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Spatial growth of fundamental solutions for certain perturbations of the harmonic oscillator

Arne Jensen* and Kenji Yajima†

Abstract

We consider the fundamental solution for the Cauchy problem for perturbations of the harmonic oscillator by time dependent potentials, which grow at spatial infinity slower than quadratic, but faster than linear functions, and whose Hessian matrices have a fixed sign. We prove that the fundamental solution at resonant times grows indefinitely at spatial infinity with the algebraic growth rate, which increases indefinitely, when the growth rate of perturbations at infinity decrease from the near quadratic to the near linear ones.

1 Introduction

We consider d-dimensional time dependent Schrödinger equations

$$i\frac{\partial u}{\partial t} = \left(-\frac{1}{2}\Delta + V(t,x)\right) u(t,x), \quad (t,x) \in \mathbf{R}^1 \times \mathbf{R}^d. \tag{1}$$

We assume throughout this paper that $V(t,x)$ is smooth with respect to the x variables, and $V(t,x)$ and its derivatives $\partial_x^\alpha V(t,x)$ are continuous. Under the conditions to be imposed on $V(t,x)$ in what follows Eqn. (1)

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generates a unique unitary propagator $\{U(t,s)\colon t,s \in \mathbf{R}\}$ in the Hilbert space $\mathcal{H} = L^2(\mathbf{R}^d)$, so that the solution in $\mathcal{H}$ of (1) with the initial condition

$$u(s,x) = \varphi(x) \in \mathcal{H}$$

is uniquely given by $u(t) = U(t,s)\varphi$. The distribution kernel $E(t,s,x,y)$ of $U(t,s)$ is called the fundamental solution (FDS for short) of the equation.

We begin with a brief review on properties of the FDS laying emphasis on its smoothness and boundedness with respect to the x variables. We denote the classical Hamiltonian and Lagrangian corresponding to (1) by

$$H(t,x,p) = p^2/2 + V(t,x) \quad \text{and} \quad L(t,q,v) = v^2/2 - V(t,q)$$

and $(x(t,s,y,k), p(t,s,y,k))$ is the solution of the initial value problem for Hamilton's equations

$$\dot{x}(t) = \partial_p H(t,x,p), \quad \dot{p}(t) = -\partial_x H(t,x,p)\ ;\ x(s) = y, \quad p(s) = k. \tag{2}$$

When $s = 0$, we write $(x(t,s,y,k), p(t,s,y,k)) = (x(t,y,k), p(t,y,k))$.

Suppose first that $V(t,x)$ is at most quadratic at spatial infinity in the sense that

$$\sup_t |\partial_x^\alpha V(t,x)| \le C_\alpha, \text{ for all } \ |\alpha| \ge 2 \tag{3}$$

. Then, in the seminal work [4], Fujiwara has shown that there exists a T depending only on V such that, for the time interval $0 \le \pm(t-s) < T$, the map $\mathbf{R}^d \ni k \mapsto x(t,s,y,k) \in \mathbf{R}^d$ is a diffeomorphism for every fixed $y \in \mathbf{R}^d$ and, therefore, there exists a unique path of (2) such that $x(s) = y$ and $x(t) = x$. Let

$$S(t,s,x,y) = \int_s^t (\dot{x}(r)^2/2 - V(r,x(r)))dr$$

be the action integral of the path. The FDS $E(t,s,x,y)$ has the following form for $0 < \pm(t-s) < T$:

$$E(t,s,x,y) = \frac{e^{\mp\frac{i\pi d}{4}}}{(2\pi|t-s|)^{d/2}} e^{iS(t,s,x,y)} a(t,s,x,y) \tag{4}$$

where $a(t,s,x,y)$ is a smooth function of (x,y), such that $\partial_x\partial_y a(t,s,x,y)$ are C^1 with respect to (t,s,x,y) and

$$|\partial_x\partial_y(a(t,s,x,y) - 1)| \le C_{\alpha\beta} t^2 \tag{5}$$

(see [9] for a similar result when magnetic fields are present). In particular, $E(t,s,x,y)$ is smooth and bounded with respect to the spatial variables $(x,y) \in \mathbf{R}^d \times \mathbf{R}^d$ for every $0 < |t-s| < T$.

Under the condition (3), however, the structure (4) of the FDS breaks down at later times because the singularities of the initial data $\delta(x)$ can in general recur in finite time. Indeed, the FDS $E_0(s+t,s,x,y) = E_0(t,x,y)$ of the harmonic oscillator, viz. Eqn. (1) with $V(t,x) = x^2/2$, is given for non-resonant times, $m\pi < t < (m+1)\pi$, $m \in \mathbf{Z}$ by

$$E_0(t,x,y) = \frac{e^{-id(1+2m)\pi/4}}{|2\pi \mathrm{sin}t|^{d/2}} e^{(i/\mathrm{sin}t)(\mathrm{cos}t(x^2+y^2)/2 - x\cdot y)}, \tag{6}$$

and for resonant times $t - s = m\pi$ by

$$E_0(m\pi,x,y) = e^{-im\pi/2}\delta(x-(-1)^m y). \tag{7}$$

Note that $E_0(t,x,y)$ is smooth and spatially bounded at non-resonant times; at resonant times $t = m\pi$, singularities of $E_0(0,x,y) = \delta(x-y)$ recur at $x = (-1)^m y$, however, it is smooth and decays rapidly at spatial infinity. Actually it vanishes outside the singular point $x = (-1)^m y$.

