System Identification of Civil Engineering Structures using State Space and ARMAV Models

Andersen, P.; Kirkegaard, Poul Henning; Brincker, Rune

INSTITUTTET FOR BYGNINGSTEKNIK
DEPT. OF BUILDING TECHNOLOGY AND STRUCTURAL ENGINEERING
AALBORG UNIVERSITET • AUC • AALBORG • DANMARK

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P. Andersen, P.H. Kirkegaard & R. Brincker
Aalborg University,
Department of Building Technology and Structural Engineering.
Sohngaardsholmsvej 57, DK-9000, Denmark.

ABSTRACT

In this paper the relations between an ambient excited structural system, represented by an innovation state space system, and the Auto-Regressive Moving Average Vector (ARMAV) model are considered. It is shown how to obtain a multivariate estimate of the ARMAV model from output measurements, using the non-linear prediction error method. The performance of this algorithm is compared with two system realization estimators. These are the Matrix Block Hankel stochastic realization estimator and the Stochastic Subspace Technique. The example shows that the use of the non-linear prediction error method and the Stochastic Subspace Technique results in the most accurate modal parameter estimates and spectral densities.

1. INTRODUCTION

For analysis of ambient excited civil engineering structures the univariate Auto-Regressive Moving Average (ARMA) model has been used for many years, see Pandit et al. [1], Kozin et al. [2]. The use of this model has, however, not directly provided mode shape information but only revealed the eigen-frequencies and corresponding damping ratios. Recently, the interest has turned to the multivariate Auto-Regressive Moving Average Vector (ARMAV) model, see Pandit [3], Andersen et al. [4], Kirkegaard et al. [5], Piombo et al. [6]. The reason is that this model is directly linked to the underlying multivariate continuous-time system, and for this reason, it is also possible to extract mode shape information directly.

Because of the increased complexity of multivariate systems it is very appealing to represent such systems in state space. Using this approach the complexity is reduced to the manipulation of a few matrices. The use of stochastic state space formulations has also led to special system identification techniques that operate directly in state space. These techniques are referred to as system realization techniques. Some of the techniques are e.g. the Eigensystem Realization Algorithm, see Juang et al. [7], the Matrix Block Hankel method, see Hoen [8], and a fairly new technique referred to as the Stochastic Subspace Technique, see Van Overschee [9]. These techniques originate from system theory and statistics, see Aoki [10], and make use of numerical linear algebra. The parametrization is easy and the convergence is not iterative and therefore guaranteed. The methods do not rely on using non-linear least-squares. In contrast, the state space system can also be estimated using prediction error methods (PEM), see Ljung [11], which rely on non-linear optimization. The reasons for using these techniques compared to those previously mentioned are the appealing asymptotical properties of the estimate.

The aim of this paper is to compare the performance of the non-linear prediction error estimates of the ARMAV model with the performance of two different system realization techniques for system identification of multivariate systems. The techniques are the Matrix Block Hankel (MBH) method and the stochastic subspace technique, denoted (N4SID) for Numerical algorithm for Subspace State Space System IDentification. The performance of the methods is evaluated by

comparison of modal parameters and spectral densities.

Section two consider how to represent a continuous-time second order structural system in state space, and how the sampled version of this system is represented in discrete time. It will also consider how to account for ambient excitation and the presence of disturbance. Section three describes the relationship between state space systems and ARMAV models. It also describes how to obtain the PEM estimate of the ARMAV model. Section four briefly introduces the MBH and N4SID estimators. Section five is a simulated example that illustrates the performance of the three different algorithms. Finally, in section six, conclusions will be made.

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2. STATE SPACE DESCRIPTION OF STRUCTURAL SYSTEMS

In this section it will briefly be shown how a continuous-time structural system, described by a second-order multivariate differential equation, can be converted to an equivalent discrete-time state space realization. It is also shown how to account for the presence of disturbance in the system, and the lack of known excitation.

2.1 Continuous-time state space systems

Consider a multivariate continuous-time civil engineering second order structural system with ny degrees of freedom. The system is described by an $ny \times ny$ positive definite diagonal mass matrix M, an $ny \times ny$ symmetric semi-definite viscous damping matrix C, and an $ny \times ny$ symmetric positive definite stiffness matrix K. The system is excited by an input $u(t)$, of dimension $nu \times 1$, through a selection matrix S of dimension $ny \times nu$. Let the displacements, velocities and accelerations be described by the $ny \times 1$ vectors $z(t), \dot{z}(t), \ddot{z}(t)$ respectively. A state space realization of this system is

$$\begin{bmatrix} \dot{z}(t) \\ \ddot{z}(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix} \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}S \end{bmatrix} u(t) \ , \quad y(t) = \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} \tag{1}$$

The system output of the state space system can be observed through an $ny \times 1$ observation vector $y(t)$. The output depends on the type of measurements and the process to be measured. In the present case the output will be displacements $z(t)$. Compactly the system can be written as

$$\dot{x}(t) = Fx(t) + Eu(t) \ , \quad y(t) = Cx(t) \tag{2}$$

The $n \times 1$ vector $x(t)$ is the state vector, F is the $n \times n$ state matrix, C is the observation matrix, and the $n \times nu$ matrix E is the input matrix. Equation (2) can be solved by supplying an initial state vector x_0. By solving it, the system output can be obtained as

$$x(t) = e^{F(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{F(t-s)}Eu(s)ds \ , \quad y(t) = Cx(t) \tag{3}$$

This equation provides a complete description of the output of the system due to any excitation.

