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by

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On k-geodetic digraphs with excess one

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Abstract

A k-geodetic digraph G is a digraph in which, for every pair of vertices u and v (not necessarily distinct), there is at most one walk of length $\leq k$ from u to v. If the diameter of G is k, we say that G is strongly geodetic. Let $N(d,k)$ be the smallest possible order for a k-geodetic digraph of minimum out-degree d, then $N(d,k) \geq 1 + d + d^2 + \ldots + d^k = M(d,k)$, where $M(d,k)$ is the Moore bound obtained if and only if G is strongly geodetic. Thus strongly geodetic digraphs only exist for $d = 1$ or $k = 1$, hence for $d, k \geq 2$ we wish to determine if $N(d,k) = M(d,k) + 1$ is possible. A k-geodetic digraph with minimum out-degree d and order $M(d,k) + 1$ is denoted as a $(d,k,1)$-digraph or said to have excess 1. In this paper we will prove that if a $(d,k,1)$-digraph is always out-regular and that if it is not in-regular, then it must have 2 vertices of in-degree less than d, d vertices of in-degree $d+1$ and the remaining vertices will have in-degree d. Furthermore we will prove there exist no $(2,2,1)$-digraphs and no diregular $(2,k,1)$-digraphs for $k \geq 3$.

1. Introduction

A digraph which satisfies that for any two vertices u, v in G, there is at most one walk of length at most k from u to v, is called a *k-geodetic digraph.* If the diameter of a k-geodetic digraph G is k, we say that G is *strongly geodetic.*

Let G be a k-geodetic digraph with minimum out-degree d. What is then the smallest possible order, $N(d,k)$, of such a G? Letting n_i be the number of vertices in distance i from a vertex v for $i = 0, 1, 2, \ldots$, and realizing that

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$n_i \geq d^i$, we see that a lower bound is given as

$$N(d,k) \geq \sum_{i=0}^{k} n_i \geq \sum_{i=0}^{k} d^i = M(d,k). \quad (1)$$

The right hand side of Eq. (1) is the so called *Moore bound* for digraphs. The Moore bound is an upper theoretical bound for the so called *the degree/diameter problem*, which is the problem of finding the largest possible order of a digraph with maximum out-degree d and diameter k. A digraph with order $M(d,k)$, maximum out-degree d and diameter k is called a *Moore digraph.* If a k-geodetic digraph has $M(d,k)$ vertices, then it must be strongly geodetic, and therefore a Moore digraph. However, the only Moore digraphs are $(k+1)$-cycles ($d=1$) and complete digraphs, K_{d+1} ($k=1$), see [1] or [2], thus for $d \geq 2$ and $k \geq 2$ we are interested in knowing if the order for a k-geodetic digraph with minimum out-degree d could be $M(d,k)+1$. We say that a k-geodetic digraph G of minimum out-degree d and order $M(d,k)+1$ is a $(d,k,1)$*-digraph* or that it has *excess* one.

Notice that $(k+2)$-cycles and $(k+1)$-cycles with a vertex having an arc to a vertex on the $(k+1)$-cycle are $(1,k,1)$-digraphs and that complete digraphs K_{d+2} with at most one arc from each vertex deleted are $(d,1,1)$-digraphs. In the remaining part of this paper we will thus assume $d \geq 2$ and $k \geq 2$.

In this paper we will specify some further properties of the $(d,k,1)$-digraphs, especially we will show that they have diameter $k+1$, and that if a $(d,k,1)$-digraph is not diregular, then it is out-regular and there will be exactly d vertices of in-degree $d+1$, two vertices of in-degree less than d and the remaining vertices will have in-degree d. In the last section we will show that there exist no $(2,2,1)$-digraphs and no diregular $(2,k,1)$-digraphs.

