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Publication date:
1997

Document Version
Publisher's PDF, also known as Version of record

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DEPT. OF BUILDING TECHNOLOGY AND STRUCTURAL ENGINEERING
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DECEMBER 1997 ISSN 1395-7953 R9760

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Printed at Aalborg University

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ANALYTICAL VS SIMULATION SOLUTION TECHNIQUES FOR PULSE PROBLEMS IN NON-LINEAR STOCHASTIC DYNAMICS

Radosław Iwankiewicz * and Søren R.K. Nielsen **

* *Inst. of Materials Science and Applied Mechanics, Wrocław University of Technology, Poland*

** *Dept. of Building Technology and Structural Engineering, Aalborg University, Denmark*

Abstract

Advantages and disadvantages of available analytical and simulation techniques for pulse problems in non-linear stochastic dynamics are discussed. First, random pulse problems, both those which do and do not lead to Markov theory, are presented. Next, the analytical and analytically-numerical techniques suitable for Markov response problems such as moment equations, Petrov-Galerkin and cell-to-cell mapping techniques are briefly discussed. Usefulness of these techniques is limited by the fact that effectiveness of each of them depends on the mean rate of impulses. Another limitation is the size of the problem, i.e. the number of state variables of the dynamical system. In contrast, the applicability of the simulation techniques is not limited to Markov problems, nor is it dependent on the mean rate of impulses. Moreover their use is straightforward for a large class of point processes, at least for renewal processes.

1 Introduction

An often posed question is: whether or not it is of general interest to consider the problems of response of dynamical systems to random pulse trains. In order to justify such an interest let us realize the fact that any excitation to the dynamical mechanical system may be effectuated in either of two ways: as a continuous function of time, or by impulses (jumps in the velocity response process). A class of practical engineering problems leading to the random impulse process representation can be listed, which embraces all kinds of trains of shocks and impacts.

If it is of interest, which has been rather commonly accepted for a long time, to consider problems of response to random excitations with time continuous sample paths, it is of equally general interest to consider problems of response to impulse process excitations. However, the latter excitations cannot be treated by usual techniques, because they reveal inherently non-Gaussian nature. A Gaussian process is only an asymptotic special case of a stochastic impulse process. Although as a basic model an external impulse process excitation may be assumed, an insight into the physical nature of many problems reveals that the jump of the velocity response is the result of collision of two bodies. Consequently, the total impulse equal to the increment of the momentum of the system under consideration depends on the velocities of both colliding bodies prior to the collision and hence it depends on the state variables of the problem. This leads to the multiplicative noise (parametric excitation) problem[1]. Hence even for a simple mechanical model, quite advanced techniques may be required.

One of the earliest papers dealing with the response of a non-linear dynamical system to a random train of impulses is certainly due to Roberts.[2] Next different approaches to the problem were developed, such as the equivalent linearization[3], the improved perturbation technique to solve the generalized Fokker-Planck equation[4], techniques based on equations for moments,[5–7,11] Petrov-Galerkin method,[8] cell-to-cell mapping technique.[9,10,12] Recently Di Paola and co-workers[13] as well as Grigoriu[14] have tackled the problem.

2 Markov and non-Markov response problems

Consider a general multi-degree-of-freedom non-linear dynamical system under a random train of general pulses driven by a stochastic point process $\{\mathcal{N}(t), t \in [t_0, \infty[\}, \Pr\{\mathcal{N}(t_0) = 0\} = 1$, which is uniformly regular and has a finite number of points in a finite time interval. The counting process gives the number of time points in the interval $[t_0, t[$. The state vector of the system, $\mathbf{Z}(t)$, consisting of the structural generalized displacements and velocities augmented possibly by the state variables of the auxiliary filter, is governed by the set of equations of motion

$$d\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t), t\big)dt + \mathbf{d}\big(\mathbf{Z}(t), t, P(t)\big)d\mathcal{N}(t), \quad \mathbf{Z}(t_0) = \mathbf{z}_0, \tag{1}$$

where $d\mathcal{N}(t) = \mathcal{N}(t + dt) - \mathcal{N}(t)$ and $P(t)$ assumes the values $P(t_i) = P_i$ (random mark variables) at the times t_i of the impulses occurrences. If $\mathcal{N}(t)$ is a Poisson counting process, independent of the random mark variables which also are mutually independent, and both counting process and mark variables are independent of the initial conditions, the state vector $\mathbf{Z}(t)$ is a Markov process. It is not for all other point processes, e.g. renewal processes.

Nevertheless the differential of the function $\Phi(t, \mathbf{Z}(t))$ of the process governed by equations (1) can be evaluated from the following differential rule,[15]

$$\begin{aligned} d\Phi(t, \mathbf{Z}(t)) \;=\; & \frac{\partial \Phi(t, \mathbf{Z}(t))}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial \Phi(t, \mathbf{Z}(t))}{\partial Z_i} c_i\left(\mathbf{Z}(t), t\right) dt \\ & + \Big[\Phi\Big(t, \mathbf{Z}(t) + \mathbf{d}\big(\mathbf{Z}(t), t, P(t)\big)\Big) - \Phi\Big(t, \mathbf{Z}(t)\Big)\Big]\, d\mathcal{N}(t), \end{aligned} \tag{2}$$

where n is the number of state variables.

This differential formula is the startpoint to derive the differential equations for moments. However since the response process $\mathbf{Z}(t)$ is the functional of the random mark variables $P(t_i) = P_i$, $t_i < t$ and of the counting process $\mathcal{N}(\tau)$, $\tau < t$, this formula can only be used effectively, yielding explicit moments at the right-hand sides of equations, if the averaging of the last term is feasible, i.e. if the correlation between the expression in square brackets and $d\mathcal{N}(t)$ can be split. This is possible if the increments of the stochastic point process are independent, the random mark variables are independent and these two are also mutually independent. This is so if the underlying process is a compound Poisson process, i.e. if the augmented dynamical system is Poisson-driven. However if the increments of the regular counting process $\mathcal{N}(t)$ can be expressed as

$$d\mathcal{N}(t) = \rho\big(N(t)\big)dN(t), \tag{3}$$

where $N(t)$ is a homogeneous Poisson counting process and $\rho(N(t))$ is a suitably chosen zero-memory transformation, the problem can be converted to a Poisson-driven one at the expense of introducing a number of auxiliary state variables. Such a transformation, being a linear function of the auxiliary state variables, each of whose is an exponential transformation of $N(t)$, has been found for a class of Erlang renewal processes.[11,12,17] Then the structural state vector $\mathbf{Z}_1(t)$ augmented by the auxiliary variables $\mathbf{Z}_2(t)$ is governed by the stochastic differential equations