2.2 Sampling of the state space system

Let T denote the sampling interval, and define the following $t_0=kT$ and $t=(k+1)T$, where k is an integer. Inserting this into (3) yields

$$\boldsymbol{x}((k+1)T) = e^{\boldsymbol{F}T}\boldsymbol{x}(kT) + \int_{kT}^{(k+1)T} e^{\boldsymbol{F}((k+1)T-s)}\boldsymbol{E}\boldsymbol{u}(s)ds\ , \quad \boldsymbol{y}(kT) = \boldsymbol{C}\boldsymbol{x}(kT) \tag{4}$$

This is now a discrete-time state space system, which can be written compactly as

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{A}\boldsymbol{x}(t_k) + \boldsymbol{B}\boldsymbol{u}(t_k)\ , \quad \boldsymbol{y}(t_k) = \boldsymbol{C}\boldsymbol{x}(t_k) \tag{5}$$

with $\boldsymbol{A}$ being the discrete-time state matrix or transition matrix, and with t_{k+j} being equal to time step $(k+j)T$. The excitation $\boldsymbol{u}(t)$ has been assumed constant during the interval between two samples, and a zero-order hold approximation has been applied. By using this approximation the input matrix is defined as $\boldsymbol{B}=\boldsymbol{F}^{-1}(\boldsymbol{A}-\boldsymbol{I})\boldsymbol{E}$, which is approximately true if the sampling interval is sufficiently small.

The system in (5), however, does not account for disturbance that affects the structural parameters describing the system. This disturbance affects the states and can e.g. be enviromental variations, such as the temperature causing changes of the stiffness of the structure. This disturbance is termed process noise and is denoted $\boldsymbol{w}(t_k)$. Further, when a system is sampled there will most certainly be introduced some measurement noise $\boldsymbol{v}(t_k)$, due to the limited measurement accuracy. Both $\boldsymbol{w}(t_k)$ and $\boldsymbol{v}(t_k)$ will be described by sequences of independent identically distributed stochastic variables.

In some cases only the output $\boldsymbol{y}(t_k)$ is known. This is the typical case in system identification of ambient excited civil engineering structures. If the system that governs this output is described by a rational transfer function and excited by an independent identical distributed stochastic process, then the stochastic process $\boldsymbol{y}(t_k)$ is termed an Auto-Regressive Moving Average Vector (ARMAV) process. This process can also be represented using the state space form, by omitting the term that includes $\boldsymbol{u}(t_k)$.

By adding the noise terms $\boldsymbol{w}(t_k)$ and $\boldsymbol{v}(t_k)$ to the state space equation and the observation equation, respectively, and at the same time omitting the input term, the state space system looks as follows

$$\boldsymbol{x}(t_{k+1}) = \boldsymbol{A}\boldsymbol{x}(t_k) + \boldsymbol{w}(t_k)\ , \quad \boldsymbol{y}(t_k) = \boldsymbol{C}\boldsymbol{x}(t_k) + \boldsymbol{v}(t_k) \tag{6}$$

The measurement noise $\boldsymbol{v}(t_k)$ will in the general case be correlated with the process noise $\boldsymbol{w}(t_k)$, see Hannan et al. [12] and Ljung [11].

2.3 Innovation state space formulation

In order to use (6) for system identification, the predictor of the system output $\hat{\boldsymbol{y}}(t_k)$ needs to be defined. Provided that the noise terms are Gaussian distributed stochastic processes, the conditional expectation of $\boldsymbol{y}(t_k)$, given $\boldsymbol{y}(t_s)$, $\boldsymbol{u}(t_s)$, $s < k$, is given by, see Hannan et al. [12],

$$\hat{\boldsymbol{x}}(t_{k+1}) = \boldsymbol{A}\hat{\boldsymbol{x}}(t_k) + \boldsymbol{K}\left(\boldsymbol{y}(t_k) - \boldsymbol{C}\hat{\boldsymbol{x}}(t_k)\right)\ , \quad \hat{\boldsymbol{y}}(t_k) = \boldsymbol{C}\hat{\boldsymbol{x}}(t_k) \tag{7}$$

where the matrix K is the Kalman gain. The prediction errors $e(t_k) = y(t_k) - C\hat{x}(t_k)$ in the state equation of (7) amount to the part of $y(t_k)$ that cannot be predicted from the past data. For this reason $e(t_k)$ are also called the innovation. Inserting $e(t_k)$ into (7) yields

$$\hat{x}(t_{k+1}) = A\hat{x}(t_k) + Ke(t_k) , \quad y(t_k) = C\hat{x}(t_k) + e(t_k) \tag{8}$$

which is the state space innovation form. The innovations are assumed to be independent identically distributed with covariance matrix Λ. They should be taken as an unknown stochastic excitation of the system. For this reason (8) is also termed a stochastic state space system.