2. Results

Let an i-walk denote a walk of length i and a $\leq i$-walk denote a walk of length at most i. Furthermore, let $N_i^+(u)$ denote the multiset of all vertices which are end vertices in an i-walk starting in the vertex u, notice that $N_0^+(u) = \{u\}$ and $N_1^+(u) = N^+(u)$. Also let $T_i^+(u) = \cup_{j=0}^{i} N_j^+(u)$, thus it is the multiset of all vertices which are end vertices in a $\leq i$-walk. Notice that for k-geodetic digraphhs $N_i^+(u)$ and $T_i^+(u)$ are sets when $i \leq k$. Looking at $(d,k,1)$-digraphs, we will often depict all the $\leq (k+1)$-paths from some arbitrary vertex u, thus the vertices in the multiset $T_{k+1}^+(u)$.

The first important result is that a $(d,k,1)$-digraph G is in fact out-regular, as if we assume the contrary, that there is a vertex $u \in V(G)$ with $d^+(u) \geq d+1$, we get that

$$
\begin{aligned}
|V(G)| \geq |T_k^+(u)| & \\
&= 1 + (d+1) + (d+1)d + (d+1)d^2 + \ldots + (d+1)d^{k-1} \\
&= M(d,k) + M(d,k-1),
\end{aligned}
$$

a contradiction as $M(d,k-1) > 1$ for $k \geq 2$.

An immediate consequence of a $(d,k,1)$-digraph being out-regular, is that it has diameter $k+1$ which follows in following lemma.

Lemma 1. *Let G be a $(d,k,1)$-digraph, then*

- *for each vertex $u \in V(G)$ there exists exactly one vertex $o(u) \in V(G)$ such that $dist(u, o(u)) = k+1$,*
- *for any two vertices, $u, v \neq o(u)$ there is exactly one $\leq k$-path from u to v.*

Proof. As we know G is out-regular and the order is $M(d,k)+1$, the second statement follows. Let $u \in V(G)$ be any vertex and let $o(u)$ be the unique vertex not reachable with a $\leq k$-path from u, then we just need to prove $d^-(o(u)) > 0$. Assume the contrary, that $d^-(o(u)) = 0$, then $o(u) = o(v)$ for all $v \in V(G)\backslash\{o(u)\}$. But then $G\backslash\{o(u)\}$ will be a Moore digraph of degree $d \geq 2$ and diameter $k+2$, a contradiction. Hence $d^-(o(u)) > 0$ for all $u \in V(G)$ and thus $dist(u, o(u)) = k+1$. □

The unique vertex $o(u)$ with $dist(u, o(u)) = k+1$ will be called the *outlier* of u. So a $(d,k,1)$-digraph is out-regular of out-degree d and has diameter $k+1$. Showing that a $(d,k,1)$-digraph G is also in-regular is not as straightforward. We will prove that if it is not in-regular, then there are exactly two vertices of in-degree less than d, d vertices of in-degree $d+1$ and the remaining vertices are of in-degree d. Let $S' = \{v \in V(G) | d^-(v) > d\}$ and $S = \{v \in V(G) | d^-(v) < d\}$, then we get the following lemmas and theorem.

Lemma 2. *Let G be a $(d,k,1)$-digraph, then*

- *$|S'| \leq d$ and $d^-(v) = d+1$ for all $v \in S'$,*
- *$S' \subseteq N^+(o(u))$ for all $u \in V(G)$.*

Proof. Assume $u \in V(G)$ and $v \notin N^+(o(u))$, then as u must reach all in-neighbours of v in $\leq k$-paths, we must have $d^+(u) \geq d^-(v)$. If not, then there will exist an out-neighbour u' of u which has two $\leq k$-paths to v, a contradiction. Now, if $v \in N^+(o(u))$, then u must reach all in-neighbours of v, except $o(u)$, in a $\leq k$-path. Thus with the same arguments as before, we must have $d^+(u) \geq d^-(v) - 1$. Thus all vertices in S' must have in-degree $d + 1$ and both statements follows, as $|N^+(o(u))| = d$. □

