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Publication date:
2009

Document Version
Publisher's PDF, also known as Version of record

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R-2009-14 August 2009

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ISSN 1399–2503 ▪ On-line version ISSN 1601–7811

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August 2009

Abstract

Let $G = (V, E)$ be a graph with no isolated vertex. A set D is called total dominating in G if each vertex in G is adjacent to a vertex from D, and D is a minimal total dominating set if any subset $D' \subset D$ is not a total dominating set in G. If all minimal total dominating sets in G have the same cardinality then G is a total well dominated graph. In this paper we study composition and decomposition of total well dominated trees. By a reversible process we prove that any total well dominated tree can both be reduced to and constructed from a family of three small trees.

Keywords: total domination, decomposition, composition, total well dominated
AMS subject classification: 05C69

1 Notation

For notation and graph theory terminology we in general follow [4]. Let $G = (V, E)$ be a graph with vertex set V and edge set E. A *dominating set* of G is a set D of vertices of G such that every vertex in $V \setminus D$ is adjacent to a vertex in D. Further, if also each vertex in a dominating set D is adjacent to a vertex from D then D is a *total dominating set*. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. A total dominating set of minimum cardinality $\gamma_t(G)$ is called a γ_t-set for G. If a total dominating set D satisfies that no proper subset of D is a total dominating set then D is a *minimal total dominatig set*. The *upper total domination number* of a graph G, denoted by $\Gamma_t(G)$, is the maximum cardinality of a minimal total dominating set. If all minimal total dominating sets in a graph G have the same cardinality the graph is called *total well dominated* (or just *TWD*). Thus a graph G is TWD if and only if $\gamma_t(G) = \Gamma_t(G)$. In a TWD graph any minimal total dominating set is a γ_t-set. For a set D and a vertex $x \in D$ the *private neighbourhood* of x is defined by $pn(x, D) := \{y \in N[x] | N[y] \cap D = x\}$. A vertex of degree one is called a *leaf* and a vertex adjacent to a leaf is called a *stem*.

For two vertices x and y in a graph G we denote the distance between the vertices by $d_G(x, y)$.For $S \subseteq V(G)$ we define $d_G(x, S) := min_{s \in S}\{d_G(x, s)\}$.

If G is a graph and S is a vertex set in G, then the induced subgraph of G with vertex set S is denoted $G[S]$.

For $k \geq 1$ let A_k be the graph with vertex set $V(A_k) = \{x_1, x_2, \ldots, x_{k+1}, x, y\}$ and edge set $E(A_k) = \{x_1x_2, x_2x_3, \ldots, x_kx_{k+1}, x_kx, x_{k+1}y\}$. Thus A_k is the graph illustrated in Figure 1, $A_1 = P_4$ and A_2 is a $K_{1,3}$ with one edge subdivided.

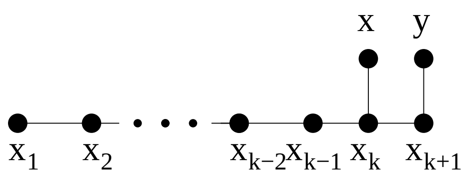

Figure 1: Illustration of A_k.

Let $\mathcal{A}_4$ be the family of graphs with the structure illustrated in Figure 2 (a) and let $\mathcal{A}_5$ be the family of graphs with the structure illustrated in Figure 2(b).

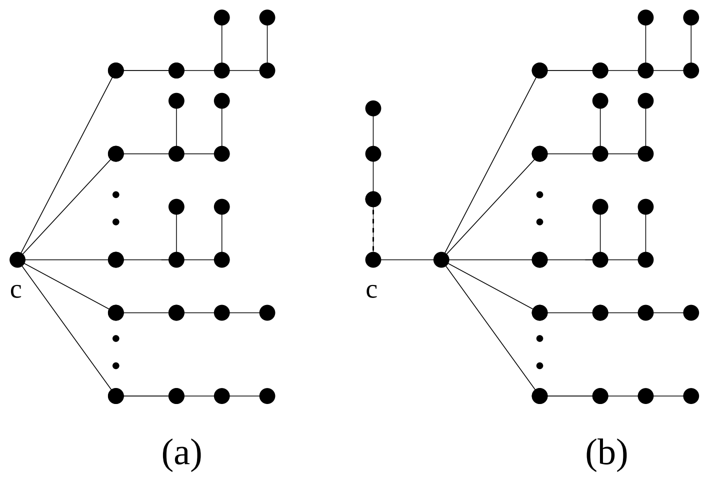

Figure 2: (a) illustrates $\mathcal{A}_4$ and (b) illustrates $\mathcal{A}_5$. In (b) the dotted line indicates that either $\deg(c) = 1$ or a path P_3 is attached to c.

In A_i we call the vertex x_1 an attachment vertex and in a graph from $\mathcal{A}_4$ or $\mathcal{A}_5$ we call the vertex c an attachment vertex. In a path P_n a vertex with smallest degree is called an attachment vertex.

For graphs H with attachment vertex a and G with a vertex v we define the graph obtained by attaching H to v in G as the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{av\}$. In the obtained graph we say that H is attached at v.

2 Introduction

Total domination, treated here, is a follow-up to domination. Graphs with all minimal dominating sets having the same cardinality are called well dominated. They form a subset of well covered graphs, surveyed by M.D. Plummer in 1993 [7] and for girth ≥ 4 characterized by Hartnell, Finbow and Nowakowski [2, 3]. For girth ≥ 6 the family of well dominated graphs equals the class of well covered graphs ([1]) and for girth ≥ 5 well dominated graphs are characterized in [2, 3].

3 Decomposition/composition-rules

In this section we give the main decomposition/composition-rules that is used for total well dominated tress. Each rule is of the kind saying that if G is a graph with some special structure then G is TWD if and only if certain subgraphs of G are TWD. We say that the graph is reduced by the rule (or the lemma with the rule).

It will turn out that any TWD tree by application of rules to be described in several lemmas below can be reduced to a smaller TWD tree of order ≤ 6, namely to P_2, P_4 or $P_3 \circ K_1$. Moreover, the rules will be invertible, so starting from these three small graphs we can by application of the rules construct any TWD tree.