2.4 Modal parameters and spectral density

Looking only at the eigenvibrations of the stochastic state space system in (8), the modal parameters can be determined from the following eigenvalue problem

$$(I\mu_j - A)\Psi_j , \quad \Phi_j = C\Psi_j , \quad j = 1, 2, \ldots , n \tag{9}$$

where Ψ_j is the jth eigenvector of the system, and μ_j is the corresponding eigenvalue. The $ny \times 1$ vector Φ_j is the observed part of the eigenvector, and referred to as the scaled mode shape. For a stable underdamped system all structural modes are represented by complex conjugated pairs of eigenvalues and corresponding mode shapes. The complex conjugated pair of eigenvalues $\{\mu_j, \mu^*_{j+1}\}$ can be equivalently expressed in terms of the eigenfrequency f_j and the damping ratio ζ_j of the mode, by converting each of them to continuous-time eigenvalues $\lambda_j = \log(\mu_j)/T$, resulting in

$$\{\lambda_j, \lambda^*_{j+1}\} = -2\pi f_j \zeta_j \pm i2\pi f_j \sqrt{1-\zeta_j^2} , \quad f_j = \frac{|\lambda_j|}{2\pi} , \quad \frac{-Re(\lambda_j)}{2\pi f_j} \tag{10}$$

The frequency response of the system can be calculated by applying the z-transform to (8) giving $z[y(t)] = C(Iz - A)^{-1}Kz[e(t)]$, where $z[\,]$ is the z-transform operator and z is a complex variable. The spectral density of the output $S_{yy}(f)$ is then given by

$$S_{yy}(f) = C(Ie^{i2\pi fT} - A)^{-1}K\Lambda K^H(Ie^{i2\pi fT} - A)^{-H}C^H \tag{11}$$

with z equal to $e^{i2\pi fT}$ and ^{H} denoting conjugate-transpose. If only the modal parameters are of interest the pair $\{A, C\}$ provides the necessary information. However, if the spectral density is desired, the triple $\{A, C, K\}$ and the innovation covariance matrix Λ must be provided.

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3. ARMAV MODELS FOR DISCRETE-TIME SYSTEMS

This section describes how to represent the state space system, described by the innovation form (8), by an ARMAV model. It is also described how to estimate this model from measured output data, and finally, how the estimated ARMAV model can be converted into the innovation state space form.

3.1 Converting the state space system to an ARMAV model

Assume that (8) is an m-dimensional minimal state space realization, having ny observed outputs. In the following it will also be assumed that the observation matrix C is regular, which essential means that all rows of C must be independent, see Gawronski et al. [13]. The observability index associated with a regular observation matrix is denoted n. For C to be a regular observation matrix the condition $n \cdot ny = m$ must be fullfilled. Note that the observation matrix defined in the previous section regular.

The first step of the conversion scheme is to transform the state space system to observability canonical form, described by the triple $\{A_{ob}, C_{ob}, K_{ob}\}$, see Aoki [10]. To do this, the state vector transformation $\hat{x}_{ob}(t_k) = Q\hat{x}(t_k)$ will be applied resulting in the following innovation state space system

$$\begin{aligned} &\hat{x}_{ob}(t_{k+1}) = A_{ob}\hat{x}_{ob}(t_k) + K_{ob}e(t_k) \;, \quad y(t_k) = C_{ob}\hat{x}_{ob}(t_k) + e(t_k) \\ &A_{ob} = QAQ^{-1} \;, \quad C_{ob} = CQ^{-1} \;, \quad K_{ob} = QK \end{aligned} \tag{12}$$

The distribution of the innovations $e(t_k)$ is unaffected by this linear transformation, which means that they are still described by the covariance matrix Λ. The $m \times m$ non-singular transformation matrix $Q = \left[C^T \;\; (CA)^T \;\; . \;\; . \;\; (CA^{n-1})^T\right]^T$ is equal to the observability matrix of the state space system to be transformed with observability index n, see Gawronski et al. [13].