$$d\mathbf{Z}(t) = \mathbf{c}\big(\mathbf{Z}(t),t\big)dt + \mathbf{d}\big(\mathbf{Z}(t),t,P(t)\big)dN(t), \quad \mathbf{Z}(t_0) = \mathbf{z}_0 \tag{4}$$

$$\begin{aligned} \mathbf{Z}(t) &= \begin{bmatrix} \mathbf{Z}_1(t) \\ \mathbf{Z}_2(t) \end{bmatrix}, \ \mathbf{c}\big(\mathbf{Z}(t),t\big) = \begin{bmatrix} \mathbf{c}_1\big(\mathbf{Z}_1(t),t\big) \\ \mathbf{0} \end{bmatrix}, \\ \mathbf{d}\big(\mathbf{Z}(t),t,P(t)\big) &= \begin{bmatrix} \mathbf{d}_1\big(\mathbf{Z}_1(t),t\big)g\big(\mathbf{Z}_2(t)\big)P(t) \\ \mathbf{d}_2\big(\mathbf{Z}_2(t),t\big) \end{bmatrix}, \end{aligned} \tag{5}$$

where $P(t)$ assumes the values $P(t_i) = P_i$ at the times t_i of the Poisson events and P_i are mutually independent and identically distributed as P. Further, $g\big(\mathbf{Z}_2(t)\big) = \rho\big(N(t)\big)$ is a known function of the auxiliary variables.

Although the structural state vector $\mathbf{Z}_1(t)$ is a non-Markov process, the augmented state vector $\mathbf{Z}(t)$ becomes a non-diffusive Markov process.

3 Analytical solution techniques

3.1 Equations for moments and modified closure approximations

For a general Poisson-driven pulse problem governed by equations (4) and (5) the equations for the mean values, the second-, third- and fourth-order joint central moments of the response, are obtained as[11,15]

$$\dot{\mu}_i(t) = E\left[c_i(\mathbf{Z}(t),t)\right] + \nu(t)E\left[d_i(\mathbf{Z}(t),t,P)\right] \tag{6}$$

$$\begin{aligned} \dot{\kappa}_{ij}(t) &= 2\left\{E\left[Z_i^0\left(c_j^0(\mathbf{Z}^0(t),t) + \nu(t)d_j(\mathbf{Z}(t),t,P)\right)\right]\right\}_s \\ &\quad +\nu(t)E\left[d_i(\mathbf{Z}(t),t,P)d_j(\mathbf{Z}(t),t,P)\right], \end{aligned} \tag{7}$$

$$\begin{aligned} \dot{\kappa}_{ijk}(t) &= 3\left\{E\left[Z_i^0 Z_j^0\left(c_k^0(\mathbf{Z}^0(t),t) + \nu(t)d_k(\mathbf{Z}(t),t,P)\right)\right]\right\}_s \\ &\quad +3\nu(t)\left\{E\left[Z_i^0 d_j(\mathbf{Z}(t),t,P)d_k(\mathbf{Z}(t),t,P)\right]\right\}_s \\ &\quad +\nu(t)E\left[d_i(\mathbf{Z}(t),t,P)d_j(\mathbf{Z}(t),t,P)d_k(\mathbf{Z}(t),t,P)\right], \end{aligned} \tag{8}$$

$$\begin{aligned} \dot{\kappa}_{ijkl}(t) &= 4\left\{E\left[Z_i^0 Z_j^0 Z_k^0\left(c_l^0(\mathbf{Z}^0(t),t) + \nu(t)d_l(\mathbf{Z}(t),t,P)\right)\right]\right\}_s \\ &\quad +6\nu(t)\left\{E\left[Z_i^0 Z_j^0 d_k(\mathbf{Z}(t),t,P)d_l(\mathbf{Z}(t),t,P)\right]\right\}_s \\ &\quad +4\nu(t)\left\{E\left[Z_i^0 d_j(\mathbf{Z}(t),t,P)d_k(\mathbf{Z}(t),t,P)d_l(\mathbf{Z}(t),t,P)\right]\right\}_s \\ &\quad +\nu(t)E\left[d_i(\mathbf{Z}(t),t,P)d_j(\mathbf{Z}(t),t,P)d_k(\mathbf{Z}(t),t,P)d_l(\mathbf{Z}(t),t,P)\right], \end{aligned} \tag{9}$$

where $Z_i^0(t) = Z_i(t) - \mu_i(t)$ and $c_j^0(\mathbf{Z}^0(t), t) = c_j(\mathbf{Z}(t), t) - E\left[c_j(\mathbf{Z}(t), t)\right]$ denote the components of the zero-mean (centralized) state vector and drift vector, and $\{...\}_s$ denotes the Stratonovich symmetrizing operator.[16]

It is no doubt that the accuracy of the results obtained with the help of a closure technique depends on the ability of the tentative probability density function for the evaluation of the unknown expectations entering the moment equations, to qualitatively model the actual density function, i.e. it should have freedom to represent possible multimodal or multipeak shapes and discrete probability components.

Consider the dynamical system subjected to a random train of impulses and to initial conditions $\mathbf{Z}(t_0) = \mathbf{z}_0$. If in the time interval $[t_0, t[$ no impulse occurred, the system has performed the deterministic drift motion from the initial state $\mathbf{z}_0$ at the time t_0 to the state $\mathbf{z}(t) = \mathbf{e}(t|\mathbf{z}_0, t_0)$ at the time t, or it has been at rest, in the case of zero initial conditions. Notice that $\mathbf{e}(t_0|\mathbf{z}_0, t_0) = \mathbf{z}_0$.

If the train of impulses is driven by a homogeneous Poisson process, the probability P_0 of no impulse occurrence in the time interval $[t_0, t[$ is expressed as

$$P_0(t|t_0) = \Pr\{N(t) = 0 | N(t_0) = 0\} = \exp(-\nu(t - t_0)). \tag{10}$$

The probability $P_0(t|t_0)$ may be high, close to the unity, if the length $t - t_0$ of the time interval is small, i.e. at the early transient stage, especially if also the mean arrival rate ν is small.