For the decomposing/composing we shall use some special vertices and admissible sets, therefore these concepts are defined in the following.

Definition 1 *Let G be a TWD graph and let $v \in V(G)$. If the graph $G' = (V(G) \cup \{x\}, E(G) \cup \{vx\})$ is TWD and $\gamma_t(G) = \gamma_t(G')$ then v is called a* special vertex.

Thus, in a TWD graph G the vertex v is special if and only if G has a γ_t-set containing v. In $P_6 = x_1 x_2 \cdots x_6$ the vertex x_3 is special.

Definition 2 *Let G be a graph and let $S \subseteq V(G)$. The set S is called* admissible *if*
(i) neither $G[S]$ nor $G - N[S]$ have isolated vertices
and
(ii) $\forall v \in S$: either v is a stem in $G[S]$ or $pn(v, S \cup (V(G) \backslash N[S])) \neq \emptyset$.

From the definition of an admissible set it can be seen that a set S is admissible in G if and only if $S \cup (V(G) \backslash N[S])$ is a total dominating set in G and $(S \cup (V(G) \backslash N[S])) \backslash \{s\}$ is not a total dominating set in G for any vertex $s \in S$.

Observation 1 *If S is a admissible set in a graph G, then G has no isolated vertex and for each minimal total dominating set S' in $G - N[S]$, the set $S \cup S'$ is a minimal total dominating set in G. I.e., an admissible set can be extended to a minimal domination set. Also, if G is TWD then $G - N[S]$ is TWD when S is admissible.*

Lemma 1 *A graph G is TWD if and only if each component of G is TWD.*

Lemma 2 *If a vertex $v \in V(G)$ is adjacent to two leaves l_1 and l_2 then G is TWD if and only if $G - l_1$ is TWD.*

Lemma 3 *Let G be a graph containing two adjacent stems s_1 and s_2. If L denotes the leaves adjacent to s_1 or s_2 and $C_1, \ldots, C_k$ is the components of $G - (\{s_1, s_2\} \cup L)$ then G is TWD if and only if $G_i := G - \{C_1, C_2, \ldots, C_{i-1}, C_{i+1}, C_{i+2} \ldots, C_k\}$ is TWD for each $i \in \{1, \ldots, k\}$.*

Proof. The statement is trivial for $k \leq 1$, so assume $k \geq 2$. Since $\delta(G) \geq 1$ if and only if $\delta(G_i) \geq 1$ for each i, $1 \leq i \leq k$, we see that G can be totally dominated if and only each G_i can. As v_1 and v_2 are stems they must be contained in any total domination set for any of the graphs $G, G_i, 1 \leq i \leq k$. Let $D_i \subseteq V(G_i), 1 \leq i \leq k$ and $D = \bigcup_{i=1}^{k} D_i$. We note that D is a total dominating set for G and, in fact, also that D is a minimal total dominating set for G if and only if each D_i is a minimal total dominating set for $G_i, \forall i$. We have $|D| = \sum_{i=1}^{k} |D_i| - 2 \cdot (k-1)$. That implies that G is TWD if and only if G_i is TWD for each $i, 1 \leq i \leq k$. $\square$

Lemma 4 *Let G be a graph with a path $l_1 s_1 v_1 v_2 s_2 l_2$ and assume that $deg(l_1) = deg(l_2) = 1$. Then G is TWD if and only if $G - v_1 v_2$ is TWD.*

Proof. Since s_1 and s_2 are stems these vertices must be in all total dominating sets in G and $G - v_1 v_2$. Thus a total dominating set of G is also a total dominating set in $G - v_1 v_2$ and a total dominating set of $G - v_1 v_2$ is trivially a total dominating set of G. Thus the result follows. $\square$

Lemma 5 *Let T be a TWD tree reduced by Lemma 2, i.e., without multiple leaves, and let s be a stem of T having valency at least 3. Then T contains another stem at distance at most 3 from s.*

Proof. Let s be a stem in T and assume that $\deg(s) \geq 3$ and that no other stem in T is within distance 3 from s. Since s is adjacent to exactly one leaf l there must exists a path $P : abcsdef$ in T. For each vertex x from P let C_x denote the union of all components of $T - x$ not containing vertices from P.

Since no stem $s' \neq s$ is within distance 3 from s the set $V(C_x) \backslash N[x]$ is a total dominating set for C_x when $x \in \{b, e\}$. Thus there is a minimal total dominating set D_x of C_x not containing a vertex adjacent to x.

For $x \in \{c, d\}$ let D_x^+ be a minimal total dominating set of C_x and D_x^- be a minimal total dominating set of $C_x - N[x]$.

For $x \in \{a, f\}$ let y be the vertex from P adjacent to x. Further let D_x^+ be a minimal total dominating set of $G[V(C_x) \cup \{x, y\}]$ not containing y and D_x^- be a minimal total dominating set of $G[V(C_x) \cup \{x\}]$ not containing x.

Further let D_s be a minimal total dominating set of $C_s - N[s]$ and let $S := D_s \cup D_b \cup D_e$.

Now let $A := \{c, s, d\} \cup D_a^- \cup D_c^- \cup D_d^- \cup D_f^- \cup S$, $B := \{s, d\} \cup D_a^+ \cup D_c^+ \cup D_d^- \cup D_f^- \cup S$, $C := \{c, s\} \cup D_a^- \cup D_c^- \cup D_d^+ \cup D_f^+ \cup S$ and $D := \{s, l\} \cup D_a^+ \cup D_c^+ \cup D_d^+ \cup D_f^+ \cup S$. By construction all of these sets are minimal total dominating sets, and since T is TWD $|A| = |B| = |C| = |D|$. Since $|A| = |B|$ we obtain $|D_a^-| + |D_c^-| + 1 = |D_a^+| + |D_c^+|$ and since $|A| = |C|$ we obtain $|D_d^-| + |D_f^-| + 1 = |D_d^+| + |D_f^+|$. But then

$$|D| = 2 + |D_a^+| + |D_c^+| + |D_d^+| + |D_f^+| + |S| = 4 + |D_a^-| + |D_c^-| + |D_d^-| + |D_f^-| + |S| = |A| + 1.$$

