at  $x_4$  and if  $x_3$  is not a stem it follows from Lemma 7 that a  $P_3$  cannot be attached at  $x_4$ . Thus it can be assumed that  $\deg(x_4) = 2$ .

Consider the graphs that can be attached at  $x_5$ . Since  $T$  is reduced only the graphs  $P_3, P_4, A_1$  and  $A_2$  can be attached at  $x_5$ . Since  $T$  is reduced Lemma 4 implies that  $x_5$  cannot be adjacent to a stem if  $x_3$  is a stem, and Lemma 8 implies that  $x_5$  cannot be adjacent to a stem if  $x_3$  is not a stem. Thus  $x_5$  is not adjacent to a stem, and therefore  $A_1$  is not attached at  $x_5$ . If a  $P_3 : av_1v_2$  is attached at  $x_5$  and  $ax_5 \in E(T)$  then  $\{v_1, v_2, x_2, x_3, x_4\}$  is contained in a minimal total dominating set  $D$  and  $D' := (D \setminus \{x_4, v_2\}) \cup \{a\}$  is a total dominating set. Since  $|D'| < |D|$  we obtain a contradiction since  $T$  is TWD, so no  $P_3$  is attached at  $x_5$ . Further, Lemma 11 implies that  $A_2$  is not attached at  $x_5$  when  $x_3$  is a stem. So we may assume that  $\deg(x_5) = \deg(x_4) = 2$ .

Since  $T$  is reduced by Lemma 10 we may assume that  $x_3$  is a stem and  $\deg(x_4) = \deg(x_5) = 2$ . I.e., for any leaf in  $T$  which is the origin of a sufficiently long path the structure near  $P$  must be as illustrated in Figure 7 when  $k \geq 6$ .

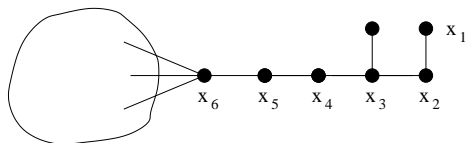


Figure 7: Illustration of structure in  $T$  when  $k \geq 6$ .

Consider the graphs attached at  $x_6$ . Since  $T$  is reduced Lemma 13 implies that  $A_1$  is not attached at  $x_6$  and from Lemma 12 it follows that  $A_3$  is not attached at  $x_6$ . Thus only  $P_3, P_4$  and  $A_2$  can be attached at  $x_6$ . If a  $P_3 : av_1v_2$  is attached at  $x_6$  and  $ax_6 \in E(T)$  then Lemma 4 implies that  $x_6$  is not adjacent to a stem and therefore  $\{v_1, v_2, x_4, x_5\}$  is contained in a minimal total dominating set  $D$ . But then  $D' := (D \setminus \{x_5, v_2\}) \cup \{a\}$  is a total dominating set and  $|D'| < |D|$ . Since  $T$  is TWD a contradiction is obtained if a  $P_3$  is attached at  $x_6$ . So only  $P_4$ 's and  $A_2$ 's can be attached at  $x_6$ ; and, in fact, by Lemma 11 at most one  $A_2$  can be attached to  $x_6$ .

The component of  $T - x_7$  containing  $x_1 \cdots x_6$  is a graph in  $\mathcal{A}_4$  with attachment vertex  $x_6$ . Thus Lemma 14 implies that  $A_1$  cannot be attached at  $x_7$  because otherwise  $T$  could be reduced. If a  $P_4 : av_1v_2v_3$  is attached at  $x_7$  then Lemma 8 implies  $x_7$  is not adjacent to a stem and thus  $\{a, v_2, v_3, x_7, x_4\}$  is a subset of a minimal total dominating set  $D$ . But then  $D' := (D \setminus \{a, x_4\}) \cup \{x_6\}$  is a total dominating set that satisfies  $|D'| < |D|$ . By using similar arguments we obtain that  $A_2$  cannot be attached at  $x_7$ .

Assume that  $A_3$  or a graph from  $\mathcal{A}_4$  is attached at  $x_7$  and let  $T'$  denote this subgraph. By considering  $T$  it follows that  $\{x_5, x_4, x_3, x_2\}$  is contained in a minimal total dominating set  $D$ , but  $D' := (D \setminus \{x_4, x_5, y\}) \cup \{x_6, x_7\}$  is a smaller total dominating set for a vertex  $y \in D \cap V(T')$  at distance two from  $x_7$ . Therefore, as  $T$  is TWD it can be assumed that only  $P_3$ 's can be attached at  $x_7$  and Lemma 7 implies that at most one  $P_3$  is attached at  $x_7$ . But in fact no  $P_3$  can be attached to  $x_7$ , because starting from a leaf  $x_1$  we found in Figure 7 that  $\deg x_4 = 2$ . We note that the component of  $T - x_8$  containing  $x_1 \cdots x_7$  is a graph in  $\mathcal{A}_5$  with attachment vertex  $x_7$ .