This recurrence of singularities does not take place, however, if $V(t,x)$ is increasing at infinity at a slower pace than in (3), and the FDS keeps the form (4) for any finite time with slight modifications (see [12]). More precisely, suppose that V is **subquadratic** at spatial infinity in the sense that

$$\lim_{|x|\to\infty} \sup_t |\partial_x^\alpha V(t,x)| = 0, \;\; |\alpha| = 2, \quad |\partial_x^\alpha V(t,x)| \le C_\alpha, \;\text{for all}\;\; |\alpha| \ge 3. \tag{8}$$

Then for any $T > 0$, there exists $R > 0$ such that, for any pair $(x,y) \in \mathbf{R}^d \times \mathbf{R}^d$ with $x^2 + y^2 \ge R^2$, there is unique path of (2) such that $x(s) = y$ and $x(t) = x$ and the FDS for $0 < \pm(t-s) \le T$ may be written in the form (4), where $S(t,s,x,y)$ is the action integral for (x,y) with $x^2+y^2 \ge R^2$. Moreover, we have $a(t,s,x,y) \to 1$ as $x^2+y^2 \to \infty$. In particular, $E(t,s,x,y)$ remains spatially smooth and bounded for any finite time $t \ne s$.

On the other hand, if $V(x)$ increases faster than $C\langle x\rangle^2$ at spatial infinity and $V(t,x) = V(x) \ge C|x|^{2+\varepsilon}$ as $|x| \to \infty$ for some $\varepsilon > 0$ and $C > 0$ with some additional technical assumptions on the derivatives, $E(t,0,x,y)$ is nowhere C^1 with respect to (t,x,y) ([10]). It is also shown that if $V(x) \sim C|x|^{10+\varepsilon}$, then $E(t,0,x,y)$ is unbounded with respect to (x,y) for any $t \in \mathbf{R}$. This result has been proven only in one dimension, but it is believed that the similar results hold in any dimensions.

In this way, properties of the FDS experience a sharp transition when the growth rate at spatial infinity of the potential $V(t,x)$ changes from subquadratic to superquadratic. Thus, the FDS for the borderline case, viz.

perturbations of the harmonic oscillator

$$i\frac{\partial u}{\partial t} = \left(-\frac{1}{2}\Delta + \frac{1}{2}x^2 + W(t,x)\right) u(t,x), \quad (t,x) \in \mathbf{R}^1 \times \mathbf{R}^d, \tag{9}$$

where $W(t,x)$ is subquadratic, has attracted particular interest of many authors, and the following properties of $E(t,s,x,y)$ have been established (see e.g. [13], [5], [12], [2] and [3]). We may set $s = 0$, which we will do, and we will write $E(t,x,y)$ for $E(t,0,x,y)$; $\langle x\rangle = (1+|x|^2)^{1/2}$.

(a) The structure of the FDS $E_0(t,x,y)$ at non-resonant times as stated in (6) is stable under perturbations and $E(t,x,y)$ is smooth and spatially bounded for $m\pi < t < (m+1)\pi$.

However, $E(t,x,y)$ at resonant times is more sensitive to perturbations:

(b) If W is **sublinear**, viz. $|\partial_x^\alpha W(t,x)| = o(1)$, $|\alpha| = 1$, as $|x| \to \infty$ uniformly with respect to t, then the recurrence of singularities at resonant times $m\pi$, $m \in \mathbf{Z}$, persists (WF_x denotes the wavefront set):

$$\mathrm{WF}_x E(m\pi, x, y) = \{(-1)^m(y,\xi) \colon \xi \in \mathbf{R}^d \setminus \{0\}\},$$

and it decays rapidly at spatial infinity, viz. for any N,

$$|E(m\pi,x,y) \le C_N\langle x-y\rangle^{-N}, \quad |x-y| \ge 1. \tag{10}$$

(c) If W is of **linear type**, viz. $|\partial_x^\alpha W(t,x)| \le C$ for $|\alpha| = 1$, singularities of $E(0,x,y)$ can propagate at resonant times. For example, if $W = a\langle x\rangle$, then with $\hat{\xi} = \xi/|\xi|$,

$$\mathrm{WF}_x E(m\pi, x, y) = \{(-1)^m(y + 2am\hat{\xi}, \xi) \colon \xi \in \mathbf{R}^d \setminus \{0\}\},$$

but it remains to decay rapidly at spatial infinity:

$$|E(m\pi,x,y) \le C_N\langle x-y\rangle^{-N}, \quad |x-y| \ge 1. \tag{11}$$

(d) If W is superlinear and satisfies the following sign condition on the Hessian matrix $\partial_x^2 W = (\partial^2 W/\partial x_j \partial x_k)$ that

$$C_1\langle x\rangle^{-\delta} \le \partial_x^2 W(t,x) \le C_2\langle x\rangle^{-\delta}, \quad (t,x) \in \mathbf{R}^1 \times \mathbf{R}^d \tag{12}$$

for some constants $0 < \delta < 1$ and $0 < C_1 < C_2 < \infty$ or $-\infty < C_1 < C_2 < 0$, then $E(m\pi, x, y)$, $m \in \mathbf{Z}$, is C^∞ with respect to (x,y), viz. singularities at resonant times $t = m\pi$ are swept away.