Thus we obtain a contradiction. $\square$

Lemma 6 *Let G be a graph with a vertex v adjacent to two stems s_1 and s_2. If G is reduced by Lemma 2, 3 and 4 then $G \cong P_3 \circ K_1$ or G is not TWD.*

Proof. Assume G is TWD and reduced by Lemma 2, 3 and 4. Let l_i be a leaf adjacent to s_i for $i \in \{1, 2\}$. First assume that $A := \{s_1, s_2, l_1, l_2\}$ is a admissible set. Then there is a minimal total dominating set D such that $A \subseteq D$. But then $(D \backslash \{l_1, l_2\}) \cup \{v\}$ is a total dominating set of smaller cardinality. Thus it can be assumed that A is not admissible and thus s_1 or s_2 must be adjacent to a stem. Since G is reduced by Lemma 4 that stem must be v, and since G is reduced by Lemma 2 and Lemma 3 we obtain that $G \cong P_3 \circ K_1$. □

Removing all but one attached P_3 from a vertex does not change the property of being TWD.

Lemma 7 *Let G be a graph and let $v \in V(G)$. If G_i is the graph obtained from G by attaching i P_3's to the vertex v then G_1 is TWD if and only if G_2 is TWD.*

Proof. Let $v_1v_2v_3$ and $u_1u_2u_3$ be the two P_3's added to G to obtain G_2 and assume that $\{v_1v, u_1v\} \subseteq E(G_2)$. Since $\{u_2, u_3\}$ is an admissible set in G_2 it follows that $G_1 = G_2 - N[\{u_2, u_3\}]$ is TWD if G_2 is TWD. Now assume G_1 is TWD and let D be any minimal total dominating set in G_2. Assume WLOG that $d_{G_2}(v, \{v_1, v_2, v_3\} \cap D) \leq d_{G_2}(v, \{u_1, u_2, u_3\} \cap D)$ then, if $u_1 \in D$ we can replace it by u_3, so we may assume $\{u_2, u_3\} \subseteq D$, $u_1 \notin D$. Then $D \backslash \{u_2, u_3\}$ is a minimal total dominating set in $G_2 - N_{G_2}[\{u_2, u_3\}] \cong G_1$. Since $|D \cap \{u_1, u_2, u_3\}| = 2$ and G_1 is TWD it follows that $|D| = \gamma_t(G_1) + 2$ and thus G_2 is TWD. □

Lemma 8 *Let G be a connected graph with a P_4 attached at a vertex v and let H be the graph obtained by removing the attached P_4. If v is adjacent to a stem in G then G is TWD if and only if H is TWD and v is special in H.*

Proof. First assume that G is TWD and let $v_1v_2v_3v_4$ be the P_4 attached at v such that $vv_1 \in E(G)$. Since $\{v_2, v_3\}$ and $\{v_3, v_4\}$ are admissible sets $H = G - N[\{v_2, v_3\}]$ is TWD, and $G - N[\{v_3, v_4\}]$ is TWD which proves that v is special in H. Now assume conversely that H is TWD and v is a special vertex in H and consider a minimal total dominating set D in G. If $\{v_3, v_4\} \subseteq D$ then $D \backslash \{v_3, v_4\}$ is just a minimal total dominating set in $G - \{v_2, v_3, v_4\}$ and since v is special we obtain $|D| = 2 + \gamma_t(H)$ in this case. Otherwise $D \cap \{v_1, v_2, v_3, v_4\} = \{v_2, v_3\}$ since v is adjacent to a stem in G. We see that $D \backslash \{v_2, v_3\}$ is a minimal total dominating set in H and, as H is TWD, that $|D \backslash \{v_2, v_3\}| = \gamma_t(H)$, implying that $|D| = 2 + \gamma_t(H)$. It follows that G is TWD. □

Lemma 9 *Let G be a graph with a $P_2 : v_1v_2$ attached at a nonstem $v \in V(G)$ such that $vv_1 \in E(G)$ and $deg(v) \geq 3$. Let $C_1, \ldots, C_k$ be the components of $G - \{v, v_1, v_2\}$. If G_i denotes the graph $G[V(C_i) \cup \{v, v_1, v_2\}]$ for $i \in \{1, \ldots, k\}$ then G is TWD if and only if $G_1, \ldots, G_k$ is TWD.*

Proof. Let D be a minimal total dominating set in G. If $v \in D$ then v is the only vertex that dominates v_1 and it follows that $D \cap G_i$ is a minimal total dominating set in G_i. Since

$|D \cap \{v, v_1, v_2\}| = 2$ it follows that G is TWD if $G_1, \ldots, G_k$ is TWD. Assume that G is TWD, we choose an index i and we shall prove that G_i is TWD. Let D_i be a minimal total dominating set in G_i. Since v is not a stem G has a minimal total dominating set D such that $D \cap G_i = D_i$. If $v_2 \in D_i$ then $(D \cup \{v\}) \backslash \{v_2\}$ must be a minimal total dominating set in G since G is TWD and therefore $(D_i \cup \{v\}) \backslash \{v_2\}$ must be a minimal total dominating set in G_i. Thus it follows that G_i is TWD if all of the minimal total dominating sets in G_i not containing v_2 have the same cardinality. Assume that D_i and D_i' are minimal total dominating sets in G_i not containing v_2. Since G is TWD and the sets D and $(D \backslash D_i) \cup D_i'$ are minimal total dominating sets in G it follows that $|D_i| = |D_i'|$. Thus G_i is TWD and consequently $G_1, \ldots, G_k$ are TWD if G is TWD. $\square$

Lemma 10 *Let T be a tree reduced by Lemma 4 and 8 with a vertex v such that all components of $T - v$ except one namely C, $C \not\cong P_1$, are components isomorphic to P_4 or A_2 attached at v and at least one P_4 is attached at v. Let x denote the vertex from $V(C) \cap N[v]$, let $C_1, \ldots, C_k$ be the components of $C - N[x]$ and let v_i denote the vertex in C_i adjacent to a vertex from $C - C_i$. Then T is TWD if and only if each of $C_1, \ldots, C_k$ are TWD, v_i is a special vertex in C_i adjacent to a stem in C_i and $C_i \not\cong P_2$.*

