

## **On the Optimal Location of Sensors for Parametric Identification of Linear Structural Systems**

Kirkegaard, Poul Henning; Brincker, Rune

*Publication date:*  
1992

*Document Version*  
Publisher's PDF, also known as Version of record

[Link to publication from Aalborg University](#)

*Citation for published version (APA):*  
Kirkegaard, P. H., & Brincker, R. (1992). *On the Optimal Location of Sensors for Parametric Identification of Linear Structural Systems*. Dept. of Building Technology and Structural Engineering, Aalborg University. Fracture and Dynamics Vol. R9239 No. 40

### **General rights**

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal -

### **Take down policy**

If you believe that this document breaches copyright please contact us at [vbn@aub.aau.dk](mailto:vbn@aub.aau.dk) providing details, and we will remove access to the work immediately and investigate your claim.

3004  
ex3

---

**INSTITUTTET FOR BYGNINGSTEKNIK**  
DEPT. OF BUILDING TECHNOLOGY AND STRUCTURAL ENGINEERING  
AALBORG UNIVERSITETSCENTER • AUC • AALBORG • DANMARK

---

**FRACTURE AND DYNAMICS**  
**PAPER NO. 40**

**Presented at the "17th International Seminar on Modal Analysis and Structural Dynamics", Leuven, Belgium, September 21-25, 1992**

---

**P. H. KIRKEGAARD & R. BRINCKER**  
**ON THE OPTIMAL LOCATION OF SENSORS FOR PARAMETRIC IDENTIFICATION OF LINEAR STRUCTURAL SYSTEMS**  
**DECEMBER 1992**

---

**ISSN 0902-7513 R9239**



The FRACTURE AND DYNAMICS papers are issued for early dissemination of research results from the Structural Fracture and Dynamics Group at the Department of Building Technology and Structural Engineering, University of Aalborg. These papers are generally submitted to scientific meetings, conferences or journals and should therefore not be widely distributed. Whenever possible reference should be given to the final publications (proceedings, journals, etc.) and not to the Fracture and Dynamics papers.

---

**INSTITUTTET FOR BYGNINGSTEKNIK**  
DEPT. OF BUILDING TECHNOLOGY AND STRUCTURAL ENGINEERING  
AALBORG UNIVERSITETSCENTER • AUC • AALBORG • DANMARK

---

FRACTURE AND DYNAMICS  
PAPER NO. 40

Aalborg Universitetsbibliotek

530003856681



Presented at the "17th International Seminar on Modal Analysis and Structural Dynamics", Leuven, Belgium, September 21-25, 1992

---

P. H. KIRKEGAARD & R. BRINCKER  
ON THE OPTIMAL LOCATION OF SENSORS FOR PARAMETRIC IDENTIFICATION OF LINEAR STRUCTURAL SYSTEMS  
DECEMBER 1992

---

ISSN 0902-7513 R9239



# On the Optimal Location of Sensors for Parametric Identification of Linear Structural Systems

P. H. Kirkegaard & R. Brincker

*Department of Building Technology and Structural Engineering*

*Aalborg University*

*Sohnsgaardsholmsvej 57, 9000 Aalborg, Denmark*

ABSTRACT: A survey of the field of optimal location of sensors for parametric identification of linear structural systems is presented. The survey shows that few papers are devoted to the case of optimal location of sensors in which the measurements are modelled by a random field with non-trivial covariance function. Most often it is assumed that the results of the measurements are statistically independent random variables. In an example the importance of considering the measurements as statistically dependent random variables is shown. The example is concerned with optimal location of sensors for parametric identification of modal parameters for a vibrating beam under random loading. The covariance of the modal parameters expected to be obtained is investigated to variations of number and location of sensors. Further, the influence of the noise on the optimal location of the sensors is investigated.

## 1. INTRODUCTION

When performing experiments one faces the problem of choosing the experimental conditions (test signals, sampling strategy, location of sensors etc.) so that the information provided by the experiment is maximized. The problem of experiment design has been given much attention in the literature. The theory of the design of static experiments originated in the early thirties, see e.g. ref.1, and has been considerably developed in the statistical literature after the second world war, ref.2 and ref.3 can be mentioned as basic references. However, the models considered in the statistical literature are generally static and their applicability to dynamic models has become clear only recently. Design of experiments for parametric identification of dynamic systems has been a subject of research during the last decades mainly developed in relation to identification of electrical systems.

The problem of experiment design can be regarded as a generalisation of the problem of optimal input signal design that has been comprehensively treated in the literature, see e.g. ref.4. Representative surveys of the problem of experiment design for dynamic system identification are given in the system identification textbooks ref.5, ref.6, ref.7, ref.8 and ref.9. Beyond these textbooks many research papers exist, mainly on the problem of optimal input design for system identification. Especially, the paper ref.10 may be noticed as a contribution to the literature concerned with experiment design for dynamic system identification.

Design of experiments in relation to structural problems seems a subject which only has received little attention during the last decade and will be a subject of research in the future, see ref.11. Just recently papers have appeared about optimal number and location of sensors. The choice of the location of sensors can have a significant influence on the quality of the results of the experiment. Therefore, ideally, the number of sensors, usually limited to minimize the cost of the instrumentation, is always located in a way regarded as optimal.

In the past, the optimal sensor location problem (OSLP) has often found its solution from practical considerations, e.g. as closely as possible to the antinodes of the lower frequency mode shapes (problem of signal-to-noise ratio) or deck level in offshore structures (problem of cost). However, if there is more than one mode and, above all, more than one sensor, the OSLP is very difficult to solve by the antinode technique. Based on the antinode technique and experience ref.12 has given guidelines for placement of sensors for measuring the earthquake response of buildings.

The aim of this paper is to survey the field of optimal location of sensors for parametric identification of linear structural systems. In section 2 several mathematical solutions to the OSLP proposed in the literature are reviewed and classified according to their main characteristics. It will be seen that few researchers have tried to solve the problems mentioned above. Especially, the problem of estimating the optimal number of sensors seems nearly unsolved. In section 3 an example is given concerned with optimal location of sensors for parametric identification of modal parameters for a vibrating beam subjected to the action of a transverse random load. The covariance of the modal parameters expected to be obtained is investigated to variations of number and location of sensors. Further, the influence of the noise on the optimal location of the sensors is investigated. At last in section 4 conclusions are given.