This paper is concerned with the properties of the FDS $E(t,s,x,y)$, when $t-s \in \pi\mathbf{Z}$, viz. when t and s are at resonant times. We show that, in the last case (d) above, $E(m\pi,x,y)$ increases indefinitely as $|x| \to \infty$ at the algebraic rate $C|x|^{d\delta/(2-2\delta)}$, exhibiting a sharp contrast to the decay result (10) or (11) for the case, when W is at most linearly increasing at spatial infinity. More precisely:

Theorem 1.1. *Suppose that $W(t,x)$ is subquadratic and satisfies the sign condition* (12) *for some $0<\delta<1$ and $0<C_1<C_2<\infty$ or $-\infty<C_1<C_2<0$. Let $m \in \mathbf{Z}$. Let $\chi \in C_0^\infty(\mathbf{R}^d \setminus \{0\})$ be such that $\chi(x)=1$ for $a \le |x| \le b$, $0<a<b<\infty$ being constants. Then there exist constants $0<M_1<M_2$, independent of $R \ge 1$, such that*

$$M_1 R^{d\delta/(2-2\delta)} \le \left(\int_{\mathbf{R}^d} |E(m\pi,0,x,y)|^2 \chi\left(\frac{x}{R}\right)^2 \frac{dx}{R^d}\right)^{1/2} \le M_2 R^{d\delta/(2-2\delta)}. \quad (13)$$

It is interesting to note that, when δ increases from 0 to 1, the growth rate as $|x| \to \infty$ of $W(t,x)$ decreases (hence $W(t,x)$ becomes weaker), whereas that of $E(m\pi,x,y)$ as $|x-y| \to \infty$, $r(\delta) = d\delta/(2-2\delta)$, increases from 0 indefinitely to infinity. This seemingly contradictory behavior may be understood via the semi-classical picture as follows. For functions $a(x)$ and $b(x)$ on Ω, $a \sim b$ means that $A_1 a(x) \le b(x) \le A_2 a(x)$, $x \in \Omega$, for constants $0<A_1<A_2$. At time 0 consider the ensemble Γ of classical particles in the phase space $\mathbf{R}^d \times \mathbf{R}^d$ sitting on the linear Lagrangian manifold $\{(x,p) \in \mathbf{R}^d \times \mathbf{R}^d \colon x = y, p \in \mathbf{R}^d\}$ with uniform momentum distribution $(2\pi)^{-d/2}dp$. Semiclassically, this is described by the wave function $\delta(x-y) = E(0,x,y)$. After time $m\pi$, Γ will be transported by the Hamilton flow (2) to the Lagrangian manifold $\{(x(m\pi,y,k), p(m\pi,y,k)) \colon k \in \mathbf{R}^d\}$. As we shall see below, we have $|p(m\pi,y,k)| \sim |k|$ and $|x(m\pi,y,k)| \sim |k|^{1-\delta}$ as $|k| \to \infty$. It follows at least semiclassically

$$|E(m\pi,x,y)| \sim \left|\det\left(\frac{\partial x}{\partial p}\right)\right|^{-1/2} \sim |k|^{d\delta/2} \sim |x|^{d\delta/(2-2\delta)}, \quad |x| \to \infty,$$

which is consistent with (13). Here is another remark, which clarifies that Theorem 1.1 is more or less consistent with the known results. We should also note that if $\delta = 0$, then $W = c\langle x\rangle^2$, and $m\pi$ is no longer a resonant time for $V = x^2/2 + W$, and the corresponding $E(m\pi,x,y)$ is bounded as $|x-y| \to \infty$; on the other hand, if $\delta = 1$, then $W = c\langle x\rangle$ and, as in (c) above, a large portion of $E(m\pi,x,y)$ is concentrated in a bounded domain $|x-y| \le 2cm$, which may be represented as the extreme case of $C\langle x\rangle^{d\delta/(2-2\delta)}$ as $\delta \to 1$.

We mention here that the result of the theorem has been conjecture by Martinez and the second author in [7], where a similar problem is studied in the semi-classical setting. More precisely, they consider the FDS of the semi-classical Schrödinger equation

$$ih\frac{\partial u}{\partial t} = \left(-\frac{h^2}{2}\Delta + \frac{1}{2}x^2 + h^\mu W(x)\right)u,$$

where $W(x)$ is t independent and satisfies the same conditions as in this paper, (8) and (12); and they prove that the FDS at the resonant times may be written in the form

$$E(n\pi, x, y) = h^{-d(1+\nu)/2}a(x, y, h)e^{iS(x,y)/h}, \quad \nu = \mu/(1-\delta), \tag{14}$$

where $S(x, y)$ is the action integral of the path of (2) connecting $x(0) = y$ and $x(n\pi)$ and $a(x, y, h)$ satisfies $C^{-1} \le |a(x, y, h)| \le C$ uniformly with respect h on every compact subset K of $\mathbf{R}^{2d}\setminus\{(x,(-1)^n x) \colon x \in \mathbf{R}^d\}$. Thus, $E(n\pi, x, y)$ has the extra growing factor $h^{-d\nu/2}$ as $h \to 0$ compared to $E(t, x, y)$ at non-resonant times $t \neq n\pi$ and they remark that, if their arguments applied for non-smooth potentials, (14) would imply the estimate (13) of Theorem 1.1 for the homogeneous potential $W(x) = C|x|^{2-\delta}$.