Joint probability density function of the state vector $\mathbf{Z}(t)$ can be represented in form of the sum of the continuous and discrete parts as

$$\begin{aligned} f_{\mathbf{Z}}(\mathbf{z}, t) &= f_{\mathbf{Z}}(\mathbf{z}, t \mid N(t) = 0) \Pr\{N(t) = 0\} + f_{\mathbf{Z}}(\mathbf{z}, t \mid N(t) > 0) \Pr\{N(t) > 0\} \\ &= P_0(t|t_0) \prod_{i=1}^{n} \delta\Big(z_i - e_i(t|\mathbf{z}_0, t_0)\Big) + (1 - P_0(t|t_0)) f_{\mathbf{Z}}^0(\mathbf{z}, t), \end{aligned} \tag{11}$$

where $e_i(t|\mathbf{z}_0, t_0)$ denotes the drift from the initial state $\mathbf{z}_0$ at $t = t_0$ obtained from (1) for $d\mathcal{N}(t) = 0$. Hence the system must be at the position $z_i = e_i(t|\mathbf{z}_0, t_0)$ with probability one as specified by the delta spikes, if no impulses have arrived. So, $f_{\mathbf{Z}}^0(\mathbf{z}, t) = f_{\mathbf{Z}}(\mathbf{z}, t \mid N(t) > 0)$ denotes the continuous joint probability density function on condition that at least one impulse has occurred during the preceding time interval $[t_0, t[$. Expectations evaluated with respect to $f_{\mathbf{Z}}^0(\mathbf{z}, t)$ are denoted as $E[\cdots]_0$. In particular, the conditional mean value function and the conditional joint central moments of the order r are denoted as $\mu_i^0(t)$ and $\kappa_{i_1 i_2 \cdots i_r}^0(t)$. The relationships between unconditional and conditional moments read[15]

$$\mu_i(t) = P_0(t|t_0) e_i(t|\mathbf{z}_0, t_0) + (1 - P_0(t|t_0)) \mu_i^0(t), \tag{12}$$

$$\kappa_{i_1 i_2 \cdots i_r}(t) = E\left[\prod_{j=1}^{r} Z_{i_j}^0(t)\right] = P_0 \prod_{j=1}^{r}\left(e_{i_j} - \mu_{i_j}(t)\right) + (1-P_0)\, E\left[\prod_{j=1}^{r} Z_{i_j}^0(t)\right]_0 =$$
$$P_0 \prod_{j=1}^{r}\left(e_{i_j} - \mu_{i_j}(t)\right) + (1-P_0)\, E\left[\prod_{j=1}^{r}\left(Z_{i_j}(t) - \mu_{i_j}^0(t) + \frac{P_0}{1-P_0}(\mu_{i_j}(t) - e_{i_j})\right)\right]_0 =$$
$$\left(P_0 + \frac{(-P_0)^r}{(1-P_0)^{r-1}} \prod_{j=1}^{r}\left(e_{i_j} - \mu_{i_j}(t)\right)\right) + \tag{13}$$
$$(1-P_0)\sum_{j=0}^{r-1}\binom{r}{j}\left\{\kappa_{i_1 i_2 \cdots i_{r-j}}^0(t) \prod_{k=r-j+1}^{r}(\mu_{i_k}(t) - e_{i_k})\right\}_s \cdot \left(\frac{P_0}{1-P_0}\right)^j,$$

where (12) has been used. Further, the arguments of $P_0(t|t_0)$ and $e_i(t|\mathbf{z}_0, t_0)$ have been omitted for ease of notation. The inverse relationship can be similarly derived in few steps

$$\kappa_{i_1 i_2 \cdots i_r}^0(t) = E\left[\prod_{j=1}^{r}\left(Z_{i_j}(t) - \mu_{i_j}^0(t)\right)\right]_0 =$$
$$-\frac{P_0}{1-P_0}\prod_{j=1}^{r}\left(e_{i_j} - \mu_{i_j}^0(t)\right) + \frac{1}{1-P_0} E\left[\prod_{j=1}^{r}\left(Z_{i_j}(t) - \mu_{i_j}^0(t)\right)\right] =$$
$$-\frac{P_0}{(1-P_0)^{r+1}}\prod_{j=1}^{r}\left(e_{i_j} - \mu_{i_j}(t)\right) + \frac{1}{1-P_0} E\left[\prod_{j=1}^{r}\left(Z_{i_j}^0(t) + \frac{P_0}{1-P_0}(e_{i_j}(t) - \mu_{i_j})\right)\right] = \tag{14}$$
$$\frac{P_0^r - P_0}{(1-P_0)^{r+1}}\prod_{j=1}^{r}\left(e_{i_j} - \mu_{i_j}(t)\right) +$$
$$\frac{1}{1-P_0}\sum_{j=0}^{r-1}\binom{r}{j}\left\{\kappa_{i_1 i_2 \cdots i_{r-j}}(t) \prod_{k=r-j+1}^{r}\left(e_{i_k} - \mu_{i_k}(t)\right)\right\}_s \left(\frac{P_0}{1-P_0}\right)^j.$$

For systems with polynomial drift vectors the following modified cumulant neglect closure scheme may be used. In case of closure at the order N all centralized moments $\kappa_{i_1 i_2 \cdots i_r}^0(t)$, $r > N$ with respect to the continuous joint probability density function $f_{\mathbf{Z}}^0(\mathbf{z}, t)$ are first expressed in terms of corresponding centralized moments of the order $j \leq N$ by means of the ordinary cumulant neglect closure approximations. This will work if $f_{\mathbf{Z}}^0(\mathbf{z}, t)$ is monomodal and not deviating too much from a multivariate normal distribution, since the joint cumulants are zero in the latter case. Then, the corresponding unconditional moments $\kappa_{i_1 i_2 \cdots i_r}(t)$, $r > N$ may be expressed in terms of the centralized moments $\kappa_{i_1 i_2 \cdots i_j}^0(t)$, $j \leq N$ by means of (13). Finally, all joint moments $\kappa_{i_1 i_2 \cdots i_j}^0(t)$, $j \leq N$ within this expression can be expressed in terms of $\kappa_{i_1 i_2 \cdots i_j}(t)$, $j \leq N$ by means of (14), and the required closure scheme is obtained. In case of closure at the order $N = 4$ the explicit closure approximations for the 5th and 6th order joint centralized moments have been derived for the case $e_i(t|\mathbf{z}_0, t_0) = 0$.[5,15]

The modified cumulant-neglect closure technique proved to be effective in the case of evaluating transient response moments for low mean rates of impulses, i.e. for sparse trains of impulses[5]. Stationary moments, even though the mean rate is low, can often be evaluated

with the help of ordinary cumulant-neglect closure approximations, because for very long time intervals the spike of the probability density function becomes rather insignificant. However, as the experience of the authors shows, numerical integration scheme combined with ordinary cumulant-neglect closure approximations runs into instability at the early transient stage if the mean rate of impulses is low.

