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Analysis of stress updates in the material-point method

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Summary The material-point method (MPM) is a new numerical method for analysis of large strain engineering problems. The MPM applies a dual formulation, where the state of the problem (mass, stress, strain, velocity etc.) is tracked using a finite set of material points while the governing equations are solved on a background computational grid. Several references state, that one of the main advantages of the material-point method is the easy application of complicated material behaviour as the constitutive response is updated individually for each material point. However, as discussed here, the MPM way updating and integrating stresses in time is problematic. This is discussed using an example of the dynamical collapse of a soil column.

Introduction

The material-point point method is a new computational method for modelling large-stain dynamical engineering problems. The material-point method was originally developed by Sulsky and coworkers [1, 2]. An important extension, known as the generalized material point method (GIMP), is presented by Bardenhagen and Kober [3].

Theory

In the MPM a continuum problem is discretized by representing the domain of the problem, Ω, by a finite set $p = 1, ..N_p$ material points. Each material point is assigned a mass, stress, velocity and density, denoted $m_p, \boldsymbol{\sigma}_p, \mathbf{v}_p, \rho_p$, respectively. The domain associated with the material point, p, is denoted Ω_p and the volume of this domain is denoted V_p. In addition, a finite set grid nodes $i = 1, ..N_n$, are defined where the governing equations are solved. In the original MPM formulation the material points are represented using the Dirac delta function when forming the governing equations on the grid. Hence, the interpolation between the material points and the mesh is governed by the nodal shape functions $N_i(\mathbf{x})$. In the GIMP, further a particle characteristic function, $\chi_p(\mathbf{x})$, is defined for each material point.

The governing equation is the balance of momentum

$$\rho \frac{d\mathbf{v}}{dt} = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{b}, \tag{1}$$

where $\rho = \rho(\mathbf{x}, t)$ is the current density, $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ is the spatial velocity, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, t)$ is the Cauchy stress tensor and $\mathbf{b} = \mathbf{b}(\mathbf{x}, t)$ is the specific body force. Utilizing the GIMP formulation, the discrete equation becomes

$$m_i \frac{d\mathbf{v}_i}{dt} = \mathbf{f}_i^{int} + \mathbf{f}_i^{ext}, \tag{2}$$

where $m_i \frac{d\mathbf{v}_i}{dt} = \sum_p m_p \frac{d\mathbf{v}_p}{dt} \bar{N}_{ip}$ is the nodal momentum rate of change,

$$\mathbf{f}_i^{int} = -\sum_p \boldsymbol{\sigma}_p V_p \frac{\partial \bar{N}_{ip}}{\partial \mathbf{x}} \quad \text{and} \quad \mathbf{f}_i^{ext} = \int_{\partial \Omega_\tau} N_i \boldsymbol{\tau} dS + \sum_p m_p \mathbf{b}_p \bar{N}_{ip} \tag{3}$$

is the internal force and external forces, respectively.

The weighting and the gradient weighting function are defined by

$$\bar{N}_{ip} = \frac{1}{V_p} \int_{\Omega_p \bigcup \Omega} N_i \chi_p dV \quad \text{and} \quad \frac{\partial \bar{N}_{ip}}{\partial \mathbf{x}} = \frac{1}{V_p} \int_{\Omega_p \bigcup \Omega} \frac{\partial N_i}{\partial \mathbf{x}} \chi_p dV. \tag{4}$$

For comparison, if the particle function is defined by $\chi_p(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_p)$ the original MPM formulation is retrieved. Another comparison relevant for this discussion is that the MPM formulation is similar to a finite element formulation, where the Gauss points are replaced by material points.

Time-integration scheme

Equation 2 is the basic equation that using appropriate boundary conditions is integrated in time in order to find the evolution of the state variables, that are defined at the material points. This constitutes to updating the position, velocity, stress and density at the material points. The typical procedure applied here and in the references [1, 2] is to use explicit time integration. Hence, the time domain is divided into a finite set of timesteps and the size of the time-steps has to satisfy the Courant criterion.