Consider any graph  $G'$  obtained by attaching a graph from  $\mathcal{A}_5$  to the center of a star  $K_{1,t}$  for some  $t \geq 0$ . Since  $G'$  is not TWD it follows there must be a component  $C$  of  $T - x_8$ , not containing the vertex  $x_1$ , such that  $C$  does not contain an admissible set  $D$  that satisfies  $y \notin D$  and either  $y \in N[D]$  or  $N(y) \cap C \subseteq N[D]$  where  $y$  is the vertex from  $C$  adjacent to  $x_8$ . If this were not the case then the union of such admissible sets in all components of  $T - x_8$  not containing  $x_1$  would be an admissible set  $D$  such that  $T - N[D]$  had a component isomorphic to a graph like  $G'$ . Since  $T$  is TWD it follows from Observation 1 and Lemma 1 that this is a contradiction.

Let  $C$  be such a component and consider a path  $yv_1v_2$  in  $C$  such that  $yx_8 \in E(T)$ . Since  $\{v_1, v_2\}$  cannot be an admissible set  $G - N[\{v_1, v_2\}]$  must have an isolated vertex. Since  $T$  is reduced Lemma 15 implies  $y$  is not a stem and therefore  $v_1$  or  $v_2$  must be adjacent to a stem in  $C - \{y, v_1, v_2\}$ . If  $v_1$  is adjacent to such a stem  $u$  then since  $T$  is reduced Corollary 1 shows that  $u$  must be adjacent to a stem  $z$  but then  $\{v_1, u, z\}$  is contained in an admissible



set in  $C \setminus \{y\}$  which is a contradiction to the choice of  $C$ . Thus  $v_2$  must be adjacent to a stem  $u \neq v_1$ . Thus Corollary 1 and Lemma 7 imply that  $T$  must have the structure illustrated in Figure 8. Let the set  $A$  be as illustrated in Figure 8. Further, let  $D$  be all vertices from  $V(C) \setminus \{y\}$  at distance at least three from  $A$ . Now  $A \cup D$  is a total dominating set of  $C - y$  and thus there is a minimal total dominating set  $D'$  of  $C - y$  such that  $D' \subseteq A \cup D$ . If no  $A_3$ -graph is attached at  $y$  then  $D'$  is an admissible set in  $T$  contradicting the choice of  $C$ . Thus we may assume that an  $A_3$  is attached at  $y$ .

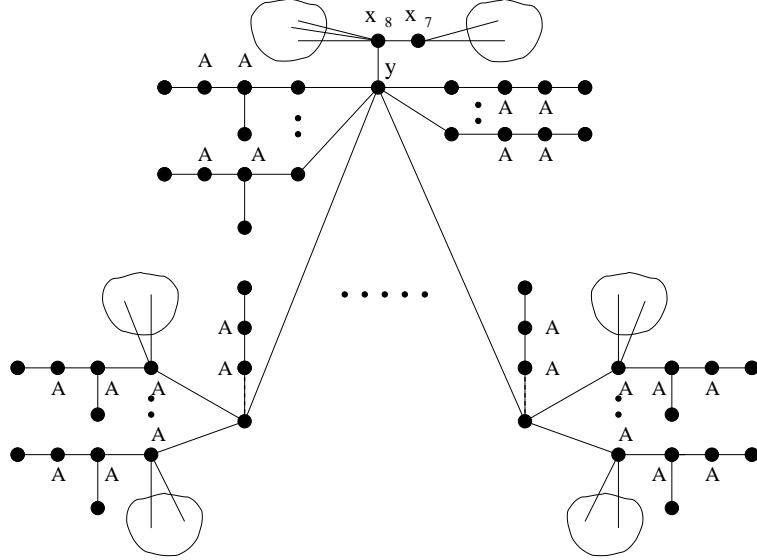


Figure 8: Illustration of  $T$ .

Now consider the graphs attached at  $x_8$ . Only  $P_3, P_4, A_1, A_2, A_3$  or a graph from  $\mathcal{A}_4 \cup \mathcal{A}_5$  can be attached at  $x_8$ . By Lemma 16 a graph from  $\mathcal{A}_4$  is not attached at  $x_8$  and by Lemma 15  $A_1$  cannot be attached at  $x_8$ . If one of the graphs  $P_4, A_2, A_3$  is attached at  $x_8$  and  $a$  denotes the attachment-vertex from such a graph, then since  $x_8$  is not adjacent to a stem then  $\{a, x_2, x_3, x_4, x_5, x_8\}$  is contained in a minimal total dominating set  $D$ . Further,  $D' := (D \setminus \{a, x_5\}) \cup \{x_7\}$  is a total dominating set. Since  $T$  is TWD none of these graphs can be attached at  $x_8$ . Thus  $T$  has the structure as the graph from Lemma 17 and since  $T$  is reduced this is a contradiction.

Now let  $P$  be a subpath of a diametrical path in  $T$ . By the above arguments  $k \leq 7$  and  $k \geq 3$  since  $T \not\cong P_2$ . Thus there must be a graph attached at  $x_k$  and the information about graphs attached at  $x_3, x_4, x_5, x_6$  and  $x_7$  implies that  $T \in \mathcal{A}_4$ .

This proves the statement since all the graphs from  $\{P_2, P_4, P_3 \circ K_1\}$  are reduced graphs and Lemma 4, Lemma 10, Lemma 11 and Lemma 14 imply that no graph from  $\mathcal{A}_4$  is reduced.  $\square$

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