Lemma 3. *If $S' \neq \emptyset$, then $|S'| = d$.*

Proof. As a $(d, k, 1)$-digraph is out-regular, its average in-degree must be d and thus $\sum_{v \in S'}(d^-(v) - d) = \sum_{v \in S}(d - d^-(v)) = |S'|$. Now let $v \in S'$, then we know $|N^-(v)| = |N_1^-(v)| = d + 1$ and $|N_t^-(v)| \geq d|N_{t-1}^-(v)| - \epsilon_t$ for $1 < t \leq k$, where $\epsilon_2 + \epsilon_3 + \ldots \epsilon_k \leq |S'|$. As all vertices in $T_k^-(v)$ are distinct, it implies that

$$|V(G)| \geq \sum_{i=0}^{k} |N_i^-(v)|. \tag{2}$$

Estimating the above sum, we get a safe lower bound by letting $\epsilon_2 = |S'|$ and $\epsilon_t = 0$ for all $3 \leq t \leq k$, thus

$$\begin{aligned}
|V(G)| &\geq 1 + |N^-(v)| + |N_2^-(v)| + |N_3^-(v)| + \ldots + |N_k^-(v)| \\
&\geq 1 + (d + 1) + ((d + 1)d - |S'|)(1 + d + \ldots + d^{k-2}) \\
&= 2 + d + d^2 + \ldots + d^k + (d - |S'|)(1 + d + \ldots d^{k-2}) \\
&= M(d, k) + 1 + (d - |S'|)M(d, k - 2).
\end{aligned}$$

But as G is a $(d, k, 1)$-digraph, we have $|V(G)| = M(d, k) + 1$, which together with the preceding inequality and Lemma 2 gives $|S'| = d$. □

A consequence of the above proof, is also that $S \subseteq N^-(v)$ for all $v \in S'$.

Theorem 1. *Let G be a $(d, k, 1)$-digraph. Then, if G is not diregular, we have $S = \{z, z'\}$ where $o(u) \in S$ for all $u \in V(G)$.*

Proof. Assume G is not diregular, thus we can assume $S' = \{u_1, u_2, \ldots, u_d\}$ where $d^-(u_i) = d+1$ and $o(u) \in N^-(u_j)$ for all $u \in V(G)$ and $j = 1, 2, \ldots, d$ according to Lemmas 2 and 3. Moreover, from the proof of Lemma 3 we see that $dist(v, u_i) \leq k$ for all $v \in G$ and $i = 1, 2, \ldots, d$.

Now let $N^-(u_1) = \{z_1, z_2, \ldots, z_{d+1}\}$ where $z_1 = o(u_1)$. Then $S' \cap T_{k-1}^-(z_1) = \emptyset$, as otherwise $(z_1, u_j, \ldots, z_1)$ will be a $\leq k$-cycle for some $j = 1, 2, \ldots, d$. Also, no two vertices u_i and u_j can belong to the same

$T^-_{k-1}(z_l)$ for $1 \leq l \leq d+1$, as if they did, $(z_1, u_i, \ldots, z_l)$ and $(z_1, u_j, \ldots, z_l)$ would be two distinct $\leq k$-paths. Thus we can assume $S' \cap T^-_{k-1}(z_l) = \{u_l\}$ for $2 \leq l \leq d$ and $dist(u_l, z_l) = k-1$, as otherwise there will be two $\leq k$-walks $(z_1, u_l, \ldots, z_l, u_1)$ and (z_1, u_1). As $(o(u), u_i)$ is an arc for all $u \in V(G)$ and $i = 1, 2, \ldots, d$ none of the vertices $z_2, z_3, \ldots, z_d$ can be the outlier of any vertex in G, as otherwise $(o(u) = z_l, u_l, \ldots, z_l)$ will be a k-cycle. Thus $o(u) \in \{z_1, z_{d+1}\}$ for all $u \in V(G)$.