3.2 Petrov-Galerkin method to solve the forward and backward Kolmogorov-Feller equations

The forward Kolmogorov-Feller equation for the joint probability density $f_{\mathbf{Z}}(\mathbf{z},t)$ of the state vector $\mathbf{Z}(t)$ reads in the case of absorbtion on a part of the boundary[15,18]

$$\left.\begin{array}{lll} \frac{\partial}{\partial t} f_{\mathbf{Z}}(\mathbf{z},t) = \mathcal{K}_{\mathbf{z},t}[f(\mathbf{z},t)] \quad , \quad \forall\, t \in]t_0,t_1] & , & \forall\, \mathbf{z} \in S_t \\ f_{\mathbf{Z}}(\mathbf{z},t_0) = f_0(\mathbf{z}) & , & \forall\, \mathbf{z} \in S_{t_0} \\ f_{\mathbf{Z}}(\mathbf{z},t) = 0 \quad , \quad \forall\, t \in]t_0,t_1] & , & \forall\, \mathbf{z} \in \partial S_t^{(0)} \cup \partial S_t^{(2)} \end{array}\right\}, \tag{15}$$

where S_t is the solution set at the time t, bounded by the surface $\partial S_t = \partial S_t^{(0)} \cup \partial S_t^{(1)} \cup \partial S_t^{(2)}$. $\partial S_t^{(2)}$ is the non-accessible (natural) part of the boundary, whereas the accessible boundary is made up of the exit part $\partial S_t^{(1)}$ and the entrance part $\partial S_t^{(0)}$. As indicated, absorbtion boundary conditions must be specified on $\partial S_t^{(0)}$.

The forward integro-differential Kolmogorov-Feller operator is given by [15,18]

$$\begin{aligned} &\mathcal{K}_{\mathbf{z},t}[f_{\mathbf{Z}}(\mathbf{z},t)] = \\ &-\sum_i \frac{\partial}{\partial z_i}[c_i(\mathbf{z},t) f_{\mathbf{Z}}(\mathbf{z},t)] + \nu(t) \int_{\mathcal{P}} \left[f_{\mathbf{Z}}\big(\mathbf{a}(\mathbf{z},p,t),t\big) \frac{1}{|\mathcal{J}|} - f_{\mathbf{Z}}(\mathbf{z},t) \right] f_P(p)\, dp, \end{aligned} \tag{16}$$

where

$$\mathcal{J} = \det\left(\mathbf{I} + \frac{\partial \mathbf{d}(\mathbf{a}(\mathbf{z},p,t),t)}{\partial \mathbf{y}^T} \right), \tag{17}$$

and $\mathbf{a} = \mathbf{a}(\mathbf{z},p,t)$ is the inverse transformation of

$$\mathbf{z} = \mathbf{a} + \mathbf{d}(\mathbf{a},p,t) \tag{18}$$

$\frac{\partial \mathbf{d}}{\partial \mathbf{y}^T}$ is the gradient of $\mathbf{d}(\mathbf{y},p,t)$ with respect to $\mathbf{y}$ and $f_P(p)$ is the probability density of the mark variable P.

The backward Kolmogorov-Feller equation with absorbtion boundary conditions is [15]

$$\left.\begin{array}{lll} \frac{\partial}{\partial t} f_{\mathbf{Z}}(\mathbf{z},t) + \mathcal{K}^T_{\mathbf{z},t}[f_{\mathbf{Z}}(\mathbf{z},t)] = 0 \quad , \quad \forall\, t \in [t_0,t_1[& , & \forall\, \mathbf{z} \in S_t \\ f_{\mathbf{Z}}(\mathbf{z},t_1) = f_1(\mathbf{z}) & , & \forall\, \mathbf{z} \in S_{t_1} \\ f_{\mathbf{Z}}(\mathbf{z},t) = 0 \quad , \quad \forall\, t \in [t_0,t_1[& , & \forall\, \mathbf{z} \in \partial S_t^{(1)} \cup \partial S_t^{(2)} \end{array}\right\}, \tag{19}$$

where $f_1(\mathbf{z})$ is the terminal value of the unknown function $f_{\mathbf{Z}}(\mathbf{z},t)$. In this case the absorbtion boundary condition is specified on $\partial S_t^{(1)}$.

The backward integro-differential Kolmogorov-Feller operator is given by

$$\mathcal{K}^T_{\mathbf{z},t}[f_{\mathbf{Z}}(\mathbf{z},t)] =$$

$$\sum_i c_i(\mathbf{z},t)\frac{\partial}{\partial z_i}f_{\mathbf{Z}}(\mathbf{z},t) + \nu(t)\int_{\mathcal{P}}\Big[f_{\mathbf{Z}}\Big(\mathbf{z}+\mathbf{d}(\mathbf{z},p,t)\Big) - f_{\mathbf{Z}}\Big(\mathbf{z},t\Big)\Big]f_P(p)dp. \tag{20}$$

The Galerkin method for solving the boundary and initial value problems (15) and (19) consists in expanding the unknown function $f_{\mathbf{Z}}(\mathbf{z},t)$ in series of approximating shape functions and expanding the variational field in series of weighting functions. For the problem (15) the shape functions must fulfil the boundary condition $N_i(\mathbf{z}) = 0$, $\mathbf{z} \in \partial S_t^{(0)} \cup \partial S_t^{(2)}$, whereas the weighting functions $V_i(\mathbf{z}) = 0$, $\mathbf{z} \in \partial S_t^{(1)}$. In contrast, for the problem (19) the shape functions fulfil the boundary condition $N_i(\mathbf{z}) = 0$, $\mathbf{z} \in \partial S_t^{(1)} \cup \partial S_t^{(2)}$, and the weighting functions $V_i(\mathbf{z}) = 0$, $\mathbf{z} \in \partial S_t^{(2)}$. Further the shape and weighting functions must be sufficiently smooth that $\mathcal{K}_{\mathbf{z},t}[...]$ and $\mathcal{K}^T_{\mathbf{z},t}[...]$ may become adjoint operators when integrated over S_t. In order to achieve numerical stability due to the large Courant number in part of the mesh an upwind differencing in the weighting function becomes necessary as performed in the Petrov-Galerkin variational method.