Proof. First assume that T is TWD. Let $x_1x_2x_3x_4x_5 = vx_6 = xx_7 \ldots x_a$ be a path in T such that $x_1x_2x_3x_4$ is a P_4 attached at v and $x_4v \in E(G)$. We have $x_8 = v_i$ for some i. Since G is reduced by Lemma 8 the vertex x cannot be a stem. Neither is x_7 a stem, for assume otherwise that x_7 is a stem and let D be a minimal total dominating set in the component of $T - xx_7$ containing x_7. Since x_7 is a stem, D is an admissible set in T containing x_7 and by Observation 1 all components of $T - N[D]$ are TWD. But the component of $T - N[D]$ containing v is not TWD so we have a contradiction. Thus we may assume that x_7 is not a stem and therefore $\{v, x\}$ is an admissible set. Since $T - N[\{v, x\}]$ contains $C_1, \ldots, C_k$ as components each of these is TWD. Let C_i be the component containing $x_8 = v_i$, we shall show that C_i has a stem adjacent to x_8 and that $C_i \not\cong P_2$. Assume otherwise that either $C_i \cong P_2$ or C_i does not have a stem adjacent to x_8. Let C' be the component of $T - xx_7$ containing x_7. By the assumptions the set $D'' := V(C') \backslash (N(x_8) \cap V(C_i))$ is a total dominating set for C'. Let D' be a minimal total dominating set of C' such that $D' \subseteq D''$. Since $N(x_8) \cap D'' = \{x_7\}$ the vertex x_7 must be in D' and D' is an admissible set in T. But the component of $T - N[D']$ containing v is not TWD. This contradiction proves that in C_i there is a stem adjacent to x_8 and that $C_i \not\cong P_2$. Thus it just remains to prove that x_8 is a special vertex in C_i. Since T is reduced by Lemma 4 and x_8 is adjacent to a stem in C_i the vertex x_7 cannot be adjacent to a stem in $G - C_i$. Since $\{x_1, x_2, v, x\}$ and $\{x_1, x_2, x_4, v\}$ are admissible sets all components of $C - x$ must be TWD and for each such component H the graph obtained by removing the vertex adjacent to x must be TWD and have the same total domination number as H.

Let D be a minimal total dominating set in $C' \cap (C_1 \cup C_2 \cup \cdots \cup C_{i-1} \cup C_{i+1} \cup C_{i+2} \cup \cdots \cup C_k)$ not containing any of the vertices $v_1, \ldots, v_k$. Now $D \cup \{v, x\}$ and $D \cup \{x_4, v\}$ are admissible sets and thus all components of $T - N[D \cup \{v, x\}]$ and $T - N[D \cup \{x_4, v\}]$ is TWD. Since $|D \cup \{v, x\}| = |D \cup \{x_4, v\}|$ we have $\gamma_t(T - N[D \cup \{x_4, v\}]) = \gamma_t(T - N[D \cup \{v, x\}])$ and by considering the components of these graphs we obtain that C_i and the graph obtained by attaching a P_1 to x_8 in C_i must be TWD and have the same total domination number, i.e. x_8 is special in C_i.

Now assume conversely that T can be constructed as described in the lemma, we shall prove that T is TWD. Let D be a minimal total dominating set in T. Since $v_1, \ldots, v_k$ is adjacent to a stem in $C_1, \ldots, C_k$ the set $D \cap C_i$ is a minimal total dominating set in C_i and if $v_i \in D$ then also in the graph obtained from C_i by attaching a P_1 to v_i. Now consider the set $D' := D \backslash \{V(C_1) \cup \cdots \cup V(C_k)\}$. If D' does not contain isolated vertices it is a minimal total dominating set in $T - V(C_1) - \cdots - V(C_k)$. Otherwise D' contains exactly one vertex y from $N(x) \backslash \{v\}$, $\{x, v\} \cap D' = \emptyset$ and $D' \backslash \{y\}$ is a minimal total dominating set in the component of $T - x$ containing v. By considering minimal total dominating sets not containing v in this component it can be observed that they all have cardinality $\gamma_t(T - C_1 - \cdots - C_k) - 1$ and thus we obtain that G is TWD. $\square$

Corollary 1 *Let T be a TWD tree reduced by Lemma 2, 3, 4, 8, 9 and 10. For any leaf v in T we have that either $T \in \{P_2, P_4, P_3 \circ K_1\}$ or T has the structure as one of the graphs from Figure 3 where v is a leaf in $T - T'$.*

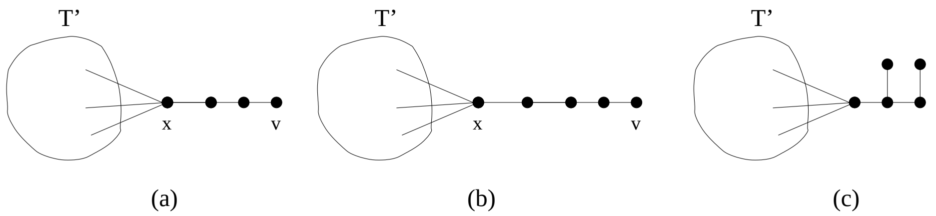

Figure 3: Illustration of structure near leaf. $\deg(x) \geq 3$ in (a) and (b) .

Remark. Let T be a reduced TWD tree. The only such trees with ≤ 4 vertices are P_2 and P_4. Let v be a leaf in T. Assume v is attached to a stem x_2 with $\deg_T(x_2) \geq 3$. Then T by Lemma 5 contains another stem at distance at most 3 from x_2. By Lemma 4 it cannot be at distance 3 from x_2, so the distance is 1 or 2. If the other stem is at distance 2 from x_2 then $T = P_3 \circ K_1$ by Lemma 6. So otherwise T has two adjacent stems, one of which is x_2 with $\deg_T(x_2) \geq 3$. But then we obtain from Lemma 3 that T has a subtree A_2 as described on Figure 1. Here, in Figure 3(c) we have $V(A_2) = \{x_1, x_2, x_3, v, y\}$ where v is a leaf with stem x_3, $\deg_T(x_3) = 2$; y is a leaf with stem x_2, $\deg_T(x_2) \geq 3$; and x_1 is the attachment vertex in A_2 to the rest of T, $\deg_T(x_1) \geq$2.