Finally we wish to show that $S = \{z_1, z_{d+1}\}$. Assume the contrary, thus for some $2 \leq l \leq d$ we have $d^-(z_l) < d$ and $o(u) \neq z_l$ for all $u \in V(G)$, as $S \subseteq N^-(u_1)$. But then

$$\begin{aligned} |V(G)| &\leq 1 + (d-1)(1 + d + d^2 + \ldots + d^{k-1}) + 1 \\ &= M(d,k) - M(d,k-1) + 1 \\ &< M(d,k) + 1 \end{aligned}$$

as $dist(u_l, z_l) = k-1$ and $dist(u_j, z_l) \geq k$ for all $j \neq l$. Thus $S \subseteq \{z_1, z_{d+1}\}$ and as $\sum_{v \in S'}(d^-(v) - d) = d = \sum_{v \in S}(d - d^-(v))$ and $d^-(u) > 0$ for all $u \in V(G)$ the result follows.

□

If G is diregular, we get the following useful lemma.

Lemma 4. *Let G be a diregular $(d,k,1)$-digraph, then the mapping $o : V(G) \mapsto V(G)$ is an automorphism.*

Proof. Let A be the adjacency matrix of G, then due to the properties of G we get

$$I + A + A^2 + \ldots + A^k = J - P, \tag{3}$$

where J is the matrix with all entries equal to 1 and P is a permutation matrix with entry $P_{ij} = 1$ if $o(i) = j$ and $P_{ij} = 0$ otherwise.

Now, as we know G is diregular, we know that $AJ = JA$, and as the left hand side of Eq. (3) is a polynomial in A, we must also have $PA = AP$, thus o is an automorphism. □

Notice that if G is diregular there will be exactly d $(k+1)$-paths from a given vertex u to $o(u)$, as all u's out-neighbours must reach $o(u)$ in k-paths and if there were more than d $(k+1)$-paths, one of u's out-neighbours would have more than one $\leq k$-path to $o(u)$, a violation of the definition of $(d,k,1)$-digraphs.

3. $(2, k, 1)$-digraphs

In this section we will assume $d = 2$ and prove the none-existence of $(2, 2, 1)$-digraphs and diregular $(2, k, 1)$-digraphs.

Theorem 2. *There are no* $(2, 2, 1)$*-digraphs.*

Proof. Assume G is a $(2, 2, 1)$-digraph, then it has 8 vertices and we can depict the relationship between the vertices in $T_3^+(1)$ as in Fig. 1, where we can see $o(1) = 8$.

Assume G isn't diregular, then we know from Theorem 1 that $d^-(8) = 1$ and there exist another vertex $z \in V(G)$ with $d^-(z) = 1$ and $o(3) = o(6) = z$. Furthermore we know $N^+(8) = N^+(z) = \{u_1, u_2\}$ with $d^-(u_i) = 3$ for $i = 1, 2$. Notice that $6 \notin \{u_1, u_2\}$, as otherwise G would contain a 2-cycle, $(6, 8, 6)$. As the diameter of G is 3, we must have $dist(2, 6) = 2$ for 2 to reach 8 and thus $o(2) = 8$. Assume without loss of generality that $6 \in N^+(4)$. Then for 5 to reach 8 we must have $3 \in N^+(5)$, as $N^-(6) = \{3, 4\}$ and $4 \notin N^+(5)$, as otherwise $(2, 4)$ and $(2, 5, 4)$ will be two distinct ≤ 2-paths. The only vertices which 2 cannot reach are 1 and 7. If $7 \in N^+(5)$ we have $(5, 7)$ and $(5, 3, 7)$ as ≤ 2-paths, which is a contradiction. If instead $1 \in N^+(5)$ then we have the ≤ 2-paths $(5, 1, 3)$ and $(5, 3)$ another contradiction.