The transformation leads to the observability canonical state space realization, with A_{ob} defined as

$$A_{ob} = \begin{bmatrix} 0 & I & . & . & 0 & 0 \\ 0 & 0 & . & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & 0 & I \\ -A_n & -A_{n-1} & . & . & -A_2 & -A_1 \end{bmatrix} \tag{13}$$

which implies that there are n auto-regressive coefficient matrices A_i. These coefficient matrices can be extracted using the relation $[A_n \;\; A_{n-1} \;\; . \;\; . \;\; A_2 \;\; A_1] = CA^nQ^{-1}$. It is seen that the observability index n associated with the observation matrix C controls the order of the ARMAV model. The moving average parameters C_i can be extracted from K_{ob} using the following relation

$$\begin{bmatrix} C_1 \\ C_2 \\ . \\ C_n \end{bmatrix} = \begin{bmatrix} I & 0 & . & . & 0 & 0 \\ A_1 & I & . & . & 0 & 0 \\ . & . & . & . & . & . \\ A_{n-1} & A_{n-2} & . & . & A_1 & I \end{bmatrix} K_{ob} + \begin{bmatrix} A_1 \\ A_2 \\ . \\ A_n \end{bmatrix} \tag{14}$$

resulting in n moving average coefficient matrices. The ARMAV model will then given by

$$y(t_k) = -\sum_{i=1}^{n} A_i y(t_{k-i}) + e(t_k) + \sum_{i=1}^{n} C_i e(t_{k-i}) \tag{15}$$

The dimension of the state space system, and therefore also the order of the ARMAV model, is dependent upon the condition of the output measurements $y(t_k)$, the number of observed outputs of the system, and the total number of degrees of freedom present in the system. Also, the actual excitation of the structure, and the type of noise present in the system play an important role.

3.2 System identification using ARMAV models

In this section the estimation procedure will be a non-linear prediction error method, operating on the ARMAV model, defined in (15). The PEM estimate of the model is calculated by minimizing a quadratic loss function V of the prediction errors $\varepsilon(t_k)$, giving

$$V = \frac{1}{2}\frac{1}{N}\sum_{k=1}^{N} \varepsilon^T(t_k) W^{-1} \varepsilon(t_k) = \frac{1}{2}\frac{1}{N}\sum_{k=1}^{N} \left[y(t_k) - \hat{y}(t_k|t_{k-1})\right]^T W^{-1} \left[y(t_k) - \hat{y}(t_k|t_{k-1})\right] \tag{16}$$

where W is a weight matrix, and N is the number of output measurements. $\hat{y}(t_k|t_{k-1})$ is the one step-ahead predictor of the model, see Ljung [11]

$$\hat{y}(t_k|t_{k-1}) = -\sum_{i=1}^{n} A_i y(t_{k-i}) + \sum_{i=1}^{n} C_i \varepsilon(t_{k-i}) = \phi^T(t_k)\theta \tag{17}$$

$\phi^T(t_k)$ is the multivariate regression matrix of dimension $ny \times 2n \cdot ny^2$

$$\phi^T(t_k) = \varphi^T(t_k) \otimes I \,, \quad \varphi^T(t_k) = \left[-y^T(t_{k-1}) \;\; . \;\; . \;\; -y^T(t_{k-n}) \;\; \varepsilon^T(t_{k-1}) \;\; . \;\; . \;\; \varepsilon^T(t_{k-n})\right]^T \tag{18}$$

where I is an $ny \times ny$ identity matrix, and $\otimes$ is the Kronecker product. The $2n \cdot ny^2 \times 1$ parameter vector θ contains the stacked autoregressive coefficient matrices A_i and moving average coefficient matrices C_i, in the following way $\theta = col([A_1 \; \ldots \; A_n \; C_1 \; \ldots \; C_n])$. Here $col(\cdot)$ means stacking of all columns of the argument matrix on top of each other. The numerical search procedure is the damped Gauss-Newton, see Ljung [11], defined as

$$\theta^{(i+1)} = \theta^{(i)} + \mu \left[\frac{1}{N}\sum_{k=1}^{N} \psi(t_k) W^{-1} \psi^T(t_k)\right]^{-1} \left[\frac{1}{N}\sum_{k=1}^{N} \psi(t_k) W^{-1} \varepsilon(t_k)\right] \tag{19}$$

where the superscript $^{(i)}$ denotes iteration number and μ is a bisection constant. W is taken as the covariance matrix Λ of the previously estimated prediction errors. $\psi(t_k)$ is the gradient of the predictor $\hat{y}(t_k|t_{k-1})$.

Because of the complexity of the ARMAV model it is vital that $\psi(t_k)$ is calculated in a recursive manner. The jth element of $\psi(t_k)$ is obtained by differentiating (17) with respect to the jth element of θ, and can be formulated as a multivariate autoregressive process in the moving average

coefficient matrices as

$$\psi_j(t_k) = \frac{\partial \hat{y}(t_k|t_{k-1})}{\partial \theta_j} = -\sum_{i=1}^{n} C_i \frac{\partial \hat{y}(t_{k-i}|t_{k-i-1})}{\partial \theta_j} - \sum_{i=1}^{n} \frac{\partial A_i}{\partial \theta_j} y(t_{k-i}) + \sum_{i=1}^{n} \frac{\partial C_i}{\partial \theta_j} \varepsilon(t_{k-i}) \quad (20)$$

Principally, for each element of θ there is a corresponding multivariate autoregressive filter. But in practice there will only be $2ny^2$. The reason for this is, that the gradients of the elements of A_{i+1} and C_{i+1}, are the gradients of the elements of A_i and C_i at the previous time step.