## 2. ON THE OPTIMAL SENSOR LOCATION PROBLEM

In what follows attention is focused on the best choice of location of sensors in the problem of parametric identification of the parameter vector  $\bar{\theta}$  from the output of the sensors include in the vector  $\bar{y}^m(t)$  given by the measuring equation

$$\bar{y}^m(t) = \bar{y}(t|\bar{\theta}) + \bar{e}(t) \quad (1)$$

where  $\bar{y}(t|\bar{\theta})$  denotes a prediction of  $\bar{y}^m(t)$  based on a model and the parameter vector  $\bar{\theta}$ . The additive noise  $\bar{e}(t)$  is normally assumed to be Gaussian stationary white noise both in space and time parameters.

Since the estimate  $\hat{\bar{\theta}}_N$  of the parameter vector  $\bar{\theta}$  to be estimated from the experiment is dependent on random processes the accuracy of  $\hat{\bar{\theta}}_N$  must be considered in a statistical sense. For experiment design purposes, it is normally assumed that the accuracy of the parameter estimate is most conveniently expressed in terms of the parameter covariance matrix  $\bar{C}_{\hat{\bar{\theta}}_N}$ . Many authors postulate the existence of an asymptotically efficient unbiased estimator as a basis for the experiment design. This implies that there is a lower bound, the Cramer-Rao lower bound, on the achievable covariance of the estimate  $\hat{\bar{\theta}}_N$  irrespective of the estimator algorithm used, provided it is unbiased. This leads to a great simplification, since the minimum variance given by the Cramer-Rao lower bound can be easily computed in several estimation problems. The Cramer-Rao lower bound is given by

$$\bar{C}_{\hat{\bar{\theta}}_N} \leq \bar{J}^{-1} \quad (2)$$

$$\bar{J} = E_{\bar{Y}^m|\bar{\theta}} \left[ \left( \frac{\partial \log f_{\bar{Y}^m}(\bar{y}^m|\bar{\theta})}{\partial \bar{\theta}} \right) \left( \frac{\partial \log f_{\bar{Y}^m}(\bar{y}^m|\bar{\theta})}{\partial \bar{\theta}} \right)^T \right] \quad (3)$$

where  $\bar{J}$  is the Fisher information matrix, see e.g. ref.5, which depends on the experimental conditions, e.g. the optimal location of the sensors.  $f_{\bar{Y}^m}(\bar{y}^m|\bar{\theta})$  is the conditional joint probability density function of  $\bar{Y}^m$ .  $E_{\bar{Y}^m|\bar{\theta}}[\cdot]$  is an expectation operator.

For comparing different informative experiments it is necessary to have a measure of the applicability of the experiment. A logical approach is to choose a measure related to the expected accuracy of the parameter estimates to be obtained from the data collected. Clearly, the parameter accuracy depends on both experimental conditions  $\mathcal{H}$  and the parameter estimator. Formally, the problem of optimal experiment for parametric identification experiment design could be



stated as

$$\min_{\mathcal{H}} \mathcal{A}(\overline{\overline{C}}_{\hat{\theta}_N}(\mathcal{H})) = \min_{\mathcal{H}} \mathcal{A}(\overline{\overline{J}}^{-1}(\mathcal{H})) \quad (4)$$

where  $\mathcal{A}(\cdot)$  is a scalar function of the covariance matrix. Typically, such scalar functions are e.g. the determinant (D-OPTIMUM), the trace (A-OPTIMUM) or the maximal eigenvalue of the covariance matrix (E-OPTIMUM), see e.g. ref.10, ref.4, ref.7 and ref.13. In ref.14 a detailed discussion of design criteria related to experiment design is given. It is seen from (4) that the basic idea underlying the experiment design theory is that a design should be chosen to make a scalar measure of the inverse of the Fisher information matrix as small as possible. By doing so a design is chosen to get as much information as possible about the parameter vector  $\bar{\theta}$  in a Fisherian sense.

The approach, mentioned above, for determination of the optimal experimental design has been studied in few papers with regard to optimal location of sensors for parametric identification of structural systems, see e.g. ref.15, ref.16 and ref.17. In these papers the Fisher information matrix is established for a multi-degrees of freedom structural system.

In ref.15 an example is given to investigate the properties of the solution to the OSLP. The example considers the influence of the changes in prior parameter estimates and the input characteristics on the optimal sensor location. From the results shown in ref.15, it is noted that the optimal sensor location for estimation of  $\bar{\theta}$  actually depends not only on the actual values of parameters not to be identified, but also on the values of the parameter  $\bar{\theta}$  itself which is to be identified. Thus it is necessary to have some a priori information about the system parameters to be able to ascertain the optimal sensor location. The conclusion of the example is that design of an experiment on purely heuristic grounds may be difficult since the example has yielded results showing that the OSLP appears to depend in a more or less complex manner on the actual parameter values of the system and the excitation.

In ref.16 optimal parameter identification experiment design has been considered for lightly damped flexible structures. The main result of the paper is a separation principle that decouples the problems of optimal input design and optimal sensor location design such that each is solved independently utilising simplified criteria. The decoupling effect is seen to give a significant simplification of the experimental design. The decoupling results indicate that for lightly damped structures the sensors can be optimally placed by utilising mode shape information only. This is significant since sensors can be roughly placed based on mode shape information which is less uncertain prior to the measurements than information about the modal frequencies and damping ratios.

In ref.17 an analytical expression for the determinant of the Fisher information matrix is given for a multi-degrees of freedom system. Further, the relation between necessary number of measuring points and the number of excitation frequencies is established. At last the information that can be gained only by dynamic testing with regard to model parameters is investigated.