It has been found that for a two-dimensional problem, i.e. the Duffing oscillator under trains of impulses with high to moderate mean rate, the Petrov-Galerkin method provides very accurate solution of the backward Kolmogorov-Feller equation [8]. Unfortunately it has not been possible to devise the Petrov-Galerkin technique suitable for sparse trains of pulses. Another drawback of the method is that, despite todays technology, the solution is not feasible for state vectors of dimension larger than 4 or 5, not even with parallellization of the calculations.

3.3 Cell-to-cell mapping technique

The problem is discretized in time and space. The time axis is divided into small time intervals Δt and $f_{\mathbf{Z}}(\mathbf{z_0},t_i)$ denotes the first order probability density function at the time $t_i = t_0 + i\Delta t$, $i = 0,1,2,\ldots$. The probability density function at the subsequent instant t_{i+1}, is given by the convolution integral

$$f_{\mathbf{Z}}(\mathbf{z},t_{i+1}) = \int_{S_{t_i}} q_{\mathbf{Z}}(\mathbf{z},t_{i+1}|\mathbf{z}_0,t_i) f_{\mathbf{Z}}(\mathbf{z}_0,t_i)d\mathbf{z}_0, \tag{21}$$

where $q_(\mathbf{z},t_{i+1}|\mathbf{z}_0,t_i)$ is the transition probability density function of the state vector from the state $\mathbf{Z}(t_0) = \mathbf{z}_0$ at the time t_i to the state $\mathbf{Z}(t) = \mathbf{z}$ at the time t_{i+1} and S_{t_i} is the sample space.

If the time interval Δt is short enough and if the mean rate is low, it follows from the Poisson law that the probability of occurrence of more than one impulse in this time interval may be neglected and the following asymptotic form of the transition probability density may be assumed,[10]

$$q_{\mathbf{Z}}(\mathbf{z}, t_{i+1}|\mathbf{z}_0, t_i) = P_0(t_{i+1}|t_1) q_{\mathbf{Z}}^{(0)}(\mathbf{z}, t_{i+1}|\mathbf{z}_0, t_i) + \Big(1 - P_0(t_{i+1}|t_i)\Big) q_{\mathbf{Z}}^{(1)}(\mathbf{z}, t_{i+1}|\mathbf{z}_0, t_i) + \mathrm{O}\Big((\nu\Delta t)^n\Big), \tag{22}$$

where

$$P_0(t_{i+1}|t_i) = \Pr\{N(t_{i+1}) = 0 | N(t_i) = 0\} = \exp\left(-\nu(t_{i+1} - t_1)\right) \tag{23}$$

and

$$q_{\mathbf{Z}}^{(0)}(\mathbf{z}, t_{i+1}|\mathbf{z}_0, t_i) = \delta\Big(z_1 - e_1(t_{i+1}|\mathbf{z}_0, t_i)\Big) \cdots \delta\Big(z_n - e_n(t_{i+1}|\mathbf{z}_0, t_i)\Big), \tag{24}$$

is the transition probability density conditional on no impulse arrival and $\mathbf{e}(t_{i+1}|\mathbf{z}_0, t_i)$ has the same meaning as in equation (11). Surely, (22) is fulfilled at best for sparse pulse trains where ν is small. Hence the method is expected to work at best in this case. The transition probability density $q_{\mathbf{Z}}^{(1)}(\mathbf{z}, t_{i+1}|\mathbf{z}_0, t_i)$ conditional on one impulse arrival is of continuous type. Hence the expansion (22) is based on pretty much the same idea as the modified cumulant neglect closure scheme derived from (11). Algorithms have been devised for evaluation of $q_{\mathbf{Z}}^{(1)}(\mathbf{z}, t_{i+1}|\mathbf{z}_0, t_i)$.[10] Especially a method based on a Taylor expansion in the impulse magnitude P is tractable.[9,12]

The sample space is divided into a finite number M of small volumes (cells) [19], where the volume $\Delta\mathbf{z}_k$ of the mesh element is centered at $\mathbf{z}_k$. Assume that $\Delta\mathbf{z}_k$ is sufficiently small for $q_{\mathbf{Z}}(\mathbf{z}_j, t_i + \Delta t|\mathbf{z}_k, t_i)$ and $f_{\mathbf{Z}}(\mathbf{z}_k, t_i)$ to be approximately constant throughout the cell. The probability of being in the kth cell at the time t_i is

$$\pi_k^{(i)} = f_{\mathbf{Z}}(\mathbf{z}_k, t_i)\Delta\mathbf{z}_k. \tag{25}$$

The probability $\pi_j^{(i+1)}$ of being in the jth cell at the time t_{i+1} is then given by

$$\pi_j^{(i+1)} = \sum_{k=1}^{M} Q_{jk}\pi_k^{(i)}, \quad j = 1, \ldots, M, \tag{26}$$

$$Q_{jk} = \Delta\mathbf{z}_k q_{\mathbf{Z}}(\mathbf{z}_j, t_i + \Delta t|\mathbf{z}_k, t_i). \tag{27}$$

where Q_{jk} is the the element in the jth row and kth column of the transition probability matrix. The transition probability Q_{jk} will not depend on time t_i if the Markov process is stationary.

The transition of states can be represented by the matrix equation

$$\boldsymbol{\pi}^{(i+1)} = \mathbf{Q}\boldsymbol{\pi}^{(i)} = \mathbf{Q}^i\boldsymbol{\pi}^{(0)}, \tag{28}$$

where $\mathbf{Q}^i = \mathbf{Q}\cdots\mathbf{Q}$ ($\mathbf{Q}$ multiplied by itself i times), $\boldsymbol{\pi}^{(i)}$ is an M-dimensional vector of the state probabilities $\pi_k^{(i)}$ after ith transition, and $\boldsymbol{\pi}^{(0)}$ denotes the initial distribution at the time t_0. The stationary distribution $\boldsymbol{\pi}^{(\infty)}$ may be obtained after infinite many transitions as $i \to \infty$. Obviously, $\boldsymbol{\pi}^{(\infty)} = \mathbf{Q}\boldsymbol{\pi}^{(\infty)}$, determining $\boldsymbol{\pi}^{(\infty)}$ as the normalized eigenvector related to the eigenvalue $\lambda = 1$ of the matrix $\mathbf{Q}$.