It is well known that the boundedness of $E(t, s, x, y)$ with respect to (x, y) implies the so called L^p-L^q estimates of the propagator $U(t, s)$ (hence, also finite time Strichartz's estimates). There are examples of Schrödinger equations with smooth coefficients, which exhibit break down of the estimates, e.g. the harmonic oscillator at resonant times. However, to the best knowledge of the authors, in all known examples they are broken because of local singularities and, Theorem 1.1 is the first example in which they are broken because of the growth at spatial infinity of the FDS (see [8] for L^p-L^q estimates for potentials which are singular but decay at infinity). For the micro-local smoothing estimate which may be applied for proving the smoothness of the FDS, see for example [1] or [6].

The rest of the paper is devoted to the proof of this theorem. We prove it only in the $m = 1$ case. The proof for the other cases is similar. In section 2, we recall several known facts, which will be used in section 3, where the theorem is proved. We often omit some of the variables of functions, if no confusion is to be feared. For functions f of several variables, we write $f \in C^k(x)$ or $f \in C^k(t, x)$ etc., if f is of class C^k with respect to x or (t, x) etc.

2 Preliminaries

Recall that we are assuming that W is subquadratic. We set the initial time $s = 0$ and omit the variable s. The solutions $(x(t), p(t)) = (x(t, y, k), p(t, y, k))$ of (2) satisfy the integral equations

$$x(t) = y\text{cos}t + k\text{sin}t - \int_0^t \sin(t-s)\partial_x W(s, x(s))ds, \tag{15}$$

$$p(t) = -y\text{sin}t + k\text{cos}t - \int_0^t \cos(t-s)\partial_x W(s, x(s))ds. \tag{16}$$

Since the subquadratic condition implies $|\dot{x}(t)|+|\dot{p}(t)| \le C(1+|x(t)|+|p(t)|)$ for a constant $C > 0$ and, hence,

$$e^{-C|t|}(1 + |y| + |k|) \le (1 + |x(t)| + |p(t)|) \le e^{C|t|}(1 + |y| + |k|), \tag{17}$$

it follows, as $y^2 + k^2 \to \infty$, uniformly with respect to t in compact intervals, that

$$|x(t) - (y\text{cos}t + k\text{sin}t)| = o(|y| + |k|), \tag{18}$$

$$|p(t) - (-y\text{sin}t + k\text{cos}t)| = o(|y| + |k|). \tag{19}$$

We fix $m \in \mathbf{Z}$, $m \ne 0$, and $0 < \varepsilon < \pi/2$, and consider t in the interval $I = [m\pi - \varepsilon, m\pi + \varepsilon]$. Then, the following results have been proved in Lemma 2.3, Lemma 2.5 and Lemma 3.5, respectively, of [12] by using the integral equations (15) and (16).

(i) For any α and β, as $R^2 = y^2 + k^2 \to \infty$

$$\|\partial_y^\alpha \partial_k^\beta(\partial_y x(t) - (\text{cos}t)\mathbf{1})\| \to 0, \quad \|\partial_y^\alpha \partial_k^\beta(\partial_k x(t) - (\text{sin}t)\mathbf{1})\| \to 0, \tag{20}$$

$$\|\partial_y^\alpha \partial_k^\beta(\partial_y p(t) + (\text{sin}t)\mathbf{1})\| \to 0, \quad \|\partial_y^\alpha \partial_k^\beta(\partial_k p(t) - (\text{cos}t)\mathbf{1})\| \to 0 \tag{21}$$

uniformly with respect to $t \in I$. Here $\mathbf{1}$ is the $d \times d$ identity matrix.

(ii) Let $R > 0$ be sufficiently large. Then, for any $t \in I$ and $(\xi, y) \in \mathbf{R}^{2d}$ with $\xi^2 + y^2 \ge R^2$, there exists a unique $k \in \mathbf{R}^d$ such that the solution $(x(s, y, k), p(s, y, k))$ of (2) satisfies

$$p(t, y, k) = \xi. \tag{22}$$

(iii) Let R be as in (ii) and define $\varphi(t, \xi, y)$ for $t \in I$ and $\xi^2 + y^2 > R^2$ by

$$\varphi(t, \xi, y) = x(t, y, k) \cdot \xi - \int_0^t L(s, x(s, y, k), \dot{x}(s, y, k))ds,$$

where (y,k) is determined by (22). Then $\varphi \in C^\infty(\xi,y)$ and $\partial_\xi^\alpha \partial_y^\beta \varphi \in C^1(t,\xi,y)$ for any α,β; φ is a generating function of the canonical map $(p(t,y,k),y) \mapsto (x(t,y,k),k)$:

$$(\partial_\xi \varphi)(t,p(t,y,k),y) = x(t,y,k), \quad (\partial_y \varphi)(t,p(t,y,k),y) = k, \tag{23}$$

and φ satisfies the Hamilton-Jacobi equation $\partial_t \varphi = \xi^2/2 - V(t,\partial_\xi \varphi)$. Moreover, as $\xi^2+y^2 \to \infty$, $\partial_\xi^\alpha \partial_y^\beta \varphi$ approaches the corresponding function of the harmonic oscillator whenever $|\alpha+\beta| \geq 2$:

$$\sup_{t\in I} \left| \partial_\xi^\alpha \partial_y^\beta \left(\varphi(t,\xi,y) - \frac{(\xi^2+y^2)\mathrm{sin}t + 2\xi\cdot y}{2\mathrm{cos}t} \right) \right| \to 0.$$