Determination of the optimal location of sensors for parametric identification of structural systems has also been studied in ref.18, ref.19 and ref.20. One of the first solutions to the OSLP based on the parameter covariance matrix seems to be given in ref.18 (1977). In brief, a linear relationship between small perturbations in a finite dimensional representation of the system parameters  $\bar{\theta}$  to be estimated and a finite sample of observations of the system response is used to determine approximately the covariance matrix. In ref.19 and ref.20 it is adopted that the optimal solution to the OSLP is the one giving the best value of a scalar measure of the covariance matrix of the Bayes parameter estimates.

Papers concerned with optimal location of sensors for parametric identification of parameter distributed systems where the design is based on the Fisher information matrix have also been produced. In ref.21 the Fisher information matrix is associated with the system parameters while ref.22 considers the system eigenvalues, i.e. parameters representing the natural frequencies of the undamped system. Beyond these papers the OSLP for parameter distributed systems has also been considered in ref.23. The basic idea in that paper is to place sensors in a distributed parameter system described by the diffusion equation such that the identification error sensitivity with respect to the location of a new sensor is maximized. So ref.23 does not really deal with a method for optimal location of sensors.

Beyond the papers concerned with optimal location of sensors for parametric identification the OSLP has also been considered from the standpoint of a researcher who does experimental modal testing, see e.g. ref.24, ref.25, ref.26 and ref.27.

The literature mentioned above deals with the OSLP assuming the measurements to be statistically independent random variables. However, the OSLP becomes more realistic and more interesting if the measurements are assumed to be statistically dependent random variables. Such problems have been considered in e.g. ref.28, ref.29 and ref.30. In ref.28 the aim of the experiment is parametric identification of continuous mechanical systems subjected to random load, where the response can be given, by a model, which is linear in the parameters.

The most recent results (and a large literature) concerning the OSLP can be found in e.g. ref.11, ref.23, ref.28 and ref.29. A throughout review of the sensor

placement literature concerning distributed system is given in ref.31.

It may be noticed that the OSLP for parametric identification is closely related to the problem of optimal location of sensors and controllers for control of systems. This problem seems extensively studied, especially for large space structures. References considering the problem of optimal location of sensor and controller are e.g. ref.32, ref.33, ref.34, ref.35, ref.36, ref.37, ref.38 and ref.39. Further, the OSLP is also related to the problem of optimal location of sensors for failure detection of systems by vibration monitoring considered in e.g. ref.40 and ref.41.

### 3. EXAMPLE: OPTIMAL LOCATION OF SENSORS IN A VIBRATING BEAM

This example is concerned with optimal location of sensors for parametric identification of modal parameters for a vibrating beam subjected to the action of a transverse random load. The covariance of the modal parameters expected to be obtained is investigated to variations of number and location of sensors. Further, the influence of the noise on the optimal location of the sensors is investigated.

It is assumed that the equation of motion for the beam is given by

$$EI \frac{\partial^4 y(z, t)}{\partial z^4} + C_d \frac{\partial y(z, t)}{\partial t} + \rho_m \frac{\partial^2 y(z, t)}{\partial t^2} = u(z, t) \quad (5)$$

where  $y(z, t)$  is the deflection of the beam at the time  $t$  and distance  $z$  from its end.  $L$  is the beam length,  $\rho_m$  is the beam mass per unit length,  $C_d$  is the viscous damping coefficient per unit length and  $EI$  is the bending stiffness of the beam assumed to be constant along the length of the beam. The boundary conditions describing the simply supported beam are

$$\frac{\partial^2 y(0, t)}{\partial z^2} = \frac{\partial^2 y(L, t)}{\partial z^2} = y(0, t) = y(L, t) = 0 \quad (6)$$

The beam load  $u(z, t)$  is modelled as a zero-mean stationary Gaussian stochastic process  $\{U(z, t)\}$  with a covariance given by

$$E[U(z_1, t_1)U(z_2, t_2)] = \delta(t_1 - t_2)\delta(z_1 - z_2) \quad (7)$$

where  $\delta$  is the Dirac delta function, i.e. it is assumed that the stochastic load is white noise in both time and space with variance 1. The load is modelled as a stochastic load, due to the inevitable fact that loadings acting on structural systems are stochastic. Due to the system linearity the solution for the displacement



$y(z, t)$  is as follows if a modal approach is used

$$y(z, t) = \sum_{j=1}^{\infty} q_j(t) \phi_j(z) \quad (8)$$

where  $q_j(t)$  is a generalised coordinate and  $\phi_j(z)$  is the mode shape of the  $j$ th mode. See e.g. ref.42 for a solution of  $q_j(t)$  and  $\phi_j(z)$ . The response problem is therefore in principle solved once the modal displacements are determined. The mode shape of the  $j$ th mode is given by

$$\phi_j(z) = \sin \frac{j\pi z}{L} \quad (9)$$

The mode shapes satisfy the following orthogonality relations

$$\int_0^L \phi_i(z) \phi_j(z) dz = 0 \quad \forall : i \neq j \quad (10)$$

$$\int_0^L \phi_i(z) \phi_j(z) dz = M_j = \rho_m \frac{L}{2} \quad \forall : i = j \quad (11)$$

$M_j$  is the generalized modal mass.  $q_j(t)$  is the solution of the following second order differential equation

$$\ddot{q}_j(t) + 2\zeta_j \omega_j \dot{q}_j(t) + \omega_j^2 q_j(t) = \frac{p_j(z, t)}{M_j} \quad (12)$$

where  $p_j(z, t)$  is the generalized modal loads given by

$$p_j(z, t) = \int_0^L \phi_j(z) u(z, t) dz \quad (13)$$

$\zeta_j$  and  $\omega_j$  are the modal damping and the undamped frequencies of the  $j$ th mode, respectively, given by

$$\omega_j^2 = \frac{EI}{\rho_m} \left( \frac{j\pi}{L} \right)^4 \quad \zeta_j = \frac{1}{2} \frac{C_d}{\rho_m \omega_j} = \frac{1}{2} \sqrt{\frac{C_d^2}{\rho_m EI}} \left( \frac{L}{j\pi} \right)^2 \quad (14)$$

The beam is assumed to be modelled so that the lowest undamped eigenfrequency  $\omega_1 = 2.0$  rad/sec implying that  $\zeta_1 = 0.04$ .