The cell-to-cell mapping technique described above has been applied to a Duffing oscillator under Poisson[10] and renewal impulses [12], hence for the problems two and three (one auxiliary) state variables, respectively. The stationary marginal displacement and velocity response probability densities[10,12] and reliability function [10] have been evaluated. The results obtained confirmed the congesture that the method is highly effective for sparse pulse trains.[9,10,12]

4 Simulation technique

The sample path of the random train of impulses is obtained by generating the sequence of interarrival times and the sequence of impulses magnitudes. The interarrival times may be sampled directly from the given probability distribution, or in the case of an Erlang process, with the help of a generated train of Poisson distributed points. The impulses magnitudes are generated as sample values of a random variable P with a prescribed probability distribution.

The response sample curve is next obtained by numerical integration, of the homogeneous governing equation of motion between the impulses arrival times, whereas at every time point of an impulse occurrence the velocity response is increased by a jump, which gives the updated initial condition for the next interarrival time interval. Usually a standard 4th order Runge-Kutta technique may be used for numerical integration.

Usually an ensemble of 50 000 response sample curves was generated for the problems considered by the authors, and in the case of ergodic sampling[12] the generated response sample curve had a length of 4 000 000 natural periods of a comparative linear oscillator.

The simulation technique is quite straightforward to apply for a large class of regular point processes if the probability distribution of the interarrival times is specified. This class certainly embraces all the renewal processes.

Possible extensions of the simulation technique can be performed for the problems in which the interarrival times and the impulse magnitudes are correlated random variables.

5 Illustrative numerical results

Consider a Duffing oscillator under a Poisson driven train of impulses. Let $Y(t)$ and $\dot{Y}(t)$ denote, respectively, the displacement and velocity responses of the Duffing oscillator. Then $\mathbf{Z}(t)$, $\mathbf{c}(\mathbf{Z}(t),t)$ and $\mathbf{d}(\mathbf{Z}(t),t)$ of equation (4) are given by

$$\mathbf{Z}(t)=\begin{bmatrix} Y(t) \\ \dot{Y}(t) \end{bmatrix}, \mathbf{c}(\mathbf{Z}(t),t)=\begin{bmatrix} \dot{Y}(t) \\ -2\zeta\omega_0\dot{Y}(t)-\omega_0^2 Y(t)-\varepsilon\omega_0^2 Y^3(t) \end{bmatrix}, \mathbf{d}=\begin{bmatrix} 0 \\ P(t). \end{bmatrix}. \quad (29)$$

where ζ is the damping ratio, ω_0 is the natural frequency of the linear oscillator corresponding to the Duffing oscillator and ε is the non-linearity parameter. The data assumed for the Duffing oscillator is: $\omega_0 = 1s^{-1}$, $\zeta = 0.05$, $\varepsilon = 0.5$.

Computations have been performed for two different values of the mean arrival rates fo impulses: $\nu = 0.1\omega_0$ and $\nu = 0.01\omega_0$.

The random magnitudes of impulses have been assumed as centralized, Rayleigh distributed random variables, calibrated in such a way that in both cases $\nu E[P^2]$ has the same value and that the variance of the stationary response of a comparative linear oscillator has unit value (cf. e.g. [12]).

A uniform 50×50 mesh has been used with the limits $[-6\sigma_Y, 6\sigma_Y] \times [-6\sigma_{\dot{Y}}, 6\sigma_{\dot{Y}}]$, where σ_Y and $\sigma_{\dot{Y}}$ are standard deviations of the stationary response of a comparative linear oscillator. The length of a transition time interval has been assumed as $\Delta t = T_0/2$, where $T_0 = 2\pi/\omega_0$ is the natural period of the comparative linear oscillator. The transient marginal probability density functions of the displacement and the velocity response evaluated at the time points:

$t = T_0$, $t = 2T_0$ and $t = 10T_0$ for $\nu = 0.1\omega_0$ and $\nu = 0.01\omega_0$ are shown in Figures 1 and 2, respectively. The solid line (——) and dashed line (- - - -) represent, respectively, the analytical and simulation results obtained in the case of non-zero initial conditions: $Y(0) = 1$, $\dot{Y}(0) = 0$. The dotted line (.....) and the dashed-dotted line (-.-.-.-) represent, respectively, the analytical and simulation results obtained in the case of zero initial conditions.

The simulation results have been based on the ensemble of 100 000 of the response sample functions obtained by numerical integration, with the help of 4th order Runge-Kutta technique, of the homogeneous governing equation of motion (1) in the time intervals between the impulses arrivals, with the updated initial conditions for each time interval, due to the jump in the velocity response process at each impulse arrival time.

It is seen that the agreement between the analytical and simulation results is certainly very good in the case of zero-mean initial conditions and in both cases $\nu = 0.1\omega_0$ and $\nu = 0.01\omega_0$. It is so in the very early transient stage, i.e. at $t = T_0$, $t = 2T_0$ as well as at $t = 10T_0$, in which case the response is almost stationary as shows the zero-mean value of the velocity response. The discrepancy of the peak values in the case of very sharp spikes of the density curves, should be attributed to the fact that in these cases the mesh is not fine enough.

In the case of non-zero mean initial conditions, for $\nu = 0.1\omega_0$, the analytically predicted probability density curves are also sufficiently close to the simulated curves except only, as before, for the peak values. For $\nu = 0.01\omega_0$ the analytical prediction is only accurate enough in the nearly stationary case of the displacement response (Fig. 2 e)). Significant discrepancy between the analytical nad simulation results in other cases may be explained by the fact that analytical, piece-wise linear, probability density functions have been obtained with a mesh too coarse to idealize sharp spikes of the probability density. Unfortunately assuming a finer mesh, e.g. 100×100, resulting in a $10^4 \times 10^4$ transition probabilities matrix, turned out to be a computational task excessively large for the computers available.

6 Conclusions

A brief review of analytically -numerical techniques, having been developed by the authors and co-workers, for non-linear dynamical systems under random impulses is done. Their advantages and shortcomings are discussed. Concluding, authors wish to express the opinion that the avenue of further development of the solution techniques for pulse problems in stochastic dynamics should be directed onto optimization of the simulation techniques.