The update of position and velocity is performed by

$$\mathbf{x}_p^{k+1} = \mathbf{x}_p^k + \Delta t \sum_{i=1}^{N_n} \bar{N}_{ip} \frac{m_i \mathbf{v}_i + \Delta t (\mathbf{f}_i^{int,k} + \mathbf{f}_i^{ext,k})}{m_i} \tag{5}$$

and

$$\mathbf{v}_p^{k+1} = \mathbf{v}_p^k + \Delta t \sum_{i=1}^{N_n} \bar{N}_{ip} \frac{(\mathbf{f}_i^{int,k} + \mathbf{f}_i^{ext,k})}{m_i}. \tag{6}$$

where the nodal mass and velocity are determined by

$$m_i^k = \sum_{p=1}^{N_p} m_p^k \bar{N}_{ip} \quad \text{and} \quad \mathbf{v}_i^k = \frac{\sum_{p=1}^{N_p} \mathbf{v}_p^k m_p \bar{N}_{ip}}{m_p}, \tag{7}$$

respectively.

Similarly, the strain increments at the material points are found by

$$\Delta \boldsymbol{\varepsilon}_p = \frac{\Delta t}{2} \sum_{i=1}^{N_n} \left(\nabla \bar{N}_{ip} \mathbf{v}_i + (\nabla \bar{N}_{ip} \mathbf{v}_i)^T \right). \tag{8}$$

Then the stress increments are found individually at each material point.

Problems regarding stresses

When considered at a specific time-step, the material point method is similar to a finite element method for the grid, but where the integration in now performed using the set of material points instead of using Gauss points. This leads to an inaccurate integration. In most implementations of MPM, the grid nodes are fixed spatially, while the material points will move in a dynamical problem. Hence, the material points move relatively to the grid between the different time steps, which leads to a further loss of precision. Thus although initially optionally discretized, the integration involved in assembling Eq. 3 will not be complete. An effect of the relative motion is the co called grid-crossing error. Typically, the gradients of nodal shape-functions are discontinuous across element boundaries. Hence, as a material point that changes cell between to time-steps between two time-steps will contribute significantly differently to the internal force, when it moves relatively to the grid. The definition a particle characteristic function in the GIMP somewhat limit the grid-crossing error. As a result of this the material point will experience an unphysical acceleration according to Eq. 6 and an unphysical strain increment according to Eq. 8. However, due to the nature of the explicit time integration these effects will be smoothed spatially over time, and as has been shown realistic results can still be obtained.

However, for general problems the incremental stress update defined at the individual material points leads to unrealistic stresses. This is due to the sum over stress increments determined from strain increments which may generally be erroneous for the individual material points. The effect of this is shown in the following numerical example.

Example: Collapsing soil column

A rectangular soil column placed on a frictional surface is considered. The column is 8 metres high and 4 metres long and plane strain conditions are considered. The soil column is unsupported along the vertical boundaries. Further, the stresses are assumed to increase linearly with the distance from the top. As these stresses cannot be sustained on the vertical sides, a plastic collapse will occur. An elasto-plastic material model based on the Mohr-Coulomb yield criterion, using an explicit return mapping scheme as described by Clausen et al. [4] is applied to enforce the yield criterion.

The soil is described using the following set of material properties:

$$E = 20\mathrm{MPa}, \quad \nu = 0.42, \quad \rho_0 = 10^3\mathrm{kg/m}^3, \quad c = 1\mathrm{kPa} \quad , \phi = 42^o \quad \text{and} \quad \psi = 0^o. \tag{9}$$

A frictional coefficient $\mu = 0.6$ is prescribed at the lower boundary.