The parameters assumed to be estimated are the modal parameters of the  $n$ -modes

$$\bar{\theta}^T = \{\zeta_1, \omega_1, \zeta_2, \omega_2, \dots, \zeta_n, \omega_n\} \quad (15)$$

In the following the optimal locations of two sensors are determined by assuming the measurements to be modelled as independent random variables and dependent random variables, respectively.

First, the Fisher information matrix will be given assuming the measurements modelled as independent random variables. The Fisher information matrix  $\bar{\bar{J}}$ , in continuous time, associated with identification of the parameters in the parameter vector  $\bar{\theta}$  associated with a system described by a partial differential equation is given as follows for a measuring time  $T_f$ , see ref.5

$$J^{ij} = \sum_{k=1}^{N_s} \int_0^{T_f} \frac{\partial y(z_k, t)}{\partial \theta_i} \frac{\partial y(z_k, t)}{\partial \theta_j} \lambda_\epsilon^{-1} dt \quad (16)$$

Each element of  $J^{ij}$  represents the cross-sensitivity of a measurement with respect to the response  $y(z_k, t)$  at the location  $z_k$ . Since it is assumed above that the system is lightly damped the optimal location of  $N_s$  sensors can be determined by maximizing the determinant of the Fisher information matrix which is similar to, see ref.16

$$\max_{\bar{z}} \det \bar{\bar{J}} \sim \max_{\bar{z}} \left( \prod_{i=1}^n \sum_{j=1}^{N_s} \frac{1}{\lambda_\epsilon} (\phi_i(z_j)^2) \right) \quad (17)$$

It is assumed that the variance of the noise  $\lambda_\epsilon$  is equal in each measuring point.

If it is assumed that the  $N$  measurements included in the vector  $\bar{Y}$  are dependent then by taking the expectation in (3) the Fisher information matrix becomes

$$\bar{\bar{J}} = - \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{\partial^2 \log f_{\bar{Y}}(\bar{y}|\bar{\theta})}{\partial \bar{\theta}^2} f_{\bar{Y}}(\bar{y}|\bar{\theta}) d\bar{y} \quad (18)$$

It is seen from (18) that a  $N \times N_s$ -dimensional integral has to be solved if the Fisher information matrix is to be estimated directly from the definition which is much more cumbersome than using (17). To see the difference between calculating the Fisher information matrix from (17) and from (18) it is assumed that an observation is taken only once at each measuring point simultaneously. Assuming two measuring points the integral in (18) becomes 2-dimensional. Since the response is assumed to be Gaussian with zero means the 2-dimensional probability density function for the two measuring points can now be written by a 2-dimensional Gaussian joint density function of the two continuous random variables  $Y_1$  and  $Y_2$

$$f_{Y_1, Y_2}(y_1, y_2|\bar{\theta}) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{12}^2}} \exp\left(-\frac{1}{2(1-\rho_{12}^2)}\left(\left(\frac{y_1}{\sigma_1}\right)^2 - 2\rho_{12}\frac{y_1y_2}{\sigma_1\sigma_2} + \left(\frac{y_2}{\sigma_2}\right)^2\right)\right) \quad (19)$$

where  $\sigma_1$  and  $\sigma_2$  are the standard deviations of the response at the two measuring points, respectively.  $\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$  is the correlation coefficient and  $\sigma_{12}$  is the covariance between  $Y_1 = Y_1(z_1, t)$  and  $Y_2 = Y_2(z_2, t)$ , respectively. The variances ( $\sigma_1^2, \sigma_2^2$ ) and covariance ( $\sigma_{12}$ ) of the stochastic response processes  $\{Y_1(z_1, t)\}$  and  $\{Y_2(z_2, t)\}$  are given by the cross-correlation function  $R_{YY}(z_1, t_1; z_2, t_2)$  of the response in the two points  $z_1$  and  $z_2$ . By using (19) and an analytical expression for the cross-correlation function it is easy to calculate the derivatives in (18) and thus the Fisher information matrix, see ref.11.

In figure 1a and 1b the determinant of the Fisher information matrix estimated by (17) and (18), respectively, is shown for different locations of the two sensors. It is assumed that only the modal parameters of the first two modes are of interest. The points in figure 1b where  $z_1 = z_2$  are determined by using a 1-dimensional probability density function in (18) instead of the 2-dimensional function. It may be noticed that the Fisher information matrix is not defined for  $z_1 = z_2 = 0 = L$ . The points  $z_i = 0$  for  $z_j \neq 0, L$  are determined in the same way as the points  $z_i = z_j$ .

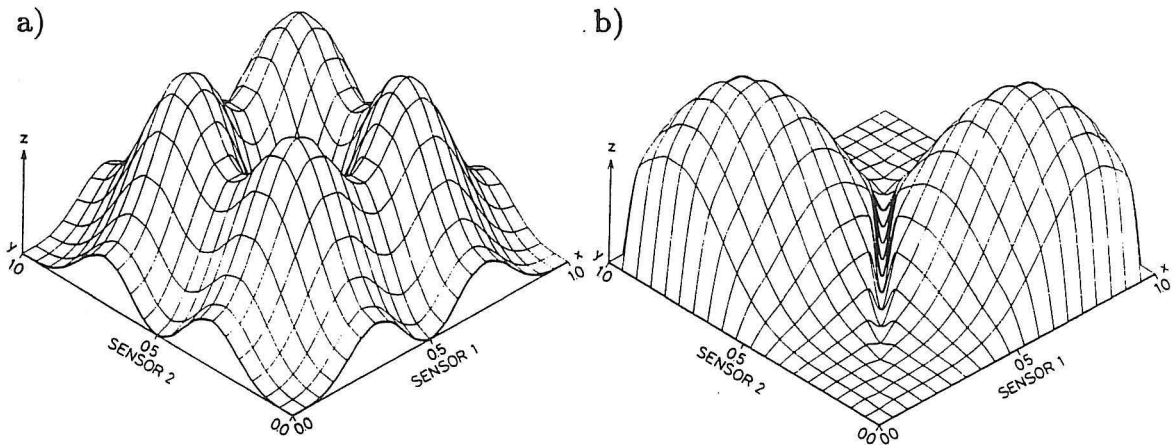


Figure 1: The determinant of the Fisher information matrix shown as a function of the location of two sensors. a) based on (17), b) based on (18).