Now assume that G is diregular and recall that then o is an automorphism, thus we can assume $8 \in N^+(5)$ as $o(2) \neq 8$. Then we see that $o(2) \neq 6$, as otherwise there would be a 2-cycle $(6, 8, 6)$ as o is an automorphism, a contradiction. So there will be a $\leq$ 2-path from 2 to 6, but $6 \notin N^+(5)$ as otherwise there are two $\leq$ 2-paths from 5 to 8, namely $(5, 8)$ and $(5, 6, 8)$. Thus $6 \in N^+(4)$, and in the same manner we see that $5 \in N^+(7)$. Let u and v be the other out-neighbour of 4 and 5 respectively, and w and z the other out-neighbour of 6 and 7 respectively.

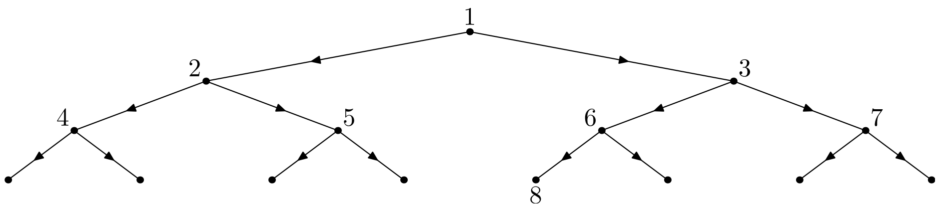

Figure 1: $T_3^+(1)$.

As 2 has to reach vertex $1, 3$ and 7 and at most one of them can be the outlier of 2, we must have $u \in \{1, 7\}$ and $v \in \{1, 3\}$, as if $u = 3$ there will exist two ≤ 2-paths from 4 to 6, namely $(4,6)$ and $(4,3,6)$ and if $v = 7$ we will get a 2-cycle, $(7,5,7)$. Similar we see $z \in \{1,4\}$ and $w \in \{1,2\}$.

Now assume $o(2) = 1$, hence $o(3) \neq 1$ and $(o(1), o(2)) = (8,1)$ is an arc. Then $u = 7$ and $v = 3$, and as o is an automorphism, we must have $z = 1$, as if $w = 1$ we will have the two ≤ 2-paths, $(6,1)$ and $(6,8,1)$. But then $(7,1,3)$ and $(7,5,3)$ are both 2-paths from 7 to 3, a contradiction.

Instead assume $o(2) = 3$, thus $u = 7$ and $v = 1$ and $(o(1), o(2)) = (8,3)$ is an arc. But then $(5,1,3)$ and $(5,8,3)$ are both 2-paths from 5 to 1. So we can safely assume $o(2) = 7$, thus $u = 1$ and $v = 3$, but then $(5,3,7)$ and $(5,8,7)$ are both 2-paths from 5 to 7, another contradiction.

□

Theorem 3. *No diregular $(2,k,1)$-digraph exists for $k \geq 2$.*

Proof. Due to Theorem 2 we can assume $k > 2$ and we label the vertices in $T^+_{k+1}(1)$ as in Fig. 2. First of all, notice that for all $u \in V(G)$ we obviously have $o(u) \notin T^+_k(u)$, so we must have $o(2) \in T^+_{k-1}(3) \cup \{1\}$. We also see that $o(2) \notin T^+_{k-2}(6)$, as otherwise there will be two $\leq k$-paths from 6 to $o(2)$, the one in $T^+_{k-2}(6)$ and $(6, 12, \ldots, 3 \cdot 2^{k-1}, 2^{k+1} = o(1), o(2))$, a contradiction.