The iterative search for a lower value of the loss function in (16) is started by supplying initial values to θ and Λ. The initial values are, in the present case, constructed using a two-stage iterative linear least-squares algorithm, see Piombo et al. [6]. In each iteration the prediction errors and gradients $\psi(t_k)$ are calculated, and the parameter vector θ is updated. The search gradient is bisected up to ten times in order to find a lower value of V. At the end of each iteration the prediction error covariance matrix Λ is updated, so it corresponds to the new prediction errors. The iterations stops when it is impossible to find a lower value of V, or if the maximum number of iterations is exceeded.

3.3 Converting the ARMAV model to state space

In order to use (10) and (11) for the calculation of the modal parameters and the spectral densities, it is necessary that the estimated ARMAV model is converted to innovation state space form. The multi-step predictor $\hat{y}(t_{k+m}|t_k)$ of the ARMAV model can be written in the following recursive form

$$\hat{y}(t_{k+m}|t_k) = \hat{y}(t_{k+m}|t_{k-1}) + \boldsymbol{h}(m)\boldsymbol{e}(t_k) \quad (21)$$

where $\boldsymbol{h}(m)$ is the impulse response function of the model at time step m. A special case of this equation is the one-step ahead predictor, which is obtained by setting m to zero and keeping in mind that $\hat{y}(t_k|t_k) = y(t_k)$. However, the predictor can also be expressed in terms of the auto-regressive coefficient matrices as

$$\hat{y}(t_{k+m}|t_k) = -\sum_{i=1}^{n} A_i \hat{y}(t_{k+m-i}|t_{k-1}) + \boldsymbol{h}(m)\boldsymbol{e}(t_k) \quad (22)$$

Define the state vector $\hat{x}(t_k)$ in terms of the predictor $\hat{y}(t_{k+m}|t_{k-1})$, for m=0 to n-1, see Hannan et al. [16], as

$$\hat{x}(t_k) = \left[\hat{y}^T(t_k|t_{k-1}) \;\; \hat{y}^T(t_{k+1}|t_{k-1}) \;\; . \;\; . \;\; \hat{y}^T(t_{k+n-2}|t_{k-1}) \;\; \hat{y}^T(t_{k+n-1}|t_{k-1})\right]^T \quad (23)$$

Let the system output be $y(t_k)$. The one-step ahead prediction can be extracted from the first element of $\hat{x}(t_k)$, giving rise to the following observation equation

$$y(t_k) = [I \;\; 0 \;\; . \;\; . \;\; 0]\hat{x}(t_k) + e(t_k) = C\hat{x}(t_k) + e(t_k) \quad (24)$$

with C being the observation matrix. By combining (21) and (22), with increasing prediction horizon, the following set of equations is obtained

$$\begin{aligned}
\hat{y}(t_{k+1}|t_k) &= \hat{y}(t_{k+1}|t_{k-1}) + \boldsymbol{h}(1) \\
\hat{y}(t_{k+2}|t_k) &= \hat{y}(t_{k+2}|t_{k-1}) + \boldsymbol{h}(2) \\
&\vdots \\
\hat{y}(t_{k+n-1}|t_k) &= \hat{y}(t_{k+n-1}|t_{k-1}) + \boldsymbol{h}(n-1) \\
\hat{y}(t_{k+n}|t_k) &= -A_1\hat{y}(t_{k+n-1}|t_{k-1}) - \ldots - A_n\hat{y}(t_k|t_{k-1}) + \boldsymbol{h}(n)
\end{aligned} \tag{25}$$

which in matrix notation is equal to

$$\hat{\boldsymbol{x}}(t_{k+1}) = A\hat{\boldsymbol{x}}(t_k) + K\boldsymbol{e}(t_k) \tag{26}$$

with A defined as in (13) and the Kalman gain $K=[\boldsymbol{h}^T(1)\ \boldsymbol{h}^T(2)\ \ldots\ \boldsymbol{h}^T(n\text{-}1)\ \boldsymbol{h}^T(n)]^T$, defined in terms of the impulse response matrices for $k = 1$ to n. Regarding the innovations, they are of course unaffected of the conversion and are still described by the covariance matrix Λ.

---

4. SYSTEM REALIZATION TECHNIQUES

This section briefly presents two state space system identification techniques which can be used to identify a system only from output measurements. The durability of these techniques for system identification of civil engineering structures are evaluated in Kirkegaard et al. [14]. The techniques rely on system realization theory and compared with ARMAV identification they have some computational advantages since solutions are found using numerical linear algebra, see e.g. Aoki [10], Van Overschee et al. [9] and Juang [7]. The results of the estimation techniques are the innovation state space triple $\{A, C, K\}$ and the innovation covariance matrix Λ, from which the modal parameters and the spectral densities can be obtained using (10) and (11).