It is seen from figure 1a that the same information will be obtained from the measurements if the sensors are placed at the same point or with the one sensor at this point and the other placed at the symmetric point. Further, it is seen that the 3D-curve is very flat near its minima. This causes difficulties in the precise choice of the optimal design on the one hand, but it also means that some imperfections in the design or in the practical positioning of the sensors results in



relatively small increase of error. This result implies that the following question can be asked: Is it correct that the optimal locations of the two sensors are at the same point? By investigating figure 1b it is seen that it is not optimal to place two sensors at the same measuring point. This disagrees with the result shown in figure 1a. The disagreement is due to the assumption that the measurements can be considered as statistically independent random variables. This implies that the spatial correlation is not taken into account in the calculations of the results shown in figure 1a. The results shown in figure 1b are obtained from calculations where the spatial correlation is taken into account. The optimal locations of the two sensors are  $z_1 = 0.242L$  and  $z_2 = 0.758L$ . Intuitively, it also seems more correct to get information from two different measuring points instead of information from one measuring point. It is also seen from the above that it is much simpler to calculate the Fisher information matrix in the case where it can be assumed that the measurements are statistically independent. However, as it is shown above, it is not a good approach when a continuous mechanical system subjected to a random load is considered. If the measurements are encumbered with noise it can be interesting to look into the optimal location of the sensors is sensitive to the variance of the random noise.

It is assumed that the measurements are given by

$$y^m(z, t) = y(z, t) + e(z, t) \quad (20)$$

where the realization  $e(z, t)$  of the noise process  $\{\mathcal{E}(z, t)\}$  only models the random measuring noise. The noise-to-signal ratio  $\gamma$  is given by

$$\gamma = \sqrt{\frac{\lambda_{\mathcal{E}}}{\sigma^2}} \quad (21)$$

$\sigma^2$  is the variance of the response for  $z = L/2$ . In figure 2 the optimal locations of the two sensors are shown as functions of the noise-to-signal ratio. The full line in figure 2 shows results where it is assumed that it is the modal parameters of the first two modes which are of interest. The dashed line shows results where it is assumed that it is the modal parameters  $(\zeta_1, \omega_1)$  of the first mode which are of interest, i.e. the Fisher information matrix is a  $2 \times 2$  matrix.

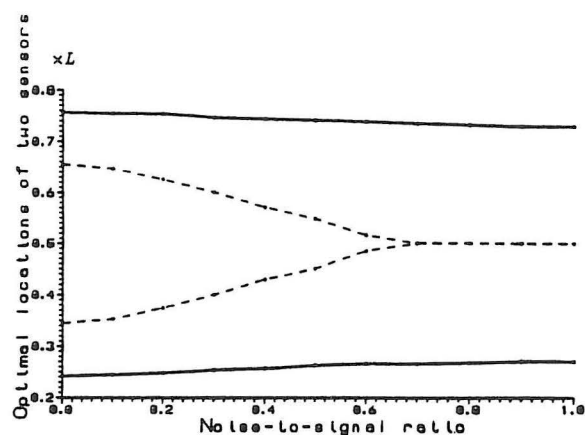


Figure 2: Optimal locations of two sensors as function of the noise-to-signal ratio  $\gamma$ .

It is seen from figure 2 that the optimal locations of the two measuring points are sensitive to the variance of the noise. It is seen that the optimal locations are more sensitive to the variance of the noise when it is the parameters of the first mode which are of interest. Then the optimal locations of the sensors are going against, as expected, the optimal location of one measuring point,  $z = L/2$ . From this result it could be expected for an increasing number of sensors that the optimal locations of the sensors become less sensitive to the variation of the noise-to-signal ratio. It may be noticed, as expected, that the amount of information from the measurements obtained from the two measuring points decreases when the measurements are encumbered with noise.

Figure 3 shows the relative change in the determinant of the Fisher information matrix, corresponding to the optimal locations of the sensors, as functions of the noise-to-signal ratio  $\gamma$ . The dashed line and the full line, respectively, correspond to the lines in figure 2.

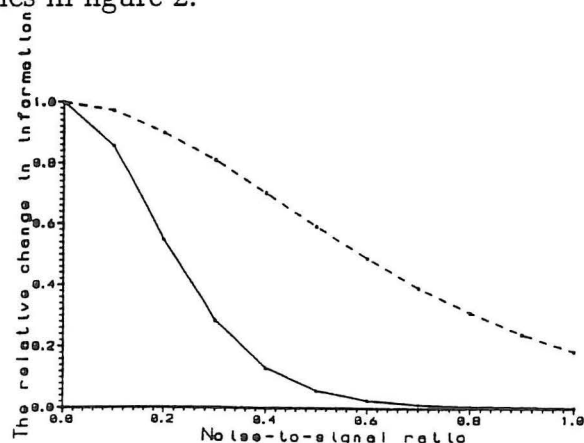


Figure 3: The change in the determinant of the Fisher information matrix as a function of the noise-to-signal ratio  $\gamma$ .

It is seen from figure 3 that the relative loss of information for increasing variance of the noise is larger when the parameters of the first two modes are of interest than when it is the parameters of the first mode which are of interest.

Figure 4 shows, for increasing noise-to-signal ratio, the ratio  $\kappa_1$

$$\kappa_1 = \frac{\det \bar{\bar{J}}_1}{\det \bar{\bar{J}}_2} \quad (22)$$

where  $\det \bar{\bar{J}}_1$  is the determinant of the Fisher information matrix for the two measuring points placed at the optimal points assuming noiseless measurements and  $\det \bar{\bar{J}}_2$  is the determinant of the Fisher information matrix for optimally located sensors, i.e. the ratio shows the loss in information by placing the sensors without taking into account that the measurements are encumbered with noise.