Furthermore, we have the following theorem [12, Theorem 1.3 (2)].

Theorem 2.1. *Let W be subquadratic. Then, for $t \in I = [m\pi - \varepsilon, m\pi + \varepsilon]$, the FDS $E(t,x,y)$ of (9) may be written in the following form*

$$E(t,x,y) = \lim_{\varepsilon\downarrow 0} \int_{\mathbf{R}^d} \frac{i^{-(m+1)d} e^{ix\cdot\xi - i\tilde{\varphi}(t,\xi,y) - \varepsilon\xi^2/2} a(t,\xi,y)}{(2\pi)^d |\mathrm{cos}t|^{d/2}} d\xi \tag{24}$$

where the integral converges in the C^∞ topology with respect to (x,y) and the functions $\tilde{\varphi}$ and a satisfy the following properties:

(a) $\tilde{\varphi} \in C^\infty(\xi,y)$, $\partial_\xi^\alpha \partial_y^\beta \tilde{\varphi} \in C^1(t,\xi,y)$ *for any α,β and*

$$\tilde{\varphi}(t,y,\xi) = \varphi(t,\xi,y) \quad \textit{for } t \in I,\ \xi^2+y^2 \geq R^2.$$

(b) $a \in C^\infty(\xi,y)$, $\partial_x^\alpha \partial_y^\beta a \in C^1(t,\xi,y)$ *for any α,β and*

$$\lim_{\xi^2+y^2\to\infty} \sup_{t\in I} |\partial_x^\alpha \partial_y^\beta (a(t,x,y)-1)| \to 0$$

for any α and β.

We call integrals of the form (24) oscillatory integrals and often write them simply as

$$\int_{\mathbf{R}^d} \frac{i^{-(m+1)d} e^{ix\cdot\xi - i\tilde{\varphi}(t,\xi,y)} a(t,\xi,y)}{(2\pi)^d |\mathrm{cos}t|^{d/2}} d\xi.$$

We also need the following result, which holds under the sign condition (12). From now on we let $m = 1$.

Proposition 2.2. *Let W be subquadratic and satisfy* (12). *Let $L > 0$. Then, there exist constants $C > 0$ and $R > 0$ depending only on L such that for every $|\xi| \geq R$ and $|y| \leq L$:*

$$C_1|\xi|^{1-\delta} \leq |\partial_\xi \varphi(\pi, y, \xi)| \leq C_2|\xi|^{1-\delta}, \tag{25}$$

$$|\partial_\xi^\alpha \varphi(\pi, y, \xi)| \leq C|\xi|^{-\delta}, \quad |\alpha| \geq 2. \tag{26}$$

Proof. The upper bound in estimate (25) is obvious from (23), (15) and (18); the lower bound is proved in [11, pages 61-63] for time independent perturbations $W(t, x) = W(x)$, and the proof applies to the time dependent case as well, if we use Lemma 2.1 and 2.2 of [12] instead of Lemma 4.2 and 4.3 of [11]. From [11, pages 61-63], we also have for $|\xi| \geq R$ and k such that $p(\pi, y, k) = \xi$

$$\|\partial_k x(\pi, y, k)\| \sim |\xi|^{-\delta}. \tag{27}$$

Differentiating $(\partial_\xi \varphi)(\pi, p(\pi, y, k), y) = x(\pi, y, k)$ with respect to k, we have

$$(\partial_\xi^2 \varphi)(\pi, \xi, y)\partial_k p(\pi, y, k) = \partial_k x(\pi, y, k) \tag{28}$$

and, applying the second result of (21) and (27), we obtain (26) for the case $|\alpha| = 2$. For higher derivatives, we further differentiate (28) and apply (20) and (21) in addition to (27). Estimate (26) follows inductively. $\square$

Lemma 2.3. *Let $L > 0$ and $0 < a < b < \infty$ be fixed arbitrarily and let $\chi \in C_0^\infty(\mathbf{R}^d)$ be supported by $\{x \in \mathbf{R}^d \colon a \leq |x| \leq b\}$. Then, there exist $R_0 > 0$ and $C_0 > 0$, such that for all $R > R_0$ and $|y| \leq L$*

$$\frac{1}{R^d} \int_{\mathbf{R}^d} |\chi(\partial_\xi \varphi(\pi, \xi, y)/R)|^2 d\xi \leq C_0 R^{d\delta/(1-\delta)}. \tag{29}$$