For the cases where v's stem v_1 has degree 2 we have by Lemma 9 that the nonleaf neighbour to v_1 has degree 2, so we obtain one of Figure 3(a) or Figure 3(b) both with $\deg(x) \geq 3$ because T has no attached P_5.

We shall later see that for trees with sufficiently large diameter the structure from a leaf as a refinement of Figure 3 can be described by Figure 7.

Lemma 11 *Let G be a graph with two A_2's attached at a vertex v and let H be the graph obtained by removing one of the attached A_2 graphs. Then G is TWD if and only if H is TWD.*

Proof. Let $vv_1v_2v_3v_4$ and $vu_1u_2u_3u_4$ be paths in G such that v_1 and u_1 are contained in different A_2 graphs attached at v. Since $\{v_2, v_3\}$ is a admissible set in G we obtain that the graph $H = G - N[\{v_2, v_3\}]$ is TWD if G is TWD. Conversely, let D be a minimal total dominating set D in G. Since v_1 and u_1 cannot both be in D we may assume that $v_1 \notin D$. Thus the A_2 attached at v containing v_1 has exactly two vertices v_2, v_3 from D. The set $D \backslash \{v_2, v_3\}$ is then a minimal total dominating set of $H = G - N[\{v_2, v_3\}]$. Thus if H is TWD then G must also be TWD. □

Lemma 12 *Let G be a graph with two A_3's attached at a nonstem v with deg$(v) \geq 3$. Then G is not TWD.*

Proof. Let $vv_1v_2v_3v_4v_5$ and $vu_1u_2u_3u_4u_5$ be paths in G such that v_1 and u_1 is contained in different A_2 graphs attached at v. Let $D := V(G)\backslash\{v, v_1, u_1\}$ then D is a total dominating set of G since v is not a stem and $\deg(v) \geq 3$. Let D' be a minimal total dominating set such that $D' \subset D$. Since $D'' := (D\backslash\{v_2, u_2\}) \cup \{v\}$ is a total dominating set and $|D''| < |D'|$ the graph cannot be TWD. □

Lemma 13 *Let H be a graph with a path $P : v_1v_2v_3$ such that deg$(v_1) \geq 2$, deg$(v_2) = 2$, deg$(v_3) = 1$. If G is the graph obtained from H by attaching an A_1 to each of v_1 and v_3 then G is TWD if and only if H is TWD.*

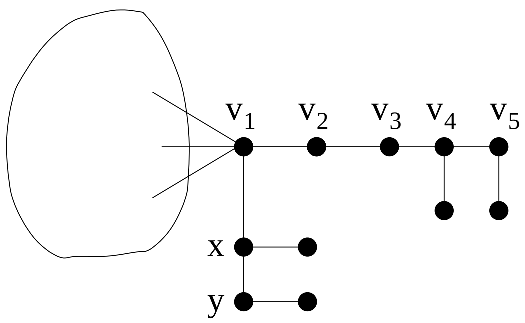

Figure 4: Illustration of G.

Proof. In the following notation is as illustrated in Figure 4. Let D be a minimal total dominating set in G. It follows that $v_2 \notin D$ and $(D \cap V(H)) \cup \{v_2\}$ is a minimal total dominating set for H and $|(D \cap V(H)) \cup \{v_2\}| = |D| - 3$. Thus G is TWD if H is TWD.

Conversely let D be a minimal total dominating set in H. Since v_2 is a stem in H we have $v_2 \in D$. Consider the set $D' := (D\backslash\{v_2\}) \cup \{x, y, v_4, v_5\}$. This set is a minimal total dominating set and $|D'| = |D| + 3$. Thus H is TWD if G is TWD. □

Lemma 14 *Let H be a graph and let $v \in V(H)$. Now let G be a graph obtained from H by attaching A_1 and a graph from $\mathcal{A}_4$ to v and let G' be the graph obtained from H by attaching P_2 to v. Then G is TWD if and only if G' is TWD.*

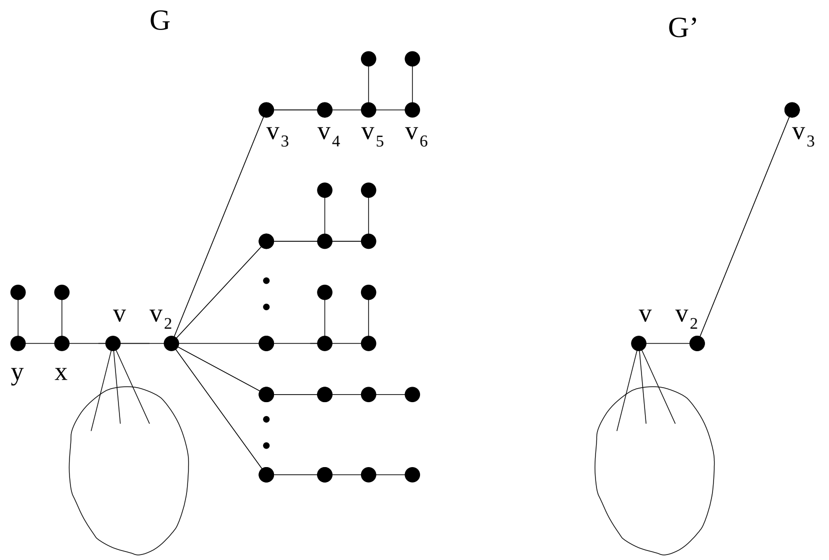

Figure 5: Illustration of G and G' from Lemma 14.

Proof. In the proof of this lemma we use the notation from Figure 5. First assume that G is TWD and let D be a minimal total dominating set of G'. Let A be the vertices at distance 2 and 3 from v_2 in $G - vv_2$ that are not leaves. Then $D' := (D \cup A \cup \{v_6, x, y\}) \setminus \{v_2\}$ is a minimal total dominating set in G. Since $|D'| - |D|$ does not depend on the choice of D the graph G' is TWD if G is TWD.