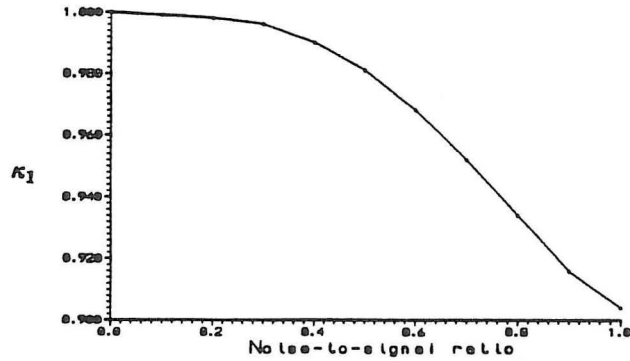


Figure 4: The ratio  $\kappa_1$  shown as a function of the noise-to-signal ratio  $\gamma$ .

It is seen from figure 4 that only a little loss of information is obtained if the sensors are placed without taking into account that the measurements are encumbered with noise. This is an important result since the noise-to-signal ratio is normally not available prior to the experiment.

Figure 5 shows the ratio  $\kappa_2$  for increasing noise-to-signal ratio  $\gamma$ ,

$$\kappa_2 = \frac{\det \bar{\bar{J}}_4}{\det \bar{\bar{J}}_3} \quad (23)$$

where  $\det \bar{\bar{J}}_3$  is the amount of information obtained if two sensors are optimally placed and  $\det \bar{\bar{J}}_4$  is the amount of information obtained if one sensor is optimally placed.



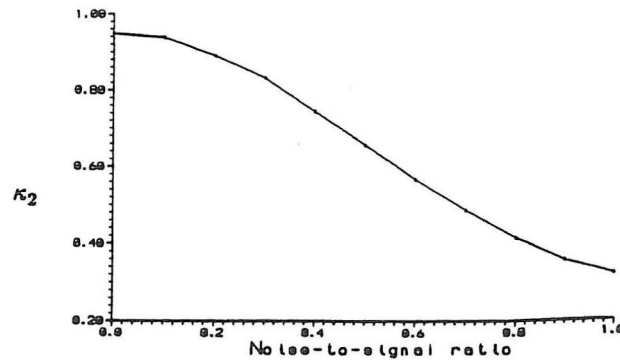


Figure 5: The ratio  $\kappa_2$  shown as a function of the noise-to-signal ratio  $\gamma$ .

Figure 5 shows, for noiseless measurements, that the amount of information obtained using one sensor is almost equal to the amount of information obtained using two sensors. It is assumed that it is the modal parameters of the first mode which are of interest. Further, it is seen for increasing noise-to-signal ratio that the ratio  $\kappa_2$  becomes smaller. This means that the amount of information obtained with two sensors becomes larger compared with the amount of information obtained with one sensor. This result raises the question: When shall additional sensors be used? To answer this question it is necessary to take into account the cost of using an additional sensor. Further, it is also necessary to consider the increase in value of information by using an additional sensor. This problem has been considered in ref.11 where a method has been proposed which can be used to determine the optimal number of sensors.

It may be noticed that the results in figure 2 indicate that the optimal location of the sensors is not sensitive to the noise-to-signal ratio if the aim of the experiment is to determine the modal parameters of more than one mode. On the other hand it is seen from figure 3-5 that good prior information about the noise-to-signal ratio is necessary if the aim is to compare the information which can be obtained using different number of sensors.

#### 4. CONCLUSIONS

In this paper a survey of different methods for determining optimal location of sensors for the parametric identification of structural systems has been given. Further, an example has been given concerned with optimal location of sensors for parametric identification of modal parameters for a vibrating beam subjected to the action of a transverse random load. The conclusions of the paper can be

stated as follows:

- It is seen that very few papers concerned with the problem of optimal location of sensors for the parametric identification of structural systems exist. Especially papers in which the subject of measurements is modelled by a random field with non-trivial covariance function.
- Design of an optimal experiment on purely heuristic grounds may be difficult since simple examples considered in different papers have yielded results showing that the optimal locations of sensors appear to depend in a more or less complex manner on the actual parameter values of the system and the excitation.
- It is evident from the example in this paper that the experimental conditions have an effect on the achievable accuracy. Thus, there is a motivation in practice to choose the appropriate sensor locations to optimise the information return from the experiment.
- The optimal locations of sensors seem to become less sensitive to e.g. the noise-to-signal ratio for increasing number of sensors.
- The question "What is the optimal number of sensors?", seems to be unsolved in the papers dealing with optimal experiment design. In order to answer the question it is necessary to take into account the cost of using an additional sensor. Further, it is also necessary to consider the increase in the value of information by using an additional sensor.

## 5. REFERENCES

- (1) Fisher, R. A.: *The Design of Experiments*. Oliver and Boyd, Edinburgh, 1935.
- (2) Fedorov, V. V.: *Theory of Optimal Experiments*. Academic Press, 1977.
- (3) Silvey, S. D.: *Optimal Design*. Monographs on Applied Probability and Statistics, Clapman and Hall, 1980.
- (4) Mehra, R. K.: Choice of Input Signals. In *Trends and Progress in Systems Identification* (P. Eykhoff, ed.). Pergamon Press, Elmsford, New York, 1981.
- (5) Goodwin, G. C. & R. L. Payne: *Dynamic System Identification: Experiment Design and Data Analysis*. Academic Press, 1977.
- (6) Kalaba, R. & K. Springarn: *Control, Identification and Input Optimization*. Plenum, New York, 1982.
- (7) Ljung, L.: *System Identification: Theory for the User*. Prentice Hall, Englewood Cliffs, 1987.
- (8) Norton, J. P.: *An Introduction to Identification*. Academic Press, New York, 1986.
- (9) Söderström, T. & P. Stoica: *System Identification*. Prentice Hall, 1987.