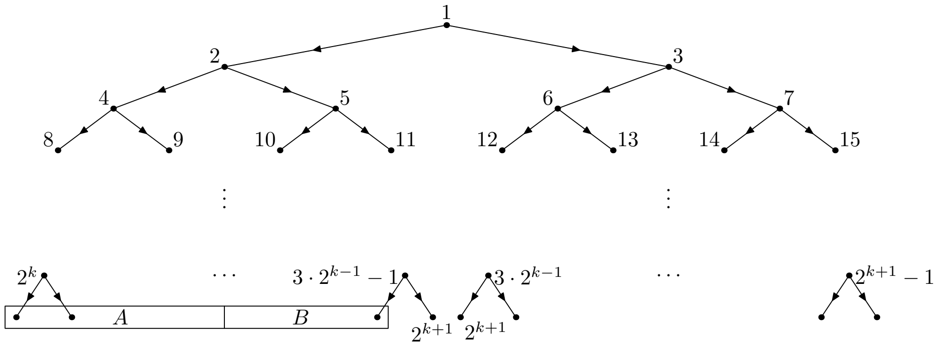

Figure 2: $T^+_{k+1}(1)$.

Now, let $A = N^+_{k-1}(4)$ and $B = N^+_{k-1}(5) \backslash \{2^{k+1}\}$, so $|A| = 2^{k-1}$ and $|B| = 2^{k-1} - 1$. Then we will look at how $(\{1\} \cup T^+_{k-1}(3)) \backslash o(2)$ is distributed on A and B. For any arc (u,v) in G, we must have that u and v will not both be in A and not both in B, as otherwise there would be two $\leq k$-paths from either 4 or 5 to v. We observe that $3 \cdot 2^{k-1} \notin B$, as otherwise there would be two $\leq k$-paths from 5 to 2^{k+1}, namely $(5, 11, \ldots 3 \cdot 2^{k-1} - 1, 2^{k+1})$ and $(5, \ldots, 3 \cdot 2^{k-1}, 2^{k+1})$. So we must have $3 \cdot 2^{k-1} \in A$, $3 \cdot 2^{k-2} \in B$, $3 \cdot 2^{k-3} \in A$, and so on, until we reach vertex 6. A consequence of this is that $N^+_{k-2}(6) \in A$, $N^+_{k-3}(6) \in B$, $N^+_{k-4}(6) \in A$ and so on, until we get either $6 \in A$ if k is even or $6 \in B$ if k is odd.

Let $a = |A \cap T^+_{k-2}(6)|$ and $b = |B \cap T^+_{k-2}(6)|$, so $a + b = 2^{k-1} - 1$. Now, if k is even we let

$$a_e = a = \sum_{i=0}^{\frac{k}{2}-1} 2^{2i} = -\frac{1}{3} + \frac{2}{3} \cdot 2^{k-1}$$

and

$$b_e = b = \sum_{i=0}^{\frac{k}{2}-2} 2^{2i+1} = -\frac{2}{3} + \frac{1}{3} \cdot 2^{k-1}.$$

Similarly, if k is odd we let

$$a_o = a = \sum_{i=0}^{\frac{k-3}{2}} 2^{2i+1} = -\frac{2}{3} + \frac{2}{3} \cdot 2^{k-1}$$

and

$$b_o = b = \sum_{i=0}^{\frac{k-3}{2}} 2^{2i} = -\frac{1}{3} + \frac{1}{3} \cdot 2^{k-1} = \frac{1}{2} a_o.$$

We start by assuming that $o(2) = 1$, then if k is even we see that vertex 3 must be in B, so $7 \in A$, $\{14, 15\} \subseteq B$,..., $N^+_{k-2}(7) \subseteq A$. Thus

$$|A| = 2 \cdot a_e = 2\left(-\frac{1}{3} + \frac{2}{3} \cdot 2^{k-1}\right) > 2^{k-1}$$

as $k > 2$, a contradiction. If k is odd, we see that vertex 3 must be in A, so $7 \in B$, $\{14, 15\} \subseteq A$,..., $N^+_{k-2}(7) \subseteq A$, thus

$$|A| = 2a_o + 1 = 2\left(-\frac{2}{3} + \frac{2}{3} \cdot 2^{k-1}\right) + 1 > 2^{k-1}$$

as $k > 2$, yet a contradiction. So we know due to symmetry that $1 \notin \{o(2), o(3)\}$.