An initial K_0-stress state is specified with the vertical and horizontal normal stress given by

$$\sigma_{yy}^0 = -dg\rho_0 \quad \text{and} \quad \sigma_{xx}^0 = \sigma_{zz}^0 = -dg\rho_0 K_0, \tag{10}$$

where $g = 9.8m/s^2$ is the gravity and d is the distance from the top soil surface. where the earth pressure coefficient is given by $K_0 = \nu/(1-\nu)$. Finally, $\sigma_{xy}^0 = 0$ is prescribed for all material points.

In order to visualize the collapse of the soil-column, each material is assigned a regular domain in the initial configuration. Further, a deformation tensor is prescribed for each material point with $\mathbf{F}_p^0 = \mathbf{I}$ at the start. The deformation tensor is integrated in time using the nodal velocities to track the deformation of the initially rectangular domains. The dynamic simulation is performed with a time step of $\Delta t = 0.001\mathrm{s}$. The simulation is performed until the soil has reached a state of vanishing velocities. For the present model the time of the collapse is $t = 2.5\mathrm{s}$. The initial configuration consists of 1800 material points. An adaptive scheme for splitting the material points in case of localized deformation is employed. The final configuration consists of 9402 material points.

In the MPM, the material points may be at arbitrary locations of the elements defined by the set of grid-nodes. The individual stresses may be unrealistic. Hence, combined with the effect of grid crossing this completely degenerates the stress at an individual material point. In order to better understand the problems, a new way of visualizing the stresses are introduced. Firstly, grid-node stress tensors are defined by

$$\boldsymbol{\sigma}_i = \sum_{p=1}^{N_p} \frac{\boldsymbol{\sigma}_p \Phi_{ip} m_p}{m_i}, \tag{11}$$

where $\boldsymbol{\sigma}_i$ is the stress tensor, associated with grid node i, $\boldsymbol{\sigma}_p$ is the stress of material point p, Φ_{ip} is the interpolation function while m_i and m_p is the nodal and material point mass, respectively. Using the nodal stresses, a smoothed material point stress tensor is defined as

$$\boldsymbol{\sigma}_p^{smooth} = \sum_{i=1}^{N_n} \boldsymbol{\sigma}_i \Phi_{ip}, \tag{12}$$

Figure 1 shows the vertical normal stress during the collapse of the soil column. The left side of the figure shows the stresses at the individual material points, while the right side shows the stresses calculated by Eq. (12) in order to provide a better visualization.

The first thing to observe is that the deformation occur in a realistic fashion. As the pressure cannot be obtained, the soil column collapses until, it has reached a state, where its satisfies the yield condition (in a global sense) and the kinetic energy has dissipated due to the bottom friction and plastic dissipation. In the final configuration, it is still possible to observe the initial corners due to the small amount of cohesion present. As observed, the vertical normal stress during and at the end of the collapse varies in a very erratic fashion. This leads to a principle question: Can we trust the simulation, when the stresses at the individual material points are so unrealistic?

As seen, the deformation occurs as physically expected although the stresses are completely erroneous at the individual material points. From the mentioned analogy to the finite element method the material points plays the role as integration points when solving the governing equation of motion. Further, from finite element analysis it is common knowledge that special care need to be made regarding when interpreting stresses, as stresses as may only be realistic at certain locations within an element. As observed, the mapping via the grid nodes determines a stress field, that is physically realistic. Hence, in terms of the grid-nodes, where the equations are solved, the stress field is realistic.

Concluding remarks

The material-point method is a new promising numerical method for large strain continuum mechanic problems. As illustrated it is successfully able to capture in a realistic fashion problems involving very large deformations. However, this note provides an illustration of problematic issue for the method. This pertains to the fact that stress fields varies in a unrealistic fashion at the individual material points. As illustrated the overall solution may still be realistic, as the stresses are realistic at the grid nodes and realistic displacement and velocity fields are observed. However, in more complex problems the unrealistic stresses pose real problems as localized effects may be difficult to capture. Hopefully the presented results eventually can lead to a better algorithms for handling stresses within the MPM.