- (10) Goodwin, G. C.: Identification: Experiment Design. In *Systems and Control Encyclopaedia* (M. Singh, ed.). Pergamon Press, Oxford, 1987.
- (11) Kirkegaard, P. H.: *Optimal Design of Experiments for the Parametric Identification of Civil Engineering Structures*. Ph.D-thesis, Aalborg University, Denmark, 1991.
- (12) Rojahn, C. & R. B. Matthiesen: *Earthquake Response and Instrumentation of Buildings*. Journal of the The Technical Councils of ASCE Vol. 103, No. TCI, pp. 1-12, 1977.
- (13) Zarrop, M. B.: *Optimal Experiment Design for Dynamic System Identification*. Springer-Verlag, New York, 1979.
- (14) Pazman, A.: *Foundation of Optimum Experimental Design*. D. Reidel, Dordrecht, 1986.
- (15) Udwadia, F. E.: *Optimal Sensor Locations for Geotechnical and Structural Identification*. In: Proc. of the 8th World Conf. of Earthquake Engineering, San Francisco, 1984.
- (16) Bayard, D. S., F. Y. Hadaegh & D. R. Meldrum: *Optimal Experiment Design for Identification of Large Space Structures*. Automatica, pp. 357-364, 1988.
- (17) Cottin, N.: *On the Experimental Design for the Parametric Identification of linear Elastomechanical Systems*. Proc. of Eurodyn 90, 1990.
- (18) Shah, P. & F. E. Udwadia: *A Methodology for Optimal Sensor Locations for Identification of Dynamic Systems*. Journal of Applied Mechanics, ASME, Vol. 45, pp. 188-196, 1978.
- (19) Sprandel, J. K.: *Structural Parameter Identification of Member Characteristics in a Finite- Element Model*. Ph.D-Thesis, Purdue University, 1979.
- (20) Vestroni, F. & D. Capecchi: *Aspects of the Application of Structural Identification in Damage Evaluation*. In: Application of Structural Identification, 1988.
- (21) Qureshi, Z. H., T. S. Ng & G. C. Goodwin: *Optimum Experimental Design for Identification of Distributed Systems*. Int. J. Control, Vol. 31, No. 1, pp. 21-29, 1980.
- (22) Rafajlowicz, E.: *Design of Experiments for Eigenvalue Identification in Distributed Parameter Systems*. Int. J. Control, Vol. 34, No. 6, pp. 1079-1094, 1981.
- (23) Le Pourhiet, A. & L. Le Letty: *Optimization of Sensor Locations in Distributed Parameter System Identification*. In: Identification and System Parameter Estimation, (Rajbman, ed.). North-Holland Publishing Company, 1978.
- (24) Kammer, D. C. & M. J. Triller *Efficient Sensor Placement for On-Orbit Modal Identification of Sequentially Assembled Large Space Structures*. 10th International Modal Analysis Conference, February, 1992.
- (25) Penney, J. E. T., M. I. Friswell & S. D. Garvey: *The Automatic Choice of Measurement Locations for Dynamic Testing*. 10th International Modal Analysis Conference, February, 1992.
- (26) Flanigan, C. C. & C. D. Botos: *Automated Selection of Accelerometer Locations for Modal Survey Tests*. 10th International Modal Analysis Conference, February, 1992.
- (27) Ibáñez, P., R. Vasudevan & C. B. Smith: *Experimental and Theoretical Analysis of Building*. ASCE-EMD Specialty Conference, UCLA Extension, pp. 412-430, 1976.

- (28) Kazimierczyk, P.: *Optimal Experiment Design; Vibrating Beam under Random Loading*. Eur. J. Mech., A/Solids, Vol. 8, No. 3, pp. 161-184, 1989.
- (29) Kotulski, Z.: *Optimization of Sensors' Location in A Stochastic Extrapolation Problem*. Journal of Sound and Vibration 138(3), pp. 351-363, 1990.
- (30) Kubrusly, C.S. & H. Malebranche: *Sensors and Controllers Location in Distributed System - A Survey*. Automatica, Vol. 21, pp. 117-128, 1985.
- (31) Sobczyk, K.: *Theoretic Information Approach to Identification and Signal Processing*. Proc. IFIP Conf. on Reliability and Optimization of Structural Systems, 1987.
- (32) Norris, G. A. & R. E. Skelton: *Placing Dynamic Sensors and Actuators on Flexible Space Structures*. In NASA Langley Research Center, Proceedings of the 4th Annual Workshop, 1988.
- (33) Chang, M. I. J. & T. T. Soong: *Optimal Controller Placement in Modal Control of Complex Systems*. Journal of Mathematical Analysis and Application, Vol 75, pp. 340-358, 1980.
- (34) Soong, T. T. & M. I. J. Chang: *On Optimal Control Configuration in Theory of Modal Control*. North-Holland Publishing Company & SM Publications. IUTAM, 1980.
- (35) Schulz, G. & G. Heimbold: *Dislocated Actuator/Sensor Positioning and Feedback Design for Flexible Structures*. J. Guidance, Control , Vol. 6, No. 5, pp. 361-367, 1983.
- (36) Juang, J. N. & P. Rodriguez: *Formulations and Applications of Large Structure Actuator and Sensor Placements*. 2nd VPI & SU/AiAA Symp. Dynam. Contr. Large Flexible Spacecraft, Blacksburg, 1979.
- (37) Wu, Y-W. A.: *Optimization of Large Structure Sensor Designs*. Asilomar Conf. on Circuits, Systems and Computers, pp. 385-388, 1979.
- (38) Munak, A.: *Optimal Sensor Allocation for Identification of Unknown Parameters in a Bubble-Column Loop Bioreactor*. Int. Conf. on Analysis and Optimization of Systems, 6, Nice, France, 1984.
- (39) Vander Velder, W. E. & C. R. Carigan: *Number and Placement of Control System Components Considering Possible Failure*. J. Guidance, Control , Vol. 7, No. 6, pp. 703-709, 1984.
- (40) Watanabe, K., M. Sasaki & D. M. Himmelblau: *Determination of Optimal Measuring Sites for Fault Detection of Non-linear Systems*. Int. J. Systems SCI, Vol. 16, No. 11, pp. 1345-1363, 1985.
- (41) Ferretti, C., M. Mazzon, K. Skaar & M. Michaelsen: *Platform Damage Identification Through an Optimised Methodology of Forced Vibration Measurement*. TECNOMARE, Italy.
- (42) Thomsen, W. T.: *Theory of Vibration with Applications*. Prentice-Hall, 1981.