Now, assume that $o(2) \neq 3$. Then we know the distribution of all the vertices in $T^+_{k-1}(3) \cup \{1\}$ except for those in $T^+_i(o(2))$, where i is given by $dist(3, o(2)) = k-1-i$. Assume $i = 0$, thus $o(2) \in N^+_{k-2}(7)$, or that $N^+(o(2))$

is in the same set (A or B) as $N^+_{k-1-i}(6)$, then we see that $|A| \geq 2a > 2^{k-1}$, a contradiction. So we can assume there exist vertices u and v, such that $N^+(o(2)) = \{u, v\} \in T^+_{k-2}(7)$ and that not both u and v are in the same set (A or B) as $N^+_{k-1-i}(6)$.

Let for i even c_e denote the number of vertices in every second layer of $T^+_i(o(2))$ such that $N^+_i(o(2))$ is not one of those layers, then

$$c_e = \sum_{j=0}^{\frac{i}{2}-1} |N^+_{2j+1}(o(2))| = 2(1 + 2^2 + \ldots + 2^{i-2}) = \frac{2}{3} \cdot 2^i - \frac{2}{3}.$$

Let d_e denote the number of vertices in the remaining layers, thus

$$d_e = \sum_{j=0}^{\frac{i}{2}-1} |N^+_{2j+2}(o(2))| = 2c_e.$$

Similar for i odd we let c_o denote the number of vertices in every second layer, where $N^+_i(o(2))$ is not one of those layers, thus

$$c_o = \sum_{j=0}^{\frac{i-3}{2}} |N^+_{2j+2}(o(2))| = \frac{1}{3}(2^{i+1} - 1) - 1 = \frac{1}{3} \cdot 2^{i+1} - \frac{4}{3}$$

and the number of vertices in the remaining layers are then

$$d_o = \sum_{j=0}^{\frac{i-1}{2}} |N^+_{2j+1}(o(2))| = 2c_o + 2.$$

We will now count the number of vertices in A depending on whether k and i are even or odd, and which set (A or B) u and v are in, a total of 8 different scenarios. Notice that exactly one of 1 and 3 will be in A. We will obtain contradictions in some of the scenarios and in the remaining we will obtain that $o(2) = 7$. Thus we will have proven that $o(2) \in \{3, 7\}$.

If k is even, we get following scenarios:

- i **even:**
  - $u, v \in A$**:** Then

$$\begin{aligned} |A| &= 2a_e + 1 + c_e - d_e - 1 \\ &= 2\left(-\frac{1}{3} + \frac{2}{3} \cdot 2^{k-1}\right) - c_e \\ &= \frac{2}{3} \cdot 2^k - \frac{2}{3} \cdot 2^i. \end{aligned}$$

Now as we already know $|A| = 2^{k-1}$, we must have $i = k-2$, and thus $o(2) = 7$.

– **$u \in A$, $v \in B$:** Then half of the vertices in $T_i^+(o(2))\backslash\{o(2)\}$, thus $2^i - 1$ vertices, will be in A and the other in B, hence

$$\begin{aligned} |A| &= 2a_e + 1 - d_e - 1 + 2^i - 1 \\ &= 2\left(-\frac{1}{3} + \frac{2}{3} \cdot 2^{k-1}\right) - \frac{4}{3}(2^i - 1) + 2^i - 1 \\ &= -\frac{1}{3} + \frac{2}{3} \cdot 2^k - \frac{1}{3} \cdot 2^i \end{aligned}$$

a contradiction with $|A| = 2^{k-1}$.

- **i odd:**

– **$u, v \in B$:** Similar to before, we see that

$$\begin{aligned} |A| &= 2a_e + 1 + c_o - d_o \\ &= 2\left(-\frac{1}{3} + \frac{2}{3} \cdot 2^{k-1}\right) + 1 + c_o - 2c_o - 2 \\ &= -\frac{2}{3} + \frac{4}{3} \cdot 2^{k-1} - \left(\frac{1}{3} \cdot 2^{i+1} - \frac{4}{3}\right) - 1 \\ &= -\frac{1}{3} + \frac{4}{3} \cdot 2^{k-1} - \frac{1}{3} \cdot 2^{i+1}, \end{aligned}$$

again a contradiction to the fact that $|A| = 2^{k-1}$.