4.1 Matrix Block Hankel stochastic realization estimator

The Matrix Block Hankel stochastic realization estimator is a covariance driven estimation technique. The present algorithm is based on Aoki [10], whereas the name of the technique is due to Hoen [8]. Based on (8), the covariance function of the output $\Sigma_k=E[\boldsymbol{y}(t_i)\boldsymbol{y}^T(t_{i+k})]$ at time lag k is

$$\Sigma_k = CA^kPC^T + CA^{k-1}K\Lambda = CA^{k-1}M\ ,\quad k>0 \quad \wedge \quad \Sigma_0 = CPC^T + \Lambda\ ,\quad k=0 \tag{27}$$

where P is the covariance matrix of the state vectors $E[\boldsymbol{x}(t_k)\boldsymbol{x}^T(t_k)]$, and $M=APC^T+K\Lambda$ is the cross-covariance $E[\boldsymbol{x}(t_{k+1})\boldsymbol{y}^T(t_k)]$. In the following, the state matrix A is assumed to be stable, which implies that $\boldsymbol{x}(t_k)$ is covariance stationary, i.e. that P is constant. In this case P satisfies the Ricatti equation, see Aoki [10]

$$P = APA^T + K\Lambda K^T = APA^T + (M - APC^T)(\Sigma_0 - CPC^T)^{-1}(M - APC^T)^T \tag{28}$$

where the definitions of M and Σ_0 in (27) have been used to express K and Λ. This implies that

having estimated the triple $\{A, C, M\}$ the Kalman gain matrix K and the innovation covariance matrix Λ can be determined from the positive definite solution of the Riccati equation. In order to solve the equation it is required that Λ has to be positive definite.

The estimation of the triple $\{A, C, M\}$ is based on a decomposition of a block Hankel matrix of the form

$$H_{jk}(p) = \begin{bmatrix} \Sigma_p & \Sigma_{p+1} & \cdot\ \cdot & \Sigma_{p+k-1} \\ \Sigma_{p+1} & \Sigma_{p+2} & \cdot\ \cdot & \Sigma_{p+k} \\ \cdot & \cdot & \cdot\ \cdot & \cdot \\ \Sigma_{p+j-1} & \Sigma_{p+j} & \cdot\ \cdot & \Sigma_{p+j+k-2} \end{bmatrix} \tag{29}$$

which for $p=1$ is the product of the observability matrix O_j and another matrix called Ω_k, defined respectively as

$$O_j = \left[C^T \ (CA)^T \ (CA^2)^T \ \cdot\ \cdot \ (CA^{j-1})^T\right]^T \quad \wedge \quad \Omega_k = \left[M \ AM \ A^2M \ \cdot\ \cdot \ A^{k-1}M\right] \tag{30}$$

First the singular value decomposition USV^T of the Hankel matrix $H_{jk}(1)$ is calculated. In this matrix the theoretical covariance matrices Σ_k of (27) have been replaced by sampled covariance matrices. By a proper choice of coordinate system an internal balanced estimate of the triple is then obtained

$$\hat{A} = S^{-½}U^T H_{jk}(2)VS^{-½}, \quad \hat{C} = H_{1k}(1)VS^{-½}, \quad \hat{M} = S^{-½}U^T H_{j1}(1) \tag{31}$$

The constants j and k will in general be equal to the state space dimension divided by the number of observed outputs. For minimal systems the estimates are unique, because the Gramians $O_j^T O_j$ and $\Omega_k\Omega_k^T$ are non-singular. What remains, in order to obtain a complete estimate of the triple $\{A, C, K\}$ and the innovation covariance matrix Λ, is to estimate K and Λ by the solution of the above Riccati equation. The minimality of the realization should always be checked, by verifying that the realization is both observable and reachable. It should also be checked that Λ has full rank and is positive definite. If this is not the case or if the realization is non-minimal, the state space dimension, and therefore also j and k, should be reduced. The spectral densities and the modal parameters can then be determined using (10) and (11).

4.2 Stochastic Subspace Technique

Subspace algorithms for identification of linear dynamic systems have recently been considered in a number of papers, see Van Overshee et al. [9] and De Moor et al. [15]. The main theorem of the subspace theory demonstrates how the Kalman filter states can be obtained from input-output data using linear algebra tools (QR and SVD). Once these states are known, the identification problem becomes a linear least-squares problem in the unknown matrix pair $\{A, C\}$ in (6). If the external input is unknown, a stochastic subspace technique (N4SID) is used to determine the system matrices. Compared to the stochastic realization methods in the previous section, the N4SID is data driven instead of covariance driven, so that the explicit formation of the covariance matrix is avoided. In the following the N4SID is briefly described based on Van Overschee et al. [9] and De Moor. [15].

In order to use the N4SID it is assumed that (27) is fulfilled and that the system in (6) is observable. The N4SID relies on output block Hankel matrices of the form

$$Y_{0|i-1} = \begin{bmatrix} y(0) & y(1) & . & . & y(j-1) \\ y(1) & y(2) & . & . & y(j) \\ . & . & . & . & . \\ y(i-1) & y(i) & . & . & y(i+j-2) \end{bmatrix} \tag{32}$$

where the first subscript denotes the time index of the upper left element, while the second subscript is the time index of the bottom left element. For all output block Hankel matrices, the number of columns will be j and for all theoretical derivations, it is assumed that $j \to \infty$.