Next, let D be a minimal total dominating set in G. It can be observed that the cardinality of $D \cap (V(G) \setminus (V(G') \setminus \{v, v_2, v_3\}))$ does not depend on the choice of D. If $v \in D$ then $v_3 \notin D$. Thus if $D' := (D \cap V(G')) \cup \{v_2\}$ when $v \in D$ and $D' := (D \cap V(G')) \cup \{v_2, v_3\}$ when $v \notin D$ then $|D'| - |D|$ does not depend on D. Since D' is a minimal total dominating set in G' the graph G is TWD if G' is TWD. □

Lemma 15 *Let G be a graph with a nonstem $v \in V(G)$ adjacent to a stem x. Assume that a graph from $\mathcal{A}_5$ is attached at v. If H is the graph obtained by removing the attached $\mathcal{A}_5$-graph and attaching P_7 to v then G is TWD if and only if H is TWD.*

Proof. Assume that G and H are as described in the lemma. For both graphs let P denote a path $v_1 v_2 \ldots v_7$ contained in the graph attached at v such that vv_1 is an edge. Let G' be the component of $G - vv_1$ (or $H - vv_1$) containing v. Since G contains a admissible set A such that $G' = G - N[A]$ and H contains a admissible set B such that $G' = H - N[B]$ the graph G' is TWD if either G or H is TWD.

Assume H is TWD and let D be a minimal total dominating set in G. It can easily be observed that $|D \cap (V(G) \backslash V(G'))|$ does not depend on the choice of D. Further $D' := D \cap V(G')$ must be a minimal total dominating set of G'. If this is not the case then v is in D and its sole purpose is to dominate v_1, so that the sets $D' \cup \{v_3, v_4, v_6, v_7\}$ and $(D' \cup \{v_2, v_3, v_6, v_7\}) \backslash \{v\}$ are both minimal total dominating sets in H. But since H is TWD this is a contradiction. Thus it follows that G is TWD if H is TWD. By similar arguments it can be proven that H is TWD if G is TWD. $\square$

Lemma 16 *Let G be a graph. If a graph from $\mathcal{A}_4$ and a graph from $\mathcal{A}_5$ are attached at a vertex $v \in V(G)$ then G is TWD if and only if the graph H obtained by removing the attached graph from $\mathcal{A}_5$ is TWD.*

Proof. First assume that G is TWD. It can be observed that all graphs from $\mathcal{A}_5$ have a total dominating set D such that its attachment vertex is not contained in D and each vertex from D is adjacent to exactly one vertex from D. If D is such a set in the graph from $\mathcal{A}_5$ attached to v then D is an admissible set in G and thus $H = G - N[D]$ is TWD.

Now assume that H is TWD and let D be a minimal total dominating set in G. Let G' be the graph from $\mathcal{A}_5$ attached at v. By considering G' it can be observed that $|D \cap V(G')| = \gamma_t(G')$. Assume first that $D'' := D \cap H$ is not a minimal total dominating set in H. Let $v_1 \dots v_6$ be a path in the attached graph from $\mathcal{A}_4$ such that $vv_1 \in E(G)$.

If $v \in D$ then D'' must dominate H but the only neighbour to v contained in D is the vertex from $N[v] \cap V(G')$. Thus v must be an isolated vertex in $H[D'']$, $v_3 \in D''$ and $(D'' \backslash \{v_3\}) \cup \{v_1\}$ is a minimal total dominating set in H. If $v \notin D$ then v is not dominated by D'' and $v_3 \in D$. Thus $(D'' \backslash \{v_3\}) \cup \{v_1\}$ is a minimal total dominating set in H. In all cases $|D''| = \gamma_t(H)$ and we obtain that G is TWD if H is TWD. $\square$

Lemma 17 *Let G be a graph with the structure illustrated in Figure 6 and assume $N[x]$ does not contain any stems and all vertices at distance two from x in $G - vx$ is adjacent to a stem. Let H be the component of $G - xv$ containing x. Then G is TWD if and only if H is TWD and x is special in H.*

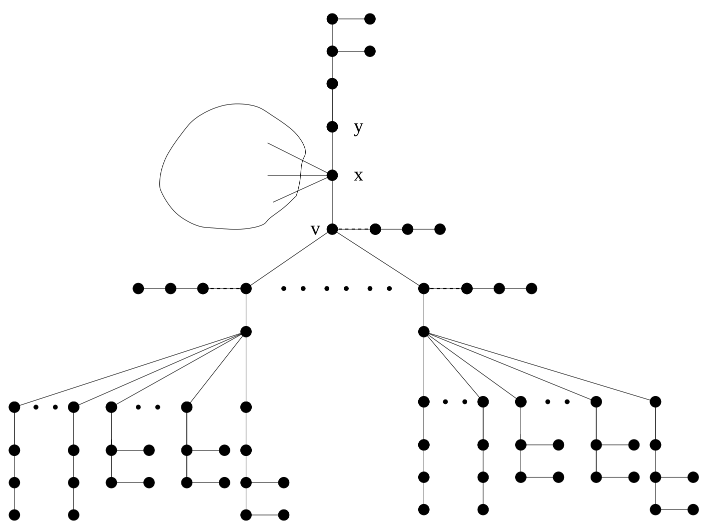

Figure 6: Illustration of G.

Proof. First assume that G is TWD. Let G' be the union of all components in $G - v$ not containing x. By considering G' it can be seen that it has minimal total dominating sets A and B such that $N[A] \cap v \neq \emptyset$, $N[B] \cap v = \emptyset$, $|A| = |B|$ and for $X \in \{A, B\}$ each vertex from X is adjacent to exactly one vertex from X. Each of the sets A and B are admissible in G. Thus $H = G - N[A]$ is TWD and $G - N[B]$ is TWD proving that x is special in H.