If $\chi(x) > \delta > 0$ for $a_1 < |x| < b_1$, $a < a_1 < b_1 < b$, then we also have the lower bound:

$$C_1 R^{d\delta/(1-\delta)} \leq \frac{1}{R^d} \int_{\mathbf{R}^d} |\chi(\partial_\xi \varphi(\pi, \xi, y)/R)|^2 d\xi. \tag{30}$$

Proof. For sufficiently large $R > 0$, we have by virtue of (19) that $1/2 \leq |p(\pi, y, k)|/|k| \leq 2$ for $|y| \leq L$ and $k \geq R$, and (25) implies

$$C_1|k|^{1-\delta} \leq |x(\pi, y, k)| \leq C_2|k|^{1-\delta}.$$

It follows that, if $\chi(x(\pi, y, k)/R) \neq 0$, then $aR/C_2 \leq |k|^{1-\delta} \leq bR/C_1$. Hence, whenever $\chi(\partial_\xi \varphi(\pi, \xi, y)/R) \neq 0$, we have

$$D_1 R^{1/(1-\delta)} \leq |\xi| \leq D_2 R^{1/(1-\delta)}.$$

and

$$\frac{1}{R^d}\int |\chi(\partial_\xi\varphi(\pi,\xi,y)/R)|^2 d\xi \le CR^{d\delta/(1-\delta)}.$$

A similar argument yields the lower bound in the second case. We omit the obvious details. $\square$

3 Proof of Theorem 1.2

Before starting the proof we remark the following: If we were able to prove the faster decay as $|\xi|\to\infty$ for the higher derivatives $\partial_\xi^\alpha\varphi$, say,

$$|\partial_\xi^\alpha\varphi(\pi,y,\xi)| \le C|\xi|^{-\delta-|\alpha|}, \tag{31}$$

then the standard stationary phase method combined with a change of scale would yield the pointwise estimate

$$|E(m\pi,x,y)| \sim C|x|^{d\delta/(2-2\delta)} \quad \text{as } |x|\to\infty. \tag{32}$$

However, (31) does not seem to hold in general and this required a weaker formulation of the theorem and a little complicated proof given below.

We need to estimate

$$I(R) \equiv \frac{1}{R^d}\int_{\mathbf{R}^d}\left|\chi\left(\frac{x}{R}\right)E(\pi,x,y)\right|^2 dx. \tag{33}$$

In what follows we omit the variable π, the domain of integration $\mathbf{R}^d$ from integral signs and write $\tilde{\varphi}$ as φ. Since y is fixed in the following computation, we sometimes omit the variables y as well. This should not cause any confusion. Then, by virtue of (24), (33) may be written as an oscillatory integral

$$\begin{aligned}
I(R) &= \lim_{\varepsilon\downarrow 0}\frac{1}{(2\pi)^{2d}R^d}\int\left|\chi\left(\frac{x}{R}\right)\int e^{i(x\cdot\xi-\varphi(\xi))-\varepsilon\xi^2/2}a(\xi)d\xi\right|^2 dx\\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{(2\pi)^{2d}R^d}\int\chi\left(\frac{x}{R}\right)^2 e^{ix\cdot(\xi-\eta)+i(\varphi(\eta)-\varphi(\xi))-\varepsilon(\xi^2+\eta^2)/2}a(\xi)\overline{a(\eta)}d\xi d\eta dx\\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{(2\pi)^{d}}\int\hat{\chi_2}(R(\eta-\xi))e^{i(\varphi(\eta)-\varphi(\xi))-\varepsilon(\xi^2+\eta^2)/2}a(\xi)\overline{a(\eta)}d\xi d\eta,
\end{aligned} \tag{34}$$

where we wrote $\chi_2(x)=\chi^2(x)$ and we defined the Fourier transform by

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) = \frac{1}{(2\pi)^d}\int e^{-ix\cdot\xi}f(x)dx.$$

In what follows we omit the limit sign $\lim_{\varepsilon\downarrow 0}$ and the damping factors which arise from $\exp(-\varepsilon(\xi^2+\eta^2)/2)$. In the right side of (34), we change variables η to $\zeta=\eta-\xi$ and expand by Taylor's formula as

$$a(\xi+\zeta)=\sum_{|\alpha|\le N}\frac{1}{\alpha!}\zeta^\alpha a^{(\alpha)}(\xi)+\sum_{|\alpha|=N+1}\frac{1}{\alpha!}\zeta^\alpha b_\alpha(\xi,\zeta)$$

in the resulting formula, where $a^{(\alpha)}=\partial_\xi^\alpha a$ and where we wrote

$$b_\alpha(\xi,\zeta)=\int_0^1(1-\theta)^N a^{(\alpha)}(\xi+\theta\zeta)d\theta.$$

This expresses $I(R)$ as

$$\sum_{|\alpha|\le N}\frac{1}{(2\pi)^d\alpha!}\int\hat{\chi}_2(R\zeta)\zeta^\alpha e^{i\varphi(\xi+\zeta)-i\varphi(\xi)}a(\xi)\overline{a^{(\alpha)}(\xi)}d\xi d\zeta+B_N(R), \tag{35}$$

where $B_N(R)$ is the sum over α with $|\alpha|=N+1$ of constants times

$$\begin{aligned}\iint\hat{\chi}_2(R\zeta)\zeta^\alpha e^{i(\varphi(\xi+\zeta)-\varphi(\xi))}a(\xi)\overline{b_\alpha(\xi,\zeta)}d\xi d\zeta&\\ =\int e^{-i\varphi(\xi)}a(\xi)\left(\int e^{i\varphi(\xi+\zeta)}\hat{\chi}_2(R\zeta)\zeta^\alpha\overline{b_\alpha(\xi,\zeta)}d\zeta\right)d\xi.&\end{aligned}$$