An orthogonal projection Z_i of the row space of $Y_{i|2i-1}$ (the future) onto the row space of $Y_{0|i-1}$ (the past) is introduced

$$Z_i \equiv Y_{i|2i-1} \Big/ Y_{0|i-1} \tag{33}$$

By singular value decomposition of this projection it can be proved that

$$Z_i = \boldsymbol{Q}_i \hat{\boldsymbol{X}}_i \tag{34}$$

which is the product of the extended observability matrix $\boldsymbol{Q}_i$ and $\hat{\boldsymbol{X}}_i \equiv [\hat{\boldsymbol{x}}(t_i)\ \hat{\boldsymbol{x}}(t_{i+1})\ \ldots\ \hat{\boldsymbol{x}}(t_{i-j-1})]$, which is the Kalman state sequence. Further, it can be proved that another projection Z_{i+1} is defined as

$$Z_{i+1} \equiv Y_{i+1|2i-1} \Big/ Y_{0|i} \tag{35}$$

implying that

$$Z_i = \boldsymbol{Q}_{i-1} \hat{\boldsymbol{X}}_{i+1} \tag{36}$$

From (34) and (36) the Kalman states can be obtained from output data using singular value decomposition techniques which means that the matrix pair $\{\boldsymbol{A}, \boldsymbol{C}\}$ can be estimated from the following set of linear equations

$$\begin{bmatrix} \hat{\boldsymbol{X}}_{i+1} \\ \boldsymbol{Y}_{i|i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix} \hat{\boldsymbol{X}}_i + \begin{bmatrix} \rho_w \\ \rho_e \end{bmatrix} \tag{37}$$

where the last term concist of residual matrices. A numerically efficient way to solve this set of linear equations is described in Van Overschee et al. [9]. This algorithm also gives an estimate of $\boldsymbol{M}$ from which estimates of $\boldsymbol{K}$ and Λ can be obtained by the solution of the Riccati equation (28). The spectral densities and the modal parameters can then be calculated using (10) and (11).

5. EXAMPLE

In the following example the performance of the ARMAV prediction error estimator is compared with the performance of the two system realization estimators, described in the previous section. The reference system is a 4-DOF linear system excited by Gaussian white noise. The estimated modal parameters of all structural modes, and the autospectral densities of the estimates, will be compared with the ones of this reference system.

The reference system has been converted, using a sampling interval of 0.015 seconds, from a continuous-time description to a covariance equivalent discrete time ARMAV model, see Andersen et al. [4], from which 4000 samples have been simulated. Gaussian white noise has been added to the output of the system amounting to 10 % of the mean of the standard deviations of the simulated output. The eigenfrequencies and damping ratios of the system are listed in the second and third column of table 1, respectively.

The optimal ARMAV model has two autoregressive coefficient matrices and two moving average coefficient matrices. This model order is equivalent to a state space dimension of 8, which is also the optimal state space dimension for the N4SID estimator. However, the optimal state space dimesion for the MBH estimator is 12. This dimension results in the estimation of some spurious modes, which are not presented in the following.

The estimated eigenfrequencies and damping ratios are shown in columns four to nine in table 1. Both the eigenfrequencies and the damping ratios has been identified very well. Especially the ARMAV and the N4SID estimates are very close to the modal parameters of the reference system.In figure 1, the autospectral densities of all outputs of the three estimates are plotted together with the FFT autospectral densities of the simulated output measurements. It is seen that the spectral densities of the ARMAV and the N4SID estimates agrees very well, whereas the FFT and MBH spectral densities diviates. In each plot the peak around the first eigenfrequency has been zoomed in. In these plots the agreements between ARMAV and N4SID becomes even more clear.

All three estimators returns agreeable results. Especially the results obtained from the ARMAV and the N4SID estimators are of high quality. However, it should be noted that the analysis is based on simulated data. When used on real data the ARMAV prediction error estimator returns the most accurate results of these two estimators, see Kirkegaard et al. [14]. In this case the estimates obtained from the N4SID can be applied as excellent initial estimates for the PEM estimator.

6. CONCLUSION

In this paper it has been shown how to represent a multivariate second-order ambient excited structural system as a discrete-time innovation state space system. Based on this system it has then been considered how to represent it using an Auto-Regressive Moving-Average Vector (ARMAV) model.

The formulation of this model has provided an efficient recursive approach for the calculation of predictor gradient for use in a non-linear multivariate prediction error method (PEM). It has briefly been adressed how to estimate multivariate ambient excited systems directly in state space. Two estimators has been presented; the Matrix Block Hankel (MBH) stochastic estimator, and the stochastic subspace technique (N4SID).

The performance of the three multivariate estimators has been compared in a simulated example. The results of this example shows agreement of the modal parameter estimates of the three estimators. Especially the PEM and the N4SID algorithms reveals a high degree of agreement with the modal parameters of the reference system.