Now assume that H is TWD and let D be a minimal total dominating set of G. Let G' be the union of all the graphs from $\mathcal{A}_5$ attached at v (not containing x) and let $G'' := G - V(G')$. By considering G' it can be observed that $|D \cap V(G')| = \gamma_t(G')$. In the following we prove that $|D \cap V(G'')|$ does not depend on the choice of D. If $v \notin D$ then let $D' := D \cap V(H)$ and otherwise let $D' := ((D \cap V(H)) \backslash \{v\}) \cup \{y\}$. Since $N[x]$ does not contain any stems and all vertices at distance two from x in H is adjacent to a stem the set D' must be a minimal total dominating set for H or the graph obtained by attaching a P_1 to x in H. Since we assume that x is special in H we have $|D'| = \gamma_t(H)$. Thus $|D \cap V(G'')|$ does not depend on the choice of D since $|D \cap V(G'')| = |D'|$ if no P_3 is attached at v and $|D \cap V(G'')| = |D'| + 2$ if a P_3 is attached at v. Thus the graph G is TWD if H is TWD. □

4 Main Result

In this we section consider TWD trees that cannot be reduced by any of the decomposition/composition rules. Such a tree is called *reduced* and the following theorem proves that $\{P_2, P_4, P_3 \circ K_1\}$ is the family of reduced trees.

The idea of the proof will be to traverse a diametrical path from an end towards its center and examine which subtrees it can, or rather cannot, have attached.

Theorem 1 *Let T be a TWD tree. Then T is reduced if and only if $T \in \{P_2, P_4, P_3 \circ K_1\}$.*

Proof. Assume that T is a reduced TWD and $T \notin \{P_2, P_4, P_3 \circ K_1\}$. In the following we consider a path $P : x_1 \dots x_k$ in T such that

1. x_1 is a leaf.
2. Any path $x_k x_{k-1} u_1 u_2 \dots u_l$ in T has length at most $k-1$.
3. If C is the center-vertices in T and P' is a path between C and x_1, then $V(P) \subseteq V(C) \cup V(P')$.
4. No path $x_1 \dots x_k x_{k+1}$ satisfy conditions 1-3.

In the following we only say that a graph H with attachment vertex a is attached at x_i if a longest path $x_i a u_1 \dots u_l$ where $\{a, u_1, \dots, u_l\} \subseteq V(H)$ has length at most $i-1$ and $a \neq x_{i-1}$.

The only vertex from P to which a P_1 can be attached is x_3. Since T cannot be reduced it follows from Corollary 1 that a P_1 is not attached at x_i for $i \geq 5$, because x_i should then have degree two or be adjacent to a stem of degree two as on Figure 3(c), and that is not the case. Since T cannot be reduced Lemma 2 implies that a P_1 is not attached at x_2. If a P_1 is attached at x_4 then $T \cong P_3 \circ K_1$ by Lemma 6. No vertex of P can have a P_2 attached, since it follows from Corollary 1 that a P_2 cannot be attached at x_i for $i \geq 5$ because x_i should then either have degree two or have a leaf attached, neither of which can occur. Since T is reduced Lemma 6 implies that a P_2 is not attached at x_3 and Lemma 4 implies that a P_2 is not attached at x_4.

Now consider the vertex x_4. Since T is reduced only a P_3 or the graph A_1 can be attached at x_4. By Lemma 4 the graph A_1 cannot be attached at x_4. If x_3 is a stem then it follows from Lemma 4 that a P_3 is not attached at x_4 and if x_3 is not a stem it follows from Lemma 7 that a P_3 cannot be attached at x_4. Thus it can be assumed that $\deg(x_4) = 2$.

Consider the graphs that can be attached at x_5. Since T is reduced only the graphs P_3, P_4, A_1 and A_2 can be attached at x_5. Since T is reduced Lemma 4 implies that x_5 cannot be adjacent to a stem if x_3 is a stem, and Lemma 8 implies that x_5 cannot be adjacent to a stem if x_3 is not a stem. Thus x_5 is not adjacent to a stem, and therefore A_1 is not attached at x_5. If a $P_3 : a v_1 v_2$ is attached at x_5 and $a x_5 \in E(T)$ then $\{v_1, v_2, x_2, x_3, x_4\}$ is contained in a minimal total dominating set D and $D' := (D \backslash \{x_4, v_2\}) \cup \{a\}$ is a total dominating set. Since $|D'| < |D|$ we obtain a contradiction since T is TWD, so no P_3 is attached at x_5. Further, Lemma 11 implies that A_2 is not attached at x_5 when x_3 is a stem. So we may assume that $\deg(x_5)$=$\deg(x_4)$=2.

Since T is reduced by Lemma 10 we may assume that x_3 is a stem and $\deg(x_4)$=$\deg(x_5)$=2. I.e., for any leaf in T which is the origin of a sufficiently long path the structure near P must be as illustrated in Figure 7 when $k \geq 6$.

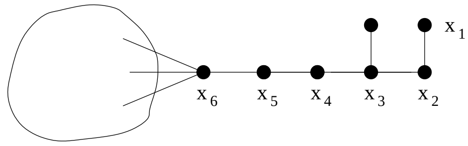

Figure 7: Illustration of structure in T when $k \geq 6$.

Consider the graphs attached at x_6. Since T is reduced Lemma 13 implies that A_1 is not attached at x_6 and from Lemma 12 it follows that A_3 is not attached at x_6. Thus only P_3, P_4 and A_2 can be attached at x_6. If a $P_3 : av_1v_2$ is attached at x_6 and $ax_6 \in E(T)$ then Lemma 4 implies that x_6 is not adjacent to a stem and therefore $\{v_1, v_2, x_4, x_5\}$ is contained in a minimal total dominating set D. But then $D' := (D \backslash \{x_5, v_2\}) \cup \{a\}$ is a total dominating set and $|D'| < |D|$. Since T is TWD a contradiction is obtained if a P_3 is attached at x_6. So only P_4's and A_2's can be attached at x_6; and, in fact, by Lemma 11 at most one A_2 can be attached to x_6.