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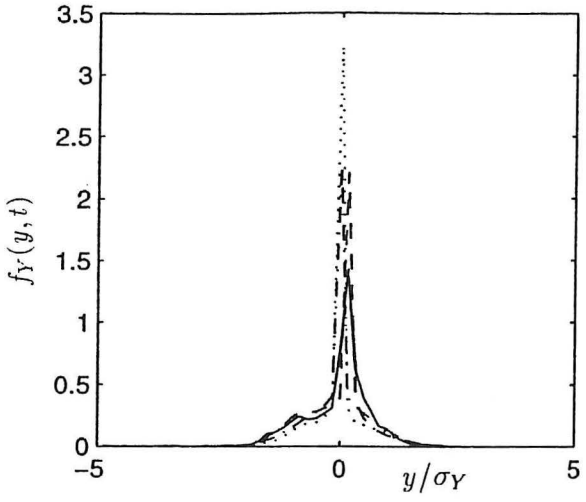

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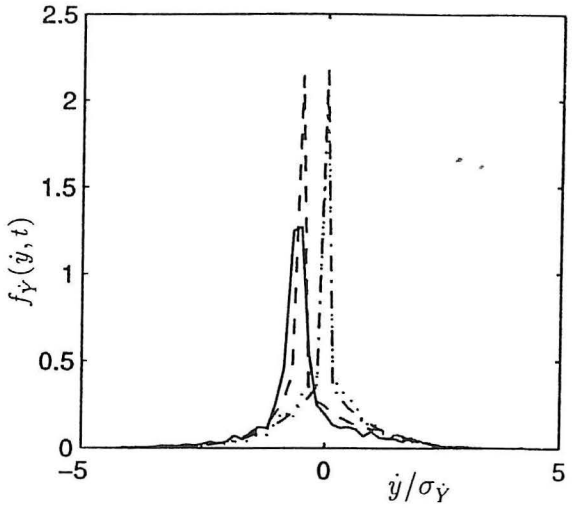

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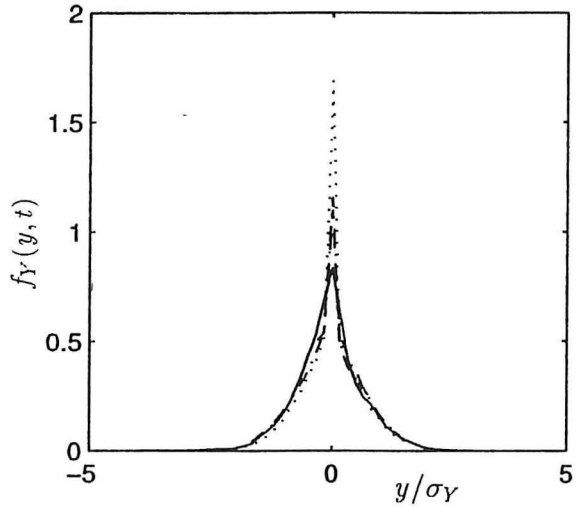

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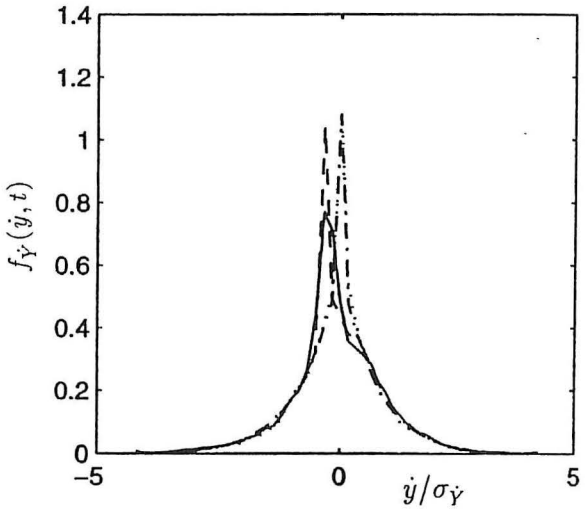

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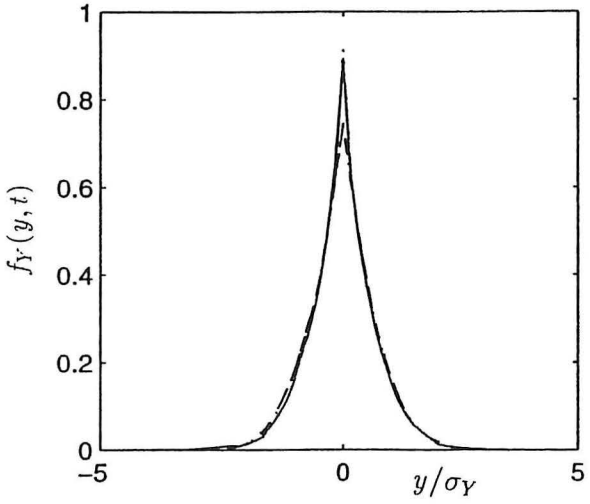
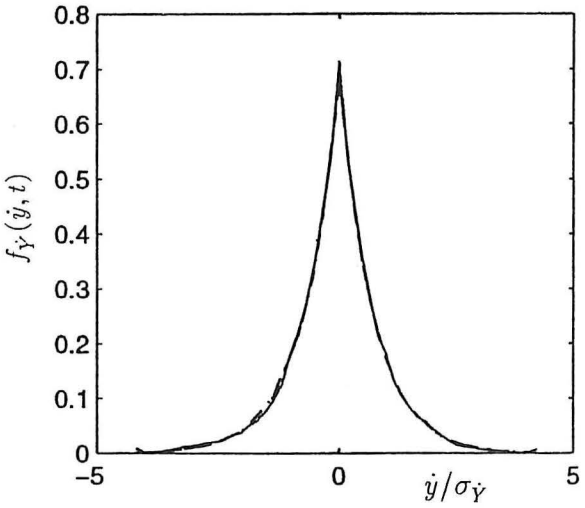
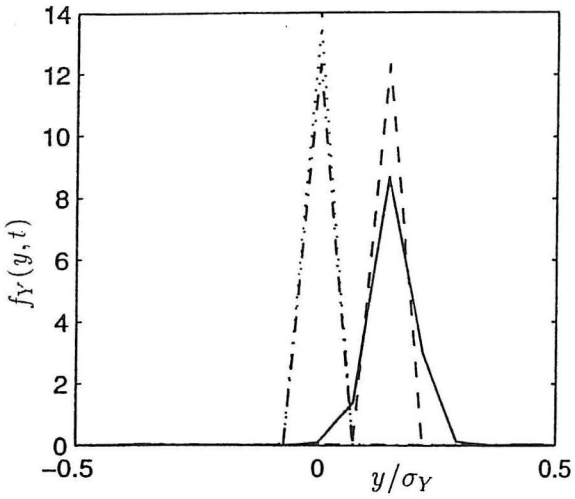


e) Displacement response, $t = 10T_0$.



f) Velocity response, $t = 10T_0$.