We take $\ell\in\mathbf{N}$ such that $\ell(1-\delta)>d$ and apply integration by parts ℓ times to the inner integral, which we denote by $I(\xi,R)$, by using the identity

$$\left\{\frac{1-i\partial_\zeta\varphi(\xi+\zeta)\cdot\partial_\zeta}{1+(\partial_\zeta\varphi(\xi+\zeta))^2}\right\}e^{i\varphi(\xi+\zeta)}=e^{i\varphi(\xi+\zeta)}.$$

Thus, if we write M for the transpose of the differential operator on the left, we have

$$I(\xi,R)=\int e^{i\varphi(\xi+\zeta)}M^\ell(\hat{\chi}_2(R\zeta)\zeta^\alpha\overline{b_\alpha(\xi,\zeta)})d\zeta. \tag{36}$$

Since M has the form

$$M=\frac{1}{1+(\partial_\zeta\varphi)^2}+i\mathrm{div}_\zeta\left(\frac{\partial_\zeta\varphi}{1+(\partial_\zeta\varphi)^2}\right)+\frac{i\partial_\zeta\varphi}{1+(\partial_\zeta\varphi)^2}\cdot\partial_\zeta,$$

$\partial_\zeta^\alpha\varphi$ are bounded for $|\alpha|\ge 2$ and since

$$C^{-1}\langle\xi+\zeta\rangle^{2(1-\delta)}\le 1+(\partial_\xi\varphi(\xi+\zeta))^2\le C\langle\xi+\zeta\rangle^{2(1-\delta)}.$$

by virtue of (25), M^ℓ is an ℓ-th order differential operator with respect to ∂_ζ whose coefficients are bounded by $C\langle \xi+\zeta\rangle^{-\ell(1-\delta)}$. Hence

$$|I(\xi,R)| \le C \sum_{|\beta+\gamma+\delta|\le \ell} \int \langle \xi+\zeta\rangle^{-\ell(1-\delta)} R^{|\beta|} |(\partial_\zeta^\beta \hat{\chi_2})(R\zeta)||\zeta^{\alpha-\gamma}||\partial_\zeta^\delta b_\alpha(\xi,\zeta)|d\zeta.$$

Since $\hat{\chi_2}(\zeta)$ is rapidly decreasing and $\partial_\zeta^\delta b_\alpha(\xi,\zeta)$ are bounded, the integrand is bounded for any $L>0$ by a constant times

$$\langle \xi\rangle^{-\ell(1-\delta)}\langle \zeta\rangle^{\ell(1-\delta)}\langle R\zeta\rangle^{-L} R^{|\beta|}|\zeta|^{N+1-|\gamma|}.$$

It follows, by changing variables ζ to ζ/R, and by taking L large enough, that for $R>1$

$$\begin{aligned}|I(\xi,R)| &\le CR^{-N-1-d+\ell}\langle \xi\rangle^{-\ell(1-\delta)} \int \langle \zeta/R\rangle^{\ell(1-\delta)}\langle\zeta\rangle^{N+1-L}d\zeta \\ &\le C'R^{-N-1-d+\ell}\langle\xi\rangle^{-\ell(1-\delta)}.\end{aligned}$$

Thus, for ℓ such that $\ell(1-\delta)>d$ we may estimate the remainder $B_N(R)$ in (35) by

$$|B_N(R)| \le \frac{C}{R^{N+d+1-\ell}} \int |a(\xi)|\langle\xi\rangle^{-\ell(1-\delta)}d\xi \le \frac{C'}{R^{N+d+1-\ell}},$$

and we may ignore $B_N(R)$ by taking N large enough. We have next to deal with the first terms in (35), which are sum over $|\alpha|\le N$ of

$$A_\alpha = \frac{1}{(2\pi)^d\alpha!}\int \left(\int \hat{\chi_2}(R\zeta)\zeta^\alpha e^{i(\varphi(\xi+\zeta)-\varphi(\xi))}d\zeta\right) a(\xi)\overline{a^{(\alpha)}(\xi)}d\xi. \tag{37}$$