## FRACTURE AND DYNAMICS PAPERS

PAPER NO. 12: Lise Gansted, Rune Brincker & Lars Pilegaard Hansen: *Numerical Cumulative Damage: The FM-Model*. ISSN 0902-7513 R8920.

PAPER NO. 13: Lise Gansted: *Fatigue of Steel: Deterministic Loading on CT-Specimens*.

PAPER NO. 14: Jakob Laigaard Jensen, Rune Brincker & Anders Rytter: *Identification of Light Damping in Structures*. ISSN 0902-7513 R8928.

PAPER NO. 15: Anders Rytter, Jakob Laigaard Jensen & Lars Pilegaard Hansen: *System Identification from Output Measurements*. ISSN 0902-7513 R8929.

PAPER NO. 16: Jens Peder Ulfkjær: *Brud i beton - State-of-the-Art. 1. del, brudforløb og brudmodeller*. ISSN 0902-7513 R9001.

PAPER NO. 17: Jakob Laigaard Jensen: *Full-Scale Measurements of Offshore Platforms*. ISSN 0902-7513 R9002.

PAPER NO. 18: Jakob Laigaard Jensen, Rune Brincker & Anders Rytter: *Uncertainty of Modal Parameters Estimated by ARMA Models*. ISSN 0902-7513 R9006.

PAPER NO. 19: Rune Brincker: *Crack Tip Parameters for Growing Cracks in Linear Viscoelastic Materials*. ISSN 0902-7513 R9007.

PAPER NO. 20: Rune Brincker, Jakob L. Jensen & Steen Krenk: *Spectral Estimation by the Random Dec Technique*. ISSN 0902-7513 R9008.

PAPER NO. 21: P. H. Kirkegaard, J. D. Sørensen & Rune Brincker: *Optimization of Measurements on Dynamically Sensitive Structures Using a Reliability Approach*. ISSN 0902-7513 R9009.

PAPER NO. 22: Jakob Laigaard Jensen: *System Identification of Offshore Platforms*. ISSN 0902-7513 R9011.

PAPER NO. 23: Janus Lyngbye & Rune Brincker: *Crack Length Detection by Digital Image Processing*. ISSN 0902-7513 R9018.

PAPER NO 24: Jens Peder Ulfkjær, Rune Brincker & Steen Krenk: *Analytical Model for Complete Moment-Rotation Curves of Concrete Beams in bending*. ISSN 0902-7513 R9021.

PAPER NO 25: Leo Thesbjerg: *Active Vibration Control of Civil Engineering Structures under Earthquake Excitation*. ISSN 0902-7513 R9027.

PAPER NO. 26: Rune Brincker, Steen Krenk & Jakob Laigaard Jensen: *Estimation of correlation Functions by the Random Dec Technique*. ISSN 0902-7513 R9028.

PAPER NO. 27: Jakob Laigaard Jensen, Poul Henning Kirkegaard & Rune Brincker: *Model and Wave Load Identification by ARMA Calibration*. ISSN 0902-7513 R9035.



## FRACTURE AND DYNAMICS PAPERS

PAPER NO. 28: Rune Brincker, Steen Krenk & Jakob Laigaard Jensen: *Estimation of Correlation Functions by the Random Decrement Technique*. ISSN 0902-7513 R9041.

PAPER NO. 29: Poul Henning Kirkegaard, John D. Sørensen & Rune Brincker: *Optimal Design of Measurement Programs for the Parameter Identification of Dynamic Systems*. ISSN 0902-7513 R9103.

PAPER NO. 30: L. Gansted & N. B. Sørensen: *Introduction to Fatigue and Fracture Mechanics*. ISSN 0902-7513 R9104.

PAPER NO. 31: R. Brincker, A. Rytter & S. Krenk: *Non-Parametric Estimation of Correlation Functions*. ISSN 0902-7513 R9120.

PAPER NO. 32: R. Brincker, P. H. Kirkegaard & A. Rytter: *Identification of System Parameters by the Random Decrement Technique*. ISSN 0902-7513 R9121.

PAPER NO. 33: A. Rytter, R. Brincker & L. Pilegaard Hansen: *Detection of Fatigue Damage in a Steel Member*. ISSN 0902-7513 R9138.

PAPER NO. 34: J. P. Ulfkjær, S. Krenk & R. Brincker: *Analytical Model for Fictitious Crack Propagation in Concrete Beams*. ISSN 0902-7513 R9206.

PAPER NO. 35: J. Lyngbye: *Applications of Digital Image Analysis in Experimental Mechanics*. Ph.D.-Thesis. ISSN 0902-7513 R9227.

PAPER NO. 36: J. P. Ulfkjær & R. Brincker: *Indirect Determination of the  $\sigma - w$  Relation of HSC Through Three-Point Bending*. ISSN 0902-7513 R9229.

PAPER NO. 37: A. Rytter, R. Brincker & P. H. Kirkegaard: *An Experimental Study of the Modal Parameters of a Damaged Cantilever*. ISSN 0902-7513 R9230.

PAPER NO. 38: P. H. Kirkegaard: *Cost Optimal System Identification Experiment Design*. ISSN 0902-7513 R9237.

PAPER NO. 39: P. H. Kirkegaard: *Optimal Selection of the Sampling Interval for Estimation of Modal Parameters by an ARMA-Model*. ISSN 0902-7513 R9238.

PAPER NO. 40: P. H. Kirkegaard & R. Brincker: *On the Optimal Location of Sensors for Parametric Identification of Linear Structural Systems*. ISSN 0902-7513 R9239.

Department of Building Technology and Structural Engineering  
The University of Aalborg, Sohngaardsholmsvej 57, DK 9000 Aalborg  
Telephone: 45 98 15 85 22    Telefax: 45 98 14 82 43