Fig. 1. Transient marginal probability densities $f_Y(y,t)$ of the displacement response and $f_{\dot{Y}}(\dot{y},t)$ of the velocity response of a Duffing oscillator to a Poisson impulse process with mean rate $\nu = 0.1\omega_0$.



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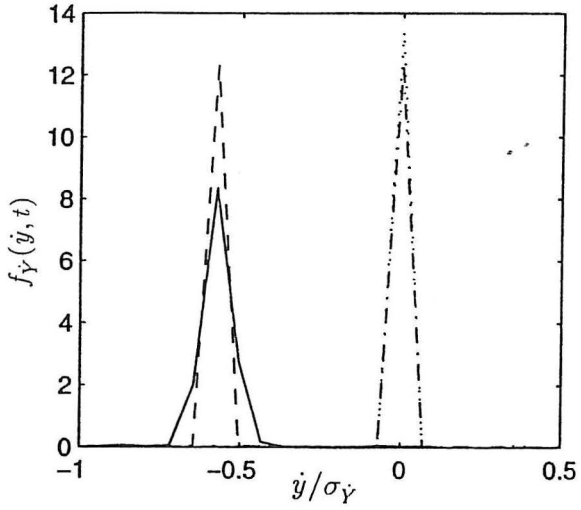

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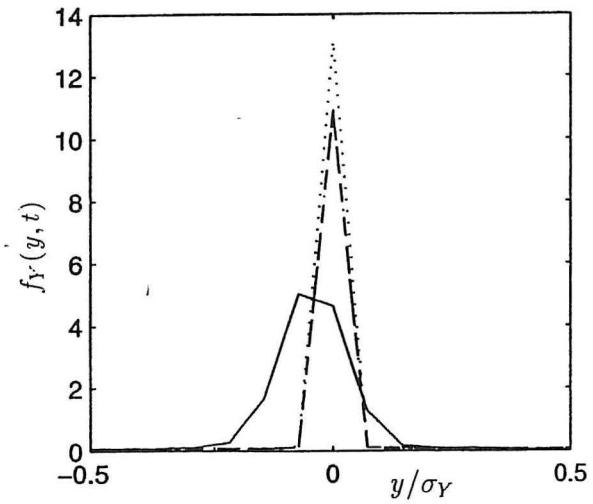

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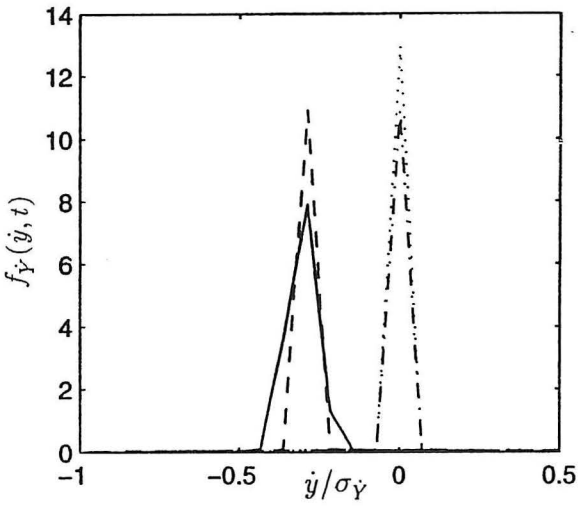

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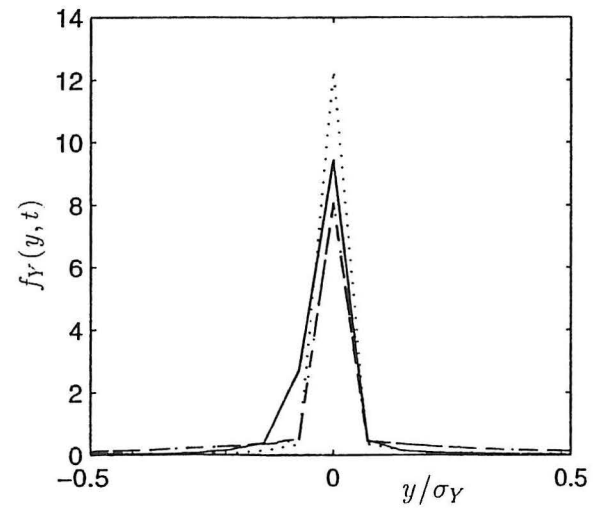

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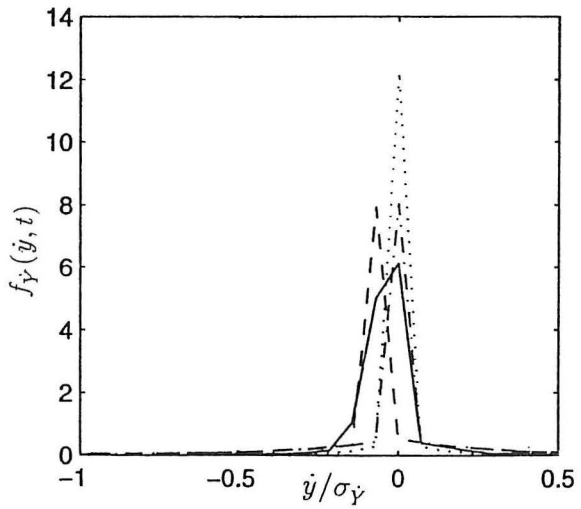

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Fig. 2. Transient marginal probability densities $f_Y(y,t)$ of the displacement response and $f_{\dot{Y}}(\dot{y},t)$ of the velocity response of a Duffing oscillator to a Poisson impulse process with mean rate $\nu = 0.01\omega_0$.

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Department of Building Technology and Structural Engineering
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