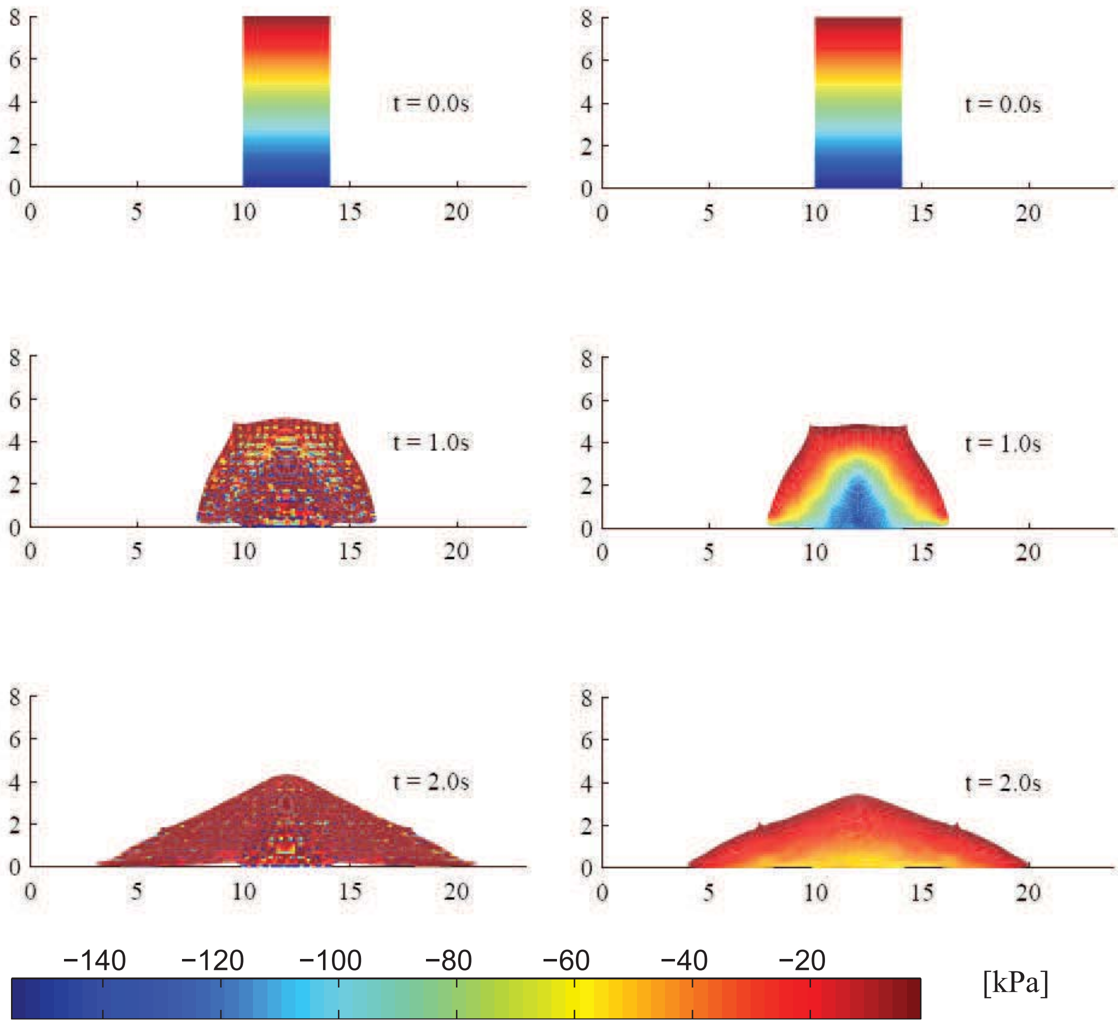

Figure 1: Vertical normal stresses during the collapse of the soil column. Left: The stresses at the individual material points. Right: A smoothing using Eq. (12) is introduced for a better visualization.

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