By using Taylor's formula, we write

$$\begin{aligned} e^{i(\varphi(\xi+\zeta)-\varphi(\xi))} &= e^{i\zeta\cdot\partial_\xi\varphi(\xi)}e^{i\Psi(\xi,\zeta)}, \\ \Psi(\xi,\zeta) &= \zeta\cdot\left(\int_0^1 (1-\theta)\frac{\partial^2\varphi}{\partial\xi^2}(\xi+\theta\zeta)d\theta\right)\zeta,\end{aligned}$$

and expand $e^{i\Psi}$ via Taylor's formula:

$$e^{i(\varphi(\xi+\zeta)-\varphi(\xi))} = e^{i\zeta\cdot\partial_\xi\varphi(\xi)}\left(\sum_{m=0}^N \frac{(i\Psi)^m}{m!} + \frac{(i\Psi)^{N+1}}{N!}\int_0^1 (1-\theta)^N e^{i\theta\Psi}d\theta\right),$$

where we take N large enough so that $(N+1)\delta>d$. We then insert this into the right of (37). Note that

$$|\Psi(\xi,\zeta)| \le C\langle\xi\rangle^{-\delta}\langle\zeta\rangle^\delta|\zeta|^2$$

by virtue of (26). It follows that the contribution to A_α of the term containing $(i\Psi)^{N+1}/(N+1)!$ is bounded by taking L such that $L > (2+\delta)(N+1)+|\alpha|+d$ by

$$
\begin{aligned}
C_{LN} \iint & \langle R\zeta\rangle^{-L}|\zeta|^{2(N+1)+|\alpha|}\langle\xi\rangle^{-(N+1)\delta}\langle\zeta\rangle^{(N+1)\delta}d\xi d\zeta \\
& \le C_{LN}R^{-2(N+1)-|\alpha|-d}\cdot\int\langle\zeta\rangle^{-L+(N+1)\delta}|\zeta|^{2(N+1)+|\alpha|}d\zeta\cdot\int\langle\xi\rangle^{-(N+1)\delta}d\xi \\
& \le CR^{-(d+|\alpha|+2N+2)}.
\end{aligned}
$$

Thus, we may again ignore this term and we are left for A_α with

$$
\frac{1}{(2\pi)^d\alpha!}\sum_{m=0}^{N}\frac{1}{m!}\int\left(\int e^{i\zeta\cdot\partial_\xi\varphi(\xi)}\hat{\chi}_2(R\zeta)\zeta^\alpha(i\Psi(\xi,\zeta))^m d\zeta\right)a(\xi)\overline{a^{(\alpha)}(\xi)}d\zeta d\xi.
$$

Here we repeat the same argument as in the first step to the inner integral. We expand $\Psi(\xi,\zeta)$ further by Taylor's formula:

$$
\begin{aligned}
\Psi(\xi,\zeta) &= \sum_{2\le|\alpha|\le N}\frac{\zeta^\alpha}{\alpha!}\varphi^{(\alpha)}(\xi)+L_N(\xi,\zeta), \\
L_N(\xi,\zeta) &= \sum_{|\alpha|=N+1}C_\alpha\zeta^\alpha\left(\int_0^1(1-\theta)^N\varphi^{(\alpha)}(\xi+\theta\zeta)d\theta\right)
\end{aligned}
$$

and expand the product $\Psi(\xi,\zeta)^m$. We estimate the contribution to A_α of the terms which contain L_N, by performing integration by parts ℓ times, $(1-\delta)\ell > d$, by using the identity

$$
\left\{\frac{1-i\partial_\xi\varphi(\xi)\cdot\partial_\zeta}{1+|\partial_\xi\varphi(\xi)|^2}\right\}e^{i\zeta\partial_\xi\varphi(\xi)} = e^{i\zeta\partial_\xi\varphi(\xi)}
$$

and the estimate (25). This yields the bound $CR^{-2(N+1)-d+\ell}$ for the contribution and we ignore them. The rest is a sum of the terms of the form

$$
C_{\beta_1\ldots\beta_m}\zeta^\beta\varphi^{(\beta_1)}(\xi)\ldots\varphi^{(\beta_m)}(\xi), \quad \beta=\beta_1+\cdots+\beta_m
$$

and their contributions to A_α are given by constants times

$$
\begin{aligned}
&\int e^{i\zeta\partial_\xi\varphi(\xi)}\hat{\chi}_2(R\zeta)\zeta^{(\alpha+\beta)}\varphi^{(\beta_1)}(\xi)\ldots\varphi^{(\beta_m)}(\xi)a(\xi)\overline{a^{(\alpha)}(\xi)}d\xi \\
&= \frac{1}{(iR)^{|\alpha|+|\beta|}R^d}\int(\partial_\zeta^{\alpha+\beta}\chi_2)(\partial_\xi\varphi(\xi)/R)\varphi^{(\beta_1)}(\xi)\ldots\varphi^{(\beta_m)}(\xi)a(\xi)\overline{a^{(\alpha)}(\xi)}d\xi.
\end{aligned}
$$

By virtue of Lemma 2.3, this integral is bounded in modulus by

$$\frac{C}{R^{|\alpha|+|\beta|}R^d}\int |(\partial_\zeta^{\alpha+\beta}\chi_2)(\partial_\xi\varphi(\xi)/R)|d\xi \leq C' R^{d\delta/(1-\delta)}R^{-|\alpha+\beta|}.$$

Thus the main contribution to $I(R)$ is given by the one with $\alpha=\beta=0$:

$$\frac{1}{(2\pi)^d}\frac{1}{R^d}\int \chi(\partial_\xi\varphi(\xi)/R)^2|a(\xi)|^2 d\xi.$$

Since $a(\xi)\to 1$ as $|\xi|\to\infty$, this is comparable with $CR^{d\delta/(1-\delta)}$ for large R by virtue of Lemma 2.3. The theorem follows. $\square$

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