The component of $T - x_7$ containing $x_1 \cdots x_6$ is a graph in $\mathcal{A}_4$ with attachment vertex x_6. Thus Lemma 14 implies that A_1 cannot be attached at x_7 because otherwise T could be reduced. If a $P_4 : av_1v_2v_3$ is attached at x_7 then Lemma 8 implies x_7 is not adjacent to a stem and thus $\{a, v_2, v_3, x_7, x_4\}$ is a subset of a minimal total dominating set D. But then $D' := (D \backslash \{a, x_4\}) \cup \{x_6\}$ is a total dominating set that satisfies $|D'| < |D|$. By using similar arguments we obtain that A_2 cannot be attached at x_7.

Assume that A_3 or a graph from $\mathcal{A}_4$ is attached at x_7 and let T' denote this subgraph. By considering T it follows that $\{x_5, x_4, x_3, x_2\}$ is contained in a minimal total dominating set D, but $D' := (D \backslash \{x_4, x_5, y\}) \cup \{x_6, x_7\}$ is a smaller total dominating set for a vertex $y \in D \cap V(T')$ at distance two from x_7. Therefore, as T is TWD it can be assumed that only P_3's can be attached at x_7 and Lemma 7 implies that at most one P_3 is attached at x_7. But in fact no P_3 can be attached to x_7, because starting from a leaf x_1 we found in Figure 7 that $\deg x_4 = 2$. We note that the component of $T - x_8$ containing $x_1 \cdots x_7$ is a graph in $\mathcal{A}_5$ with attachment vertex x_7.

Consider any graph G' obtained by attaching a graph from $\mathcal{A}_5$ to the center of a star $K_{1,t}$ for some $t \geq 0$. Since G' is not TWD it follows there must be a component C of $T - x_8$, not containing the vertex x_1, such that C does not contain an admissible set D that satisfies $y \notin D$ and either $y \in N[D]$ or $N(y) \cap C \subseteq N[D]$ where y is the vertex from C adjacent to x_8. If this were not the case then the union of such admissible sets in all components of $T - x_8$ not containing x_1 would be an admissible set D such that $T - N[D]$ had a component isomorphic to a graph like G'. Since T is TWD it follows from Observation 1 and Lemma 1 that this is a contradiction.

Let C be such a component and consider a path yv_1v_2 in C such that $yx_8 \in E(T)$. Since $\{v_1, v_2\}$ cannot be an admissible set $G - N[\{v_1, v_2\}]$ must have an isolated vertex. Since T is reduced Lemma 15 implies y is not a stem and therefore v_1 or v_2 must be adjacent to a stem in $C - \{y, v_1, v_2\}$. If v_1 is adjacent to such a stem u then since T is reduced Corollary 1 shows that u must be adjacent to a stem z but then $\{v_1, u, z\}$ is contained in an admissible set in $C\backslash\{y\}$ which is a contradiction to the choice of C. Thus v_2 must be adjacent to a stem $u \neq v_1$. Thus Corollary 1 and Lemma 7 imply that T must have the structure illustrated in Figure 8. Let the set A be as illustrated in Figure 8. Further, let D be all vertices from $V(C)\backslash\{y\}$ at distance at least three from A. Now $A \cup D$ is a total dominating set of $C - y$ and thus there is a minimal total dominating set D' of $C - y$ such that $D' \subseteq A \cup D$. If no A_3-graph is attached at y then D' is an admissible set in T contradicting the choice of C. Thus we may assume that an A_3 is attached at y.

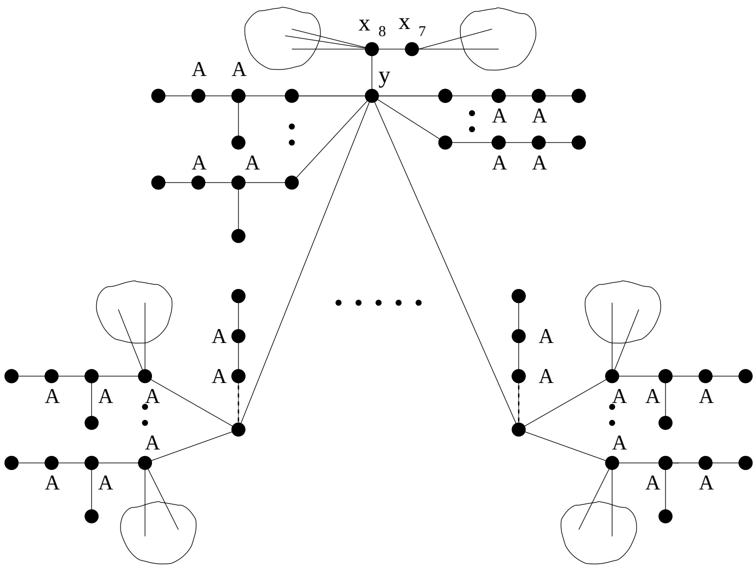

Figure 8: Illustration of T.

Now consider the graphs attached at x_8. Only P_3, P_4, A_1, A_2, A_3 or a graph from $\mathcal{A}_4 \cup \mathcal{A}_5$ can be attached at x_8. By Lemma 16 a graph from $\mathcal{A}_4$ is not attached at x_8 and by Lemma 15 A_1 cannot be attached at x_8. If one of the graphs P_4, A_2, A_3 is attached at x_8 and a denotes the attachment-vertex from such a graph, then since x_8 is not adjacent to a stem then $\{a, x_2, x_3, x_4, x_5, x_8\}$ is contained in a minimal total dominating set D. Further, $D' := (D\backslash\{a, x_5\}) \cup \{x_7\}$ is a total dominating set. Since T is TWD none of these graphs can be attached at x_8. Thus T has the structure as the graph from Lemma 17 and since T is reduced this is a contradiction.

Now let P be a subpath of a diametrical path in T. By the above arguments $k \leq 7$ and $k \geq 3$ since $T \not\cong P_2$. Thus there must be a graph attached at x_k and the information about graphs attached at x_3, x_4, x_5, x_6 and x_7 implies that $T \in \mathcal{A}_4$.

This proves the statement since all the graphs from $\{P_2, P_4, P_3 \circ K_1\}$ are reduced graphs and Lemma 4, Lemma 10, Lemma 11 and Lemma 14 imply that no graph from $\mathcal{A}_4$ is reduced. $\square$

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