7. ACKNOWLEDGEMENTS

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	Reference		ARMAV		MBH		SST	
Mode	f_i	ζ_i	f_i	ζ_i	f_i	ζ_i	f_i	ζ_i
1	6.9553	4.37	6.9648	4.77	6.9387	5.11	6.9624	4.73
2	13.2298	8.31	13.2449	7.57	13.2687	7.75	13.2517	7.52
3	18.2093	11.44	18.1630	11.03	17.8075	10.99	18.1810	11.22
4	21.4063	13.45	21.5220	13.05	21.3812	13.06	21.4724	12.77

Table 1: Reference and estimated natural eigenfrequencies f_i, [Hz], *and damping ratios ζ_i,* [%].

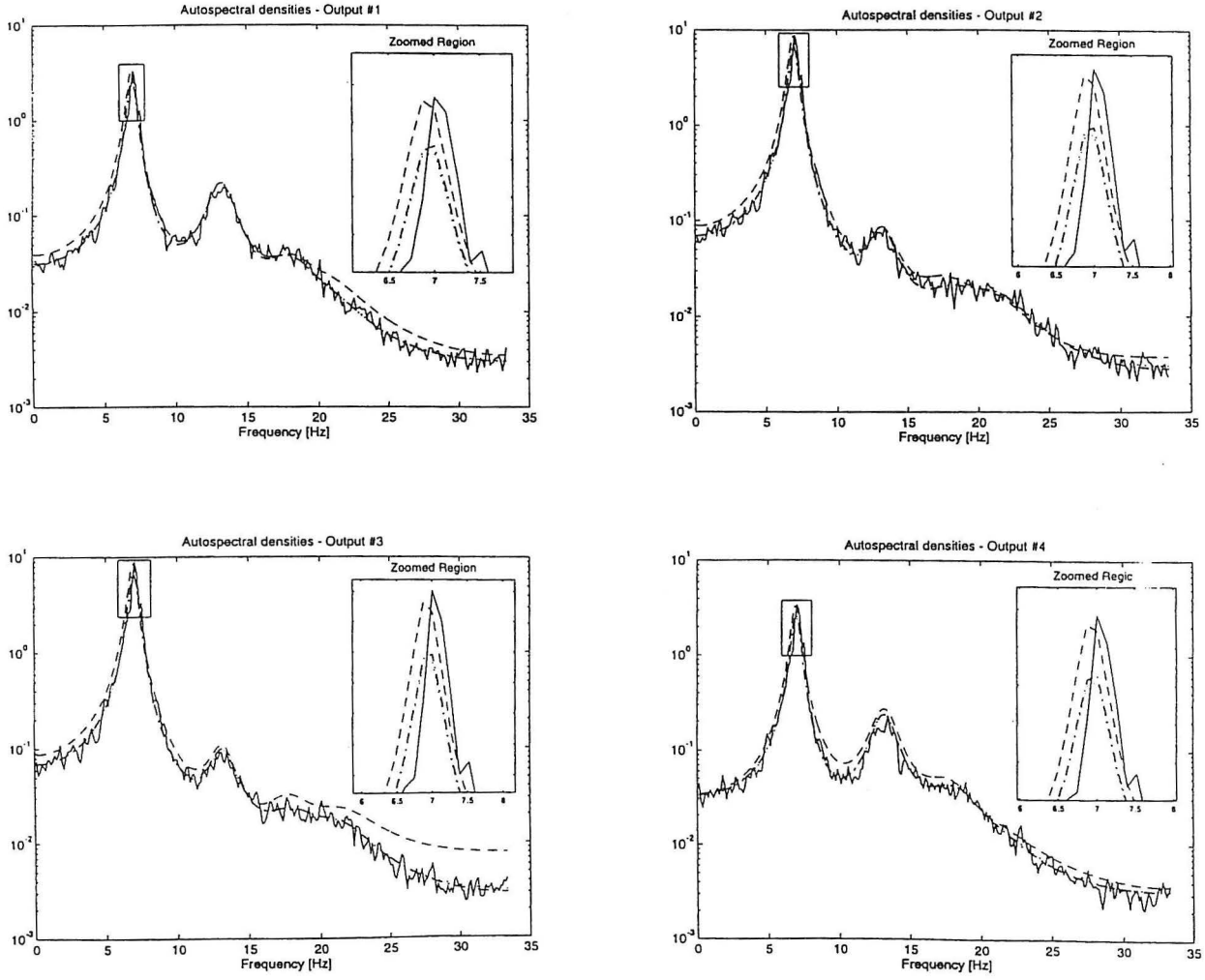

Figure 1: Autospectral densities of all four output. The legends are:
FFT: [_____], ARMAV: [_ · _ · _], MBH: [_ _ _ _], and SST: [. . . . .].

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Department of Building Technology and Structural Engineering
Aalborg University, Sohngaardsholmsvej 57, DK 9000 Aalborg
Telephone: +45 98 15 85 22 Telefax: +45 98 14 82 43