– **$u \in A$, $v \in B$:** We see

$$\begin{aligned} |A| &= 2a_e + 1 + 2^i - 1 - d_o \\ &= 2\left(-\frac{1}{3} + \frac{2}{3} \cdot 2^{k-1}\right) + 1 + 2^i - 1 - \frac{2}{3}(2^{i+1} - 1) \\ &= \frac{2}{3} \cdot 2^k - \frac{1}{3} \cdot 2^i. \end{aligned}$$

As $|A| = 2^{k-1}$, this implies $i = k-1$, but then $o(2) = 3$, a contradiction to our assumption.

If k is odd we have:

- **i even:**

- $u, v \in A$**:** Then

$$
\begin{aligned}
|A| &= 2a_o + 1 + c_e - d_e - 1 \\
&= 2\left(-\frac{2}{3} + \frac{2}{3} \cdot 2^{k-1}\right) - c_e \\
&= -\frac{2}{3} + \frac{2}{3} \cdot 2^k - \frac{2}{3} \cdot 2^i,
\end{aligned}
$$

yet a contradiction to $|A| = 2^{k-1}$.

- $u \in A$**,** $v \in B$**:** We see

$$
\begin{aligned}
|A| &= 2a_o + 1 - d_e - 1 + 2^i - 1 \\
&= 2\left(-\frac{2}{3} + \frac{2}{3} \cdot 2^{k-1}\right) - \frac{4}{3}(2^i - 1) + 2^i - 1 \\
&= -1 + \frac{2}{3} \cdot 2^k - \frac{1}{3} \cdot 2^i,
\end{aligned}
$$

a contradiction to $|A| = 2^{k-1}$ and $i \neq 0$.

- i **odd:**
  - $u, v \in B$**:** Similar to before, we see that

$$
\begin{aligned}
|A| &= 2a_o + 1 + c_o - d_o \\
&= 2\left(-\frac{2}{3} + \frac{2}{3} \cdot 2^{k-1}\right) + 1 + c_o - 2c_o - 2 \\
&= -\frac{4}{3} + \frac{4}{3} \cdot 2^{k-1} - \left(\frac{1}{3} \cdot 2^{i+1} - \frac{4}{3}\right) - 1 \\
&= -1 + \frac{4}{3} \cdot 2^{k-1} - \frac{1}{3} \cdot 2^{i+1},
\end{aligned}
$$

  yet another contradiction to the fact that $|A| = 2^{k-1}$.

  - $u \in A$**,** $v \in B$**:** We see

$$
\begin{aligned}
|A| &= 2a_o + 1 + 2^i - 1 - d_o \\
&= 2\left(-\frac{2}{3} + \frac{2}{3} \cdot 2^{k-1}\right) + 2^i - \frac{2}{3}(2^{i+1} - 1) \\
&= -\frac{2}{3} + \frac{2}{3} \cdot 2^k - \frac{1}{3} \cdot 2^i.
\end{aligned}
$$

  Then we must have $k = 3$ and $i = 1$, thus $o(2) = 7$.

To summarize the above, we have $o(2) \in \{3,7\}$ and $o(3) \in \{2,4\}$. Using similar arguments we observe $o(4) \in \{5,10\}$, as $(11,\ldots,2^{k+1} = o(1), o(2), o(4))$ is a k-path. Now, if $o(2) = 3$ we get $o(4) \in N^{+}(o(2)) = \{6,7\}$, but this is a contradiction to our observation. On the other hand, if $o(2) = 7$ we must have $o(4) \in \{14,15\}$ again a contradiction.

□

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